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Spin glasses in the limit of an infinite number of spin components

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We consider spin glass models in which the number of spin components m is infinite. In the formulation of the problem appropriate for numerical calculations proposed by several authors, we show that the order parameter defined by the long-distance limit of the correlation functions is actually zero and there is only "quasi-long-range order" below the transition temperature. Nonetheless, there can be a finite temperature phase transition where the decay of correlations changes from exponential to power law. We also show that the spin glass transition temperature is zero in three dimensions so power-law behavior only occurs at $T=0$ in this case. We also argue that the order of limits, $m\to\infty$ and $N\to\infty$ is important, where N is the number of spins.

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I. INTRODUCTION

It is of interest to study a spin glass (SG) model in which the number of spin components m is infinite, because it provides some simplifications compared with Ising ($m=1$) or Heisenberg ($m=3$) models. For example, in mean field (MF) theory (i.e., for the infinite range model) there is no "replica symmetry breaking" [1] so the ordered state is characterized by a single order parameter q, rather than by an infinite number of order parameters [encapsulated in a function $q(x)$] which are needed [2] for finite m.

There has recently been renewed interest [3,4] in the $m=\infty$ model, and the interesting result emerged from these studies that the *effective* number of spin components depends on the system size N and is only really infinite in the thermodynamic limit. One motivation for the present study is to investigate some consequences of this result.

Further motivation for our present study comes from earlier work by two of us [5] which argued that the isotropic *XY* ($m=2$) and Heisenberg spin glasses have a finite spin glass transition temperature T_{SG} in three dimensions, like the Ising spin glass. The results of Ref. [5] also indicate that T_{SG} is very low compared with the mean field transition temperature T_{SG}^{MF} and *decreases* with increasing m; see Table I. The data in Table I suggest that T_{SG}/T_{SG}^{MF} may be zero in the $m=\infty$ limit in three dimensions, and we investigate this possibility here.

In this paper, we study the $m=\infty$ SG model; both the infinite range version and the short-range model in three and two dimensions. We find that we need to carefully specify the order in which the limits $m\to\infty$ and the thermodynamic limit $N\to\infty$ are taken. In Ref. [1], the $N\to\infty$ limit is taken first (since a saddle-point calculation is performed) and the $m\to\infty$ limit is taken at the end. However, in the formulation of the $m=\infty$ problem which has been proposed for numerical implementation in finite dimensions [3,4,8,9] and which we use here, the limit $m\to\infty$ is taken first for a lattice of finite size. In the latter case, we find that for $T<T_{SG}$ the spin glass correlations decay with a *power* of the distance r and tend to zero for $r\to\infty$, so the order parameter, defined in terms of the long-distance limit of the correlation function, is actually zero. Nonetheless, there can still be a transition at T_{SG} separating a high temperature phase, where the correlations decay exponentially, from the low temperature phase where they decay with a power law. In particular for the infinite range model $T_{SG}=T_{SG}^{MF}=1$. By contrast, if one takes $N\to\infty$ first with m finite, the power law decay eventually changes to a constant (of order of $1/m$) at large r and so, multiplying by m, a nonzero spin glass order parameter can be defined, which is equivalent to that of Ref. [1].

Overall we conclude that to obtain sensible physical results, the limit $N\to\infty$ should be taken *first*. As a result, the approach of Refs. [3,4,8,9], which performs the $m\to\infty$ limit first, should be considered as the zeroth order term in a $1/m$ expansion which needs to be resummed [10] in order to get results for large but *finite m*. The latter would avoid the inconsistencies in the strictly $m=\infty$ results.

We give phenomenological arguments for these conclusions and back them up (for the case where $m\to\infty$ is taken first) by numerical results at zero temperature. We also find, from numerical results at finite temperature, that $T_{SG}/T_{SG}^{MF}=0$ in three dimensions for $m=\infty$, consistent with the trend of the results in Table I.

In Sec. II we discuss the model and the methods used to study it numerically. In Sec. III we describe our results at

TABLE I. Estimates of the spin glass transition temperature, relative to the mean field value, $T_{SG}^{MF}=\sqrt{z}/m$, see Eq. (2), for different values of m for the three-dimensional simple cubic lattice ($z=6$). The factor of $1/m$ in T_{SG}^{MF} appears because the spins were normalized to unity in Refs. [5–7], rather than to $m^{1/2}$ as here. For the model used in this paper, T_{SG}^{MF} is *finite* for $m\to\infty$.

m	Model	T_{SG}^{MF}	T_{SG}	T_{SG}/T_{SG}^{MF}
1	Ising	2.45	0.97[6,7]	0.40
2	*XY*	1.22	0.34 [5]	0.28
3	Heisenberg	0.82	0.16 [5]	0.20

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$T=0$ for both short-range and the infinite-range model, while in Sec. IV we describe finite temperature results for short-range models. Our conclusions are summarized in Sec. V.

II. MODEL AND METHOD

We take the Edwards-Anderson [11] Hamiltonian

$$\mathcal{H} = -\sum_{\langle i,j\rangle} J_{ij}\mathbf{S}_i \cdot \mathbf{S}_j, \tag{1}$$

where the spins $\mathbf{S}_i$ $(i=1,\ldots,N)$ are classical vectors with m components and normalized to length $m^{1/2}$, i.e., $\mathbf{S}_i^2=m$. As we shall see, this normalization is necessary to get a finite mean-field transition temperature in the $m=\infty$. The J_{ij} are independent random variables with a Gaussian distribution with zero mean. We consider both the infinite range model and short-range models with nearest-neighbor interactions in two and three dimensions. For the infinite range model, the standard deviation is taken to be $1/\sqrt{N-1}$ while for the short-range models the standard deviation is set to be unity. According to the mean field approximation, the spin glass transition temperature is

$$T_{SG}^{MF} = \frac{\langle \mathbf{S}_i^2\rangle}{m}\left[\sum_j J_{ij}^2\right]_{\text{av}}^{1/2}, \tag{2}$$

where $[\cdots]_{\text{av}}$ indicates an average over the disorder. Hence, for the infinite range model (where mean field theory is exact), Eq. (2) gives $T_{SG}=T_{SG}^{MF}=1$, while for the short-range case it gives $T_{SG}^{MF}=\sqrt{z}$, where z is the number of nearest neighbors (four for the square lattice and six for the simple cubic lattice).

As shown in other work [3,4,8,9], the problem can be simplified for $m=\infty$. The spin-spin correlation function,

$$C_{ij} \equiv \frac{1}{m}\langle \mathbf{S}_i \cdot \mathbf{S}_j\rangle, \tag{3}$$

is given by

$$T^{-1}C_{ij} = (A^{-1})_{ij}, \tag{4}$$

where

$$A_{ij} = H_i\delta_{ij} - J_{ij}, \tag{5}$$

and the H_i have to be determined self consistently to enforce (on average) the length constraint on the spins,

$$C_{ii} = 1. \tag{6}$$

Angular brackets, $\langle\cdots\rangle$, refer to a thermal average for a given set of disorder. Equation (6) with $i=1,\ldots,N$ represents N equations which have to be solved for the N unknowns H_i. In Sec. IV we will solve these equations numerically for a range of sizes at finite temperature. We emphasize that in Eqs. (3)–(6) the limit $m\rightarrow\infty$ has been taken with N finite. This is the opposite order of limits from that in the analytical work of Ref. [1] where $N\rightarrow\infty$ was taken before $m\rightarrow\infty$. As we shall see, the results from the two orders of limits are different.

Equations (3)–(6) are not well defined at $T=0$. However, Aspelmeier and Moore [4] pointed out that one can solve the $m=\infty$ problem *directly* at $T=0$, using the following method. At zero temperature there are no thermal fluctuations so each spin lies parallel to its local field, i.e.,

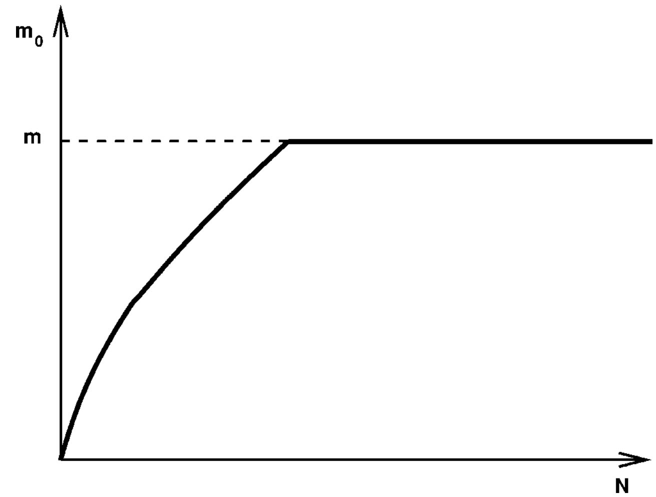

FIG. 1. A plot of the average effective number of spin components, m_0, as a function of system size N for a fixed, finite number of spin components m. For small N, $m_0 \sim N^{\mu}$, but once m_0 hits the actual number of spin components m, it sticks at m as N is further increased.

$$\mathbf{S}_i = H_i^{-1}\sum_j J_{ij}\mathbf{S}_j, \tag{7}$$

where $m^{1/2}H_i$ is the magnitude of the local field on site i. Remarkably, it was shown by Hastings [3] that these local fields are precisely the zero temperature limit of the H_i in Eq. (5). Hastings [3] also showed that the average number of independent spin components which are nonzero in the ground state, which we call m_0, cannot be arbitrarily large, but satisfies the bound

$$m_0 < \sqrt{2N}. \tag{8}$$

This means that one can always perform a global rotation of the spins such that only m_0 components have a nonzero expectation value and the remaining $m-m_0$ components vanish. Thus one can think of m_0 as the *effective number of spin components*. If m is finite, then, at some value of N, m_0 would equal the actual number of spin components m. At this point, all spin components are used so m_0 "sticks" at the value m as N is further increased; see Fig. 1.

More generally we can write Eq. (8) as

$$m_0 \sim N^{\mu} \quad (m_0 < m) \tag{9}$$

and the bound in Eq. (8) gives $\mu \leqslant 1/2$. Later, we will determine μ numerically for several models. For Eqs. (3)–(6) to be valid we need $m > m_0$ which corresponds to the curved part of the line in Fig. 1. As discussed above, this corresponds to taking the limit $m\rightarrow\infty$ *first*, followed by the limit $N\rightarrow\infty$. Since m_0 increases with N one needs larger values of m for larger lattice sizes. This will be important in what follows.

We therefore see that we can numerically solve the $m=\infty$ problem at $T=0$ on a finite lattice by taking a number of spin components which is *finite* but greater than m_0, and solving Eqs. (7). To do this we cycle through the lattice, and at site i, say, we calculate H_i from

$$H_i = \frac{1}{m^{1/2}}\left|\sum_j J_{ij}\mathbf{S}_j\right|. \tag{10}$$

we then set $\mathbf{S}_i$ to the value given by Eq. (7) so it lies parallel to its instantaneous local field. This is repeated for each site i, and then the whole procedure is iterated to convergence. Although spin glasses with finite m have many solutions of Eqs. (7), it turns out that for $m=\infty$ (in practice this means $m>m_0$) there is a unique stable solution [12], so the numerical solution of Eqs. (7) is straightforward. We will discuss our numerical results at $T=0$ using Eqs. (7) in Sec. III, and here we simply note that we do indeed find a unique solution of these equations.

Next we consider the order parameter in spin glasses for $m=\infty$. In the absence of a symmetry breaking field, one defines the long range order parameter q by the behavior of the spin-spin correlation function $[C_{ij}^2]_{\rm av}$ at large distances, i.e.,

$$q^2 = \lim_{R_{ij}\to\infty}[C_{ij}^2]_{\rm av} \quad \text{(short range)}, \tag{11}$$

where $R_{ij}=|\mathbf{R}_i-\mathbf{R}_j|$. For the infinite-range model, any distinct pair of sites will do, and so

$$q^2 = [C_{ij}^2]_{\rm av}\ (i\neq j) \quad \text{(infinite range)}. \tag{12}$$

We now give phenomenological arguments, which will be supported by numerical data in Sec. III, that q obtained from Eqs. (11) and (12), in which C_{ij} is determined by Eqs. (3)–(6), is actually *zero* for $m=\infty$, and that, at best, spin correlations have only "quasi-long-range order." For the short range case, this means that $[C_{ij}^2]_{\rm av}$ decays with a power of the distance R_{ij}, while for the infinite range model the correlation function in Eq. (12) tends to zero with a power of N.

To see why this is the case, we take $T=0$ and consider first the infinite-range model. For a given N, the spins "splay out" in $m_0\sim N^{\mu}$ directions. We expect the spins to point, on average, roughly equally in all directions in this m_0-dimensional space. Now C_{ij} in Eq. (3) is equal to $\cos\theta_{ij}$ where θ_{ij} is the angle between $\mathbf{S}_i$ and $\mathbf{S}_j$. We take the square and average equally over all directions. To do the average, take a coordinate system with the polar axis along $\mathbf{S}_i$, so $\theta_{ij}=\theta_j$ the polar angle of $\mathbf{S}_j$. Then we have

$$q^2 = [C_{ij}^2]_{\rm av} = \langle\cos^2\theta_j\rangle = \frac{1}{\mathbf{S}^2}\langle S_z^2\rangle \sim \frac{1}{m_0}\frac{\sum_{\alpha=1}^{m_0}\langle S_\alpha^2\rangle}{\mathbf{S}^2} = \frac{1}{m_0}\sim N^{-\mu}, \tag{13}$$

where we used the result that the average is roughly the same for all the m_0 spin components. Since μ will turn out to be nonzero it follows that *the order parameter tends to zero* with a power of the size of the system. The same will be true at temperatures $T<T_{SG}$, while above T_{SG} the order parameter as defined here will vanish faster, as $1/N$.

How can we reconcile this vanishing order parameter with earlier results [1] that the order parameter in the infinite-range model is nonzero below $T_{SG}=1$, and in particular is unity at $T=0$. The difference comes in part because q^2 in Ref. [1], which we call $q_{\rm AJKT}^2$, is m times our q^2, and so

$$q_{\rm AJKT}^2 = mq^2 \sim \frac{m}{m_0} \quad (T=0). \tag{14}$$

Note that since we take $m\to\infty$, the difference between q and $q_{\rm AJKT}$ is not just a "trivial" scale factor. More precisely, since m is taken to infinity before $N\to\infty$, m_0 is just an N-dependent constant and the right-hand side of Eq. (14) *does not exist* with this order of limits. However, Ref. [1] performs the limit $N\to\infty$ first, for which $m_0=m$, see Fig. 1, and the limit of Eq. (14) is well defined if m is then allowed to tend to infinity. Hence the difference between our results and those of Ref. [1] is that they take $N\to\infty$ first, whereas we take $m\to\infty$ first.

Going back to the calculation of C_{ij}, if one sums $[C_{ij}^2]_{\rm av}$ for the infinite range model over all pairs of sites we find that the spin glass susceptibility $\chi_{\rm SG}$ at $T=0$ is given by

$$\chi_{\rm SG} = \frac{1}{N}\sum_{i,j}[C_{ij}^2]_{\rm av} = 1+(N-1)q^2 \simeq Nq^2 \sim N^{1-\mu}. \tag{15}$$

Turning now to the short-range case, we expect that $\chi_{\rm SG}\sim N^{1-\mu}$ will still be true, which implies that correlations decay with a power of distance. Assuming that $[C_{ij}^2]_{\rm av}\sim 1/R_{ij}^y$ for some exponent y, then integrating over all $\mathbf{r}$ up to $r=L$ (where $N=L^d$) and requiring that the result goes as $N^{1-\mu}$, gives $y=d\mu$, i.e.,

$$[C_{ij}^2]_{\rm av} \sim \frac{1}{R_{ij}^{d\mu}}. \tag{16}$$

Such power law decay is often called quasi-long-range order. We expect that Eq. (16) will be true quite generally at $T=0$ and everywhere below T_{SG} if $T_{SG}>0$. Note that this implies that $q=0$ according to Eq. (11). Above T_{SG}, $[C_{ij}^2]_{\rm av}$ will decay to zero exponentially with distance.

If m is large but finite, then $[C_{ij}^2]_{\rm av}$ will saturate when R_{ij} is sufficiently large that all the spin components are used. This happens when $[C_{ij}^2]_{\rm av}\sim 1/m$, i.e., for $R_{ij}\gtrsim m^{1/d\mu}$. In this case, $q_{\rm AJKT}^2=mq^2$ will be finite according to Eq. (11). Although, according to our definition, q is always zero, there can be a finite temperature transition at $T=T_{SG}$ which separates the region $T>T_{SG}$ where correlations decay exponentially, from the region $T<T_{SG}$ where correlations decay with a power law. Mathematically this is the same behavior as occurs in the Kosterlitz-Thouless-Berezinskii theory of the two-dimensional XY ferromagnet.

In Secs. III A and III B we will provide numerical support for Eq. (15) for the infinite-range and short-range cases, respectively.

III. RESULTS AT ZERO TEMPERATURE

A. Infinite range model

We consider a range of lattice sizes up to $N=2048$ and for each size the number of samples is shown in Table II.

TABLE II. Number of samples used in the $T=0$ studies of the infinite range model.

N	N_{samp}
32	1000
64	1000
128	1000
256	1000
512	1000
1024	777
2048	302

The average number of nonzero spin components in the ground state is given by Eq. (9), for which it has been shown that [3,4]

$$\mu = 2/5 \quad \text{(infinite range)} \tag{17}$$

exactly. This result has been confirmed numerically [4]. Our results for μ are shown in Fig. 2 and indeed give μ close to $2/5$. The small deviation is presumably due to corrections to scaling.

We also calculated q^2 at $T=0$ from Eq. (12). In Eq. (3) the thermal average, $\langle \cdots \rangle$, is unnecessary, and the spin directions are determined by solving Eqs. (7) and (10). The results for are shown in Fig. 3, showing that it vanishes with exponent $-\mu$ as a function of N, as expected from Eq. (13).

B. Short-range models

First of all we describe our results for three dimensions. The number of samples is shown in Table III.

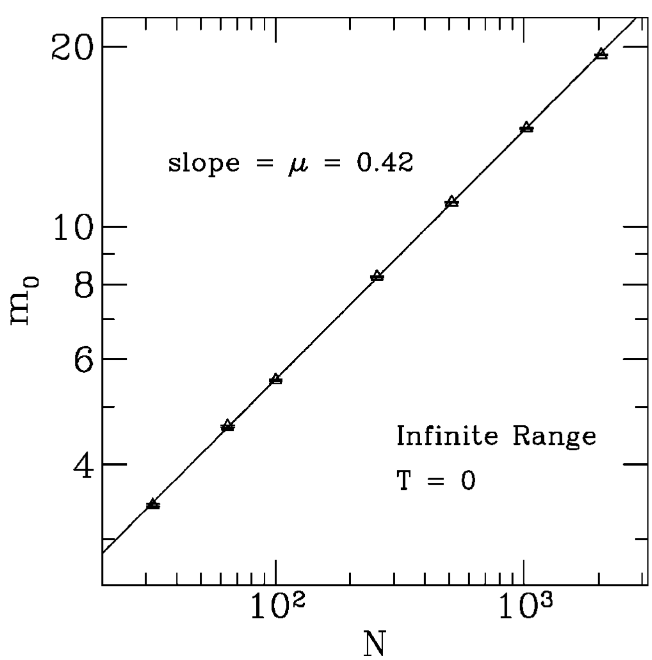

FIG. 2. The average number of nonzero spin components in the ground state m_0 as a function of N for the infinite range model. We see that m_0 increases like N^{μ} with μ close to $2/5$ as expected.

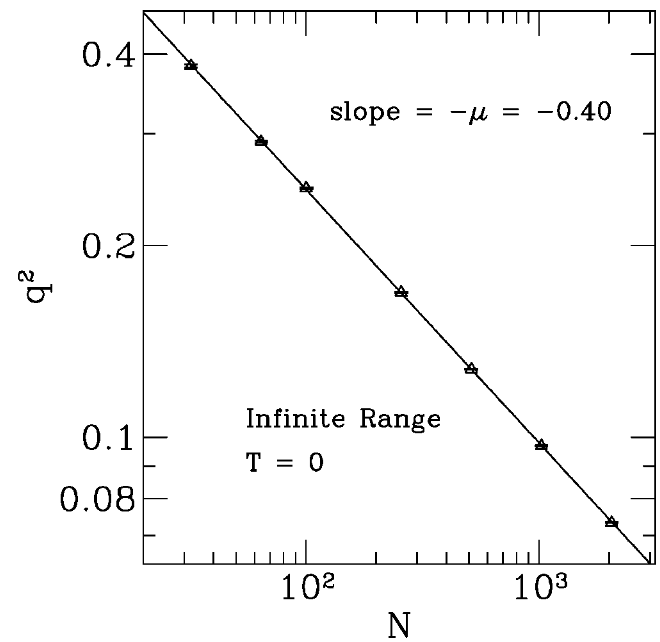

FIG. 3. The square of the order parameter at $T=0$ for infinite range model. As expected, it decreases like $N^{-\mu}$ with $\mu=2/5$.

Our results for μ are shown in Fig. 4, indicating that $\mu \simeq 0.33$, definitely different from the infinite range result of $2/5$. The results for χ_{SG} as a function of N are shown in Fig. 5. We see that χ_{SG} grows with an exponent $1-\mu$ with the same value of μ as in Fig. 4. We therefore find that $d\mu \simeq 1.0$, and so, from Eq. (16), the spin glass correlations decay as

$$[C_{ij}]_{\mathrm{av}}^2 \sim \frac{1}{R_{ij}} \quad (d=3, T=0). \tag{18}$$

(It is of course possible that power of R_{ij} may not be exactly -1.)

Next we describe our results for two dimensions. The number of samples used is shown in Table IV. Our results for μ are shown in Fig. 6, and give $\mu \simeq 0.29$. The data for χ_{SG} are shown in Fig. 7. We see that χ_{SG} increases as $N^{1-\mu}$ with

TABLE III. Number of samples used in the calculations for the short-range model in three dimensions.

	$T=0$		$T>0$
L	N_{samp} (m_0)	N_{samp} (χ_{SG})	N_{samp}
3	1000		
4	1000	1000	100
6	1000	1000	100
8	1000	1000	100
10	1000		
12	1105	1105	100
16	785	785	
24		500	

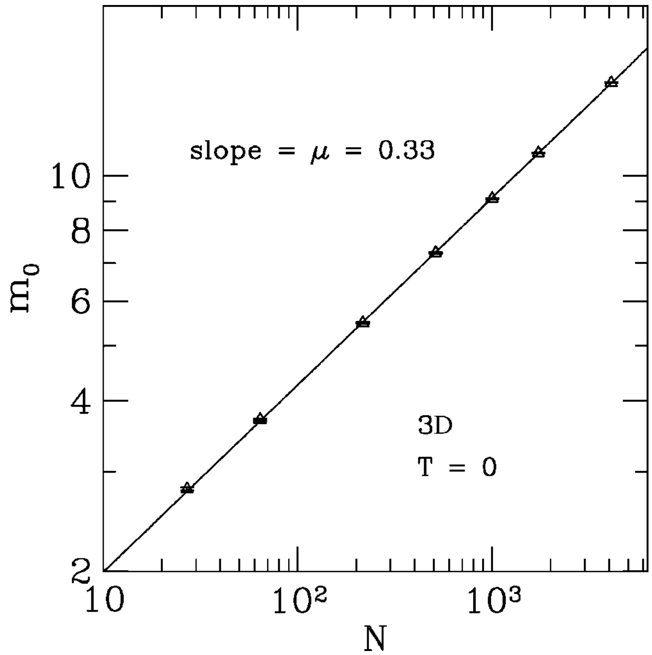

FIG. 4. The average number of nonzero spin components in the ground state m_0 as a function of N for the short-range model in $d=3$. We see that m_0 increases like N^{μ} with $\mu \simeq 0.33$.

the same μ as determined from Fig. 6. We therefore find that $d\mu \simeq 0.58$, and so, from Eq. (16), the spin glass correlations decay as

$$[C_{ij}]^2_{\text{av}} \sim \frac{1}{R_{ij}^{0.58}} \quad (d=2, T=0). \tag{19}$$

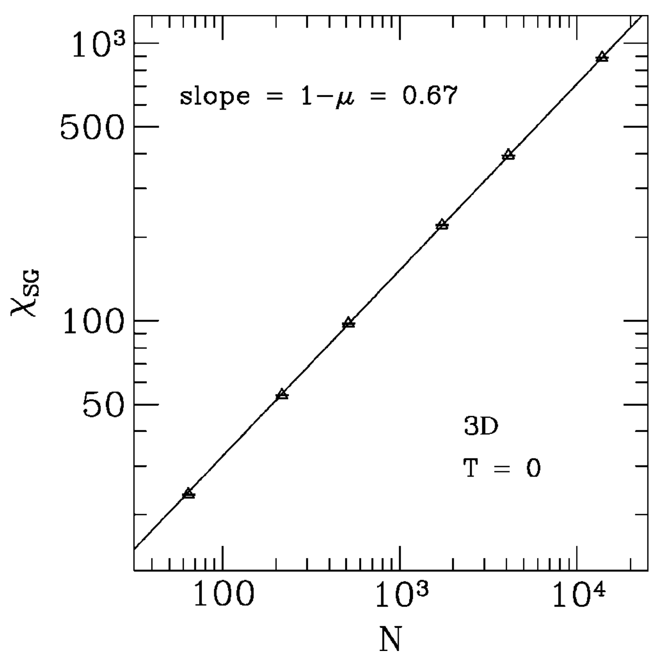

FIG. 5. The spin glass susceptibility for the short-range model in $d=3$ for different system sizes. As expected it varies as $N^{1-\mu}$, where $\mu \simeq 0.33$ was also found in Fig. 4.

TABLE IV. Number of samples used in the calculations for the short-range model in two dimensions.

	$T=0$		$T>0$
L	N_{samp} (m_0)	N_{samp} (χ_{SG})	N_{samp}
4	1000	1000	1000
6	1000	1000	1000
8	1000	1000	1000
10	1000		
12	1000	1000	1000
14	1000		
16	1000	1000	1000
18	1000		
20	1000		
22	1000		
24		1000	500
28	1000		
32	1000	1000	309
48		472	136
64	1016	1016	

IV. RESULTS FOR SHORT-RANGE MODELS AT FINITE TEMPERATURE

We have determined finite temperature properties by solving Eqs. (4)–(6) self-consistently using the Newton-Raphson method. We start at high temperature, $T=T_1$ say, and take our initial guess to be $H_i = 1/\beta$ which is the solution obtained perturbatively to first order in $1/T$. We then solve the equations at successively lower temperatures, $T_1 > T_2 > T_3$

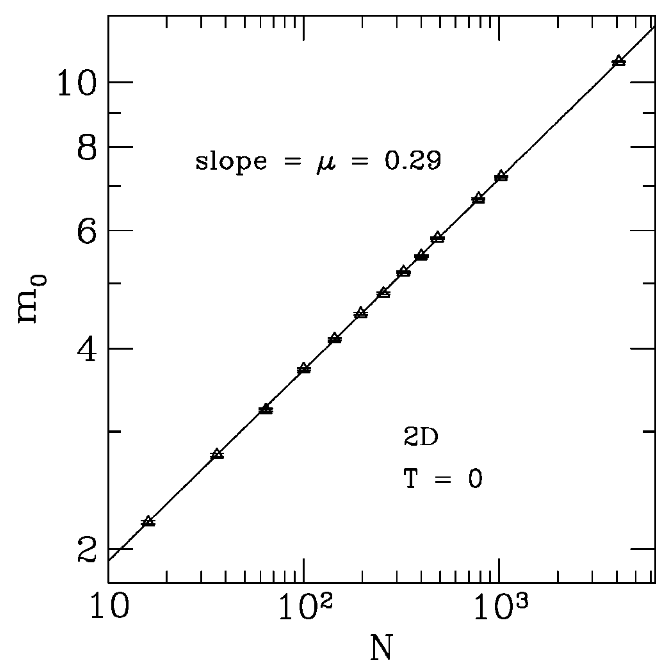

FIG. 6. The average number of nonzero spin components in the ground state m_0 as a function of N for the short-range model in $d=2$. We see that m_0 increases like N^{μ} with $\mu \simeq 0.29$.

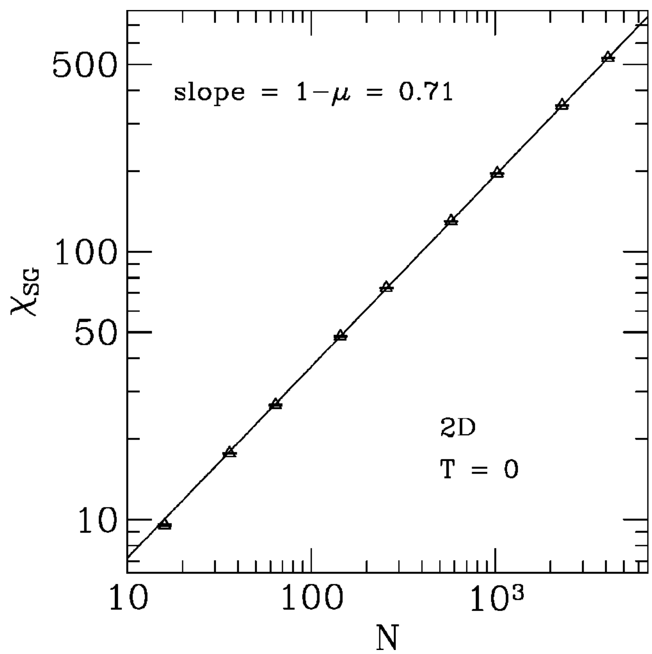

FIG. 7. The spin glass susceptibility for the short-range model in $d=2$ for different system sizes. As expected it varies as $N^{1-\mu}$, where $\mu \simeq 0.29$ was also found in Fig. 6.

$>T_4 \cdots$, and obtain the initial guess for the H_i at temperature T_{i+1} by integrating the equations (4)

$$\frac{dH_i}{d\beta} = -\sum_j (B^{-1})_{ij}, \tag{20}$$

in which

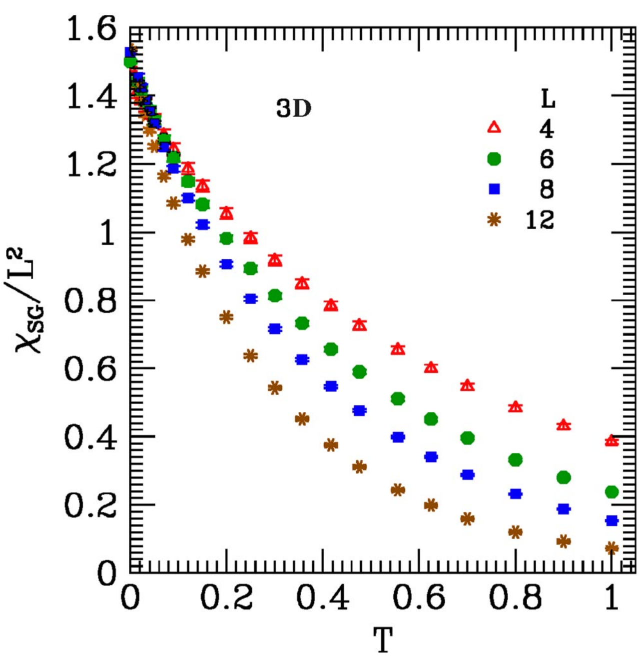

FIG. 8. The spin glass susceptibility as a function of temperature in three dimensions. The vertical axis has been divided by $L^{d(1-\mu)}$, in which we took $\mu=1/3$ in order to collapse the data at $T=0$ according to the data in Figs. 4 and 5.

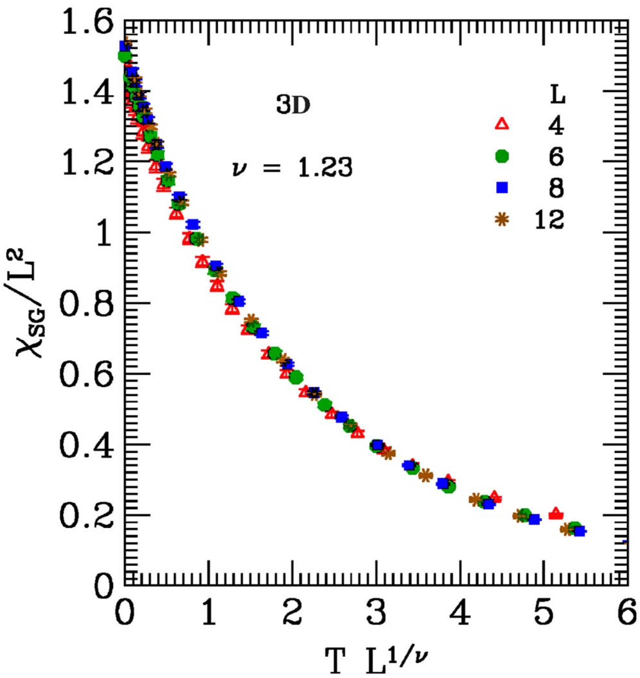

FIG. 9. A scaling plot of the spin glass susceptibility in Fig. 8 assuming a zero temperature transition.

$$B_{ij} = (\beta C_{ij})^2, \tag{21}$$

from β_i to β_{i+1} ($\beta=1/T$).

Results for χ_{SG} in $d=3$ are shown in Fig. 8, in which we scaled the vertical axis by $L^{d(1-\mu)}$ ($=L^2$) so the data collapse at $T=0$. If we assume a zero temperature transition, the data should fit the finite-size scaling form

$$\chi_{SG} = L^{d(1-\mu)} X(L^{1/\nu}T), \tag{22}$$

where $X(x) \to \text{const}$ for $x \to 0$, and the power law prefactor in front of the scaling function $X(x)$ then gets the $T=0$ limit correct. Figure 9 shows an appropriate scaling plot with $\nu=1.23$. Apart from the smallest size, $L=4$, the data clearly collapse well. By considering different values of ν we estimate

$$\nu = 1.23 \pm 0.13 \quad (d=3). \tag{23}$$

This result can be compared with that of Morris *et al.* [9] who quote $\nu=1.01\pm0.02$. Since our results cover a larger range of sizes and have better statistics, we feel that the error bars of Morris *et al.* are too optimistic. Assuming this, our result is consistent with theirs.

We should, however, also test to see if the data can be fitted with a finite value for T_{SG}. To do this, it is convenient to analyze the correlation length of the finite system, ξ_L, and plot the dimensionless ratio ξ_L/L which has the expected scaling form [5,13]

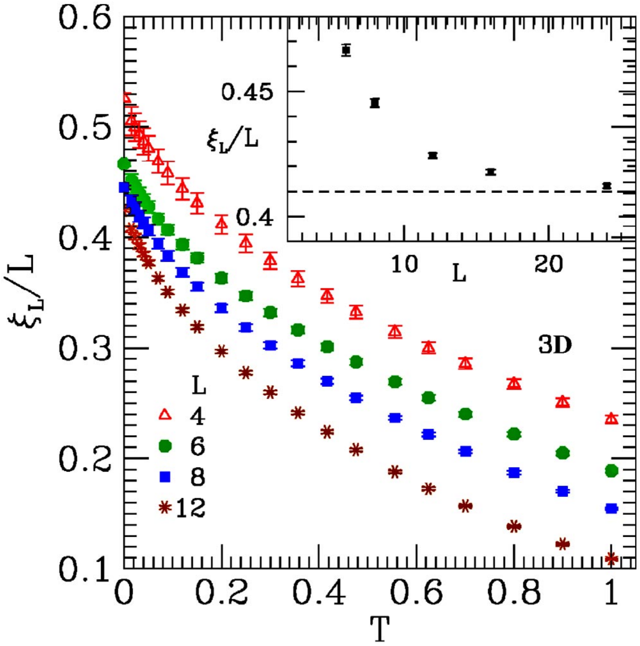

FIG. 10. The main figure is a plot of ξ_L/L against T in three dimensions. The inset shows ξ_L/L at $T=0$ as a function of L. The dashed line is a guide to the eye.

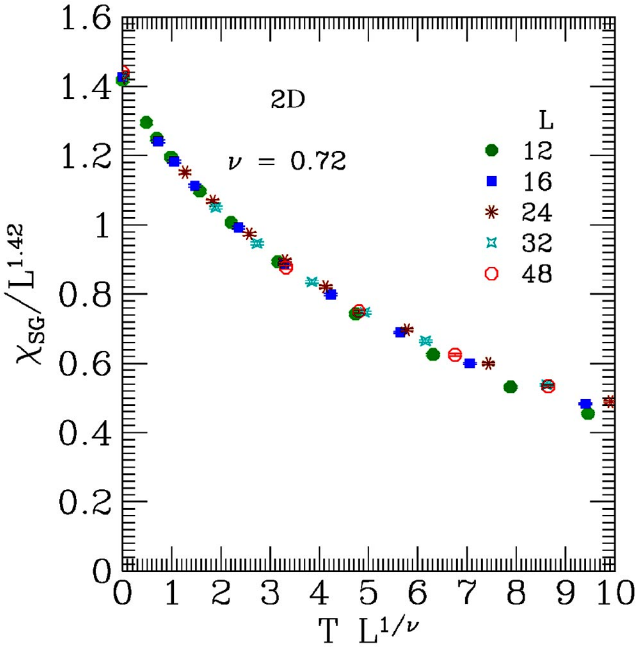

FIG. 11. A scaling plot of the data for $\chi_{\rm SG}$ in two dimensions, assuming a zero temperature transition. In the vertical axis, $\chi_{\rm SG}$ is divided by $L^{d(1-\mu)} \simeq L^{1.42}$ so that the data collapse at $T=0$.

$$\frac{\xi_L}{L} = F\big(L^{1/\nu}(T - T_{SG})\big) \tag{24}$$

without any unknown power of L multiplying the scaling function F. Hence the data for different sizes should intersect at T_{SG} and also splay out below T_{SG}. To determine ξ_L we Fourier transform $[C_{ij}^2]_{\rm av}$ to get $\chi_{\rm SG}(\mathbf{k})$ and then use [5,13]

$$\xi_L = \frac{1}{2\sin(k_{\rm min}/2)}\left(\frac{\chi_{\rm SG}(0)}{\chi_{\rm SG}(\mathbf{k}_{\rm min})} - 1\right)^{1/2}, \tag{25}$$

where $\mathbf{k}_{\rm min}=(2\pi/L)(1,0,0)$ is the smallest nonzero wave vector on the lattice.

The results are shown in the main part of Fig. 10. The data do not intersect at any temperature, but seem to be approaching an intersection at $T=0$ for the larger sizes. To test out this possibility, we have computed the correlation length directly at $T=0$, from the solution of Eqs. (7) and (10), where we can study larger sizes than in the finite-T formulation of Eqs. (3)–(6). The data are shown in the inset of Fig. 10. It indicates, fairly convincingly, that ξ_L/L approaches a constant for $L\to\infty$ at $T=0$, and hence that there is a transition at $T=0$.

In $d=2$ it is well established that $T_{SG}=0$ even for the Ising case. A scaling plot for $\chi_{\rm SG}$ for $m=\infty$ in $d=2$, corresponding to Eq. (22), is shown in Fig. 11 with $\nu=0.72$, which gives the best data collapse for larger sizes, and $d(1-\mu)=1.42$ which is obtained from the $T=0$ results in Sec. III. Again the data scale well. Overall we estimate

$$\nu = 0.72 \pm 0.05 \quad (d=2, \quad \text{from } \chi_{\rm SG}). \tag{26}$$

This is consistent with the results in Morris *et al.* [9] who quote $\nu=0.65\pm0.02$.

We have also computed the correlation length ξ_L/L in two dimensions, and show the data in Fig. 12. The curves become independent of size, for large L, at $T=0$, confirming that $T_{SG}=0$. A scaling plot of the data for the largest sizes ($L\geqslant 24$) in Fig. 13 has the best data collapse with $\nu=0.65$ and altogether we estimate

$$\nu = 0.65 \pm 0.05 \quad (d=2, \quad \text{from } \xi_L/L), \tag{27}$$

which is consistent with our estimate from $\chi_{\rm SG}$ in Eq. (26), and with the result of Morris *et al.* [9].

V. CONCLUSIONS

We have considered the spin glass in the limit where the spins have an infinite number of components. In the formulation of this problem appropriate for numerical calculations [3,4,8,9], where the limit $m\to\infty$ is taken with N finite, we find that the order parameter, defined in terms of correlation functions in zero (symmetry-breaking) field, vanishes. Instead, below T_{SG}, there is only quasi-long-range order in which the correlations decay to zero with a power of distance. The transition temperature T_{SG} can be finite; it separates the region at low temperature, where the correlations decay with a power of the distance, from the region at high temperature where correlations decay exponentially.

Whereas we define the order parameter in terms of the *long distance limit* of the correlation functions, Aspelmeier and Moore [4] define a *local* order parameter in terms of the contribution to the constraint in Eq. (6) that comes from the eigenmodes with zero eigenvalue of the matrix A_{ij}. They argue their order parameter is related to the susceptibility in the presence of a small field h, where the limit $N\to\infty$ is

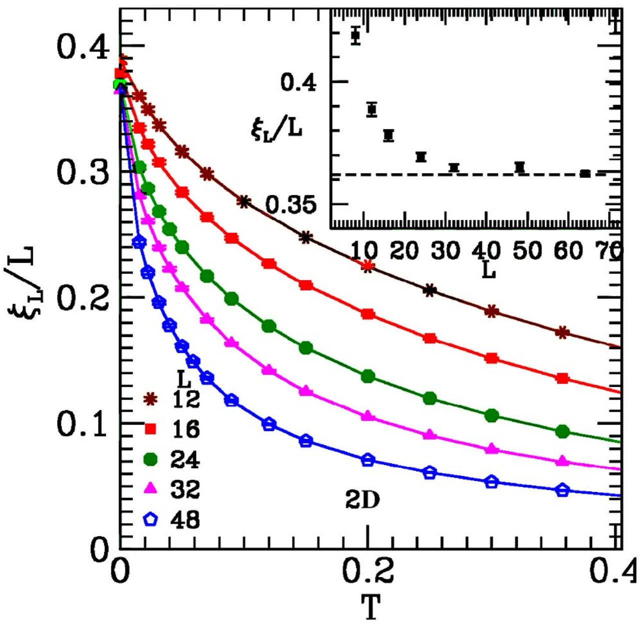

FIG. 12. Data for ξ_L/L as a function of T in two dimensions. Clearly the data for larger sizes are merging at $T=0$ indicating a transition at $T_{SG}=0$. The inset shows data for ξ_L/L at $T=0$ confirming that the data become independent of size at $T=0$. The dashed line is a guide to the eye.

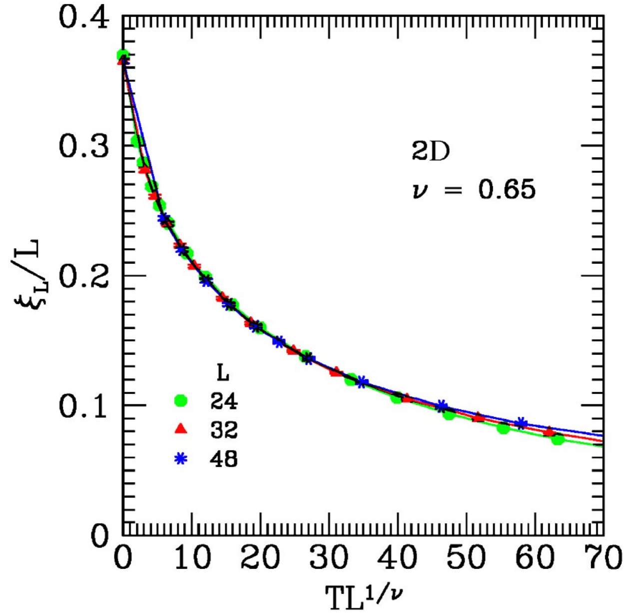

FIG. 13. A scaling plot of the data from the largest sizes for ξ_L/L in two dimensions assuming $T_{SG}=0$.

taken before the limit $h\rightarrow 0$ in order to break the symmetry. From numerics on the infinite-range model, Aspelmeier and Moore claim that their order parameter agrees with that of Almeida *et al.* [1].

However, in a sensible physical model, *any* reasonable definition of the order parameter should give the same answer. In particular, one should be able to obtain the square of the order parameter from the long distance limit of the correlation function (off-diagonal long range order) in zero field, and get the same answer as the local expectation value of the spin in the presence of a small symmetry breaking field. This does not appear to be the case for the $m=\infty$ model if the limit $m\rightarrow\infty$ is taken before $N\rightarrow\infty$.

On the other hand, if the thermodynamic limit, $N\rightarrow\infty$, is taken with m large but finite, then the correlations saturate at a value of order $1/m$ at large distance, and so a finite spin glass order parameter can be defined from the long distance limit of the correlation functions. This seems to agree with the order parameter found in the analytical work of Ref. [1], and is presumably the same as the local order parameter in a symmetry breaking field. Hence there seems to be no inconsistency if the limit $N\rightarrow\infty$ is taken first.

We have also studied the $m=\infty$ model in three dimensions, finding the transition (where correlations change from exponential to power law) to be at zero temperature, in contrast to the situation for [5,13] $m=1$, 2, and 3. We suspect that $T_{SG}=0$ only in the $m=\infty$ limit, rather than for all m less than some (nonzero) critical value m_c, since spin glasses with $m=\infty$ seem to have unique features. For example, we have already mentioned that there is only quasi-long-range order below T_{SG} in this case, in contrast to finite m. Another example of the special features of the $m=\infty$ limit is that Green *et al.* [14] find the upper critical dimension, above which the critical exponents are mean field like, to be $d_u=8$, whereas for finite m one has $d_u=6$. Our result that $T_{SG}=0$ for $m=\infty$ in $d=3$ is *consistent* with the claim of Viana [15] that the lower critical dimension (below which $T_{SG}=0$) is also $d_l=8$, but currently we cannot say anything specific about dimensions above 3.

We find, not surprisingly, that $T_{SG}=0$ also in two dimensions. Our results for the correlations length exponent at the $T=0$ transition in $d=2$ and 3 are consistent with those of Morris *et al.* [9].

Finally, we note that Aspelmeier and Moore [4] have proposed that the $m=\infty$ model is a better starting point for describing Ising or Heisenberg spin glasses in finite dimensions than the Ising model. We have argued in this paper that the spin glass with m strictly infinite is not a sensible model, but one rather needs to consider m large but finite. Hence the $m=\infty$ formulation proposed by Aspelmeier and Moore [4] and others [3,8,9] would need to be extended to a $1/m$ expansion and evaluated, at the very least, to order $1/m$. More probably an infinite resummation would be needed [10] to obtain sensible results in the spin glass phase, but this may be feasible.

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