# Electromagnetic fields in space-times with local rotational symmetry*

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The Debye-potential formalism is applied to perfect-fluid space-times with local rotational symmetry which form a subclass of the generalized Goldberg-Sachs space-times. A decoupled equation for the potentials is obtained. It is seen that this equation yields a considerable amount of information in its general form without specializing to any example of the subclass. Finally some particular space-times, namely, Kantowski-Sachs universes, Taub space, and anisotropic spatially homogeneous cosmological models, are discussed in detail.

## I. INTRODUCTION

In curved space-times the Maxwell equations form a stronger coupled system of partial differential equations than those in flat space-time. This introduces difficulty in integrating these equations as the usual methods, which are adequate in flat space-time, fail when applied to these equations. Cohen and Kegeles[1] have shown that the Hertz-potential formalism can be extended to all curved space-times and the Debye-potential (two-component Hertz-potential) formalism may be extended to the generalized Goldberg-Sachs class, namely, the class that consists of those space-times which admit a shear-free congruence of null geodesics along the repeated principal null direction of the Weyl tensor. We shall henceforth refer to the above paper by Cohen and Kegeles[1] as paper I.

We confine our investigations to perfect-fluid space-times with local rotational symmetry as given by Ellis[2] and by Ellis and Stewart,[3] which form a subclass of the generalized Goldberg-Sachs class of space-times. This subclass consists of a wide range of interesting space-times such as the Friedmann models, Kerr, Schwarzschild, Godel, Kantowski, and Sachs universes, Taub-NUT (Newman-Unti-Tamburino) anisotropic spatially homogeneous cosmological models, etc. The solutions to the Maxwell equations by this approach in curved space-times have been obtained for the Friedmann models and the Kerr and Schwarzschild solutions in paper I, while the behavior of electromagnetic fields in the Godel universe has been treated in detail by Cohen, Vishveshwara, and Dhurandhar.[4]

We adopt the Newman-Penrose formalism for our study and use the equations given in paper I and the null tetrad given by Wainwright[5] for all perfect-fluid space-times with local rotational symmetry. In Sec. II we write down the equations governing a complex scalar function $\psi$ which contains all the information about the electromagnetic field, the components of the Maxwell field tensor being obtained by differentiation of the scalar $\psi$. In Sec. III we show that it is possible to derive a considerable amount of information from the governing equations without specializing to any particular space-time, thus maintaining the discussion at a general level. In Sec. IV we treat some important space-times, namely, the Kantowski and Sachs universes, anisotropic spatially homogeneous cosmologies, and the Taub-NUT space-time, which would supplement the space-times mentioned above already studied by this method.

## II. THE GOVERNING EQUATIONS

The geometry of perfect-fluid space-times with local rotational symmetry is described by the line element

$$ds^2=-\frac{(dx^0)^2}{F^2}+X^2(dx^1)^2+Y^2[(dx^2)^2+t^2(dx^3)^2]+\frac{y}{F^2}(2dx^0-y\,dx^3)dx^3-hX^2(2dx^1-h\,dx^3)dx^3\,, \tag{2.1}$$

where $F$, $X$, and $Y$ are, in general, functions of $x^0$ and $x^1$, and $t$, $y$, and $h$ are functions of $x^2$ only. Further, $t$, $y$, and $h$ satisfy conditions listed in the references cited above.[2,3]

The null tetrad given by Wainwright for the general line element (2.1) is

$$\begin{aligned}k_a&=\frac{1}{\sqrt{2}}\left(\frac{1}{F},-X,0,Xh-\frac{y}{F}\right),\\ n_a&=\frac{1}{\sqrt{2}}\left(\frac{1}{F},X,0,-\left(Xh+\frac{y}{F}\right)\right),\\ m_a&=\frac{1}{\sqrt{2}}(0,0,Y,itY),\\ \bar{m}_a&=\frac{1}{\sqrt{2}}(0,0,Y,-itY).\end{aligned} \tag{2.2}$$

 

We have the following relations between the tetrad components,

$$k^a n_a = -m_a \bar{m}_a = -\bar{m}_a m^a = -1\,.$$

All other scalar products vanish.

Here as in the rest of the paper we follow the notation of paper I. We use the following correspondence between the numerical indexing and the contravariant tetrad vectors,

$$k^a, n^a, m^a, \bar{m}^a \leftrightarrow 1, 2, 3, 4\,.$$

The tangent vectors in the tetrad directions are then

$$\omega_1 = k^a \frac{\partial}{\partial x^a}, \quad \omega_2 = n^a \frac{\partial}{\partial x^a}, \qquad \omega_3 = m^a \frac{\partial}{\partial x^a}, \quad \omega_4 = \bar{m}^a \frac{\partial}{\partial x^a}\,. \tag{2.3}$$

The intrinsic frame derivatives which occur in the equations are given by

$$D = \omega_1, \quad \Delta = \omega_2, \quad \delta = \omega_3, \quad \bar{\delta} = \omega_4\,. \tag{2.4}$$

The dual one-forms defined by $\omega^i(\omega_j) = \delta^i_j$ are then

$$\omega^1 = -n_a dx^a, \quad \omega^2 = -k_a dx^a\,,$$
$$\omega^3 = \bar{m}_a dx^a, \quad \omega^4 = n\, dx^a\,.$$

We tabulate the nonvanishing spin coefficients for the above tetrad which will be used in setting up the equations for the Debye potentials:

$$\alpha = -\frac{1}{2\sqrt{2}}\,\frac{1}{Y}\,\frac{t_{,2}}{t} - \frac{i}{2\sqrt{2}}\,\frac{1}{tY^2}\,(yY_{,0} + hY_{,1})\,,$$

$$\beta = \frac{1}{2\sqrt{2}}\,\frac{1}{Y}\,\frac{t_{,2}}{t} + \frac{i}{2\sqrt{2}}\,\frac{1}{tY^2}\,(yY_{,0} + hY_{,1})\,,$$

$$\gamma = \frac{1}{2\sqrt{2}}\,\frac{F}{X}\left(X_{,0} + \frac{F_{,1}}{F^2}\right) + \frac{i}{4\sqrt{2}}\,\frac{1}{tY^2}\left(Xh_{,2} + \frac{y_{,2}}{F}\right),$$

$$\epsilon = -\frac{1}{2\sqrt{2}}\,\frac{F}{X}\left(X_{,0} - \frac{F_{,1}}{F^2}\right) - \frac{i}{4\sqrt{2}}\,\frac{1}{tY^2}\left(Xh_{,2} - \frac{y_{,2}}{F}\right),$$

$$\mu = \frac{1}{\sqrt{2}}\,\frac{1}{Y}\left(\frac{Y_{,1}}{X} - FY_{,0}\right) + \frac{i}{2\sqrt{2}}\,\frac{1}{tY^2}\left(Xh_{,2} + \frac{y_{,2}}{F}\right),$$

$$\rho = \frac{1}{\sqrt{2}}\,\frac{1}{Y}\left(FY_{,0} + \frac{Y_{,1}}{X}\right) - \frac{i}{2\sqrt{2}}\,\frac{1}{tY^2}\left(Xh_{,2} - \frac{y_{,2}}{F}\right), \tag{2.5}$$

$$\nu = \frac{i}{2\sqrt{2}}\,\frac{1}{tY}\left[h\left(\frac{F_{,1}}{F} + \frac{X_{,1}}{X}\right) + y\left(\frac{F_{,0}}{F} + \frac{X_{,0}}{X}\right)\right],$$

$$\pi = \frac{i}{2\sqrt{2}}\,\frac{1}{tY}\left[h\left(\frac{F_{,1}}{F} - \frac{X_{,1}}{X}\right) + y\left(\frac{F_{,0}}{F} - \frac{X_{,0}}{X}\right)\right],$$

$$\tau = \frac{i}{2\sqrt{2}}\,\frac{1}{tY}\left[h\left(\frac{F_{,1}}{F} - \frac{X_{,1}}{X}\right) + y\left(\frac{F_{,0}}{F} - \frac{X_{,0}}{X}\right)\right],$$

$$\kappa = \frac{i}{2\sqrt{2}}\,\frac{1}{tY}\left[h\left(\frac{F_{,1}}{F} + \frac{X_{,1}}{X}\right) + y\left(\frac{F_{,0}}{F} + \frac{X_{,0}}{X}\right)\right].$$

The commas denote partial differentiation with respect to the coordinates.

The perfect-fluid space-times with local rotational symmetry may be reduced to three distinct cases:

Case (i) $X = 1,\ Y = Y(x^1),\ F = F(x^1),\ h = 0$.
Case (ii) $h = y = 0$.
Case (iii) $F = 1,\ X = X(x^0),\ Y = Y(x^0),\ y = 0$.

With these specializations Wainright has shown that for all rotationally symmetric space-times with perfect fluid which are not conformally flat, the Weyl tensor is of type (2, 2), with $k^a$ and $n^a$ as the repeated principal null vectors. Further, the last four of the spin coefficients tabulated in (2.5) vanish, showing that the congruences of $k^a$ and $n^a$ are geodesic and shear free. The specializations mentioned above also simplify the expressions for the spin coefficients.

We use the decoupled equation for the scalar $\psi$ for type (2, 2) space-times (Petrov-$D$), namely, Eq. (5.10) in paper I. We give below the equation and the null tetrad components of the Maxwell field tensor in terms of $\psi$. In terms of the intrinsic frame derivatives given in (2.3) and (2.4) and the the spin coefficients, the equation is

$$[(\Delta - \gamma - \bar{\gamma} + \bar{\mu} - \mu)D - (\bar{\delta} - \alpha + \bar{\beta} - \pi - \bar{\tau})\delta]\psi = 0\,, \tag{2.6}$$

and the tetrad components of the Maxwell field tensor are

$$\varphi_0 = f_{km} = [(\delta - \bar{\alpha} - \beta + \bar{\pi})D + (D - \epsilon + \bar{\epsilon} - \bar{\rho})\delta]\bar{\psi}\,,$$

$$\varphi_1 = \tfrac{1}{2}(f_{kn} + f_{\bar{m}m})$$
$$= [(\Delta - \gamma - \bar{\gamma} + \bar{\mu} - \mu)D + (\bar{\delta} - \bar{\alpha} + \beta + \bar{\pi} + \tau)\bar{\delta}]\bar{\psi}\,,$$

$$\varphi_2 = f_{\bar{m}n} = [(\Delta + \gamma - \bar{\gamma} + \bar{\mu})\bar{\delta} + (\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau})\Delta]\bar{\psi}\,. \tag{2.7}$$

The tensor $F_{\mu\nu}$ in the standard basis is then given by the expression

$$F_{\mu\nu} = 2(\varphi_1 + \bar{\varphi}_1)n_{[\mu}k_{\nu]} + 2\varphi_2 k_{[\mu}m_{\nu]} + 2\bar{\varphi}_2 k_{[\mu}\bar{m}_{\nu]} + 2\varphi_0 \bar{m}_{[\mu}n_{\nu]} + 2\bar{\varphi}_0 m_{[\mu}n_{\nu]} + 2(\varphi_1 - \bar{\varphi}_1)m_{[\mu}\bar{m}_{\nu]}\,. \tag{2.8}$$

Equation (2.6) may be explicitly written in terms of the coordinate basis by making use of the spin coefficients listed in (2.5) and the intrinsic frame derivatives given by (2.3) and (2.4). We write the equation for each of the three cases separately. The spin coefficients and the frame derivatives then assume relatively simple forms:

*Case (i):*

$$\left\{F^2\frac{\partial^2}{\partial(x^0)^2}-\frac{\partial^2}{\partial(x^1)^2}+\frac{i}{tY^2}y_{,2}\frac{\partial}{\partial x^0}+\left(\frac{F_{,1}}{F}+\frac{i}{tY^2F}y_{,2}\right)\frac{\partial}{\partial x^1} -\frac{1}{Y^2}\left[\left(-\frac{iy}{t}\frac{\partial}{\partial x^0}+\frac{\partial}{\partial x^2}-\frac{i}{t}\frac{\partial}{\partial x^3}+\frac{t_{,2}}{t}\right)\left(\frac{iy}{t}\frac{\partial}{\partial x^0}+\frac{\partial}{\partial x^2}+\frac{i}{t}\frac{\partial}{\partial x^3}\right)\right]\right\}\psi=0, \quad (2.9)$$

*Case (ii):*

$$\left\{F^2\frac{\partial^2}{\partial(x^0)^2}+\frac{F}{X}(XF)_{,0}\frac{\partial}{\partial x^0}-\frac{1}{X^2}\frac{\partial^2}{\partial(x^1)^2}-\frac{F}{X}\left(\frac{1}{XF}\right)_{,1}\frac{\partial}{\partial x^1} -\frac{1}{Y^2}\left[\left(-\frac{iy}{t}\frac{\partial}{\partial x^0}+\frac{\partial}{\partial x^2}-\frac{i}{t}\frac{\partial}{\partial x^3}+\frac{t_{,2}}{t}\right)\left(\frac{iy}{t}\frac{\partial}{\partial x^0}+\frac{\partial}{\partial x^2}+\frac{i}{t}\frac{\partial}{\partial x^3}\right)\right]\right\}\psi=0. \quad (2.10)$$

This equation can also be obtained by the orthonormal tetrad method described in paper I. The choice of the orthonormal tetrad

$$\omega^0=\frac{1}{F}dx^0,\quad \omega^1=Xdx^1,$$

$$\omega^2=Ydx^2,\quad \omega^3=tYdx^3$$

and the "diagonalized" Hertz bivector

$$P=P_E\omega^0\wedge\omega^1+P_M\omega^2\wedge\omega^3$$

lead to an equation for $P$,

$$(d*d*+*d*d)P=dG+*dW, \quad (2.11)$$

where $d$ is the exterior derivative and $*$ the Hodge dual. $G$ and $W$ are gauge one-forms to be added which reduce (2.11) to a single decoupled equation satisfied by both $P_E$ and $P_M$. The gauge one-forms retain the property of the source-free Maxwell equations. In this case one chooses

$$G=2\frac{Y_{,1}}{XY}P_E\omega^0+2F\frac{Y_{,0}}{Y}P_E\omega^1,$$

$$W=2\frac{Y_{,1}}{XY}P_M\omega^0+2F\frac{Y_{,0}}{Y}P_M\omega^1. \quad (2.12)$$

Then each of the equations for $P_E$ and $P_M$ reduces to (2.10).

*Case (iii):*

$$\left\{\frac{\partial^2}{\partial(x^0)^2}-\frac{1}{X^2}\frac{\partial^2}{\partial(x^1)^2}+\left(\frac{X_{,0}}{X}+i\frac{h_{,2}}{t}\frac{X}{Y^2}\right)\frac{\partial}{\partial x^0} +i\frac{h_{,2}}{tY^2}\frac{\partial}{\partial x^1}-\frac{1}{Y^2}\left[\left(-\frac{ih}{t}\frac{\partial}{\partial x^1}+\frac{\partial}{\partial x^2}-\frac{i}{t}\frac{\partial}{\partial x^3}+\frac{t_{,2}}{t}\right) \times\left(\frac{ih}{t}\frac{\partial}{\partial x^1}+\frac{\partial}{\partial x^2}+\frac{i}{t}\frac{\partial}{\partial x^3}\right)\right]\right\}\psi=0. \quad (2.13)$$

## III. GENERAL DISCUSSION OF THE GOVERNING EQUATIONS

The general discussion of Eqs. (2.9), (2.10), and (2.13) falls into two distinct divisions as each of these equations is separable into a "temporal-radial" and an "angular" part. The temporal-radial part, as the name suggests, involves the operators $\partial/\partial x^0$ and $\partial/\partial x^1$, while the angular part involves essentially $\partial/\partial x^2$ and $\partial/\partial x^3$. In the equations for each of the cases, the angular part is enclosed within square brackets, and the rest of the operator contributes to temporal-radial development of the Debye potential. Though the operators $\partial/\partial x^0$ in case (i) and $\partial/\partial x^1$ in case (iii) appear in the angular operator, they are Killing vectors for each of the respective cases. The solutions for $\psi$ will then have dependence in the former case of the type $e^{i\omega x^0}$, and in the latter case of the type $e^{ikx^1}$, where $\omega$ and $k$ are constants. Therefore, the operators $\partial/\partial x^0$ and $\partial/\partial x^1$ appearing in each of the respective cases are reduced to constants which merely multiply the scalar function $\psi$.

### A. The solutions of the angular operator

We treat cases (i), (ii), and (iii) separately. We observe that the angular operators appearing in cases (i) and (iii) are of the same form if we substitute in case (i) $\psi=\chi e^{i\omega x^0}$, $\chi$ independent of $x^0$, and in case (iii) $\psi=\chi e^{ikx^1}$, $\chi$ independent of $x^1$. Therefore, it is sufficient to study only one of the cases (i) and (iii). We choose to treat case (iii) with hindsight as we will be investigating a specific example of this case in the following section.

The prominent function appearing in the angular operator is $t(x^2)$. From the conditions on the function $t(x^2)$ given in Refs. 2 and 3, it can assume the following functional forms:

(a) $t=\sin x^2$,

(b) $t=\sinh x^2$,

(c) $t=x^2$,

(d) $t=\text{const}$.

The functions $h$ and $y$ of $x^2$ are then obtained from $t$ by the relations

$h_{,2}=ct$ and $y_{,2}=c't$,

where $c$ and $c'$ are constants.

*Case (ii)*. We separate the $x^2, x^3$ dependence by setting

$$\psi=Z(x^0,x^1)\chi(x^2,x^3)$$

in Eq. (2.10). The separated equation for $\chi$ is

$$\left[\frac{1}{t}\frac{\partial}{\partial x^2}\left(t\frac{\partial}{\partial x^2}\right)+\frac{1}{t^2}\frac{\partial^2}{\partial(x^3)^2}\right]\chi+\alpha\chi=0\,, \qquad (3.1)$$

where $\alpha$ is the separation constant. We now propose to discuss the above equation for the various functional forms of $t(x^2)$.

(a) $t=\sin x^2$: The solutions of (3.1) are then the familiar spherical harmonics $Y_{lm}(x^2,x^3)$. The separation constant $\alpha=l(l+1)$.

(b) $t=\sinh x^2$: The solutions to (3.1) are of the form $P_l^m(\cosh x^2)e^{imx^3}$. The regularity requirement condition at $x^2=0$ excludes $Q_l^m$ and the regularity condition at $x^2=\infty$ restricts $l$ to lie in the range $-\frac{1}{2}<l<0$. Here $\alpha$ is not an integer.

(c) $t=x^2$: In this case we use the Killing symmetry in the $\partial/\partial x^3$ direction of these space-times to simplify (3.1). We write the solution as

$$\chi(x^2,x^3)=\Theta(x^2)e^{imx^3}\,,$$

and substitute in (3.1) to obtain an equation for

$$\frac{d^2\Theta}{d(x^2)^2}+\frac{1}{x^2}\frac{d\Theta}{dx^2}+\left(\alpha-\frac{m^2}{(x^2)^2}\right)\Theta=0\,. \qquad (3.2)$$

The solutions of these equations are Bessel functions of order $m$, with argument $\sqrt{\alpha}\,x^2$. The general solution of (3.2) is a linear combination of the Bessel functions $J_m(\sqrt{\alpha}\,x^2)$ and $Y_m(\sqrt{\alpha}\,x^2)$. In particular, the "outgoing" and "ingoing" solutions are the Hankel functions.

(d) $t=$const: The constant value of $t$ may be taken to be unity as any other value of the constant may be effectively reduced to unity by rescaling the coordinate $x^3$. Equation (3.1) reduces to

$$\left(\frac{\partial}{\partial(x^2)^2}+\frac{\partial}{\partial(x^3)^2}+\alpha\right)\chi=0\,. \qquad (3.3)$$

The solutions of (3.3) are plane waves and are of the form

$$\chi=e^{\pm ik_2x^2\pm ik_3x^3}\,,$$

where $k_2$ and $k_3$ are constants. The separation constant $\alpha$ has the form

$$\alpha={k_2}^2+{k_3}^2\,.$$

*Case (iii)*. Setting $\psi=Z(x^0)\chi(x^2,x^3)e^{ikx^1}$ in (3.1), we obtain the equation for $\chi(x^2,x^3)$, i.e.,

$$\left(\frac{\partial}{\partial x^2}+k\frac{h}{t}-\frac{i}{t}\frac{\partial}{\partial x^3}+\frac{t_{,2}}{t}\right)\times\left(\frac{\partial}{\partial x^2}-k\frac{h}{t}+\frac{i}{t}\frac{\partial}{\partial x^3}\right)\chi+\alpha\chi=0\,, \qquad (3.4)$$

where $\alpha$ is the separation constant. We now consider the different functional forms of $t$ and $h$ and substitute them in Eq. (3.4).

(a) $t=\sin x^2$, $h=-c\cos x^2$: Then (2.4) reduces to

$$\left[\frac{\partial}{\partial x^2}-\frac{i}{\sin x^2}\frac{\partial}{\partial x^3}+(1-ck)\cot x^2\right]\times\left(\frac{\partial}{\partial x^2}+\frac{i}{\sin x^2}\frac{\partial}{\partial x^3}+ck\cot x^2\right)\chi+\alpha\chi=0\,. \qquad (3.5)$$

Setting $u=\cos x^2$ and $\chi=(1-x)^{\alpha'/2}(1+x)^{\beta'/2}y(u)e^{imx^3}$, we obtain

$$(1-u^2)\frac{d^2y}{du^2}+[\beta'-\alpha'-(\alpha'+\beta'+2)u]\frac{dy}{du}+n(n+\alpha'+\beta'+1)y=0\,, \qquad (3.6a)$$

where $\alpha'=|ck-m|$, $\beta'=|ck+m|$, and $n$ is the solution of

$$n(n+\alpha'+\beta'+1)+\tfrac{1}{2}(\alpha'+\beta')+\tfrac{1}{4}(\alpha'+\beta')^2+ck(1-ck)=\alpha\,. \qquad (3.6b)$$

Regularity restricts $n$ to be a non-negative integer and then the solutions to (3.6a) are the Jacobi polynomials $P_n^{(\alpha',\beta')}(u)$. The restriction on the value of $n$ brings constraints on the allowed values of the separation constant $\alpha$. Four cases arise accordingly as $\alpha'=\pm(ck-m)$ and $\beta'=\pm(ck+m)$. We may write all the cases concisely by setting $\alpha'=\epsilon_1(ck-m)$ and $\beta'=\epsilon_2(ck+m)$, with $\epsilon_1,\epsilon_2=\pm1$. Then,

$$\alpha=(n+s+1)(n+s)+ck(1-ck)\,,$$

where

$$s=-\tfrac{1}{2}\epsilon_1\epsilon_2(\alpha'+\beta')\,.$$

(b) $t=\sinh x^2$, $h=-c\cosh x^2$: The substitutions

$$z=-\sinh^2 x^2$$

and

$$\chi=2^{(\alpha'+\beta')/2}z^{(\alpha')/2}(1-z)^{\beta'/2}y(z)e^{imx^3}\,,$$

where $\alpha'=|ck-m|$ and $\beta'=|ck+m|$, yield the solutions in terms of hypergeometric functions,

$$\chi=2^{(\alpha'+\beta')/2}z^{\alpha'/2}(1-z)^{\beta'/2}e^{imx^3}\times\begin{cases}F(-n,n+\alpha'+\beta'+1,\,\alpha'+1;z)\\ U(-n,n+\alpha'+\beta'+1,\,\alpha'+1;z)\,,\end{cases} \qquad (3.7)$$

where $n$ satisfies Eq. (3.6b). The regularity conditions impose constraints on the parameters appearing in the solutions. We shall not discuss the details here.

(c) $t=x^2$, $h=\frac{1}{2}(x^2)^2+\beta$: For this case and the next, we simplify (3.4) by using the additional fact that $\partial/\partial x^3$, is a Killing vector. The solution may then be written in the form

$$\psi = Z(x^0)\chi(x^2)e^{ikx'}e^{imx^3}. \tag{3.8}$$

Equation (3.4), with the use of (3.8), gives the equation for

$$\left(\frac{d}{dx^2}+\frac{1}{t}(kh+m+t_{,2})\right)\times\left(\frac{d}{dx^2}-\frac{1}{t}(kh+m)\right)\chi+\alpha\chi=0. \tag{3.9}$$

Substituting for $t$ and $h$ we obtain

$$\frac{d^2\chi}{d(x^2)^2}+\frac{1}{x^2}\frac{d\chi}{dx^2}+\left(\alpha-k-\frac{1}{(x^2)^2}[\tfrac{1}{2}k(x^2)^2+m]^2\right)\chi=0. \tag{3.10}$$

Setting $y=\frac{1}{2}k(x^2)^2$, we obtain the two linearly independent solutions in terms of confluent hypergeometric functions:

$$\chi=e^{-k(x^2)^2/4}x^{2m}\times\begin{cases}F(1+m-\alpha/2k,\ 1+m;\tfrac{1}{2}k(x^2)^2)\\ U(1+m-\alpha/2k,\ 1+m;\tfrac{1}{2}k(x^2)^2).\end{cases}$$

(d) $t=\text{const}=A$, $h=Bx^2+C$: Equation (3.4) reduces to

$$\frac{d^2\chi}{d(x^2)^2}+[\alpha-A'-(A'x^2+\beta')^2]\chi=0, \tag{3.11}$$

where

$$A'=kB/A,\quad B'=(kC+m)/A.$$

In the particular case wherein $h$ is a constant, i.e., $A'=0$, the solutions for $\chi$ are of the form $e^{\pm i(\alpha-B'^2)^{1/2}x^2}$. In the general case we define a new coordinate $z$ by

$$z=A'x^2+B.$$

Then the equation assumes the form

$$\frac{d^2\chi}{dz^2}+z^2\chi=\mu\chi, \tag{3.12}$$

where $\mu$ is a constant. Substituting $x=-iz^2$, it is possible to convert (3.12) to a confluent hypergeometric equation whose solutions are the following:

$$\chi=e^{iz^2/2}z^{3/2}\times\begin{cases}F\left(\tfrac{3}{4}-\dfrac{\mu}{4i},\ \tfrac{3}{2};\ -iz^2\right)\\ U\left(\tfrac{3}{4}-\dfrac{\mu}{4i},\ \tfrac{3}{2};\ -iz^2\right).\end{cases}$$

*Case (i).* The discussion of case (i) would run on similar lines with $h(x^2)$ replaced by $y(x^2)$ and with $k$ replaced by $\omega$. This concludes the general discussion of the angular operator in the governing equations.

### B. The solutions of the radial-temporal operator

These solutions represent the part complementary to the angular operator solutions already discussed. The separation constant $\alpha$, which occurs in each of the operators, is in some cases restricted to a fixed set of real values as it is the eigenvalue of the angular operator. This will have to be taken into account when one looks for solutions to the radial-temporal part of the operator.

In case (i), $\partial/\partial x^0$ is a Killing vector, so the operator reduces to one which is purely radial in character. The solution has the simple time dependence of the form $e^{i\omega x^0}$, where $\omega$ is a constant. The operator essentially describes the space development of the Debye potentials. In case (iii), $\partial/\partial x^1$ is a Killing vector and the solution has $x^1$ dependence of the form $e^{ikx^1}$. The operator reduces only to a temporal operator. The solutions describe the time development of the Debye potentials. In case (ii), there is no reduction of the operator to a case which is purely temporal or radial in character and, in general, one has to treat the interlocked space-time development of the electromagnetic field. But in some of the particular cases that we investigate in the following section, which fall into this category, the symmetry inherent in geometry reduces the operator to a case which is purely temporal in character.

The radial-temporal equations for cases (i) and (iii) have the same mathematical form although the physical content differs. As before, we discuss case (iii) in detail and then indicate the changes in the solutions of case (iii) to be made in order to obtain the solutions of case (i). We first list the radial-temporal equations for the three cases. The similarities in the form of the equations for the cases (i) and (iii) will become immediately apparent on listing the equations.

*Case (i).* Setting $\psi=e^{i\omega x^0}Z(x^1)\chi(x^2,x^3)$, we have a separated equation for $Z(x^1)$, i.e.,

$$\frac{d^2Z}{d(x^1)^2}-\left(\frac{F_{,1}}{F}+\frac{ic}{Fy^2}\right)\frac{dZ}{dx^1}+\left(\omega^2F^2+\frac{c\omega}{y^2}-\frac{\alpha}{y^2}\right)Z=0. \tag{3.13}$$

*Case (ii).* Setting $\psi=Z(x^0,x^1)\chi(x^2,x^3)$, the separated equation is

$$F^2\frac{\partial^2 Z}{\partial(x^0)^2}+\frac{F}{X}(XF)_{,0}\frac{\partial Z}{\partial x^0}-\frac{1}{X^2}\frac{\partial^2 Z}{\partial(x^1)^2}-\frac{F}{X}\left(\frac{1}{XF}\right)_{,1}\frac{\partial Z}{\partial x^1}+\alpha\frac{Z}{Y^2}=0. \tag{3.14}$$

*Case (iii).* Setting $\psi = Z(x^0)e^{ikx'}\chi(x^2, x^3)$, the separated equation is

$$\frac{d^2Z}{d(x^0)^2}+\left(\frac{X_{,0}}{X}+ic\frac{X}{Y^2}\right)\frac{dZ}{dx^0}+\left(\frac{k^2}{X^2}-\frac{ck}{Y^2}+\frac{\alpha}{Y^2}\right)Z=0. \quad (3.15)$$

Equation (3.13) can be transformed to (3.15) by making the following changes:

$$x' \to x^0,$$
$$F \to 1/X,$$
$$c \to -c,$$
$$\omega \to k,$$
$$\alpha \to -\alpha.$$

Hence, it is sufficient to discuss only one of Eqs. (3.13) or (3.15). We discuss Eq. (3.15) of case (iii).

*Case (iii).* Defining a new time variable $u$ by the relation

$$u=\int_a^{x^0}\frac{dx^0}{X(x^0)}, \quad (3.16)$$

and substituting in (3.15), we have the equation

$$\frac{d^2Z}{du^2}+ic\frac{X^2}{Y^2}\frac{dZ}{du}+\left(k^2-\frac{X^2}{Y^2}(ck-\alpha)\right)Z=0. \quad (3.17)$$

The term involving the first derivative in $u$ can be "transformed away" by defining a new dependent variable $\tilde{Z}$, where

$$\tilde{Z}=Z\exp\left(\frac{i}{2}c\int\frac{X^2}{Y^2}du\right). \quad (3.18)$$

The transformation (3.18) reduces Eq. (3.17) to an "effective potential" type. The transformed equation assumes the form

$$\frac{d^2\tilde{Z}}{du^2}+\left[k^2-\frac{X^2}{Y^2}\left(ck-\alpha-\frac{c^2}{4}\right)-\frac{ic}{4}\frac{d}{du}\left(\frac{X^2}{Y^2}\right)\right]\tilde{Z}=0. \quad (3.19)$$

We remark that the above equation poses difficulties when one attempts to solve it by the conventional effective potential methods. If we write the expression multiplying $\tilde{Z}$ in (3.19) as $k^2-V_{\rm eff}$, then $V_{\rm eff}$ has the following properties:

(i) $V_{\rm eff}$ is complex.

(ii) $V_{\rm eff}$ depends on $k$ through the separation constant $\alpha$.

A possible interpretation of (ii) is that it is velocity dependent. It is possible to obtain solutions to (3.19) by the WKB method in the special case wherein condition for the validity of the WKB approximation is satisfied; that is, if we write the equation (3.19) as

$$\frac{d^2\tilde{Z}}{du^2}+p^2\tilde{Z}=0, \quad (3.20)$$

where

$$p^2(u)=k^2-\frac{X^2}{Y^2}\left(ck-\alpha-\frac{c^2}{4}\right)-\frac{ic}{2}\frac{d}{du}\left(\frac{X^2}{Y^2}\right),$$

then $|(1/p^2)dp/du|\ll 1$. This condition is satisfied if the functions $X, Y$ are slowly varying functions, in which case we may write $p^2=\rho e^{i\theta}$, where $\rho$ and $\theta$ are functions of $u$. The WKB solution of (3.20) is then

$$\tilde{Z}\sim\exp\left(\pm i\int\rho^{1/2}(\cos\tfrac{1}{2}\theta+i\sin\tfrac{1}{2}\theta)du\right),$$

and hence the temporal part of case (iii) has the solution

$$Z\sim\exp\left[\pm i\int\left(\rho^{1/2}\cos\tfrac{1}{2}\theta\mp\tfrac{1}{2}c\frac{X^2}{Y^2}\right)du\right]$$
$$\times\exp\left[\mp\int\rho^{1/2}\sin\tfrac{1}{2}\theta\,du\right].$$

The factor $\exp[\mp\int\rho^{1/2}\sin\frac{1}{2}\theta\,du]$ in the solution describes the damping or the amplification according to whether the sign in the exponent is negative or positive, respectively. The particular cases which we consider in the next section are cosmological models and consequently the functions $X$ and $Y$, which are functions of the time, vary on the cosmological time scales. So if we consider waves short as compared to the cosmological time scales, the WKB approximation is fully justified. In this approximation the damping (or amplification) factor is also very close to unity. In fact, the exponent in the damping factor may be approximated to

$$\tfrac{1}{2}\theta\rho^{1/2}\sim\frac{c\dfrac{d}{du}\dfrac{X^2}{Y^2}}{4\left|k^2-\dfrac{X^2}{Y^2}(ck-\alpha-\frac{1}{4}c^2)\right|},$$

which is close to zero since $(d/du)(X^2/Y^2)\ll 1$.

*Case (ii).* Equation (3.14) is more complicated than in the other cases being a partial differential equation in $x^0$ and $x^1$. The functions $X$, $Y$, and $F$ are, in general, functions of $x^0$ and $x^1$. The equation can be solved and useful information obtained if we introduce slight specialization. The full general case seems too general to be tractable. If we assume $F=F(x^0)$, $X=X(x^1)$, and $Y$ as a function of either $x^0$ or $x^1$ but not both, the equation may be solved by the separation of variables $x^0$ and $x^1$. If we assume for the sake of definitiveness $Y=Y(x^1)$, and write $Z(x^0,x^1)=L(x^0)M(x^1)$ in (3.14), the equation may be separated as follows:

$$\frac{1}{L}\left(F^2\frac{d^2L}{d(x^0)^2}+F\frac{dF}{dx^0}\frac{dL}{dx^0}\right)$$
$$=\frac{1}{M}\left[\frac{1}{X^2}\frac{d^2M}{d(x^1)^2}+\frac{1}{X}\frac{d}{dx^1}\left(\frac{1}{X}\right)\frac{dM}{dx^1}-\frac{\alpha}{Y^2}\right]$$
$$=-\beta^2\ \ (\text{say}), \quad (3.21)$$

where $\beta^2$ is a separation constant.
The equation for $x^0$ may be further simplified by redefining the time parameter $u$,

$$u=\int \frac{dx^0}{F}, \tag{3.22}$$

which gives

$$d^2L/du^2+\beta^2L=0, \tag{3.23}$$

whose solutions are $e^{\pm i\beta u}$.

The equation for $x^1$ is a little more involved due to the presence of the additional term $\alpha/Y^2$, where $Y$ is a function of $x^1$. The equation may be simplified as before by making a similar transformation

$$v=\int X\,dx^1, \tag{3.24}$$

which gives

$$\frac{d^2M}{dv^2}+\left(\beta^2-\frac{\alpha}{Y^2}\right)M=0. \tag{3.25}$$

This equation is in the form of the effective potential one, with the effective potential $\alpha/Y^2$. This could be easily solved with the aid of standard methods in certain special cases. For example, when $\beta$ is large and $Y$ is slowly varying, the WKB method is useful and the solutions are of the form

$$\exp\left[\pm i\int\left(\beta^2-\frac{\alpha}{Y^2}\right)^{1/2}dv\right].$$

For $\beta^2>\alpha/Y^2$, the solution is oscillatory; otherwise, it is damped.

## IV. SPECIFIC EXAMPLES

In this section we treat a few particular space-times which will supplement those already investigated by the authors mentioned above. The specific examples are specializations of cases (ii) and (iii). There is an important example of case (i), that of the Godel universe, but this has already been studied in detail.[4] The following are the space-times at which we direct our attention: (1) Kantowski-Sachs universes. (2) Spatially homogeneous anisotropic cosmologies. (3) Taub space. The first two of these are examples of case (ii), while the third comes under cases (iii).

### A. Kantowski-Sachs universes

Kantowski and Sachs obtained solutions of Einstein's equations (for dust) which are homogeneous, irrotational, and anisotropic. There are essentially two types of solutions corresponding to closed and open universes. The solutions are described by the following metrics $n$.

*Case 1:*

$$ds^2=-dt^2+X^2(t)dr^2+Y^2(t)(d\theta^2+\sin^2\theta\,d\varphi^2),$$

where

$$X(t)=\epsilon+(\epsilon\eta+b)\tan\eta,$$

$$Y(t)=a\cos^2\eta,$$

$$t-t_0=a(\eta+\tfrac{1}{2}\sin^2\eta);$$

where $a$, $b$, and $\epsilon$ are constants satisfying

$$-\infty<a<\infty,\quad a\neq 0,\quad -\frac{\pi}{2}\leq b<0,\quad \epsilon=0 \text{ or } 1.$$

These solutions represent closed universes.

*Case 2:*

$$ds^2=-dt^2+X^2(t)dr^2+Y^2(t)(d\theta^2+\sinh^2\theta\,d\varphi^2).$$

The solutions are further divided into two classes:

(a) $X=\epsilon-(\epsilon\eta+b)\tanh\eta,$

$Y=a\cosh^2\eta,$

$t-t_0=a(\eta+\tfrac{1}{2}\sinh^2\eta),$

(b) $X=\epsilon-(\epsilon\eta+b)\coth\eta,$

$Y=a\sinh^2\eta,$

$t-t_0=a(\eta-\tfrac{1}{2}\sinh^2\eta),$

where

$$-\infty<a<\infty,\quad a\neq 0,\quad 0\leq b<\infty,\quad \epsilon=0 \text{ or } 1.$$

These solutions correspond to open universes. It is easily seen that all these solutions are specializations of case (ii) discussed in the previous section. The equations for the Debye potentials simplify a great deal as $X, Y$ are functions of $t$ only, and $F=1$. Also, the metric tensor is independent of $r$, so that $\partial/\partial r$ is a Killing vector and the $r$ dependence of $\psi$ is simply $e^{\pm ikr}$.

*Case 1.* We have in this case the functional form of $t$ as $t(\theta)=\sin\theta$. From the foregoing the angular operator solutions are $Y_{lm}(\theta,\varphi)$, and the separation constant $\alpha$ is $l(l+1)$. The complete solution for $\psi$ may be written as

$$\psi=Z_{lk}(t)e^{ikr}Y_{lm}(\theta,\varphi), \tag{4.1}$$

where $Z_{lk}(t)$ satisfies the equation

$$\frac{d^2Z_{lk}}{dt^2}+\frac{1}{X}\frac{dX}{dt}\frac{dZ_{lk}}{dt}+\left(\frac{k^2}{X^2}+\frac{l(l+1)}{Y^2}\right)Z_{lk}=0. \tag{4.2}$$

We have used (3.14) to arrive at the above equation. We again make the familiar transformation of the time coordinate $t$. We define $u$ as $u=\int dt/X(t)$, which immediately transforms the equation to the effective potential form

$$\frac{d^2Z_{lk}}{du^2}+\left(k^2+l(l+1)\frac{X^2}{Y^2}\right)Z_{lk}=0\,. \tag{4.3}$$

We notice that the term multiplying $Z_{lk}$ is nonnegative, from which we may conclude that the solutions of (4.3) are always oscillatory. If $k$ is sufficiently large, that is, during one oscillation period of the solution $X$ and $Y$ change very little, then one may apply the WKB method to obtain the solution.

In the Kantowski-Sachs solution, we assume the range for $\eta$ to be $(-\pi/2, \pi/2)$. The 2-spheres $r=\text{const}$, $t=\text{const}$, expand from zero radius at $\eta=-\pi/2$ to a maximum radius at $\eta=0$, and again contract to zero radius at $\eta=\pi/2$. In Fig. 1 $X^2/Y^2$ is plotted as a function of $\eta$ for $\epsilon=0$ and $\epsilon=1$ over the entire range of $\eta$, for typical values of $a$ and $b$, $a=1$, $b=-1$. One sees from the figure that $X^2/Y^2 \to \infty$ as $\eta\to\pm\pi/2$, and reaches a maximum at $\eta=0$. This shows that $Z_{lk}$ oscillates rapidly when the radius of the 2-spheres is small, that is, during the beginning and at the end of the universe. The oscillations slow down when $\eta$ is close to zero. The behavior is more pronounced for $\epsilon=1$ than for $\epsilon=0$. In the case $\epsilon=0$, when $\eta\simeq 0$ the solution for $Z_{lk}\simeq e^{iku}$ as $X^2/Y^2\simeq 0$. The behavior of $Z_{lk}$ also depends significantly on the angular momentum $l$. The oscillation of $Z_{lk}$ accelerates with increasing angular momentum.

*Case 2.* The function $t(\theta)$ in this case is $\sinh\theta$. From the discussion of angular solutions of Sec. III we may write the solution for $\psi$ in the form

$$\psi=Z_{lk}(t)e^{ikr}Y_{lm}(-i\theta,\varphi)\,. \tag{4.4}$$

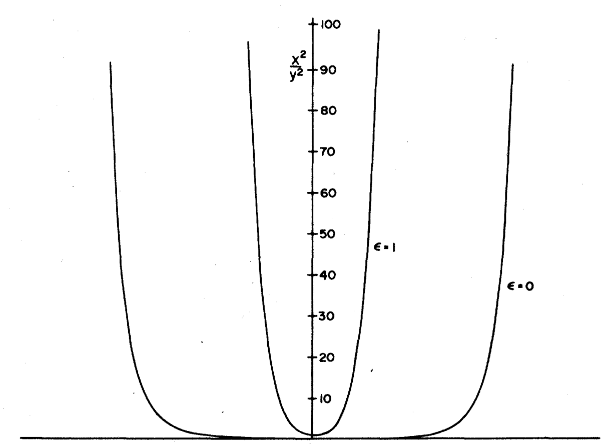


FIG. 1. The figure shows the plots of $X^2/Y^2$ as a function of $\eta$ for the values of $\epsilon=0,1$ of the case (i) of the Kantowski-Sachs universes, with the parameters $a=1$ $b=-1$. $X^2/Y^2$ tends to infinity as $\eta$ appraoches $\pm\pi/2$ for both the cases, the difference being that in the $\epsilon=1$ case, the curve is asymmetric about $\eta=0$ and steeper in the event of $\eta$ tending to $\pm\pi/2$. The actual effective potential is scaled by the factor $l(l+1)$.

The separation constant $\alpha=-l(l+1)$.

Redefining the time parameter $u$ as in case 1, we have the equation for $Z_{lk}(u)$ in this case as

$$\frac{d^2Z_{lk}}{du^2}+\left(k^2-l(l+1)\frac{X^2}{Y^2}\right)Z_{lk}=0\,. \tag{4.5}$$

A considerable amount of information may be obtained by examining the effective potential curves plotted in Figs. 2(a) and 2(b) of the function $X^2/Y^2$ vs $\eta$, with $a=1$, $b=-1$. The parameter $\eta$

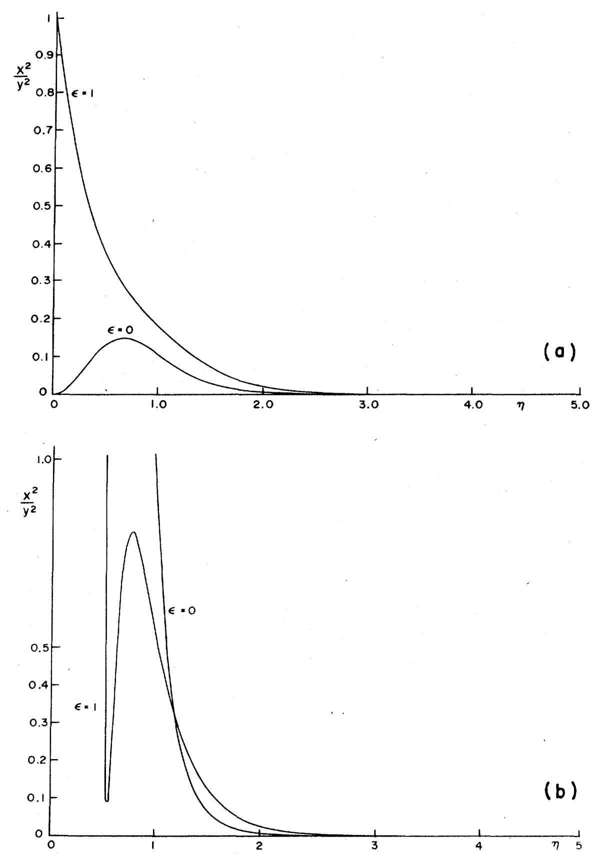


FIG. 2. (a) The figure depicts the plots of $X^2/Y^2$ (unscaled effective potential) as a function of $\eta$ for $\epsilon=0,1$ of case 2(a) of the Kantowski-Sachs universes with the parameters $a=1$, $b=-1$. In the $\epsilon=0$ case the curve has a maximum for an intermediate value of $\eta$ and tends to zero when $\eta$ tends to infinity or zero. In the $\epsilon=1$ case, $X^2/Y^2$ monotonically decreases to zero as $\eta$ ranges from zero to infinity. (b) The unscaled effective potential $X^2/Y^2$ is drawn as a function of $\eta$ for each of the cases $\epsilon=0,1$ of case 2(b) of the Kantowski-Sachs universes. The parameters $a,b$ have the values $a=1$, $b=-1$. In the $\epsilon=0$ case, $X^2/Y^2$ decreases monotonically from infinity to zero as $\eta$ ranges from zero to infinity. The $\epsilon=1$ case shows more interesting behavior in that the potential has a maximum and a minimum for intermediate values in the range of $\eta$. As $\eta$ tends to zero, $X^2/Y^2$ tends to infinity, and as $\eta$ tends to infinity, $X^2/Y^2$ tends to zero.

ranges from 0 to $\infty$. A glance at the figures in question shows that as $\eta\to\infty$, $X^2/Y^2\to 0$ and hence the solution for $Z_{lk}$ at large times may be written as $e^{\pm iku}$ for all four cases: $\epsilon=0$, 1 for cases 2(a) and 2(b). In case 2(a), where $\epsilon=0$, the effective potential has a maximum, and hence for a small enough value of $k$ and a sufficiently large value of $l$, $k^2-l(l+1)X^2/Y^2$ is negative in some interval $I=(\eta_1,\eta_2)$. $Z_{lk}$, which is oscillatory in nature in the beginning ($\eta\simeq 0$), will get damped during the time interval $I$ and after $\eta=\eta_2$, will begin oscillating. For $\epsilon=1$, $X^2/Y^2$ monotonically falls off as $\eta$ ranges from 0 to $\infty$. The oscillations of $Z_{lk}$ will slowly increase in their rapidity until they asymptotically approach the frequency of the wave $e^{\pm iku}$.

In case 2(b), it is the $\epsilon=0$ case which behaves more like the $\delta=1$ case of 2(a), except that in this case $X^2/Y^2\to\infty$ as $\eta\to 0$ and decreases monotonically to zero as $\eta\to\infty$. The decrease is more dramatic than in case 2(a), where $\epsilon=1$ for the same values of the parameters $a$ and $b$. For a given $k$ and $l$ and for sufficiently small $\eta$, the solution for $Z_{lk}$ is damped. After the parameter $\eta$ attains a certain value $\eta_0$, the root of $k^2-l(l+1)X^2/Y^2=0$, $Z_{lk}$ will start to oscillate and will rapidly reach the asymptotic wave form $e^{\pm iku}$. In the $\epsilon=1$ cases, $X^2/Y^2$ has both a maximum and a minimum which accounts for a more interesting behavior of the Debye potential. There will be essentially be two types of behavior depending on the values of $l$ and $k$:

(i) $\dfrac{k^2}{l(l+1)}\le\max\dfrac{X^2}{Y^2}$.

(ii) $\dfrac{k^2}{l(l+1)}>\max\dfrac{X^2}{Y^2}$.

The $\max(X^2/Y^2)$ does not mean the maximum value of $X^2/Y^2$ in the entire range of $\eta$ considered, but denotes the value of $X^2/Y^2$ attained when the function $X^2/Y^2$ has zero slope. In the former case there will be four zones along the $\eta$ axis: $(0,\eta_1)$, $(\eta_1,\eta_2)$, $(\eta_2,\eta_3)$, and $(\eta_0,\infty)$, where $0<\eta_1<\eta_2<\eta_3<\infty$. In the zones $(0,\eta_1)$ and $(\eta_2,\eta_3)$, $k^2/l(l+1)<X^2/Y^2$ and consequently, the solution will be damped. For the other two zones, where $k^2/l(l+1)>X^2/Y^2$, the solution will be oscillatory.

In the latter case, i.e., case (ii), there are only two zones $(0,\eta_0)$ and $(\eta_0,\infty)$, accordingly as $k^2/l(l+1)<X^2/Y^2$ or $k^2/l(l+1)>X^2/Y^2$. In the first zone the solution will be damped and after $\eta$ has exceeded $\eta_0$, the solution will become oscillatory and approach the solution $e^{\pm iku}$.

For case (2), in general, the angular momentum has a damping effect on the wave, that is, large values of angular momentum tend to slow down the oscillations and may later cause damping. This effect is the opposite of that observed in case (1).

### B. Spatially homogeneous anisotropic cosmologies

These are cosmological models which differ in symmetry properties from the above discussed models. The geometry is described by the line element

$$ds^2=-dt^2+X^2dx^2+Y^2dy^2+Z^2dz^2\,, \tag{4.6}$$

where $X$, $Y$, and $Z$ are functions of $t$. The space-time described by this line element does not in general possess local rotational symmetry unless at least two of the functions $X$, $Y$, and $Z$ are equal. Without loss of generality we may assume $Y=Z$. This introduces rotational symmetry in the geometry, thus making it possible to study the behavior of the Hertz potential by the methods described above.

The functions $X, Y$ have the following functional form:

$$X=t[\tfrac{9}{2}Mt(t+\Sigma)]^{-1/3}\,,\qquad Y=[\tfrac{9}{2}Mt(t+\Sigma)]^{2/3}\,, \tag{4.7}$$

where $\Sigma>0$ and represents departure from isotropy. The metric (4.6) fits the form of the metric given by (2.1) if we set $F=1$, $t=1$, $h=y=0$, and $X, Y$ as defined in (4.7). We use Eq. (3.14) of case (ii) with the above simplifications, and write the solution for the scalar function $\psi$ as

$$\psi=e^{i\vec{k}\cdot\vec{r}}Z(t)\,, \tag{4.8}$$

where $\vec{k}=(k_1,k_2,k_3)$ and $\vec{r}=(x,y,z)$. As done earlier, Eq. (3.14) may be written in terms of the parameter $u$ defined by

$$u=\int\frac{dt}{X(t)}\,,$$

which immediately yields the following equation for $Z$,

$$\frac{d^2Z}{du^2}+\left(k_1{}^2+\frac{X^2}{Y^2}(k_2{}^2+k_3{}^2)\right)Z=0\,, \tag{4.9}$$

where

$$\frac{X^2}{Y^2}=\left[\tfrac{2}{9}M\left(\frac{1}{t+\Sigma}\right)\right]^2.$$

Since the coefficient of $Z$ appearing in (4.9) is always non-negative, the solution is always oscillatory. For large $t$, $X^2/Y^2\to 0$, and hence the solution has the form $e^{\pm ik_1u}$. The anisotropy parameter $\Sigma$ is important only when the value of $t$ is not large compared to it. Its effect is to slow down the oscillation. When $t\to 0$, the solution for $Z$ is simply given by

$$\exp\{\pm i[k_1{}^2+(2/9M\Sigma)^2(k_2{}^2+k_3{}^2)]^{1/2}u\}.$$

### C. Taub space

The Taub space is a vacuum solution of Einstein's equations. It is described by the line element

$$ds^2 = -\frac{1}{U}dt^2 + (2n)^2 U(d\chi + \cos\theta\, d\varphi)^2 + (t^2+n^2)(d\theta^2 + \sin^2\theta\, d\varphi^2)\,, \tag{4.10}$$

where $U(t) = -1 + 2(mt+n^2)/(t^2+n^2)$, with $m, n$ as positive constants. The metric is singular at $t = t_\pm = m \pm (m^2+n^2)^{1/2}$. The metric can be extended across the surfaces $t = t_\pm$ giving the space found by Newman, Unti, and Tamburino. The transformation of coordinates for this extension is too complex to investigate the behavior of the scalar $\psi$ in the NUT region of the space-time. We therefore restrict ourselves to the Taub space defined by (4.10), and $t$ restricted to $t_- < t < t_+$. This space belongs to the class of space-times under consideration and can be admitted under case (iii). The metric (4.10) is of the same form as in (2.1) if we first make a change in the time coordinate $t$ to $x^0$ by defining

$$x^0 = \int_0^t \frac{dt}{[U(t)]^{1/2}}\,. \tag{4.11}$$

Then we set in (2.1),

$$x^1 = \chi,\quad x^2 = \theta,\quad x^3 = \varphi,\quad F = 1,\quad X = 2n\sqrt{U}\,,$$

$$Y = (t^2+n^2)^{1/2},\quad t = \sin\theta,\quad h = -\cos\theta\,.$$

From the foregoing, we may assume the solution for $\psi$ to be

$$\psi = Z(x^0)e^{ikx^1}\chi(x^2, x^3)\,.$$

From the results of Sec. III, the solutions for the angular part of the operator acting on $\psi$ are spin-weighted spherical harmonics[6] ${}_{-k}Y_{lm}(\theta,\varphi)$ for integral values of $k$. For nonintegral values of $k$, the solutions would be analytic continuations of ${}_kY_{lm}$, namely, Jacobi polynomials. The separation constant is $\alpha = -(l+k)(l-k+1)$. The equation for the temporal part may be found by making the transformations (3.16) and (3.18). The equation for $\tilde{Z}$ as defined from (3.18) with the above value of $\alpha$ is

$$\frac{d^2\tilde{Z}}{du^2} + \left[k^2 - \frac{X^2}{Y^2}[k + (l+k)(l-k+1) - \tfrac{1}{4}] - \tfrac{1}{2}i\frac{d}{du}\left(\frac{X^2}{Y^2}\right)\right]\tilde{Z} = 0\,. \tag{4.12}$$

We have $X^2/Y^2 = 4n^2U/(t^2+n^2)$, and its derivative with respect to $u$ is

$$\frac{d}{du}\left(\frac{X^2}{Y^2}\right) = \frac{16n^3U}{(t^2+n^2)^2}[m - t(1+2U)]\,. \tag{4.13}$$

We notice that for values of $t$ close to $t_\pm$, the expressions $X^2/Y^2$ and $(d/du)(X^2/Y^2)$ are very small, and hence the solution for $\tilde{Z}$ is of the form $e^{\pm iku}$. The parameter $u$ may be explicitly given in terms of $t$ by integrating the equation $du/dt = 1/2nU$ with the condition $u(0) = 0$. The integration gives

$$u = -\frac{1}{2n}\left[t + t_+\ln\left(1 - \frac{t}{t_+}\right) + t_-\ln\left(1 - \frac{t}{t_-}\right)\right]. \tag{4.14}$$

As $t \to t_\pm$, $u \to \pm\infty$.

The imaginary part in the effective potential of Eq. (4.12) appears as damping or amplification of the Debye-potential wave. In the high-frequency approximation, i.e., when the period of the wave is small as compared to the age of the Taub space, the damping or the amplification depends on the sign of $(d/du)(X^2/Y^2)$. From Eq. (4.13) we observe that $(d/du)(X^2/Y^2)$ is positive in the beginning near $t \simeq t_-$, becomes zero at some intermediate epoch $t = t_0$, $t_- < t_0 < t_+$, and then becomes negative and reaches zero at $t = t_+$. This lends itself to the following interpretation. The wave which is initially getting amplified will continue its amplification until $t$ attains the value $t_0$ and then will be damped from $t_0$ to $t_+$. The value of $t_0$ is easily found by solving the cubic $m - t(1+2U) = 0$. We get the result as

$$t_0 = m - 2(m^2+n^2)^{1/2}\cos\frac{\pi+\phi}{3}\,, \tag{4.15}$$

where

$$\phi = \tan^{-1}(n/m),\quad 0 < \phi < \pi/2\,.$$

From the rest of the effective potential the following conclusions may be drawn:

(i) For large values of $k$ the solution for $\tilde{Z}$ is oscillatory and of the form $e^{\pm ik}[1 + (X^2/Y^2)]^{1/2}$.

(ii) For large values of both $k$ and $l$, i.e., $k, l \gg 1$ and $k \sim O(l)$, the expression multiplying $\tilde{Z}$ takes the approximate form

$$k^2\left(1 + \frac{X^2}{Y^2}\right) - \frac{X^2}{Y^2}l^2\,.$$

From this we may conclude that $\tilde{Z}$ is oscillatory if $|k/l| > |X/(X^2+Y^2)^{1/2}|$; otherwise, it is damped.

(iii) For small values of $k$ and large values of $l$, the solution is heavily damped except near $t = t_\pm$, the effective potential being of the form $(X^2/Y^2)l(l+1)$. Near $t_\pm$, the function $X^2/Y^2 \simeq 0$, so that for sufficiently small departures of $t$ from $t_\pm$, $k^2$ exceeds $(X^2/Y^2)l(l+1)$ and the solution is oscillatory and of the form $\exp\{i[k^2 - (X^2/Y^2)l(l+1)]^{1/2}\}$.

(iv) When $k$ and $l$ both are small, that is, of the order of $(d/du)(X^2/Y^2)$, then the full equation has to be taken into account. It may be possible in in such a case to integrate the equation numerically.

## V. CONCLUSION

Our calculations show that the Hertz-Debye-potential formalism can be successfully applied to perfect-fluid space-times with local rotational symmetry. These include a large class of well-known space-times of which some have been discussed in the previous section. The space-times studied fall under the generalized Goldberg-Sachs class of space-times, and the techniques given in paper I have been utilized to obtain decoupled equations for the Debye potentials. With the help of the standard mathematical methods such as the WKB, Laplace transform, etc., the differential equation for the Debye potential can be solved. When the equation is too complicated—with or without approximations—it may be solved with the aid of numerical integration. From these potentials the electromagnetic fields can be investigated in its general form except for the monopole field ($l = 0$).

The foregoing studies reveal the influence of curved geometry on perturbative electromagnetic fields. In the specific examples considered, if the anisotropic cosmological models could be taken as idealizations describing the actual universe with initial anisotropies, then the nature of the superposed electromagnetic field could be of astrophysical interest.

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