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Do *H*-functions always increase during Violent Relaxation?

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Abstract. Recent work on the violent relaxation of collisionless stellar systems has been based on the notion of a wide class of entropy functions. A theorem concerning entropy increase has been proved. We draw attention to some underlying assumptions that have been ignored in the applications of this theorem to stellar dynamical problems. Once these are taken into account, the use of this theorem is at best heuristic. We present a simple counter-example.

Key words: collisionless stellar systems, violent relaxation—entropy increase

1. Introduction

A long-standing and well-known problem concerns the distribution of light (and hence of stars) in elliptical galaxies. The stars seem to be so smoothly distributed that it is widely believed that these systems are 'relaxed'. There is very little gas in ellipticals and the dynamics is governed by the mutual gravitational attraction between stars. So it is natural to look for some relaxation process that leads self-gravitating systems to some equilibrium state.

Just like the motion of molecules in gases, the motion of stars in galaxies is described by the Boltzmann equation. There are so many stars in a galaxy that the time over which collisions are effective turns out to be much longer than the galaxy's age. So galaxies are well-approximated as collisionless stellar systems and any relaxation process should be collisionless. Lynden-Bell (1967) suggested that if the stars were initially distributed in a state very far from equilibrium, large fluctuations in the galaxy's mean field could scatter stars, allow them to exchange energy and equilibrate. He called this 'violent relaxation'.

In the kinetic theory of gases, the form of the collision term in the Boltzmann equation singles out Boltzmann's *H*-function as the unique function that increases with time. While the Boltzmann equation is inherently irreversible in nature, the equation that describes galaxies does not have the collision term and is symmetric under time-reversal. Tremaine, Henon & Lynden-Bell (1986, hereinafter THL) have addressed the problem of entropy increase for collisionless stellar systems during processes like violent relaxation. THL consider entropy functions of a more general form than Boltzmann's *H*-function. They have proved a theorem concerning entropy increase. We disagree with their interpretation and its application to stellar dynamics. We give a summary of THL's proof of their theorem in Section 2 and discuss the assumptions in

Section 3. A simple example that invalidates THL's interpretation is presented in Section 4.

2. The *H*-theorem of THL

The time evolution of collisionless stellar systems is governed by the collisionless Boltzmann equation (CBE):

$$\frac{\mathrm{d}f}{\mathrm{d}t}\equiv\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}}-\frac{\partial \phi}{\partial \mathbf{x}}\cdot\frac{\partial f}{\partial \mathbf{v}}=0 \tag{1}$$

where $f(\mathbf{x}, \mathbf{v}, t)\,\mathrm{d}x\,\mathrm{d}v$ is the mass in phase volume $\mathrm{d}x\,\mathrm{d}v$ and ϕ is the mean gravitational potential

$$\phi(\mathbf{x}, t)=-G\int\frac{f(\mathbf{x}', \mathbf{v}', t)}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}v'. \tag{2}$$

Unlike the Boltzmann equation for gases, the CBE is symmetric under time-reversal. Very few properties of the time-dependent and nonlinear CBE are known. So any general result is very useful.

THL define H-functions as

$$H[f]=-\int C(f)\,\mathrm{d}x\,\mathrm{d}v \tag{3}$$

where C is a convex function with $C(0)=0$. Time evolution governed by the CBE conserves phase volumes and densities. So the contribution of any phase 'element' of density f and volume $\mathrm{d}x\,\mathrm{d}v$ to H does not change with time. Therefore $H[f]$ is conserved in time:

$$\frac{\mathrm{d}}{\mathrm{d}t}H[f]=0. \tag{4}$$

Incidentally, the convex property of C is not necessary for (4) to be true; it is enough that C is a function of f alone.

THL partition phase space into macrocells each of volume $\Delta x \Delta v$. They average $f(\mathbf{x}, \mathbf{v}, t)$ over these phase volumes to obtain a coarse-grained distribution function $F(\mathbf{x}, \mathbf{v}, t)$. The convex property of C implies that at any time

$$H[F]\geqslant H[f] \tag{5}$$

regardless of how exactly phase space is partitioned. To prove their theorem THL assume that the system was prepared in a state for which $f=F$ at some initial time $t=t_1$. This implies that

$$H[F(t_1)]=H[f(t_1)]. \tag{6}$$

At some later time $t_2>t_1$, $F(t_2)\neq f(t_2)$. So, from (5) we have

$$H[F(t_2)]\geqslant H[f(t_2)]. \tag{7}$$

From (4) we know that

$$H[f(t_2)]=H[f(t_1)] \tag{8}$$

so

$$H[F(t_2)]\geqslant H[F(t_1)]. \tag{9}$$

THL argue that inequality (9) implies that $H(t)$ is a monotonically increasing function of t and that 'the direction of the arrow of time has been determined by the assumption that $f=F$ at $t=t_1$'. They use their theorem as a basic criterion that must be satisfied by the evolution of stellar systems during processes like violent relaxation and their paper contains many examples. Further, they prove a theorem on mixing during violent relaxation. We discuss their proof and applications in the next section.

3. Comments on the H-theorem of THL

Although we agree with THL's derivation of (9), we disagree with their interpretation and applications of the same theorem. It is not clear to us that the particular choice of initial state $f=F$ is the natural one and we note that the validity of inequality (9) hinges on the assumption that $f=F$ at $t=t_1$. *Even if* $f=F$ is a reasonable initial state, we disagree with THL's conclusions that by deriving (9) they have proved that $H(t)$ increases monotonically with t (for brevity we use $H(t)$ to denote $H[F(t)]$). We note that $H(t_3) \geqslant H(t_1)$ and $H(t_2) \geqslant H(t_1)$ for $t_3 > t_2 > t_1$, *do not* imply that $H(t_3) \geqslant H(t_2)$. In other words, inequality (9) does not imply that $H(t)$ is monotonically increasing function of t, even though H takes its smallest value at $t=t_1$, and this is only because of the assumption $f=F$ at $t=t_1$. We have shown that there is no arrow of time in the problem.

We also wish to point out that if the evolution of the system was collisionless for $t<t_1$, THL's proof of (9) would imply that $H(t) \geqslant H(t_1)$ for $t<t_1$ also. Therefore the

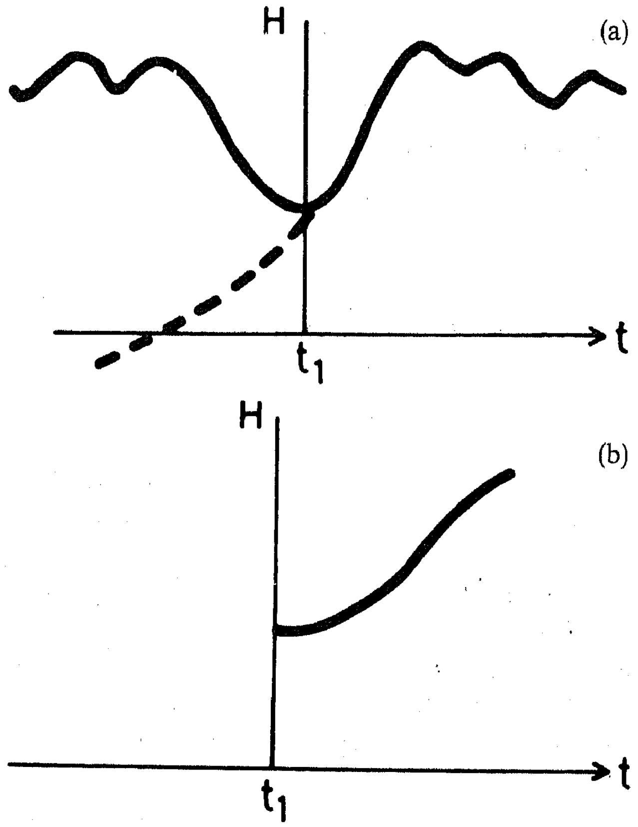

Figure 1. General behaviour of H as a function of time; (a) the solid curve is for a system showing unmixing behaviour and the dashed curve is for a system showing mixing for $t<t_1$, while (b) shows H for a mixing system.

general behaviour of H is like the solid curve in Fig. 1a rather than Fig. 1b as claimed by THL. THL do not discuss $H(t)$ for $t<t_1$. They assume that the system was formed at t_1. If the processes that brought about the special state $f=F$ were collisional (*e.g.* the formation of the stellar system from gas clouds) then the general form of H could be like the dashed curve in Fig. 1a for $t<t_1$. For $t>t_1$, H is still the solid curve. In the next section we present a simple model which might clarify the issues involved.

4. A model showing unmixing behaviour*

Let us consider an ensemble of anharmonic oscillators each of whose members is described by a Hamiltonian

$$E=E(J) \tag{10}$$

J, θ are action-angle variables. The equations of motion are

$$\dot{J}=0,\ \dot{\theta}=\frac{\mathrm{d}E}{\mathrm{d}J}. \tag{11}$$

The evolution in time of this ensemble is shown in Fig. 2 where $t_c>t_b>t_a$. It is assumed that $\mathrm{d}E/\mathrm{d}J>0$. The precise functional form of $E(J)$ is unimportant. The members of the ensemble are distributed in phase-space (*i.e.* the J–θ plane) with density $f(J, \theta, t)$. For simplicity we define coarse-graining as integration over J to get the coarse-grained distribution function

$$F(\theta, t)=\int_0^{\infty} f(J, \theta, t)\ \mathrm{d}J. \tag{12}$$

THL's coarse-graining scheme applied to our ensemble would involve averaging over some (macro)cells with some spread in both J and θ, while we have chosen to integrate over J to get the coarse-grained function. The purpose of this example is to show that H does not monotonically increase with t, *i.e.* unmixing behaviour. The differences between THL's method and our method of coarse-graining will not affect our conclusions.

H-functions are defined in a straightforward manner

$$H(t)=-\int_{-\pi}^{\pi} C[F(\theta, t)]\ \mathrm{d}\theta \tag{13}$$

where $C(F)$ is a convex function. The Hamiltonian evolution of the ensemble preserves phase volumes and densities just like the collisionless evolution of stellar systems. $H(t)$ defined in (13) takes its smallest value at $t=t_a$ when the system is in a specially prepared initial state. This is analogous to the special initial state $f=F$ at $t=t_1$ in THL's paper. It is easily verfied from Fig. 2b and Fig. 2c that $H(t_b)>H(t_a)$ and $H(t_c)>H(t_a)$ where $t_c>t_b>t_a$. This is analogous to THL's derivation of (9). We wish to point out that our model shows unmixing behaviour: $H(t_c)<H(t_b)$ even though $t_c>t_b$. This is easily verified from the functional form of F sketched in Fig. 2b and c (we note that $\int_{-\pi}^{\pi} F(\theta, t)\ \mathrm{d}\theta$ is conserved in time). So $H(t)$ is not a monotonically increasing function of t. This is a special example, but enough to show that the general form of $H(t)$ is like the solid curve in Fig. 1a rather than Fig. 1b.

* This example is due to Rajaram Nityananda.

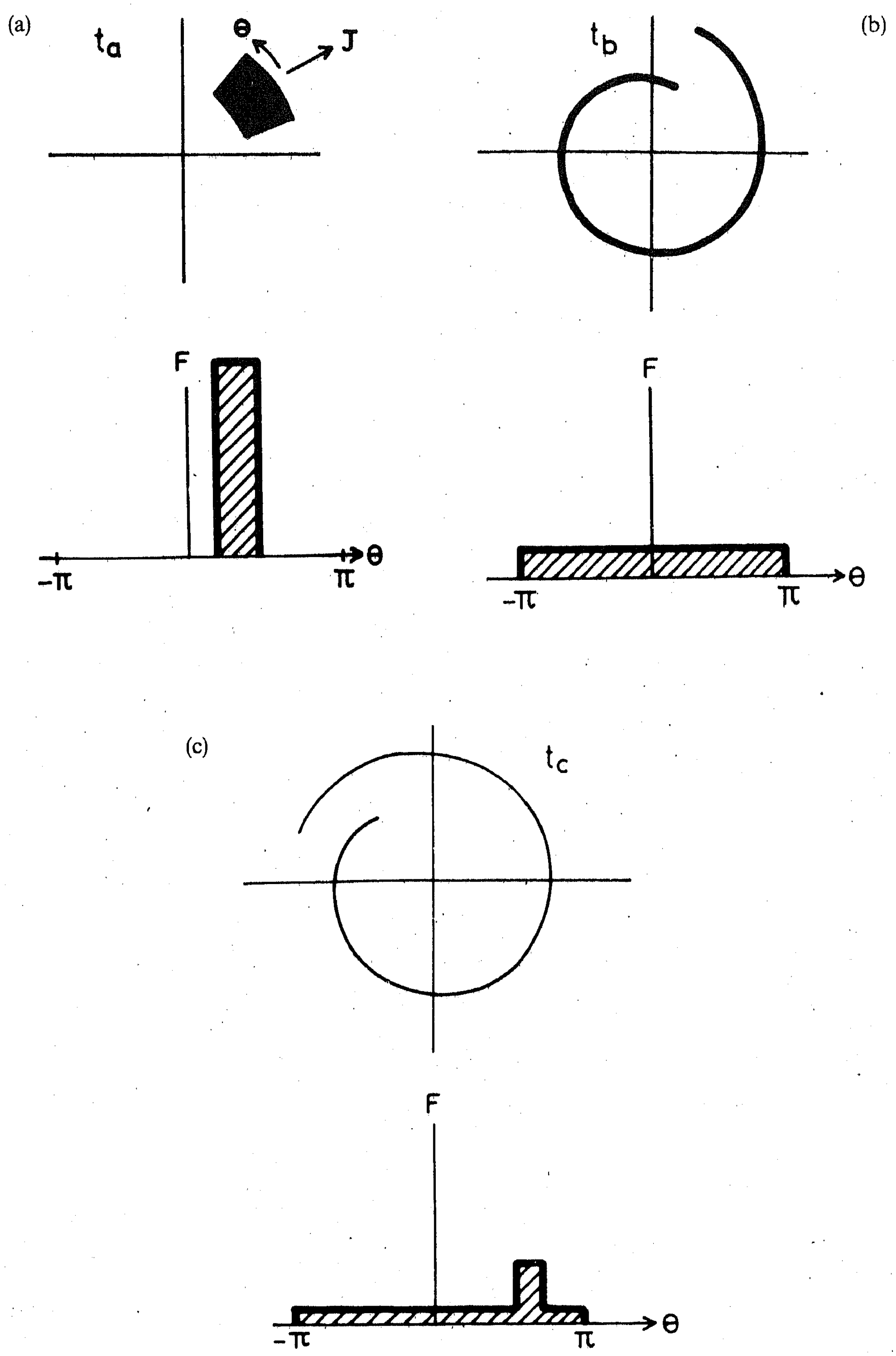

Figure 2. Evolution in time of an ensemble of anharmonic oscillators; (a) the specially prepared initial state at t_a for which H is minimum, (b) ensemble at t_b for which H is maximum, (c) ensemble at a later time t_c for which H takes an intermediate value.

5. Conclusions

THL state that all the properties of H-functions presented in their paper are well known and that only the applications to stellar dynamics is new. In our opinion THL's applications of their theorem to stellar dynamics is questionable to the extent that they have not proved that $H(t)$ increases monotonically with t even for their special choice of initial state. The situation is entirely parallel to attempts made early in this century (Tolman 1938; Ehrenfest 1912) to discuss entropy increase from Liouville's equation (which is a collisionless equation in $6N$ dimensions). All these discussions show that the general behaviour of H must be like Fig. 1a (solid curve). But this does not detract from the practical usefulness of the principle and the same could be true in the present case. Unmixing behaviour like Fig. 2 may be transient and rare but this is an additional hypothesis.

Independently of this work de Jonghe (1987) has pointed out that $H(t)$ need not be a monotonic function and noticed the difficulties with the THL interpretation of their entropy theorem.

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