Exact solutions for space-times with local rotational symmetry in which the Dirac equation separates

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The field equations for the class of perfect fluid space-times with local rotational symmetry in which the authors had earlier shown the Dirac equation separates are studied. For the vacuum and dust cases all possible solutions are exhibited. Other solutions correspond to radiation, a stiff fluid, and a fluid with negative pressure.

I. INTRODUCTION

In an earlier paper[1] (hereafter referred to as I) we investigated the problem of separability of the Dirac equation in perfect fluid space-times with local rotational symmetry and showed that separation was possible only in a certain subclass of the whole family. The geometrical properties of these space-times were also obtained but the question of the specific space-times in this subclass was left unanswered. In this paper we study the field equations for these particular space-times and attempt to isolate those exact solutions which fall in this category. For the vacuum ($p=\rho=0$) and dust ($p=0$) cases all the possible solutions are exhibited. Some exact solutions for other interesting sources like radiation ($p=\frac{1}{3}\rho$), a stiff fluid ($p=\rho$), and fluid with negative pressure ($p+\rho=0$) are also obtained. Though most of these solutions were known earlier we present a unified and systematic treatment of the different cases of particular interest as background metrics wherein our earlier separation of variables for the Dirac equation is applicable.

In the next section we set up the field equations for the relevant solutions. In Secs. III and IV we obtain all the vacuum and dust solutions, respectively. Section V contains some solutions corresponding to radiation, a stiff fluid, and a fluid with negative pressure.

II. SPACE-TIMES WITH LOCAL ROTATIONAL SYMMETRY WHEREIN THE DIRAC EQUATION SEPARATES

As demonstrated in I the space-times with local rotational symmetry in which the Dirac equation separates are of the following four types.

Case I:

$$ds^2=(1/F^2)dx^{0^2}-dx^{1^2}-Y^2(dx^{2^2}+t^2\,dx^{3^2}), \tag{1a}$$

where

$$F=F(x^1),\quad Y=Y(x^1). \tag{1b}$$

Case III:

$$ds^2=dx^{0^2}-X^2\,dx^{1^2}-Y^2(dx^{2^2}+t^2\,dx^{3^2}), \tag{2a}$$

with

$$X=X(x^0),\quad Y=Y(x^0). \tag{2b}$$

Case II a:

$$ds^2=(1/F^2)d\bar{x}^{0^2}-X^2\,d\bar{x}^{1^2}-Y^2(dx^{2^2}+t^2\,dx^{3^2}), \tag{3a}$$

where

$$F=F(\bar{x}^0),\quad X=X(\bar{x}^1),\quad Y=Y(\bar{x}^1). \tag{3b}$$

Case II b:

$$ds^2=(1/F^2)d\bar{x}^{0^2}-X^2\,d\bar{x}^{1^2}-Y^2(d\bar{x}^{2^2}+t^2\,dx^{3^2}), \tag{4a}$$

where

$$F=F(\bar{x}^0),\quad X=X(\bar{x}^1),\quad Y=Y(\bar{x}^0). \tag{4b}$$

In the above equations t is one of the four functions

$$\text{(i) } t=\text{const},\quad \text{(ii) } t=x^2,\quad \text{(iii) } t=\sin(x^2),\quad \text{(iv) } t=\sinh(x^2). \tag{5}$$

It is clear that the solutions corresponding to $t=$ const and $t=x^2$ are related trivially by transformations from Cartesian to cylindrical coordinates in the x^2-x^3 plane, i.e.,

$$x^{2'}=x^2\cos(x^3),\quad x^{3'}=x^2\sin(x^3). \tag{6}$$

Consequently, these two cases can be treated together. Further, in Cases II a and II b, by the following transformation of coordinates

$$x^0=\int\frac{d\bar{x}^0}{F(\bar{x}^0)},\quad x^1=\int X(\bar{x}^1)d\bar{x}^1, \tag{7}$$

the line elements become the following.

Case II a:

$$ds^2=dx^{0^2}-dx^{1^2}-Y^2(dx^{2^2}+t^2\,dx^{3^2}), \tag{8a}$$

where

$$Y=Y(x^1). \tag{8b}$$

Case II b:

$$ds^2=dx^{0^2}-dx^{1^2}-Y^2(dx^{2^2}+t^2\,dx^{3^2}), \tag{9a}$$

with

$$Y=Y(x^0). \tag{9b}$$

In this form Eq. (8) is a special case of (1) with $F=$ const while Eq. (9) is a special case of Eq. (2) with $X=$ const.

Choosing units $c=8\pi G=1$ and signature $(+,-,-,-)$ the field equations are

$$G_{ab}=T_{ab}, \tag{10a}$$

where for a perfect fluid

$$T_{ab}=(\rho+p)U_aU_b-pg_{ab}. \tag{10b}$$

Introducing ϵ such that

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$$\epsilon = \begin{cases} 0, & \text{for } t = \text{const}, \\ +1, & \text{for } t = \sin(x^2), \\ -1, & \text{for } t = \sinh(x^2). \end{cases} \tag{11}$$

Equation (10) for Case I becomes

$$2\frac{Y_{,11}}{Y} + \frac{Y_{,1}^2}{Y^2} - \frac{\epsilon}{Y^2} = -p, \tag{12a}$$

$$2\frac{Y_{,1}F_{,1}}{YF} - \frac{Y_{,1}^2}{Y^2} + \frac{\epsilon}{Y^2} = -p, \tag{12b}$$

$$\frac{F_{,11}}{F} - \frac{Y_{,11}}{Y} - \frac{F_{,1}}{F}\left(2\frac{F_{,1}}{F} - \frac{Y_{,1}}{Y}\right) = -p. \tag{12c}$$

For Case III one obtains

$$2\frac{X_{,0}Y_{,0}}{XY} + \frac{Y_{,0}^2}{Y^2} + \frac{\epsilon}{Y^2} = \rho, \tag{13a}$$

$$2\frac{Y_{,00}}{Y} + \frac{Y_{,0}^2}{Y^2} + \frac{\epsilon}{Y^2} = -p, \tag{13b}$$

$$\frac{Y_{,00}}{Y} + \frac{X_{,0}Y_{,0}}{XY} + \frac{X_{,00}}{X} = -p. \tag{13c}$$

As mentioned earlier, Case II a corresponds to $F = \text{const}$ in Eqs. (12) while Case II b corresponds to $X = \text{const}$ in Eqs. (13).

The field equations should be supplemented by the equation of state for the perfect fluid which we prescribe to be of the form

$$p = (\gamma - 1)\rho. \tag{14}$$

The conservation equation for T^{ab} gives

$$T^{ab}{}_{;b} = 0. \tag{15}$$

For Case I Eq. (15) gives

$$\rho_{,0} = p_{,2} = p_{,3} = 0 \tag{16a}$$

and

$$pY^4F^{\gamma/(\gamma-1)} = \text{const}, \tag{16b}$$

while for Case III we have

$$\rho(XY^2)^\gamma = \text{const} = \rho_0, \tag{17a}$$

$$p_{,1} = p_{,2} = p_{,3} = 0. \tag{17b}$$

Though not useful for the vacuum and dust cases the above "first integrals" are useful in the other cases.

III. VACUUM SPACE-TIMES ($\gamma=1; p=\rho=0$)

Case I: If $F = \text{const}$, Eqs. (12) become

$$Y_{,1}^2 = \epsilon, \quad Y_{,11} = 0. \tag{18}$$

Thus for $\epsilon = 0$, one obtains a flat space-time in Cartesian coordinates while for $\epsilon = +1$ one finds $Y^2 = (x^1)^2$, which is a flat space-time in spherical polar coordinates. There is no solution for $\epsilon = -1$.

From Eqs. (12a) and (12b) $Y = \text{const}$ is possible only if $\epsilon = 0$. In this case Eq. (12c) gives

$$F_{,11}/F - 2F_{,1}^2/F^2 = 0, \tag{19}$$

which on integration yields

$$1/F^2 = (x^1)^2 \tag{20}$$

(here and in later parts all trivial integration constants are transformed away by a suitable translation or scaling). This is just a Minkowski space-time in Rindler coordinates

$$\bar{x}^0 = x^1 \sinh(x^0), \quad \bar{x}^1 = x^1 \cosh(x^0). \tag{21}$$

In general (i.e., if $F_{,1} \neq 0$, $Y_{,1} \neq 0$) by adding Eqs. (12a) and (12b) and integrating we obtain

$$FY_{,1} = C_1. \tag{22}$$

Equation (12a) can be rewritten as

$$2Y_{,11}/(Y_{,1}^2 - \epsilon) + 1/Y = 0, \tag{23}$$

which on integration gives

$$Y(Y_{,1}^2 - \epsilon) = C_2. \tag{24}$$

Solutions of (22) and (24) satisfy (12c) identically. Hence a solution of Eq. (24) yields a solution of the field equation. From Eq. (24) for $\epsilon = 0$, we obtain

$$Y^2 = (x^1)^{4/3}, \quad F^2 = (x^1)^{2/3}, \tag{25}$$

which is the plane symmetric Taub solution[2]

$$ds^2 = z^{-1/2}(dT^2 - dz^2) - z(dx^2 + dy^2); \quad z > 0, \tag{26a}$$

as follows by the transformations

$$T = (\tfrac{3}{4})^{1/3}x^0, \quad z = (\tfrac{3}{4})^{4/3}(x^1)^{4/3}, \quad x = (\tfrac{3}{4})^{-2/3}x^2, \quad y = (\tfrac{3}{4})^{-2/3}x^3. \tag{26b}$$

For $\epsilon = 1$, the solution may be implicitly given as

$$\begin{aligned} Y &= -(c_2/2)(1 + \cosh(2\rho)), \\ F &= \pm c_1 \coth\rho, \\ \pm x^1 + c_3 &= -(c_2/2)(\sinh(2\rho) + 2\rho). \end{aligned} \tag{27}$$

On transforming to coordinates (x^0, ρ, x^2, x^3), one obtains

$$ds^2 = \tanh^2\rho\, dx^{0^2} - 4c_2^2 \cosh^4\rho\, d\rho^2 - c_2^2 \cosh^4\rho(dx^{2^2} + \sin^2 x^2\, dx^{3^2}). \tag{28}$$

This is just a Schwarzschild solution of mass $c_2/2$ as can be seen by transforming to coordinate r:

$$r = c_2 \cosh^2(\rho). \tag{29}$$

It is also one of the Levi-Civita degenerate static vacuum solution type AI (Ref. 3).

For $\epsilon = -1$ one obtains

$$\begin{aligned} Y &= c_2 \sin^2(\rho), \quad F = \pm c_1 \tan(\rho), \\ \pm x^1 + c_3 &= -(c_2/2)(\sin(2\rho) - 2\rho), \end{aligned} \tag{30}$$

which in coordinates (x^0, ρ, x^2, x^3) give

$$ds^2 = \cot^2\rho\, dx^{0^2} - c_2^2 \sin^4\rho(4\, d\rho^2 + dx^{2^2} + \sinh^2 x^2\, dx^{3^2}). \tag{31}$$

This is the degenerate static vacuum solution due to Levi-Civita[3] which in the classification of Ehlers and Kundt[3] is class AII. In terms of coordinates

$$z = c_2 \sin^2\rho, \tag{32}$$

$$ds^2 = (c_2/z - 1)dx^{0^2} - ((c_2/z) - 1)^{-1} dz^2 - z^2(dx^{2^2} + \sinh^2 x^2\, dx^{3^2}). \tag{33}$$

Case III: Let us now turn to Eqs. (13). If $X = \text{const}$ they become

$$Y_{,0}^2 = -\epsilon, \quad Y_{,00} = 0. \tag{34}$$

As for Eqs. (12) for $\epsilon = 0$ one has a flat space-time in Cartesian coordinates while for $\epsilon = -1$ one obtains $Y^2 = x^{0^2}$. This is just a Milne universe: a flat space-time in Rindler-like coordinates, as can be seen by the transformations

$$\bar{x}^0 = x^0 \cosh x^2, \quad \bar{x}^1 = x^1, \quad \bar{x}^2 = x^0 \sinh x^2 \cos x^3, \quad \bar{x}^3 = x^0 \sinh x^2 \sin x^3. \tag{35}$$

From Eqs. (13a) and (13b) $Y = \text{const}$ is possible only if $\epsilon = 0$. In this case Eq. (13c) gives

$$X_{,00} = 0, \quad \text{i.e., } X = x^0. \tag{36}$$

This again is a flat space-time in Rindler-like coordinates

$$\bar{x}^0 = x^0 \cosh x^1, \quad \bar{x}^1 = x^0 \sinh x^1. \tag{37}$$

We now consider cases when neither X nor Y is constant. As before, taking the difference of Eqs. (13a) and (13b) and integrating we obtain

$$X = c_1 Y_{,0}. \tag{38}$$

Equation (13b) on integration gives

$$Y(Y_{,0}^2 + \epsilon) = c_2. \tag{39}$$

Solutions of Eqs. (38) and (39) satisfy Eq. (13c) identically. For $\epsilon = 0$ one obtains

$$Y = x^{0^{2/3}}, \quad X = (x^0)^{-1/3}, \tag{40}$$

which is a Kasner space-time with local rotational symmetry. The Dirac equation in this case is treated in more detail elsewhere.[4]

For $\epsilon = +1$ we obtain

$$Y = c_2 \sin^2 T, \quad X = \pm c_1 \cot T, \quad \pm x^0 + c_3 = (c_2/2)(2T - \sin 2T), \tag{41}$$

which in terms of coordinates (T, x^1, x^2, x^3) gives

$$ds^2 = 4c_2^2 \sin^4 T\, dT^2 - c_1^2 \cot^2 T\, dx^{1^2} - c_2^2 \sin^4 T (dx^{2^2} + \sin^2 x^2\, dx^{3^2}). \tag{42}$$

Transforming to

$$\bar{T} = c_2 \sin^2 T, r = c_1 x^1 \tag{43}$$

yields the "inner" sector of the Schwarzschild solution, i.e. $(r < c_2)$

$$ds^2 = (c_2/\bar{T} - 1)^{-1} d\bar{T}^2 - (c_2/\bar{T} - 1) dr^2 - \bar{T}^2 (dx^{2^2} + \sin^2 x^2\, dx^{3^2}). \tag{44}$$

For $\epsilon = -1$, on the other hand,

$$Y = -c_2 \cosh^2 T, \quad X = \mp c_1 \tanh T, \quad \pm x^0 + c_3 = (c_2/2)(\sinh 2T + 2T), \tag{45}$$

which in terms of (T, x^1, x^2, x^3) yields

$$ds^2 = 4c_2^2 \cosh^4 T\, dT^2 - c_1^2 \tanh^2 T\, dx^{1^2} - c_2^2 \cosh^4 T (dx^{2^2} + \sinh^2 x^2\, dx^{3^2}). \tag{46}$$

Once again going over to

$$\bar{T} = c_2 \cosh^2 T, r = c_1 x^1, \tag{47}$$

we obtain

$$ds^2 = (1 - c_2/\bar{T})^{-1} d\bar{T}^2 - (1 - c_2/\bar{T}) dr^2 - \bar{T}^2 (dx^{2^2} + \sinh^2 x^2\, dx^{3^2}). \tag{48}$$

This solution is to the Levi-Civita static solution AII (Ref. 3), the analog of the $R < 2M$ region of the Schwarzschild solution.

IV. THE DUST SOLUTIONS ($\gamma = 1, p = 0$)

In Eq. (12) corresponding to Case I for dust, if $F_{,1} = 0$, then Eqs. (12b) and (12c) become

$$Y_{,1}^2 - \epsilon = 0, \quad Y_{,11} = 0, \tag{49}$$

which when compared with Eq. (12a), gives $\rho = 0$. Thus no dust solutions are possible in this case. Similarly, for $Y_{,1}^2 = \epsilon$ no dust solutions exist.

In general, however, Eq. (12b) gives

$$F_{,1}/F = (Y_{,1}^2 - \epsilon)/2YY_{,1}. \tag{50}$$

Differentiating (50) and substituting in Eq. (12c) one obtains

$$2Y_{,11}/Y + (Y_{,1}^2 - \epsilon)/Y^2 = 0, \tag{51}$$

which employing (12a) gives $\rho = 0$. Thus no dust solution is possible for Eq. (12). They seem to be possible only in metrics of subclass III corresponding to Eq. (13).

If $X_{,0} = 0$, Eqs. (13) yield

$$Y_{,0}^2 + \epsilon = \rho, \quad Y_{,00}/Y + (Y_{,0}^2 + \epsilon)/Y^2 = 0, \quad Y_{,00}/Y = 0, \tag{52}$$

which are consistent only for $\rho = 0$. Thus one does not have dust solutions with $X = \text{const}$. From Eq. (13b) $Y = \text{const}$ solutions are only possible for $\epsilon = 0$, which from (13a) implies $\rho = 0$. Thus one does not have such dust solutions either. If $X_{,0} \neq 0$, $Y_{,0} \neq 0$, Eq. (13b) can be rewritten as

$$2Y_{,0} Y_{,00}/(Y_{,0}^2 + \epsilon) + Y_{,0}/Y = 0, \tag{53}$$

which gives

$$Y(Y_{,0}^2 + \epsilon) = \text{const}. \tag{54}$$

For $\epsilon = 0$, Eq. (54) is solved by

$$Y = (c_1 x^0 + c_2)^{2/3}. \tag{55}$$

Replacing Y in (13c) from Eq. (55) one obtains for X, the differential equation

$$X_{,TT} - \tfrac{1}{3} X_{,T} - \tfrac{2}{9} X = 0, \tag{56a}$$

where

$$T = \log(c_1 x^0 + c^2). \tag{56b}$$

Consequently, the general solutions for X is

$$X = [c_3(c_1 x^0 + c_2) + c_4]/(c_1 x^0 + c_2)^{1/3}. \tag{57}$$

Substituting for X and Y from Eqs. (55) and (57) in Eq. (13a) yields ρ:

$$\rho = \tfrac{4}{3} c_3 c_1^2/(c_1 x^0 + c_2)[c_3(c_1 x^0 + c_2) + c_4]. \tag{58}$$

By a simple translation and scaling, the metric becomes

$$ds^2 = dx^{0^2} - ((x^0 + \alpha)/x^{0^{1/3}})^2 dx^{1^2} - x^{0^{4/3}} (dx^{2^2} + dx^{3^2}), \tag{59a}$$

where

$$\alpha = c_4/c_3 \text{ and } \rho = \tfrac{4}{3}(x^{0^2} + \alpha x^0)^{-1}. \quad (59b)$$

For the special choice of $\alpha = 0$, Eqs. (59) yield

$$ds^2 = dx^{0^2} - x^{0^{4/3}}(dx^{1^2} + dx^{2^2} + dx^{3^2}), \quad (60a)$$

$$\rho = 4/(3x^{0^2}). \quad (60b)$$

This is the Einstein–de Sitter solution for dust which has homogeneous and isotropic spatial sections. However, if $\alpha \neq 0$ we obtain a more general solution which does not seem obviously equivalent to the $\alpha = 0$ case. For $\epsilon = 1$, the solution may be written in the implicit form

$$Y = c_1 \sin^2 T, \quad \pm x^0 + c_2 = (c_1/2)[2T - \sin 2T]. \quad (61)$$

Substituting (61) in Eq. (13c) then gives

$$X_{,TT}/X - 2/\sin^2 T = 0. \quad (62)$$

By inspection $X = \cot(T)$ is a solution to the above equation. To find the other solution let

$$X = V \cot(T) \quad (63)$$

in Eq. (62) so that V satisfies

$$V_{,TT}/V_{,T} = 2 \csc^2 T/\cot T. \quad (64)$$

The above equation is integrated and finally one has

$$X = c_3(1 - T\cot T) + c_4 \cot T. \quad (65)$$

Substituting (61) and (65) in Eq. (12a) we have

$$\rho = c_3/c_1^2 \sin^4 T\,[c_3 - \cot T(c_3 T - c_4)]\,. \quad (66)$$

In terms of (T, x^1, x^2, x^3) one has

$$ds^2 = 4\sin^4 T\,dT^2 - [1 - \cot T(T - c)]^2 dx^{1^2} - \sin^4 T(dx^{2^2} + \sin^2 x^2\, dx^{3^2}), \quad (67a)$$

$$\rho = 1/\sin^4 T[1 - \cot T(T - C)]\,. \quad (67b)$$

Similarly, for $\epsilon = -1$,

$$Y = c_1 \sinh^2 T, \quad \pm x^0 + c_2 = (c_1/2)(\sinh 2T - 2T). \quad (68)$$

Substituting into Eq. (13c) gives

$$X_{,TT}/X - 2/\sinh^2 T = 0. \quad (69)$$

As before, since $X = \coth T$ is a solution of (69) we write

$$X = V \coth T, \quad (70a)$$

V is then a solution of

$$V_{,TT}/V_{,T} = (2\,\mathrm{csch}^2 T)/(\coth T), \quad (70b)$$

and consequently

$$X = c_3(T\coth T - 1) + c_4 \coth T. \quad (71)$$

For this case

$$\rho = c_3/c_1^2 \sinh^4 T\,[c_3(T\coth T - 1) + c_4 \coth T]\,. \quad (72)$$

In terms of (T, x^1, x^2, x^3) we thus have

$$ds^2 = 4\sinh^4 T\,dT^2 - [(T + \beta)\coth T - 1]^2 dx^{1^2} - \sinh^4 T(dx^{2^2} + \sinh^2 x^2\, dx^{3^2}), \quad (73a)$$

$$\rho = 1/\sinh^4 T[(T + \beta)\coth T - 1]. \quad (73b)$$

The dust solutions given by Eqs. (67) and (73) for $\epsilon = \pm 1$ are those obtained by Kantowski and Sachs.[5]

V. OTHER SOLUTIONS

If $F = \text{const}$, adding twice Eq. (12c) to (12b) one finds that the equations are consistent with (12a) only if $\rho + 3p = 0$. This case is not of physical interest. Similarly if $X = \text{const}$ adding two times (13c) to (13a) one finds that there is consistency with (13b) only for $\rho = p$, i.e., $\gamma = 2$. In this case, choosing $X = 1$ we obtain from Eq. (17)

$$\rho = c_1^2/4Y^4. \quad (74)$$

Substituting in Eq. (13a) and integrating we get

$$x^0 = \int \frac{2Y\,dY}{\sqrt{c_1^2 - 4\epsilon Y^2}}\,. \quad (75)$$

For $\epsilon = 0$, the solution after suitable scalings give

$$ds^2 = dx^{0^2} - dx^{1^2} - x^0(dx^{2^2} + x^{2^2}\, dx^{3^2}), \quad (76a)$$

$$\rho = c_1^2/4x^{0^2}. \quad (76b)$$

For $\epsilon = 1$, similarly,

$$ds^2 = dx^{0^2} - dx^{1^2} - (c_1^2/4 - x^{0^2})(dx^{2^2} + \sin^2 x^2\, dx^{3^2}), \quad (77a)$$

$$\rho = (c_1^2/4)(c_1^2/4 - x^{0^2})^{-2}, \quad (77b)$$

whereas for $\epsilon = -1$,

$$ds^2 = dx^{0^2} - dx^{1^2} - (x^{0^2} - c_1^2/4)(dx^{2^2} + \sinh^2 x^2\, dx^{3^2}), \quad (78a)$$

$$\rho = c_1^2/4/(x^{0^2} - c_1^2/4)^2. \quad (78b)$$

The solutions given by Eqs. (76), (77), (78), for a $\gamma = 2$ fluid is to our knowledge new.

Let us consider Eq. (13) for $\epsilon = 0$. Adding $(\gamma - 1)$ times Eq. (13a) to Eq. (13b) and integrating one gets

$$X = c_1(Y_{,0}^2 Y^\gamma)^{1/2(\gamma - 1)}. \quad (79)$$

Since

$$\rho = c(XY^2)^{-\gamma}, \quad (80)$$

one thus gets

$$\rho = c_\gamma = (Y_{,0})^{\gamma/(\gamma - 1)} Y^{\gamma(4 - 3\gamma)/2(\gamma - 1)}, \quad (81a)$$

where

$$c_\gamma = c/c_1^\gamma. \quad (81b)$$

Equation (13b) thus becomes

$$2Y_{,00}/Y + Y_{,0}^2/Y^2 = -(\gamma - 1)c_\gamma Y_{,0}^{\gamma/(\gamma - 1)} Y^{\gamma(4 - 3\gamma)/2(\gamma - 1)}. \quad (82)$$

The above equation will now be solved for the following interesting physical cases.

(a) $\gamma = 2$ $(p = \rho)$.

For this value the right-hand side of Eq. (82) is proportional to $Y_{,0}^2/Y^2$. Thus integrating (82) yields

$$Y = (c_2 x^0 + c_3)^{1/(1 + \alpha)}, \quad (83)$$

where

$$2\alpha = 1 + c/c_1^2.$$

Then Eq. (79) gives X as

$$X=(c_1/c_2)(1+\alpha)(c_2x^0+c_3)^{(\alpha-1)/(\alpha+1)}. \tag{84}$$

After the usual scalings one thus has

$$ds^2=dx^{0^2}-(x^0)^{2(\alpha-1)/(\alpha+1)}dx^{1^2} - (x^0)^{2/(1+\alpha)}(dx^{2^2}+dx^{3^2}), \tag{85a}$$

$$\rho=c/c_1^2(1+\alpha)^2x^{0^2}. \tag{85b}$$

This solution is identical to one of the solutions in Vajk and Eltgroth.[6]

(b) $\gamma=\frac{4}{3}$ $(p=\frac{1}{3}\rho)$.

In this case ρ is proportional to $Y_{,0}^4$ and hence Eq. (82) becomes

$$2Y_{,00}/Y+Y_{,0}^2/Y^2=-\beta Y_{,0}^4, \tag{86a}$$

where

$$\beta=\tfrac{1}{3}cc_1^{-4/3}. \tag{86b}$$

Substituting $YY_{,0}^2=u$ into the above equation and integrating one obtains

$$u=YY_{,0}^2=(c_2+\beta Y)^{-1}, \tag{87}$$

whose solution may be written as

$$Y=(c_2/\beta)\sinh^2 T,$$
$$\pm x^0+c_3=(c_2^2/16\sqrt{\beta^3})[\sinh 4T-4T], \tag{88}$$
$$X=c_1c_2\sqrt{\beta}(\cosh^3 T)/(\sinh T).$$

In terms of (T,x^1,x^2,x^3) the space-time is described by

$$ds^2=(c_2^4/4\beta^3)\sinh^4 2T\,dT^2-\cosh^4 T\coth^2 T\,dx^{1^2} - \sinh^4 T(dx^{2^2}+dx^{3^2}), \tag{89a}$$

$$\rho=16\beta^2c_\gamma/c_2^4\sinh^4 2T. \tag{89b}$$

Like the earlier case, this is also a particular solution from Vajk and Eltgroth.[6]

(c) $\gamma=0$ $(p+\rho=0)$.

For this value of γ, $\rho=\text{const}=\rho_0$ and $X=c_1Y_{,0}$. Thus Eq. (82) yields

$$2Y_{,00}/Y+Y_{,0}^2/Y^2=\rho_0. \tag{90}$$

The above equation can be integrated by letting $Y_{,0}/Y=u$. We get

$$Y=c_3[\cosh[\sqrt{3\rho_0}(x^0+c_2)/2]]^{2/3},$$
$$X=c_1c_3\sqrt{\rho_0/3}[\cosh[\sqrt{3\rho_0}(x^0+c_2)/2]]^{-1/3} \times(\sinh[\sqrt{3\rho_0}(x^0+c_2)/2]). \tag{91}$$

Thus the metric may be written as

$$ds^2=dx^{0^2}-(\cosh(\sqrt{3\rho_0}x^0/2))^{-2/3}\sinh^2(\sqrt{3\rho_0}x^0/2)dx^{1^2} - (\cosh(\sqrt{3\rho_0}x^0/2))^{4/3}(dx^{2^2}+dx^{3^2}). \tag{92}$$

To the best of our knowledge Eq. (92) is a new solution. For the various values of γ dealt with above we have not been able to obtain solutions of Eq. (13) for $\epsilon=\pm 1$ or of Eq. (12) for $\epsilon=0,+1$.

In the foregoing we have systematically obtained the various exact solutions with local rotational symmetry in which the Dirac equation is separable. As was mentioned at the outset many of them turn out to be already known solutions sometimes in terms of unconventional coordinates. Other solutions, given by Eqs. (59), (76), (77), (78), and (92), are new as far as we know. Our results, while incorporating a regular classification of these space-times would also facilitate the study of the Dirac equation in backgrounds exhibiting local rotational symmetry.

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