Multipole analysis for electromagnetism and linearized gravity with irreducible Cartesian tensors

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(Received 9 July 1990)

The relativistic time-dependent multipole expansion for electromagnetism and linearized gravity in the region outside a spatially compact source has been obtained directly using the formalism of irreducible Cartesian (i.e., symmetric trace-free) tensors. In the electromagnetic case, our results confirm the validity of the results obtained earlier by Campbell, Macek, and Morgan using the Debye potential formalism. However, in the more complicated linearized gravity case, the greater algebraic transparence of the Cartesian multipole approach has allowed us to obtain, for the first time, fully correct closed-form expressions for the time-dependent mass and spin multipole moments (the results of Campbell *et al.* for the mass moments turning out to be incorrect). The first two terms in the slow-motion expansion of the gravitational moments are explicitly calculated and shown to be equivalent to earlier results by Thorne and by Blanchet and Damour.

I. INTRODUCTION

For sources localized in a finite region of space, the multipole decomposition is one of the most convenient and useful ways of describing the external field. Both the derivation and structure of the multipole expansion are simple in the stationary scalar case (appropriate to electrostatics and Newtonian gravity) where the external solution of the Poisson equation

$$\Delta\phi=-4\pi\rho \tag{1.1}$$

can be written either as

$$\phi(\mathbf{X})=4\pi\sum_{l\geq 0}\sum_{-l\leq m\leq l}\frac{Q_{lm}}{2l+1}\frac{Y_{lm}(\Theta,\Phi)}{R^{l+1}} \tag{1.2a}$$

or

$$\phi(\mathbf{X})=\sum_{l\geq 0}\frac{(-)^l}{l!}Q_{i_1\cdots i_l}\partial_{i_1\cdots i_l}\left[\frac{1}{R}\right], \tag{1.2b}$$

where, respectively,

$$Q_{lm}=\int d^3x\, Y^*_{lm}(\theta,\varphi)r^l\rho(\mathbf{x}) \tag{1.3a}$$

or

$$Q_{i_1\cdots i_l}=\int d^3x\, x^{\langle i_1}\cdots x^{i_l\rangle}\rho(\mathbf{x})\ , \tag{1.3b}$$

where we use capital letters for the field point and lowercase ones for the source point, $R=|\mathbf{X}|$, $\partial_i=\partial/\partial X^i$, and where the angular brackets on the right-hand side of Eq. (1.3b) mean "symmetrize and take the trace-free part." All the objects appearing in Eqs. (1.2) and (1.3) are evaluated at the same "instantaneous" time. In this simple case the "Y_{lm}-type" [Eq. (1.2a)] or the equivalent (see below) "symmetric and trace-free" (STF) [Eq. (1.2b)] multipole expansions are equally simple and equally convenient. On the other hand, the relativistic time-dependent higher-spin case (which involves inhomogeneous wave equations for vectorial or tensorial fields) is more complicated and less standardized. The usual route[1] consists of (i) reducing the wave equations to Helmoltz equations via a time-Fourier transform and (ii) decomposing the higher-spin fields in vectorial or tensorial spherical harmonics. However, this approach has the double disadvantage of (i) masking the fact that the relativistic time-dependent moments involve not only a spatial integration on the source, but also a time integration, and (ii) leading to somewhat unwieldy final expressions (because of the appearance of vectorial or tensorial spherical harmonics).

In an earlier investigation Campbell, Macek, and Morgan[2] (CMM) performed the multipole decomposition for both the scalar, electromagnetic, and linearized gravity cases without a time-Fourier transform and explicitly demonstrated that the time-dependent multipole moments are the (usual type) spatial moments of some effective source functions given as particular weighted time averages over the actual source distributions. Though the scalar case can be handled directly and elegantly using standard spherical harmonics [$Y_{lm}(\theta,\varphi)$], the extension to the vector and tensor cases is more involved. Rather than using directly vectorial and tensorial harmonics, CMM have reduced the problem to an equivalent scalar one, by means of the Debye potential formalism.[3] This formalism is elegant and works with gauge-invariant fields. However, it is somewhat indirect and obliges one to add by hand to the general radiative multipole moments the lower-order static multipoles $l=0$ for electromagnetism (spin = 1) and $l=0,1$ for linear gravitation (spin = 2). On the other hand, it is well known that a convenient alternative to the use of standard spherical harmonics consists of working with irreducible Cartesian tensors.[4] In three dimensions the

Cartesian tensors irreducible under the group of spatial rotations can be expressed in terms of symmetric and trace-free tensors. Let us recall that the set of STF tensors with l indices, $T_{\langle i_1 i_2 \cdots i_l\rangle}$, $i_p=1,2,3$ constitutes a $(2l+1)$-dimensional vector space which spans the same irreducible representation of SO(3) as the set of Y_{lm}, $-l \leq m \leq l$ [the equivalence between Eqs. (1.2a) and (1.2b) follows from this]. The STF formalism has the advantage of rendering quite transparent the algebraic structure of vectorial and tensorial harmonics expansions. Its usefulness in the context of gravitation has been particularly emphasized by Thorne[5] and confirmed by several recent investigations.[6,7] For the scalar case, Blanchet and Damour[7] (BD-II) have obtained directly using the STF formalism the multipole expansion of a retarded field in the region outside its spatially compact support, as also simple closed-form STF expressions for the multipole moments. These results were originally worked on by CMM using spherical harmonics. In the linearized gravity case,[8] Thorne[5] has derived the full slow-motion expansions of the mass and spin moments both in the STF and in the Y_{lm} forms, without, however, obtaining closed-form expressions for the moments. In the first post-Newtonian (1PN) approximation of general relativity, Blanchet and Damour[7] have derived the first two terms in the slow-motion expansion of the mass moments.

In this paper we shall show how the use of STF techniques makes possible an elegant generalization of the closed-form scalar results to the vector and tensor cases without going through the Debye potential route. The analysis includes, automatically, the lower-order moments as particular cases, on the same footing as the other moments. In principle, the final results for the radiative moments obtained via the STF technique should be equivalent to those obtained via the Debye potential technique. However, the somewhat greater algebraic transparence of the STF approach will allow us to detect an error in the final results of CMM and to get, for the first time, fully correct closed-form expressions for the time-dependent linearized gravity multipole moments.

The paper is organized as follows. In the next section we recall the principal features of irreducible Cartesian tensors and summarize the relevant new formulas we need for our analysis. In Sec. III the scalar case is summarized briefly for completeness. In Sec. IV we extend the formalism to deal with the electromagnetic case, display the electric and magnetic moments, discuss the $l=0$ moment, and compare our results with CMM. Section V treats, directly by STF methods, the case of linearized gravity. The retarded linearized gravitational field is expressed, everywhere outside the source, in terms of some relativistic mass and spin moments (modulo a gauge transformation). In the slow-motion limit, both our mass and spin moments agree with the earlier results of Thorne and BD-II at 1PN level in contrast with the mass moments of CMM, which do not. We compare our results with those obtained by the Debye potential technique earlier and study the $l=0,1$ moments in detail. We display different equivalent forms of the moments, and Sec. VI contains our concluding remarks.

II. IRREDUCIBLE CARTESIAN TENSORS OR SYMMETRIC TRACE-FREE FORMALISM

In this paper we employ the multi-index notation introduced by Blanchet and Damour[6] (BD-I). An upper-case Latin letter denotes a multi-index, while the corresponding lower-case letter denotes its number of indices. Thus $L=i_1 i_2 \cdots i_l$, $P=i_1 i_2 \cdots i_p$, $Q-1=i_1 i_2 \cdots i_{q-1}$. When many multi-indices appear simultaneously, it is implicit that distinct carrier letters are used, e.g., $A_{PQ}=A_{i_1 \cdots i_p j_1 \cdots j_q}$. For repeated multi-indices, summation is implied so that $A_P B^P = A_P B_P = \sum_{i_1 \cdots i_p} A_{i_1 \cdots i_p} B_{i_1 \cdots i_p}$ [spatial indices, $i,j=1,2,3$, are freely raised or lowered by means of the Cartesian metric $\delta_{ij}=\delta^{ij}=\mathrm{diag}(+1,+1,+1)$]. We also use $r=[\sum_{i=1}^{3}(x^i)^2]^{1/2}$, $n_i=x^i/r$, $\partial_i=\partial/\partial x^i$, $x^L=x^{i_1}\cdots x^{i_l}$, $n^L=n^{i_1}\cdots n^{i_l}$, $\partial_L=\partial_{i_1}\cdots\partial_{i_l}$, $l!!=l(l-2)(l-4)\cdots(2$ or $1)$. The totally antisymmetric Levi-Civita tensor is denoted by ϵ_{ijk} (with $\epsilon_{123}=+1$).

Given a Cartesian tensor A_P, we denote its symmetric part with parentheses so that

$$A_{(P)} \equiv A_{(i_1 \cdots i_p)} \equiv \frac{1}{p!}\sum_{\sigma} A_{i_{\sigma(1)} \cdots i_{\sigma(p)}} , \tag{2.1}$$

σ running over all permutations of $(1,2,\ldots,p)$.

The symmetric and trace-free part of A_P is denoted equivalently by $\hat{A}_P \equiv A_{\langle P\rangle} \equiv A_{\langle i_1 \cdots i_p\rangle}$ and sometimes as $\underset{P}{\mathrm{STF}}(A_P)$. The explicit expression for the STF part is[9]

$$\hat{A}_P = A_{\langle P\rangle} = \sum_{k=0}^{[p/2]} a_k^p \delta_{(i_1 i_2} \cdots \delta_{i_{2k-1} i_{2k}} S_{i_{2k+1} \cdots i_p) a_1 a_1 \cdots a_k a_k} , \tag{2.2a}$$

$$S_P \equiv A_{(P)} , \tag{2.2b}$$

$$a_k^p = (-)^k \frac{p!}{(p-2k)!} \frac{(2p-2k-1)!!}{(2p-1)!!(2k)!!} , \tag{2.2c}$$

$[p/2]$ denoting the integer part of $p/2$.

In particular, for $l=2$,

$$\hat{T}_{ij} = T_{(ij)} - \tfrac{1}{3}\delta_{ij} T_{ss} , \tag{2.3}$$

while, for $l=3$,

$$\hat{T}_{ijk} = T_{(ijk)} - \tfrac{1}{5}(\delta_{ij} T_{(kss)} + \delta_{jk} T_{(iss)} + \delta_{ki} T_{(jss)}) . \tag{2.4}$$

At the heart of the STF formalism are the important results that (i) the set of all symmetric-trace-free Cartesian tensors of rank l generates an irreducible representation of weight l and dimension $2l+1$ of the group of proper rotations SO(3), and (ii) any reducible tensor of rank p can be decomposed in a sum of algebraically independent pieces each of which belongs to an irreducible representation and hence is expressible in terms of some "brick" STF tensor of rank $\leq p$.

More explicitly, one can decompose any arbitrary tensor T_P into a finite sum of terms of the form $\gamma_P^L \hat{R}_L$, where γ_P^L is a tensor invariant under SO(3), i.e., made of products of ϵ_{ijk} and δ_{ij}, and where the "brick" $\hat{R}_L$ is an irreducible STF l tensor ($l \le p$) obtained by contracting T_P with δ's and ε's before applying a STF projection. This result, a generalization of the well-known result of decomposing an arbitrary matrix into its trace, antisymmetric part, and symmetric traceless part, may be proven by induction and corresponds in group theory to repeatedly expressing a product of representations in terms of direct sums of irreducible representations: $D_s \otimes D_l = D_{|l-s|} \oplus \cdots \oplus D_{l+s}$ ($\mathbf{j}=\mathbf{l}+\mathbf{s}$ addition rule in quantum language). The simplest case ($s=1$) reads, in STF notation[6],

$$U_i \hat{T}_L = \hat{R}^{(+)}_{iL} + \frac{l}{l+1}\epsilon_{si\langle i_l}\hat{R}^{(0)}_{L-1\rangle s} + \frac{2l-1}{2l+1}\delta_{i\langle i_l}\hat{R}^{(-)}_{L-1\rangle} , \tag{2.5}$$

where

$$\hat{R}^{(+)}_{L+1} = U_{\langle i_{l+1}}\hat{T}_{L\rangle} , \tag{2.6a}$$

$$\hat{R}^{(0)}_{L} = U_a \hat{T}_{b\langle L-1}\epsilon_{i_l\rangle ab} , \tag{2.6b}$$

$$\hat{R}^{(-)}_{L-1} = U_s \hat{T}_{sL-1} . \tag{2.6c}$$

A particular case of the above formula is when $U_i = n_i$, $\hat{T}_L = \hat{n}_L$. In this case we obtain simply

$$n_i \hat{n}_L = \hat{n}_{iL} + \frac{l}{2l+1}\delta_{i\langle i_l}\hat{n}_{L-1\rangle} . \tag{2.7}$$

Related formulas of direct use in later sections are

$$r\partial_i \hat{n}_L = (l+1)n_i\hat{n}_L - (2l+1)\hat{n}_{iL} , \tag{2.8}$$

$$\partial_i(\hat{x}_L) = l\delta_{i\langle i_l}\hat{x}_{L-1\rangle} , \tag{2.9}$$

$$r\partial_a n_b = \delta_{ab} - n_a n_b . \tag{2.10}$$

To address the case of gravitation (spin 2), i.e., the symmetric tensor case, we need to generalize Eqs. (2.5) and (2.6). In group-theoretic terms, we need to decompose into irreducible pieces the product $D_1 \otimes D_1 \otimes D_l$. This can be implemented by repeated use of Eqs. (2.5) and (2.6). This leads, for the decomposition of the tensorial product of a symmetric tensor of rank 2, S_{ij}, with a STF tensor of rank l, $\hat{T}_L$, to

$$S_{ij}\hat{T}_L = H^{(+2)}_{ijL} + \underset{ij}{\mathrm{STF}}\,(\epsilon_{ai\langle i_l}H^{(+1)}_{L-1\rangle ja} + \delta_{i\langle i_l}H^{(0)}_{L-1\rangle j} + \epsilon_{ai\langle i_l}H^{(-1)}_{L-2|a|}\delta_{i_{l-1}\rangle j} + \delta_{i\langle i_l}H^{(-2)}_{L-2}\delta_{i_{l-1}\rangle j}) + \delta_{ij}K_L , \tag{2.11}$$

where the vertical bars around the index a on the right-hand side mean that this index is excluded from the bracketed (STF) operation, and where

$$H^{(+2)}_{L+2} = \hat{S}_{\langle i_{l+2}i_{l+1}}\hat{T}_{L\rangle} , \tag{2.12a}$$

$$H^{(+1)}_{L+1} = \frac{2l}{l+2}\hat{S}_{c\langle i_l}\hat{T}^{d}_{L-1}\epsilon_{i_{l+1}\rangle cd} , \tag{2.12b}$$

$$H^{(0)}_{L} = \frac{6l(2l-1)}{(l+1)(2l+3)}\hat{S}_{a\langle i_l}\hat{T}_{L-1\rangle a} , \tag{2.12c}$$

$$H^{(-1)}_{L-1} = \frac{2(l-1)(2l-1)}{(l+1)(2l+1)}\hat{S}_{ca}\hat{T}_{bc\langle L-2}\epsilon_{i_{l-1}\rangle ab} , \tag{2.12d}$$

$$H^{(-2)}_{L-2} = \frac{2l-3}{2l+1}\hat{S}_{ab}\hat{T}_{abL-2} , \tag{2.12e}$$

$$K_L = \tfrac{1}{3}S_{aa}\hat{T}_L . \tag{2.12f}$$

As a particular case, we obtain

$$n_a n_b \hat{n}_L = \hat{n}_{abL} + \frac{l}{2l+3}(\hat{n}_{a\langle L-1}\delta_{i_l\rangle b} + \hat{n}_{b\langle L-1}\delta_{i_l\rangle a}) + \frac{\delta_{ab}\hat{n}_L}{(2l+3)} + \frac{l(l-1)}{(2l+1)(2l-1)} \times \hat{n}_{\langle L-2}\delta^{a}_{i_{l-1}}\delta^{b}_{i_l\rangle} . \tag{2.13}$$

In proving Eq. (2.11) and also performing simplifications later, one needs to make explicit or "peel" a particular fixed index in an STF expression. The following "peeling" formula is useful in these circumstances:

$$(l+1)V_{\langle i}\hat{T}_{L\rangle} = V_i\hat{T}_L + l\hat{T}_{i\langle L-1}V_{i_l\rangle} - \frac{2l}{2l+1}V_a\hat{T}_{a\langle L-1}\delta_{i_l\rangle i} . \tag{2.14}$$

In the above we have collected various formulas relevant to this paper. For a more complete list, the reader is referred to Thorne[5] and to Appendix A of BD-I.

Finally, our signature is $(-+++)$, spacetime indices range from 0 to 3, and are denoted by Greek indices, while space indices (1,2,3) are denoted by Latin indices. We use the summation convention for all repeated indices irrespective of their position. The flat metric is denoted by $f_{\mu\nu}$ with components diag$(-1,+1,+1,+1)$ in Lorentzian coordinates. We denote the field point by $X^\mu=(cT,\mathbf{X})$, while the source points are denoted by $x^\mu=(ct,\mathbf{x})$, where c is the velocity of light. Beware of the fact that we use the same notation ∂_μ for $\partial/\partial X^\mu$ and $\partial/\partial x^\mu$, since it is clear from the context what we mean. We denote $\Box \equiv f^{\mu\nu}\partial_\mu\partial_\nu \equiv \Delta - c^{-2}\partial^2/\partial t^2$, G denotes Newton's gravitational constant, and the electromagnetic units are Gaussian.

III. SCALAR FIELD

In order to make our presentation self-contained, we review briefly the scalar case in this section. The standard textbook discussion of the relativistic time-dependent multipole analysis is via the time Fourier transform and may be found, in e.g., Jackson.[1] This approach, however, does not make explicit the time integration linking multipole moments to the actual evolution of the source and motivated CMM (Ref. 2) to discuss multipole decomposition without time-Fourier expansion. Their analysis exhibits the time-dependent multipole moments in a form akin to the well-known stationary-case expressions with the source functions being replaced by a weighted time average over the source. The same result was reworked directly using STF methods in BD-II, and we shall summarize their treatment. Though for the scalar case the above two treatments are similar, the generalization via the STF route to the vector and tensor cases, as shown in this paper, is more straightforward than via the Debye potential route, the latter in addition having an inherent blind spot with regard to the constant low-order multipoles (i.e., $l<s$, where s is the spin of the considered field).

Consider a source $S(\mathbf{x},t)$ that is spatially confined in a region $|\mathbf{x}|<r_0$, i.e., $S(\mathbf{x},t)=0$ if $|\mathbf{x}|>r_0$. Let V be the retarded solution of

$$\Box V=-4\pi S \ , \tag{3.1}$$

i.e.,

$$V(\mathbf{X},T)=\int \frac{d^3x}{|\mathbf{X}-\mathbf{x}|}S\left[\mathbf{x},T-\frac{1}{c}|\mathbf{X}-\mathbf{x}|\right] . \tag{3.2}$$

Then, in the region exterior to the source, $V(\mathbf{X},T)$ admits the exact multipole expansion

$$V(\mathbf{X},T)=\sum_{l\geq 0}\frac{(-)^l}{l!}\partial_L\left[\frac{F_L(T-R/c)}{R}\right] , \tag{3.3}$$

where $\partial_L\equiv\partial^l/\partial X^{i_1}\partial X^{i_2}\cdots\partial X^{i_l}$, and where the functions $F_L(U)=F_{i_1\cdots i_l}(U)$ are just the usual STF multipole moments of a particular l-dependent weighted time average of the source $S(\mathbf{x},t)$, namely,

$$F_L(U)=\int d^3x\,\hat{x}^L\bar{S}_l(\mathbf{x},U) \ , \tag{3.4}$$

where $\hat{x}^L\equiv x^{\langle i_1}\cdots x^{i_l\rangle}$,

$$\bar{S}_l(\mathbf{x},U)=\int_{-1}^{1}dz\,\delta_l(z)S\left[\mathbf{x},U+\frac{rz}{c}\right] , \tag{3.5}$$

where $r\equiv|\mathbf{x}|$, and

$$\delta_l(z)\equiv\frac{(2l+1)!!}{2^{l+1}l!}(1-z^2)^l . \tag{3.6}$$

The time average (3.5) has its physical origin in the time delays $T-|\mathbf{X}-\mathbf{x}|/c$ due to the finite velocity of propagation. From Eq. (3.6) it follows that

$$\int_{-1}^{1}dz\,\delta_l(z)=1 \tag{3.7}$$

and

$$\lim_{l\to\infty}\delta_l(z)=\delta(z) \ , \tag{3.8}$$

where $\delta(z)$ is the one-dimensional Dirac distribution. Equation (3.8) thus implies that for large l the time delays can be neglected. It may also be worth mentioning that the following expression for $\delta_l(z)$ provides the link between the CMM or BD analysis and the Fourier-transform approach:

$$Y(1-z^2)\delta_l(z)=\frac{(2l+1)!!}{2\pi}\int_{-\infty}^{+\infty}d(\omega r)e^{i\omega rz}\frac{j_l(\omega r)}{(\omega r)^l} \ , \tag{3.9}$$

where $Y(x)$ denotes Heaviside's step function.

In some applications (notably post-Newtonian expansions of gravitational fields), it is necessary to consider slow-motion or long-wavelength expansion of the time average in Eq. (3.5). Using Euler's β function, one finds[7]

$$\begin{aligned}\bar{S}_l(\mathbf{x},U)&=\sum_{p=0}^{\infty}\frac{(2l+1)!!}{(2p)!!(2l+2p+1)!!}\frac{r^{2p}}{c^{2p}}\frac{\partial^{2p}}{\partial U^{2p}}S(\mathbf{x},U)\\&=S(\mathbf{x},U)+\frac{1}{2(2l+3)}\frac{r^2}{c^2}S^{(2)}(\mathbf{x},U)\\&\quad+\frac{1}{8(2l+3)(2l+5)}\frac{r^4}{c^4}S^{(4)}(\mathbf{x},U)+\cdots \ ,\end{aligned} \tag{3.10}$$

where $S^{(n)}(\mathbf{x},U)\equiv(\partial^n/\partial U^n)S(\mathbf{x},U)$.

IV. ELECTROMAGNETIC CASE

In the Lorentz gauge ($\partial_\mu A^\mu=0$), the four-potential A^μ satisfies

$$\Box A^\mu(\mathbf{X},T)=-\frac{4\pi}{c}J^\mu(\mathbf{X},T) \ , \tag{4.1a}$$

where

$$\Box=f^{\mu\nu}\partial_\mu\partial_\nu, \quad A^\mu=(\phi,A^a), \quad J^\mu=(c\rho,J^a) . \tag{4.1b}$$

Each component of A^μ may be considered as a scalar field, and if the source J^μ is spatially compact supported, then, in the region exterior to the source, $r>r_0$, we have from Eqs. (3.3)–(3.6) the following relativistic multipole expansion of the retarded potentials A^μ:

$$\phi(\mathbf{X},T)=\sum_{l=0}^{\infty}\frac{(-)^l}{l!}\partial_L\left[\frac{F_L(U)}{R}\right] , \tag{4.2a}$$

$$A_i(\mathbf{X},T)=\frac{1}{c}\sum_{l=0}^{\infty}\frac{(-)^l}{l!}\partial_L\left[\frac{G_{iL}(U)}{R}\right] , \tag{4.2b}$$

where

$$U\equiv T-\frac{R}{c} , \tag{4.3a}$$

$$F_L(U)\equiv\int d^3x\,\hat{x}_L\int_{-1}^{1}dz\,\delta_l(z)\rho\left[\mathbf{x},U+\frac{rz}{c}\right] , \tag{4.3b}$$

$$G_{iL}(U)\equiv\int d^3x\,\hat{x}_L\int_{-1}^{1}dz\,\delta_l(z)J_i\left[\mathbf{x},U+\frac{rz}{c}\right] , \tag{4.3c}$$

and

$$\delta_l(z)\equiv\frac{(2l+1)!!}{2^{l+1}l!}(1-z^2)^l . \tag{4.3d}$$

The Cartesian tensor G_{iL} is reducible and may be decomposed as explained in Sec. II [Eqs. (2.5) and (2.6)] into three irreducible pieces denoted by U (up), C (central), D (down), and defined as

$$U_{L+1} \equiv G_{\langle L+1 \rangle} , \tag{4.4a}$$

$$C_L \equiv G_{ab\langle L-1}\epsilon_{i_l \rangle ab} , \tag{4.4b}$$

$$D_{L-1} \equiv G_{aaL-1} , \tag{4.4c}$$

so that

$$G_{iL} = U_{iL} + \frac{l}{l+1}\epsilon_{ai\langle i_l}C_{L-1\rangle a} + \frac{2l-1}{2l+1}\delta_{i\langle i_l}D_{L-1\rangle} . \tag{4.5}$$

Substituting the decomposition (4.5) in the expansion (4.2b) and simplifying the ensuing expressions, we finally obtain, after suitable changes of the summation index,

$$\phi = \sum_{l=0}^{\infty} \frac{(-)^l}{l!}\partial_L \left[\frac{F_L(U)}{R} \right] , \tag{4.6a}$$

$$A_i(\mathbf{X},T) = \frac{1}{c}\sum_{l=1}^{\infty} \frac{(-)^l}{l!} \left\{ \partial_{L-1} \left[\left[-l\frac{U_{iL-1}}{R} \right] + \frac{l}{(l+1)(2l+3)}\Delta \left[\frac{D_{iL-1}}{R} \right] \right] + \frac{l}{l+1}\epsilon_{iab}\partial_{aL-1}\left[\frac{C_{bL-1}}{R} \right] + \frac{2l-1}{2l+1}\partial_{iL-1}\left[\frac{D_{L-1}}{R} \right] \right\} , \tag{4.6b}$$

where $\Delta \equiv \partial_s \partial_s$, and where all the irreducible tensors on the right-hand side of Eq. (4.6b) are evaluated at the retarded argument $U \equiv T - R/c$.

Since $D_{iL} = D_{iL}(U)$, we have $\Delta[R^{-1}D_{iL-1}(U)] = c^{-2}R^{-1}\ddot{D}_{iL-1}$. Further, the last term in (4.6b) can be transformed away by the gauge transformation

$$A_i' = A_i - \frac{1}{c}\partial_i \left[\sum_{l=1}^{\infty} \frac{(-)^l}{l!}\frac{2l-1}{2l+1}\partial_{L-1}\left[\frac{D_{L-1}}{R} \right] \right] , \tag{4.7a}$$

$$\phi' = \phi + \frac{1}{c^2}\sum_{l=1}^{\infty} \frac{(-)^l}{l!}\frac{2l-1}{2l+1}\partial_{L-1}\left[\frac{\dot{D}_{L-1}}{R} \right] . \tag{4.7b}$$

Introducing

$$Q_L \equiv F_L - \frac{1}{c^2}\frac{2l+1}{(l+1)(2l+3)}\dot{D}_L , \quad l \geq 0 , \tag{4.8a}$$

$$K_L \equiv -lU_L + \frac{l}{(l+1)(2l+3)c^2}\ddot{D}_L , \quad l \geq 1 , \tag{4.8b}$$

$$M_L \equiv -C_L , \quad l \geq 1 , \tag{4.8c}$$

where all quantities are evaluated at the retarded argument U, and where the overdot denotes the derivative with respect to U, we obtain, after the gauge transformation (4.7),

$$\phi' = \sum_{l=0}^{\infty} \frac{(-)^l}{l!}\partial_L \left[\frac{Q_L}{R} \right] , \tag{4.9a}$$

$$A_i' = \frac{1}{c}\sum_{l=1}^{\infty} \frac{(-)^l}{l!} \left[\partial_{L-1}\left[\frac{K_{iL-1}}{R} \right] - \frac{l}{l+1}\epsilon_{iab}\partial_{aL-1}\left[\frac{M_{bL-1}}{R} \right] \right] . \tag{4.9b}$$

The expressions for F, U, C, and D in terms of the source charge ρ and current J^a follow from Eqs. (4.4), (4.3b), and (4.3c). We have

$$F_L = \int d^3x\, \hat{x}_L \int_{-1}^{1} dz\, \delta_l(z)\tilde{\rho} , \tag{4.10a}$$

$$U_L = \int d^3x \int_{-1}^{1} dz\, \delta_{l-1}(z)\hat{x}_{\langle L-1}\tilde{J}_{i_l\rangle} , \tag{4.10b}$$

$$C_L = \int d^3x \int_{-1}^{1} dz\, \delta_l(z)\tilde{J}_a\hat{x}_{b\langle L-1}\epsilon_{i_l\rangle ab} , \tag{4.10c}$$

$$D_L = \int d^3x \int_{-1}^{1} dz\, \delta_{l+1}(z)\tilde{J}_a\hat{x}_{aL} , \tag{4.10d}$$

where the left-hand sides are evaluated at the retarded argument U, and where the tilde denotes a functional dependence of the z-retarded form

$$\tilde{J}^{\mu} \equiv J^{\mu}\left[\mathbf{x}, U+\frac{rz}{c}\right] . \tag{4.11}$$

Let us now evaluate explicitly the combination $\dot{Q}_L + K_L$ (for $l \geq 1$). From Eqs. (4.8),

$$\dot{Q}_L + K_L = \dot{F}_L - lU_L - \frac{1}{c^2}\frac{1}{2l+3}\ddot{D}_L . \tag{4.12}$$

The first term on the right-hand side using Eq. (4.10a) yields

$$\dot{F}_L(U) = \int d^3x\, \hat{x}_L \int_{-1}^{1} dz\, \delta_l \dot{\rho}\left[\mathbf{x}, U+\frac{rz}{c}\right] , \tag{4.13a}$$

which on using the local conservation of charge

$$\partial_\mu J^\mu = 0 \tag{4.13b}$$

leads to

$$\dot{F}_L = \int d^3x\, \hat{x}_L \int dz\, \delta_l \left[-\frac{d}{dx^i}\tilde{J}^i + \frac{z}{c} n_i \frac{\partial}{\partial U}\tilde{J}^i\right] , \tag{4.13c}$$

where d/dx^i denotes the "total space derivative" of $\tilde{J}^k = J^k(\mathbf{x}, \tilde{t}(\mathbf{x}))$, which includes an extra term coming from the space dependence of the z-retarded time argument: $\tilde{t}(\mathbf{x}) = U + |\mathbf{x}| z/c$. On integrating by parts the first term with respect to $\mathbf{x}$ and the second term with respect to z and reorganizing the terms, we obtain

$$\dot{F}_L = l \int d^3x \int_{-1}^{1} dz\, \delta_l \hat{x}_{\langle L-1}\tilde{J}_{i_l\rangle} + \frac{1}{(2l+3)c^2}\ddot{D}_L + \frac{l}{(2l+1)(2l+3)}\int d^3x \int_{-1}^{1} dz \frac{d^2}{dz^2}(\delta_{l+1})\hat{x}_{\langle L-1}\tilde{J}_{i_l\rangle} . \tag{4.13d}$$

Substituting for $\dot{F}_L$ in Eq. (4.12) using (4.13d), we then obtain

$$\dot{Q}_L + K_L = l \int d^3x \int_{-1}^{1} dz \left[\delta_l - \delta_{l-1} + \frac{1}{(2l+1)(2l+3)}\frac{d^2}{dz^2}\delta_{l+1}\right]\hat{x}_{\langle L-1}\tilde{J}_{i_l\rangle} . \tag{4.14}$$

From the definition (4.3d) of $\delta_l(z)$, it is easy to check that the expression in parentheses vanishes identically. Thus we have proven that, for $l \geq 1$,

$$K_L = -\dot{Q}_L , \tag{4.15a}$$

and one checks easily that the same method of proof for $l = 0$ yields the conservation of charge

$$\dot{Q} = 0 . \tag{4.15b}$$

We finally obtain, from Eq. (4.9),

$$\phi'(\mathbf{X},T) = \sum_{l=0}^{\infty} \frac{(-)^l}{l!}\partial_L\left[\frac{Q_L(U)}{R}\right] , \tag{4.16a}$$

$$A_i'(\mathbf{X},T) = -\frac{1}{c}\sum_{l=1}^{\infty}\frac{(-)^l}{l!}\left[\partial_{L-1}\left[\frac{\dot{Q}_{iL-1}(U)}{R}\right] + \frac{l}{l+1}\epsilon_{iab}\partial_{aL-1}\left[\frac{M_{bL-1}(U)}{R}\right]\right] . \tag{4.16b}$$

Equations (4.16) show that, modulo a gauge transformation which preserves the Lorentz condition ($\partial_\mu A'^\mu = 0$), the exterior electromagnetic potentials can be expressed in terms of two infinite sets of STF time-dependent multipole moments: the "electric moments" $Q_{\langle i_1 \cdots i_l\rangle}(U)$ and the "magnetic" ones $M_{\langle i_1 \cdots i_l\rangle}(U)$. The structural simplicity and transparence of the STF multipole expansion (4.16) is to be contrasted with the corresponding results written with vectorial harmonics[1] or Debye potentials.[2]

From Eqs. (4.8a), (4.10a), and (4.10d), we have our final expression for the electric multipole moments:

$$Q_L(U) = \int d^3x \int_{-1}^{1} dz \left[\delta_l \hat{x}_L \tilde{\rho} - \frac{1}{c^2}\frac{2l+1}{(l+1)(2l+3)}\delta_{l+1}\hat{x}_{aL}\frac{\partial}{\partial U}\tilde{J}_a\right], \quad l \geq 0 , \tag{4.17a}$$

where we recall that the tilde denotes the functional dependence (4.11), and that $\hat{x}_L$ denotes the STF projection on all indices of $x^{i_1} \cdots x^{i_l}$ [so that one must beware that $\hat{x}_{aL} V_a \neq \hat{x}_L (x_a V_a)$].

Similarly, from Eqs. (4.8c) and (4.10c), we obtain after some straightforward manipulation the following expression for the magnetic multipole moments:

$$M_L(U)=\int d^3x \int_{-1}^{1} dz\, \delta_l(z)\hat{x}_{\langle L-1}\tilde{m}_{i_l\rangle}, \quad l\geq 1 , \tag{4.17b}$$

where the "magnetization density" m_i is given by

$$\mathbf{m}=\mathbf{x}\times\mathbf{J} . \tag{4.17c}$$

The above forms can be transformed into forms earlier obtained by CMM using the Debye potential formalism with some amount of algebra. We obtain

$$Q_L=\frac{(2l+1)!!}{2^{l+1}(l+1)!}\int d^3x\, \hat{x}_L \int_{-1}^{1} dz(1-z^2)^l\left[(l+1)\tilde{\rho}+\frac{r}{c}\left[z\dot{\tilde{\rho}}-\frac{\dot{\tilde{J}}_a n_a}{c}\right]\right], \quad l\geq 0 , \tag{4.18a}$$

$$M_L=-\frac{1}{l}\frac{(2l+1)!!}{2^{l+1}l!}\int d^3x\, \hat{x}_L \int_{-1}^{1} dz(1-z^2)^l\nabla\cdot(\mathbf{x}\times\tilde{\mathbf{J}}), \quad l\geq 1 . \tag{4.18b}$$

Unlike the corresponding result obtained by CMM, our method of proof shows that the expression (4.18a) is also valid for $l=0$ and that we do not have to append the $l=0$ multipole (i.e., the total charge) by hand. Note also that in Eq. (4.18b) the tilde operation could be performed after the spatial divergence [because $\mathbf{x}\times\nabla\tilde{t}(\mathbf{x})=0$].

For $l=0$, Eq. (4.17a), integrating by parts with respect to z, gives

$$Q=\int_{-1}^{1} dz\, \delta_0(z)\int d^3x \left[\tilde{\rho}-\frac{z}{c}n_a\tilde{J}^a\right] . \tag{4.19}$$

The above expression [in which $\delta_0(z)$ is simply $\frac{1}{2}$] for the total charge looks unfamiliar compared to the standard one:

$$Q=\int d^3x\, \rho(\mathbf{x},t) . \tag{4.20}$$

However, as J^μ is conserved, one can more generally write

$$Q=\frac{1}{c}\int_\Sigma J^\mu d\Sigma_\mu , \tag{4.21}$$

where Σ is an arbitrary spacelike hypersurface with surface element $d\Sigma_\mu$. On the surface $t=$const, Eq. (4.21) yields Eq. (4.20), while if we consider the "z-retarded cone" Σ_z defined by $t-rz/c=$const $(=U)$, Eq. (4.21) yields

$$Q=\frac{1}{c}\int_{\Sigma_z}(J^0-zn^aJ^a)_{\Sigma_z}d^3x$$

$$=\int d^3x\left[\tilde{\rho}-\frac{z}{c}n^a\tilde{J}^a\right] , \tag{4.22}$$

whose average over $z\in[-1,+1]$ leads indeed to Eq. (4.19). A direct technical proof of the z independence of the spatial integral of $J^*\equiv\tilde{\rho}-n^a\tilde{J}^a z/c$ follows from

$$\frac{\partial J^*}{\partial z}=-\frac{1}{c}\frac{d}{dx^i}(r\tilde{J}^i) , \tag{4.23}$$

which is easily seen to be a consequence of $\partial_\mu J^\mu=0$.

V. LINEARIZED GRAVITY

The field equations of linearized gravity ($g_{\mu\nu}=f_{\mu\nu}+h_{\mu\nu}$) may be conveniently written in the harmonic gauge ($\partial_\nu\bar{h}^{\mu\nu}=0$) in terms of

$$\bar{h}_{\mu\nu}\equiv h_{\mu\nu}-\tfrac{1}{2}f_{\mu\nu}h , \tag{5.1}$$

as

$$\Box\bar{h}_{\mu\nu}(\mathbf{X},T)=-\frac{16\pi G}{c^4}T_{\mu\nu}(\mathbf{X},T) . \tag{5.2}$$

As in the previous section, if the source $T^{\mu\nu}$ is compact supported, then, in the region exterior to the source, we have the following multipole expansions for $h_{\mu\nu}$:

$$\bar{h}^{00}(\mathbf{X},T)=\frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-)^l}{l!}\partial_L\left[\frac{F_L(U)}{R}\right] , \tag{5.3a}$$

$$\bar{h}^{0i}(\mathbf{X},T)=\frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-)^l}{l!}\partial_L\left[\frac{G_{iL}(U)}{R}\right] , \tag{5.3b}$$

$$\bar{h}^{ij}(\mathbf{X},T)=\frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-)^l}{l!}\partial_L\left[\frac{H_{ijL}(U)}{R}\right] , \tag{5.3c}$$

where

$$F_L(U)\equiv\int d^3x\, \hat{x}_L\int_{-1}^{1}dz\, \delta_l(z)T^{00}\left[\mathbf{x},U+\frac{rz}{c}\right] , \tag{5.4a}$$

$$G_{iL}(U)\equiv\int d^3x\, \hat{x}_L\int_{-1}^{1}dz\, \delta_l(z)T^{0i}\left[\mathbf{x},U+\frac{rz}{c}\right] , \tag{5.4b}$$

$$H_{ijL}(U)\equiv\int d^3x\, \hat{x}_L\int_{-1}^{1}dz\, \delta_l(z)T^{ij}\left[\mathbf{x},U+\frac{rz}{c}\right] . \tag{5.4c}$$

As before, we note that G_{iL} is reducible and may be decomposed into three irreducible pieces (for convenience the numerical factors are absorbed in the "bricks" in this section):

$$G^{(+)}_{L+1}\equiv G_{\langle L+1\rangle} , \tag{5.5a}$$

$$G^{(0)}_{L}\equiv\frac{l}{l+1}G_{ab\langle L-1}\epsilon_{i_l\rangle ab} , \tag{5.5b}$$

$$G^{(-)}_{L-1}\equiv\frac{2l-1}{2l+1}G_{aaL-1} , \tag{5.5c}$$

such that

$$G_{iL}=G^{(+)}_{iL}+\epsilon_{ai\langle i_l}G^{(0)}_{L-1\rangle a}+\delta_{i\langle i_l}G^{(-)}_{L-1\rangle} . \tag{5.6}$$

Similarly, H_{ijL} is also reducible and, adapting Eqs. (2.11) and (2.12), can be decomposed into six irreducible pieces:

$$H^{(+2)}_{L+2} \equiv H_{\langle L+2\rangle} , \quad (5.7a)$$

$$H^{(+1)}_{L+1} \equiv \frac{2l}{l+2} \underset{L+1}{\mathrm{STF}}(H_{\langle ci_l\rangle dL-1}\epsilon_{i_{l+1}cd}) , \quad (5.7b)$$

$$H^{(0)}_{L} \equiv \frac{6l(2l-1)}{(l+1)(2l+3)} \underset{L}{\mathrm{STF}}(H_{\langle ai_l\rangle aL-1}) , \quad (5.7c)$$

$$H^{(-1)}_{L-1} \equiv \frac{2(l-1)(2l-1)}{(l+1)(2l+1)} \underset{L-1}{\mathrm{STF}}(H_{\langle ca\rangle bcL-2}\epsilon_{i_{l-1}ab}) , \quad (5.7d)$$

$$H^{(-2)}_{L-2} \equiv \frac{2l-3}{2l+1} H_{\langle ac\rangle acL-2} , \quad (5.7e)$$

$$K_L \equiv \tfrac{1}{3} H_{aaL} , \quad (5.7f)$$

such that

$$H_{ijL} = H^{(+2)}_{ijL} + \underset{L}{\mathrm{STF}}\,\underset{ij}{\mathrm{STF}}\,(\epsilon_{aii_l}H^{(+1)}_{ajL-1} + \delta_{ii_l}H^{(0)}_{jL-1} + \delta_{ii_l}\epsilon_{aji_{l-1}}H^{(-1)}_{aL-2} + \delta_{ii_l}\delta_{ji_{l-1}}H^{(-2)}_{L-2}) + \delta_{ij}K_L . \quad (5.8)$$

Substituting the decompositions, Eqs. (5.6) and (5.8), in (5.3b) and (5.3c), respectively, and simplifying the ensuing expressions we finally obtain, after some lengthy algebra,

$$\bar{h}^{00}(\mathbf{X},T) = \sum_{l\geq 0} \partial_L[R^{-1}\mathcal{A}_L(U)] , \quad (5.9a)$$

$$\bar{h}^{0i}(\mathbf{X},T) = \sum_{l\geq 0} \partial_{iL}[R^{-1}\mathcal{B}_L(U)] + \sum_{l\geq 1} \{\partial_{L-1}[R^{-1}\mathcal{C}_{iL-1}(U)] + \epsilon_{iab}\partial_{aL-1}[R^{-1}\mathcal{D}_{bL-1}(U)]\} , \quad (5.9b)$$

$$\bar{h}^{ij}(\mathbf{X},T) = \sum_{l\geq 0} \{\partial_{ijL}[R^{-1}\mathcal{E}_L(U)] + \delta_{ij}\partial_L[R^{-1}\mathcal{F}_L(U)]\} + \sum_{l\geq 1} \{\partial_{L-1(i}[R^{-1}\mathcal{G}_{j)L-1}(U)] + \epsilon_{ab(i}\partial_{j)aL-1}[R^{-1}\mathcal{H}_{bL-1}(U)]\}$$
$$+ \sum_{l\geq 2} \{\partial_{L-2}[R^{-1}\mathcal{I}_{ijL-2}(U)] + \partial_{aL-2}[R^{-1}\epsilon_{ab(i}\mathcal{T}_{j)bL-2}(U)]\} , \quad (5.9c)$$

where

$$\mathcal{A}_L = \frac{4G}{c^4}\frac{(-)^l}{l!} F_L , \quad l \geq 0 , \quad (5.10a)$$

$$\mathcal{B}_L = -\frac{4G}{c^4}\frac{(-)^l}{l!}\frac{G^{(-)}_L}{l+1} , \quad l\geq 0 , \quad (5.10b)$$

$$\mathcal{C}_L = \frac{4G}{c^4}\frac{(-)^l}{l!}\left[-lG^{(+)}_L + \frac{l}{(l+1)(2l+1)c^2}\ddot{G}^{(-)}_L\right] , \quad l\geq 1 , \quad (5.10c)$$

$$\mathcal{D}_L = \frac{4G}{c^4}\frac{(-)^l}{l!} G^{(0)}_L , \quad l \geq 1 , \quad (5.10d)$$

$$\mathcal{E}_L = \frac{4G}{c^4}\frac{(-)^l}{l!}\frac{H^{(-2)}_L}{(l+2)(l+1)} , \quad l \geq 0 , \quad (5.10e)$$

$$\mathcal{F}_L = \frac{4G}{c^4}\frac{(-)^l}{l!}\left[K_L - \frac{1}{3}H^{(0)}_L - \frac{\ddot{H}^{(-2)}_L}{(l+2)(l+1)(2l+3)c^2}\right] , \quad l\geq 0 , \quad (5.10f)$$

$$\mathcal{G}_L = \frac{4G}{c^4}\frac{(-)^l}{l!}\left[H^{(0)}_L - \frac{2l}{(l+2)(l+1)(2l+3)c^2}\ddot{H}^{(-2)}_L\right] , \quad l\geq 1 , \quad (5.10g)$$

$$\mathcal{H}_L = -\frac{4G}{c^4}\frac{(-)^l}{l!}\frac{H^{(-1)}_L}{l+1} , \quad l\geq 1 , \quad (5.10h)$$

$$\mathcal{I}_L = \frac{4G}{c^4}\frac{(-)^l}{l!}\left[l(l-1)H^{(+2)}_L - \frac{l-1}{(2l-1)c^2}\ddot{H}^{(0)}_L + \frac{l(l-1)}{(l+2)(l+1)(2l+3)(2l+1)c^4}\frac{d^4}{dU^4}(H^{(-2)}_L)\right] , \quad l\geq 2 , \quad (5.10i)$$

$$\mathcal{T}_L = \frac{4G}{c^4}\frac{(-)^l}{l!}\left[-lH^{(+1)}_L + \frac{l-1}{(l+1)(2l+1)c^2}\ddot{H}^{(-1)}_L\right] , \quad l\geq 2 . \quad (5.10j)$$

The F's, G's, and H's on the right-hand side of Eq. (5.10) are explicitly known in terms of the source $T^{\mu\nu}$. From Eqs. (5.4), (5.5), and (5.7), we have

$$F_L(U) = \int d^3x\, \hat{x}_L \int_{-1}^{1} dz\, \delta_l \tilde{T}^{00} , \quad (5.11a)$$

$$G^{(+)}_L(U) = \int d^3x \int_{-1}^{1} dz\, \delta_{l-1}\hat{x}^{\langle L-1}\tilde{T}^{i_l\rangle 0} , \quad (5.11b)$$

$$G_L^{(0)}(U)=\frac{l}{l+1}\int d^3x\int_{-1}^{1}dz\;\delta_l\tilde{T}^{0a}\hat{x}_{b\langle L-1}\epsilon_{i_l\rangle ab}\;, \tag{5.11c}$$

$$G_L^{(-)}(U)=\frac{2l+1}{2l+3}\int d^3x\int_{-1}^{1}dz\;\delta_{l+1}\tilde{T}^{0a}\hat{x}_{aL}\;, \tag{5.11d}$$

$$H_L^{(+2)}(U)=\int d^3x\int_{-1}^{1}dz\;\delta_{l-2}\hat{x}_{\langle L-2}\tilde{\hat{T}}_{i_{l-1}i_l\rangle}\;, \tag{5.11e}$$

$$H_L^{(+1)}(U)=\frac{2(l-1)}{l+1}\int d^3x\int_{-1}^{1}dz\;\delta_{l-1}\underset{L}{\mathrm{STF}}(\tilde{\hat{T}}_{ci_{l-1}}\hat{x}_{dL-2}\epsilon_{i_lcd})\;, \tag{5.11f}$$

$$H_L^{(0)}(U)=\frac{6l(2l-1)}{(l+1)(2l+3)}\int d^3x\int_{-1}^{1}dz\;\delta_l\underset{L}{\mathrm{STF}}(\tilde{\hat{T}}_{ci_l}\hat{x}_{cL-1})\;, \tag{5.11g}$$

$$H_L^{(-1)}(U)=\frac{2l(2l+1)}{(l+2)(2l+3)}\int d^3x\int_{-1}^{1}dz\;\delta_{l+1}\tilde{\hat{T}}_{ca}\hat{x}_{bc\langle L-1}\epsilon_{i_l\rangle ab}\;, \tag{5.11h}$$

$$H_L^{(-2)}(U)=\frac{2l+1}{2l+5}\int d^3x\int_{-1}^{1}dz\;\delta_{l+2}\tilde{\hat{T}}_{ac}\hat{x}_{acL}\;, \tag{5.11i}$$

$$K_L(U)=\tfrac{1}{3}\int d^3x\;\hat{x}_L\int_{-1}^{1}dz\;\delta_l\tilde{T}^{aa}\;, \tag{5.11j}$$

where $\hat{T}_{ij}\equiv T_{ij}-\delta_{ij}T_{aa}/3$, and where the tilde denotes the same z-retarded functional dependence as in Eq. (4.11) of the previous section.

As in BD-I, we find it convenient to introduce the new STF tensors

$$M_L(U)\equiv-(\mathcal{A}_L+2\dot{\mathcal{B}}_L+\ddot{\mathcal{E}}_L+\mathcal{F}_L),\quad l\geq 0\;, \tag{5.12a}$$

$$S_L(U)\equiv+\left[\mathcal{D}_L+\tfrac{1}{2}\dot{\mathcal{H}}_L\right],\quad l\geq 1\;, \tag{5.12b}$$

$$W_L(U)\equiv-\left[\mathcal{B}_L+\tfrac{1}{2}\dot{\mathcal{E}}_L\right],\quad l\geq 0\;, \tag{5.12c}$$

$$X_L(U)\equiv-\tfrac{1}{2}\mathcal{E}_L,\quad l\geq 0\;, \tag{5.12d}$$

$$Y_L(U)\equiv+(\dot{\mathcal{B}}_L+\ddot{\mathcal{E}}_L+\mathcal{F}_L),\quad l\geq 0\;, \tag{5.12e}$$

$$Z_L(U)\equiv-\tfrac{1}{2}\mathcal{H}_L,\quad l\geq 1\;. \tag{5.12f}$$

In Eqs. (5.9) the $\bar{h}^{\mu\nu}$ are given in terms of ten functions $\mathcal{A}_L,\mathcal{B}_L,\ldots,\mathcal{T}_L$. However, not all of them are independent. To explore the various relations among them, consider $(\mathcal{C}_L-\dot{M}_L-\dot{Y}_L)$, which, employing Eqs. (5.10) and (5.12), gives

$$\mathcal{C}_L-\dot{M}_L-\dot{Y}_L\sim-lG_L^{(+)}+\dot{F}_L-\frac{\ddot{G}_L^{-}}{2l+1}\;, \tag{5.13}$$

where $\sim$ denotes equality up to an overall factor $(4G/c^4)(-)^l/l!$

Using Eqs. (5.11) and the conservation equation for $\tilde{T}^{00}$, i.e.,

$$\tilde{T}^{00}{}_{,0}=-\frac{d}{dx^i}(\tilde{T}^{0i})+\frac{z}{c}n_i\frac{\partial}{\partial U}\tilde{T}^{0i}\;, \tag{5.14}$$

we obtain, after some amount of manipulation involving integration by parts with respect to both x^a and z,

$$\mathcal{C}_L-\dot{M}_L-\dot{Y}_L\sim l\int d^3x\int_{-1}^{1}dz\;\hat{x}^{\langle L-1}\tilde{T}^{i_l\rangle 0}\left[\delta_l-\delta_{l-1}+\frac{1}{(2l+3)(2l+1)}\frac{d^2\delta_{l+1}}{dz^2}\right]\;, \tag{5.15}$$

which vanishes since as before the bracketed expression is identically zero. Note that this is the counterpart in linearized gravity of the electromagnetic identity Eq. (4.15). Thus

$$\mathcal{C}_L=\dot{M}_L+\dot{Y}_L\;. \tag{5.16}$$

Next, from Eqs. (5.10) and (5.12),

$$\mathcal{G}_L+2Y_L\sim\frac{1}{3}H_L^{(0)}+2K_L-\frac{2\dot{G}_L^{(-)}}{l+1}+\frac{2}{(l+1)(2l+3)}\ddot{H}_L^{(-2)}\;. \tag{5.17}$$

Using Eq. (5.11), the conservation equation

$$\tilde{T}^{0a}{}_{,0}=-\frac{d}{dx^b}(\tilde{T}^{ab})+\dot{\tilde{T}}^{ab}n_b\frac{z}{c}\;, \tag{5.18}$$

and suitable integration by parts with respect to x^a and z, after some algebra we obtain, for the last two terms of Eq. (5.17),

$$-\frac{2(2l+1)}{(l+1)(2l+3)}\int d^3x\,\hat{x}_L\int_{-1}^{1}dz\,\delta_l\tilde{T}^{aa}-\frac{2l(2l-1)}{(l+1)(2l+3)}\int d^3x\int_{-1}^{+1}dz\,\delta_l\tilde{T}^{ab}\hat{x}_{a\langle L-1}\delta_{i_l\rangle b}\ , \tag{5.19}$$

which is identical to the first two terms except for a sign. Thus

$$\mathcal{G}_L=-2Y_L\ . \tag{5.20}$$

Similarly,

$$-(\mathcal{T}_L+2\dot{S}_L)\sim lH_L^{(+1)}-2\dot{G}_L^{(0)}+\frac{(l+2)}{(l+1)(2l+1)}\ddot{H}_L^{(-1)}\ . \tag{5.21}$$

Using Eqs. (5.11) and (5.18) and the identity satisfied by the second derivative of δ_{l+1}, it follows

$$\mathcal{T}_L=-2\dot{S}_L\ . \tag{5.22}$$

Finally,

$$-(\mathcal{I}_L+\ddot{M}_L)\sim\ddot{F}_L-\frac{2}{l+1}\frac{d^3}{dU^3}G_L^{(-)}+\ddot{K}_L-l(l-1)H_L^{(+2)}+\frac{l-2}{3(2l-1)}\ddot{H}_L^{(0)}+\frac{3l+1}{(l+1)(2l+1)(2l+3)}\frac{d^4}{dU^4}(H_L^{(-2)})\ . \tag{5.23}$$

Using Eqs. (5.11), (2.13), and the following identity coming from Eqs. (5.14) and (5.18):

$$\tilde{T}^{00}{}_{,00}=\frac{d}{dx^i}\frac{d}{dx^j}(\tilde{T}^{ij})-2n_j\frac{z}{c}\frac{d}{dx^i}(\dot{\tilde{T}}^{ij})-\dot{\tilde{T}}^{ij}(\partial_i n_j)\frac{z}{c}+\ddot{\tilde{T}}^{ij}n_in_j\frac{z^2}{c^2}\ , \tag{5.24}$$

we obtain, after a fair amount of algebra,

$$-(\mathcal{I}_L+\ddot{M}_L)\sim-\int d^3x\frac{\hat{x}_{ijL}}{r^2}\int_{-1}^{1}dz\left[\frac{1}{(2l+3)(2l+5)}\frac{d^2\delta_{l+2}}{dz^2}+\delta_{l+1}-\delta_l\right]\ddot{\tilde{T}}^{ij}\ , \tag{5.25}$$

which vanishes because of the same δ_l identity as before.

Thus

$$\mathcal{I}_L=-\ddot{M}_L\ . \tag{5.26}$$

Summarizing, Eqs. (5.16), (5.20), (5.22), and (5.26) are the relations between the ten coefficients in Eq. (5.9):

$$\mathcal{C}_L=\dot{M}_L+\dot{Y}_L,\quad l\geq 1\ , \tag{5.27a}$$

$$\mathcal{G}_L=-2Y_L,\quad l\geq 1\ , \tag{5.27b}$$

$$\mathcal{T}_L=-2\dot{S}_L,\quad l\geq 2\ , \tag{5.27c}$$

$$\mathcal{I}_L=-\ddot{M}_L,\quad l\geq 2\ . \tag{5.27d}$$

It is easy to show that for $l=0$ and 1 the same method of proof yields

$$Y=0\ , \tag{5.28a}$$

$$\dot{M}=0\ , \tag{5.28b}$$

$$\ddot{M}_i=0\ , \tag{5.28c}$$

$$\dot{S}_i=0\ , \tag{5.28d}$$

which include the conservation laws for mass, center of mass, and angular momentum.

Adapting the treatment in BD-I to our case, Eqs. (5.27) together with the inverse of (5.12),

$$\mathcal{A}_L=-(M_L-\dot{W}_L+\ddot{X}_L+Y_L)\ , \tag{5.29a}$$

$$\mathcal{B}_L=-(W_L-\dot{X}_L)\ , \tag{5.29b}$$

$$\mathcal{D}_L=+(S_L+\dot{Z}_L)\ , \tag{5.29c}$$

$$\mathcal{E}_L=-2X_L\ , \tag{5.29d}$$

$$\mathcal{I}_L=+(\dot{W}_L+\ddot{X}_L+Y_L)\ , \tag{5.29e}$$

$$\mathcal{H}_L=-2Z_L\ , \tag{5.29f}$$

show that the general $\bar{h}^{\alpha\beta}$ can be expressed in terms of M, S, W, X, Y, and Z. We also redefine

$$M_L^{\text{new}}=-\tfrac{1}{4}c^2 l!(-)^l M_L^{\text{old}}\ , \tag{5.30a}$$

$$S_L^{\text{new}}=-\frac{1}{4}c^3\frac{(l+1)!}{l}(-)^l S_L^{\text{old}}\ . \tag{5.30b}$$

Denoting by $\mathbb{M}$ the set of STF tensors $\{M_L^{\text{new}}(U),S_L^{\text{new}}(U)\}$ and $\mathbb{W}$ the set $\{W_L(U),X_L(U),Y_L(U),Z_L(U)\}$ and denoting by brackets a functional dependence, we transform the $\bar{h}^{\alpha\beta}$ to the "canonical" form by a gauge transformation of the form

$$\bar{h}^{\alpha\beta}_{\text{can}}[\mathbb{M}]=\bar{h}^{\alpha\beta}[\mathbb{M},\mathbb{W}]+\partial^\alpha w^\beta[\mathbb{W}]+\partial^\beta w^\alpha[\mathbb{W}]-f^{\alpha\beta}\partial_\mu w^\mu[\mathbb{W}]\ , \tag{5.31a}$$

where

$$w^0[\mathbb{W}]=\sum_{l\ge 0}\partial_L[R^{-1}W_L(U)]\ ,$$

$$w^i[\mathbb{W}]=\sum_{l\ge 0}\partial_{iL}[R^{-1}X_L(U)]+\sum_{l\ge 1}\{\partial_{L-1}[R^{-1}Y_{iL-1}(U)]+\epsilon_{iab}\partial_{aL-1}[R^{-1}Z_{bL-1}(U)]\}\ . \tag{5.31b}$$

It is easily seen that the gauge transformation (5.31) preserves the harmonicity condition $\partial_\beta \bar{h}^{\alpha\beta}_{\text{can}}=0$ and yields a multipole expansion for the external canonical $\bar{h}^{\alpha\beta}$ field depending only on $\mathbb{M}=\{M_L,S_L\}$:

$$\bar{h}^{00}_{\text{can}}(\mathbf{X},T)=+\frac{4}{c^2}\sum_{l\ge 0}\frac{(-)^l}{l!}\partial_L[R^{-1}M_L(U)]\ , \tag{5.32a}$$

$$\bar{h}^{0i}_{\text{can}}(\mathbf{X},T)=-\frac{4}{c^3}\sum_{l\ge 1}\frac{(-)^l}{l!}\partial_{L-1}[R^{-1}\dot{M}_{iL-1}(U)]-\frac{4}{c^3}\sum_{l\ge 1}\frac{(-)^l l}{(l+1)!}\epsilon_{iab}\partial_{aL-1}[R^{-1}S_{bL-1}(U)]\ , \tag{5.32b}$$

$$\bar{h}^{ij}_{\text{can}}(\mathbf{X},T)=+\frac{4}{c^4}\sum_{l\ge 2}\frac{(-)^l}{l!}\partial_{L-2}[R^{-1}\ddot{M}_{ijL-2}(U)]+\frac{8}{c^4}\sum_{l\ge 2}\frac{(-)^l l}{(l+1)!}\partial_{aL-2}[R^{-1}\epsilon_{ab(i}\dot{S}_{j)bL-2}(U)]\ . \tag{5.32c}$$

From Eqs. (5.10)–(5.12) and (5.30), we have our final expression for the "mass multipole moments" on eliminating $H_L^{(0)}$ using Eqs. (5.20) and (5.17):

$$M_L(U)=G\int d^3x\int_{-1}^{1}dz\left[\delta_l\hat{x}_L\tilde{\sigma}-\frac{4(2l+1)}{c^2(l+1)(2l+3)}\delta_{l+1}\hat{x}_{aL}\frac{\partial}{\partial U}\tilde{\sigma}^a+\frac{2(2l+1)}{c^4(l+1)(l+2)(2l+5)}\delta_{l+2}\hat{x}_{abL}\frac{\partial^2}{\partial U^2}\tilde{T}^{ab}\right]\ ,\quad l\ge 0\ , \tag{5.33}$$

where, following BD-II, we have introduced

$$\sigma\equiv\frac{T^{00}+T^{ss}}{c^2}\ , \tag{5.34a}$$

$$\sigma^a\equiv\frac{T^{0a}}{c}\ , \tag{5.34b}$$

and where the tilde has the same meaning as before [Eq. (4.11)].

Similarly, using Eqs. (5.10)–(5.12) and (5.30), we get, for the "spin multipole moments,"

$$S_L(U)=G\,\underset{L}{\text{STF}}\int d^3x\int_{-1}^{1}dz\left[\delta_l\hat{x}_{L-1}\epsilon_{i_l ab}x^a\tilde{\sigma}^b-\frac{2l+1}{c^2(l+2)(2l+3)}\delta_{l+1}\epsilon_{i_l ab}\hat{x}_{acL-1}\frac{\partial}{\partial U}\tilde{T}^{bc}\right],\quad l\ge 1\ . \tag{5.35}$$

It can be noted that, both in Eqs. (5.33) and (5.35), the spatial stresses T^{ab} can be replaced by their trace-free projection $\hat{T}^{ab}=T^{ab}-\delta^{ab}T^{cc}/3$. Moreover, it will be important in future applications[10] of our results to note that, in deriving Eqs. (5.33) and (5.35), the only integral identity satisfied by $T^{\mu\nu}$ which has been used is Eq. (5.20).

The expressions for M_L and S_L [Eqs. (5.33) and (5.35), respectively], can be rewritten, after some amount of algebra and the use of the local conservation law for $T^{\mu\nu}$, in the forms

$$M_L=\frac{G}{c^2}\frac{1}{(l+1)(l+2)}\int d^3x\,\hat{x}_L\int_{-1}^{1}dz\,\delta_l\left[(l+1)(l+2)(\tilde{T}^{00}+\tilde{T}^{ss})+2\frac{r}{c}[z(\dot{\tilde{T}}^{00}+\dot{\tilde{T}}^{ss})-2(l+2)\dot{\tilde{T}}^{0a}n_a]\right.$$
$$\left.+\frac{2r^2}{c^2}(\ddot{\tilde{T}}^{00}+\ddot{\tilde{T}}^{ab}n_an_b-2z\ddot{\tilde{T}}^{0a}n_a)\right],\quad l\geq 0\ , \tag{5.36}$$

$$S_L=-\frac{G}{c}\frac{1}{l(l+2)}\int d^3x\,\hat{x}_L\int dz\,\delta_l\partial_i(\mathbf{r}\times\mathfrak{s})^i,\quad l\geq 1\ , \tag{5.37a}$$

where

$$(\mathfrak{s})^i=(l+2)\tilde{T}^{0i}+\frac{rz}{c}\dot{\tilde{T}}^{0i}-\frac{x_a}{c}\dot{\tilde{T}}^{ai}\ . \tag{5.37b}$$

It should be emphasized that, as in the electromagnetic case, both our original expressions (5.33)–(5.35), and our alternative forms (5.36) and (5.37), are valid not only for the radiative moments ($l\geq 2$), but also for the static ones ($l\leq 1$).

Though the expression for S_L matches that CMM obtained by the Debye potential formalism, the one for M_L does not. To investigate further this difference, we proceed to obtain the first two terms in the slow-motion expansion ($c^{-1}\to 0$) of the above expressions for M_L and S_L in Eqs. (5.33) and (5.35). Recalling that $T^{00}=O(c^2)$, $T^{0a}=O(c)$, and $T^{ab}=O(1)$, and expanding the retarded expressions in powers of rz/c, we find, on using Eq. (3.10),

$$M_L=\frac{G}{c^2}\int d^3x\left[\hat{x}_L(T^{00}+T^{ss})+\frac{1}{2(2l+3)}r^2\hat{x}_LT^{00}{}_{,00}-\frac{4(2l+1)}{(l+1)(2l+3)}\hat{x}_{aL}T^{0a}{}_{,0}\right]+O\left[\frac{1}{c^4}\right], \tag{5.38}$$

where now the tildes have disappeared because all the source terms on the right-hand side are evaluated at the same (coordinate) time U as the left-hand side, $M_L(U)$. The expression (5.38) for the mass moment coincides with the (linearized gravity limit of the) result of BD-II for the first post-Newtonian (1PN) radiative mass moment of weakly self-gravitating isolated systems. This confirms the validity of Eq. (5.36) in contrast with the corresponding final result in CMM which does not satisfy this necessary limit requirement. As the Debye method should, in principle, give results equivalent to the STF ones, some algebraic mistake must have crept in the analysis of CMM. However, there are not enough intermediate steps of their calculation in the paper to allow one to trace back where this occurred. Let us note in passing that it has been further shown in BD-II that Eq. (5.38), as it stands (i.e., written in terms of the *contravariant* components of $T^{\mu\nu}$), is (surprisingly) still valid when the 1PN nonlinearities are taken into account.

One can give an alternative expression for M_L (valid only in the linearized gravity limit) by exploiting the conservation law $T^{\mu\nu}{}_{,\nu}=0$. Integrating by parts and simplifying yields

$$M_L=\frac{G}{c^2}\underset{L}{\mathrm{STF}}\int d^3x\left[T^{00}\hat{x}_L+\frac{2l(l-1)}{(l+1)(2l+3)}T^{ss}\hat{x}_L\right.$$
$$\left.-\frac{6l(l-1)}{(l+1)(2l+3)}T^{si_l}\hat{x}_{L-1}x^s+\frac{l(l-1)(l+9)}{2(l+1)(2l+3)}T^{i_li_{l-1}}\hat{x}_{L-2}r^2\right], \tag{5.39}$$

in agreement with the linearized gravity limit of the otherwise formally divergent radiative mass moments derived by Thorne[5] [his Eq. (5.32a)] (see also Appendix A of BD-II).

Similarly, the first two terms of the slow-motion expansion of the spin moment expression (5.35) are

$$S_L=G\,\underset{L}{\mathrm{STF}}\int d^3x\left[\hat{x}_{aL-1}\epsilon_{i_lab}\left[\frac{T^{0b}}{c}+\frac{1}{2(2l+3)}\frac{r^2}{c^2}\frac{\ddot{T}^{0b}}{c}\right]-\frac{2l+1}{c^2(l+2)(2l+3)}\hat{x}_{acL-1}\epsilon_{i_lab}\dot{T}^{bc}\right]+O\left[\frac{1}{c^4}\right]. \tag{5.40}$$

Equation (5.40) is the spin analogue of the BD expression (5.38). Using manipulations similar to the ones above, it can be transformed into

$$S_L=G\,\underset{L}{\mathrm{STF}}\int d^3x\left[\hat{x}_{L-1}\left[\epsilon_{i_lab}x^a\frac{T^{0b}}{c}\right]-\frac{l-1}{c^2(l+2)(2l+3)}(\epsilon_{i_lab}x^a\dot{T}^{bc}x^c)\hat{x}_{L-1}\right.$$
$$\left.+\frac{1}{c^2}\frac{(l-1)(l+4)}{2(l+2)(2l+3)}\epsilon_{i_lab}x^a\dot{T}^{bi_{l-1}}r^2\hat{x}_{L-2}\right]+O\left[\frac{1}{c^4}\right], \tag{5.41}$$

which agrees with the linearized gravity limit of the formally divergent result of Thorne [his Eqs. (5.32b)–(5.33)] for radiative spin moments at the 1PN level.

Let us go back to the exact closed-form expressions for the gravitational multipoles obtained above and consider the properties of the low-order moments ($l\leq 1$). For $l=0$ the expression (5.33) reduces to (when putting $G=1$ for simplicity)

$$Mc^2=\int d^3x \int_{-1}^{1} dz \left[\delta_0(\tilde{T}^{00}+\tilde{T}^{ss})-\frac{4}{3}\delta_1 x_a \frac{\dot{\tilde{T}}^{0a}}{c}+\frac{1}{5}\delta_2 \frac{\hat{x}_{ab}\ddot{\tilde{T}}^{ab}}{c^2} \right] . \quad (5.42)$$

After some simplification, using properties of the δ_l's, we obtain

$$Mc^2=\int d^3x \int_{-1}^{1} dz\, \delta_0(\tilde{T}^{00}-zn^a\tilde{T}^{0a}) . \quad (5.43a)$$

In this form this is completely analogous to the expression (4.19) for Q in the electromagnetic case and a similar argument using the conservation equation (5.14) yields the standard expression for the total mass:

$$Mc^2=\int d^3x\, T^{00}(\mathbf{x},t) . \quad (5.43b)$$

As before, the difference in (5.43a) and (5.43b) is due to the surfaces on which one integrates: a family of z-retarded cones in (5.43a) and the constant-t surface in (5.43b). Next, we consider the mass dipole (putting also $c=1$ for simplicity)

$$M_i=\int d^3x \int_{-1}^{1} dz \left[\delta_1 x^i(\tilde{T}^{00}+\tilde{T}^{ss})-\frac{4\times 3}{2\times 5}\delta_2\hat{x}_{ai}\dot{\tilde{T}}^{0a}+\frac{2\times 3}{2\times 3\times 7}\delta_3\hat{x}_{abi}\ddot{\tilde{T}}^{ab} \right] , \quad (5.44)$$

which, after some manipulation, yields

$$M_i=\int d^3x \int_{-1}^{1} dz\, \delta_1[x^i(\tilde{T}^{00}-zn_a\tilde{T}^{0a})-rz(\tilde{T}^{0i}-zn_b\tilde{T}^{ib})] . \quad (5.45a)$$

The compatibility between Eq. (5.45a) and the usual form

$$M_i=\int d^3x\, x^i T^{00} \quad (5.45b)$$

is proven in a manner similar to the one used for Q or M, i.e., by noting that Eq. (5.45a) is an average over z of a family of integrals over z-retarded cones. We must, however, now make use not only of the conservation of four-momentum, but also of the conservation of relativistic angular momentum:

$$\partial_\nu J^{\alpha\beta\nu}=0 , \quad (5.46a)$$

with

$$J^{\alpha\beta\nu}\equiv x^\alpha T^{\beta\nu}-x^\beta T^{\alpha\nu} . \quad (5.46b)$$

Similarly, we find, with the help of Eqs. (5.46), that the total spin can be written in the equivalent forms

$$S_i=\int d^3x \int_{-1}^{1} dz\, \epsilon_{iab}(\delta_1 x_a \tilde{T}^{0b}-\tfrac{1}{5}\delta_2\hat{x}_{ac}\dot{\tilde{T}}^{bc}) \quad (5.47a)$$

$$=\int d^3x \int dz\, \delta_1[\epsilon_{iab}x_a(\tilde{T}^{0b}-zn_c\tilde{T}^{bc})] \quad (5.47b)$$

$$=\int d^3x\, \epsilon_{iab}x_a T^{0b} . \quad (5.47c)$$

VI. SUMMARY AND CONCLUSION

In this paper we have shown explicitly how the use of irreducible Cartesian tensors allowed a unified, and structurally transparent, treatment of the relativistic time-dependent multipole expansion for long-range fields, generated by a compact source, having helicities 0 (scalar), 1 (electromagnetism), or 2 (linearized gravity). Our treatment differs from previous ones in making use neither of Fourier transforms,[1] vectorial or tensorial harmonics,[1] nor Debye potentials.[2] Our direct time-domain analysis makes clear that, as in the scalar case,[2] the multipole moments are just spatial symmetric and trace-free (STF) moments of particular weighted time averages over the source distribution.

In the electromagnetic case, our results [Eqs. (4.17a) and (4.17b)] for the electric and magnetic moments are new alternative forms equivalent to, and therefore confirming, the corresponding expressions earlier obtained by Campbell, Macek, and Morgan.[2] However, our treatment allows us to encompass straightforwardly the nonradiative zeroth-order moment (total charge) instead of having to add it by hand as in Ref. 2. In the linearized gravity case, the added efficiency of the STF technique has allowed us to obtain for the first time correct closed-form expressions for both the mass and spin moments in terms of the stress-energy distribution of the source: our Eqs. (5.33)–(5.35) or, equivalently, Eqs. (5.36) and (5.37). Only the spin moments had been correctly obtained in the earlier attempt of Campbell, Macek, and Morgan. As in the helicity-1 case, our treatment encompasses directly the nonradiative low-order moments (mass, dipole, and spin) and obtains new closed-form expressions for these [Eqs. (5.42), (5.44), and (5.47a)], whose equivalence with the usual ones is discussed. Finally, we have given, in various forms, the first two terms of the slow-motion expansions of the gravitational moments and shown their equivalence with the corresponding earlier results of Thorne[5] and Blanchet and Damour.[7] We shall show in a separate publication[10] how the new form (5.40) for the slow-motion expansion of the spin moments is a useful tool in the study, within the nonlinear general-relativistic framework, of the generation of gravitational waves at the sesqui-post-Newtonian (1.5PN) approximation (i.e., one c^{-1} order beyond the recent results of Ref. 7).

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