

Quantum fields at any time

Charles G. Torre*

Department of Physics, Utah State University, Logan, Utah 84322-4415

Madhavan Varadarajan[†]

Department of Physics, University of Utah, Salt Lake City, Utah 84112

and Raman Research Institute, Bangalore 560 080, India[‡]

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The canonical quantum theory of a free field using *arbitrary foliations* of a flat two-dimensional spacetime is investigated. It is shown that dynamical evolution along arbitrary spacelike foliations is unitarily implemented on the same Fock space as that associated with inertial foliations. It follows that the Schrödinger picture exists for arbitrary foliations as a unitary image of the Heisenberg picture for the theory. An explicit construction of the Schrödinger picture image of the Heisenberg Fock space states is provided. The results presented here can be interpreted in terms of a Dirac constraint quantization of parametrized field theory. In particular, it is shown that the Schrödinger picture physical states satisfy a functional Schrödinger equation which includes a slice-dependent c -number quantum correction, in accord with a proposal of Kuchař. The spatial diffeomorphism invariance of the Schrödinger picture physical states is established. Fundamental difficulties arise when trying to generalize these results to higher-dimensional spacetimes.

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I. INTRODUCTION

The Poincaré invariant quantum theory of a free field is, for all practical purposes, completely understood [1,2,3]. Most canonical quantization treatments are in the Heisenberg picture and focus on the behavior of quantum fields relative to inertial foliations (i.e., foliations by flat time slices) of the spacetime. In particular, the energy-momentum and angular momentum of the quantum field are densely defined self-adjoint operators on a Fock space, which generate unitary dynamical evolution from one flat slice to another.

It is often assumed that the state of a quantum field in flat spacetime can be defined at *any* time, that is, upon an arbitrary spacelike hypersurface. Likewise, it is assumed that one can define unitary dynamical evolution along an arbitrary spacelike foliation of the spacetime. While such niceties are apparently unnecessary for a non-gravitational treatment of particles and their interactions, they become interesting—if not mandatory—when trying to implement some aspects of Einstein’s general theory of relativity in the quantum regime. In this context there are no preferred foliations of spacetime and general covariance requires that all spacelike foliations should be allowed in the description of dynamics. Given the technical and conceptual complexities that arise in attempts to construct a quantum theory of gravitation, it is useful to eliminate the intricate effects of the gravitational interaction and focus on the more limited—but still non-trivial—interplay between quantum field theory and general covariance in a flat spacetime. Thus it is of interest to examine free quantum field theory in the context of an arbitrary spacelike

foliation of the Minkowskian background. In this paper we focus our attention on two-dimensional spacetimes since here the investigation can be completed using standard Fock space methods, and many of the mathematical underpinnings for the investigation have already been developed in [4]. Our primary concern is to establish whether operator evolution from one arbitrary slice to another is unitarily implemented on the standard Fock space. If the evolution is unitary, then the most straightforward assignation of quantum states to slices is via the unitary image of the states in the (slice independent) Fock space. If unitarity fails (as it seems to in dimensions higher than 2), it is an open question as to how one may assign states to slices. We do not address this question, other than hinting that the algebraic approach may be one way of addressing it.

Apart from the intrinsic interest of these issues from the point of view of quantum field theory on arbitrary foliations, this investigation can be viewed in terms of a Dirac constraint quantization of parametrized scalar field theory, such as was considered by Kuchař [5]. The quantum parametrized field theory, being a field theory possessing a diffeomorphism gauge group, is often studied as a model for some issues that arise in quantum gravity. Indeed, in many “mid-superspace” models of general relativity one can identify the resulting reduced field theory with a parametrized field theory of one or more fields propagating on a fixed (often flat) spacetime (see, e.g., [9]). Successful quantization of these models thus requires one to construct a suitable quantum parametrized field theory. In the usual approach to canonical quantization of such diffeomorphism invariant field theories one aspires to use operator representatives of the classical constraint functions to define a Hilbert space of physical states. The imposition of the quantum constraints is viewed as defining unitary transformations of states corresponding to evolution from one (arbitrary) spacelike slice to

*Email address: torre@cc.usu.edu

[†]Email address: madhavan@rri.ernet.in

[‡]Permanent address.

another. Even for the parametrized theory of free fields propagating upon a two-dimensional spacetime it has been an open question whether such an approach can be rigorously implemented. We shall see that, in this case, the quantization can be completed in the desired fashion. On the other hand, it turns out that a straightforward generalization of these methods to higher-dimensional models is not available. Thus our investigation indicates that alternative approaches (e.g., algebraic approaches) to canonical quantization of generally covariant field theories become necessary already in the simplest models for canonical quantum gravity.

A succinct formulation of the problem addressed in this paper can be presented in the context of the algebraic formulation of the quantization of linear field theories on a fixed background spacetime, which is by now standard [3,22]. The C^* algebra of observables is traditionally taken to be the Weyl algebra \mathcal{A} associated with the symplectic vector space of solutions \mathcal{S} to the field equations. Quantum states are identified with positive linear functions on \mathcal{A} . Given any pair of Cauchy surfaces (Σ_1, Σ_2) , there is a symplectic transformation $\tau: \mathcal{S} \rightarrow \mathcal{S}$ which can be interpreted as classical time evolution from Σ_1 to Σ_2 . This symplectic transformation defines an automorphism of \mathcal{A} which is naturally interpreted as time evolution from Σ_1 to Σ_2 in the Heisenberg picture. Now suppose that we associate a state $\omega_1: \mathcal{A} \rightarrow \mathbb{C}$ (\mathbb{C} denotes the space of complex numbers) to the instant of time represented by Σ_1 . (An interesting, potentially thorny issue is how one explicitly prepares/determines such a state on an arbitrary slice. We hope to return to this question in future work.) By pull-back, the time evolution automorphism can be viewed as determining a new state, ω_2 , which is naturally interpreted as the Schrödinger picture state at the instant of time defined by Σ_2 . A natural question that arises is whether this dynamical evolution can be expressed in terms of a unitary transformation on a Hilbert space representation of the Weyl algebra. We will be considering a free field on Minkowski spacetime, so we focus on the standard, Poincaré invariant Fock representation of the Weyl algebra. Thus the question we wish to address in this paper is whether the automorphism of \mathcal{A} associated with a pair of arbitrary Cauchy surfaces can be realized as a unitary transformation on the Fock space representation of \mathcal{A} . Because we are restricting attention to free fields, the investigation of this issue can be given a completely equivalent mathematical formulation in terms of unitary implementability of dynamical evolution of operator valued distributions corresponding to Cauchy data (canonical coordinates and momenta) along an arbitrary foliation of spacetime by Cauchy surfaces. For free fields, the spatially smeared canonical coordinates and momenta are observables in the sense that they are densely defined self-adjoint operators on Fock space obtained by a limiting procedure from the Weyl observables. We must leave open the physical issues regarding the sense in which the quantum field on an arbitrary hypersurface is to be interpreted, measured, etc. We should also point out that there is no compelling evidence to suggest that, for Poincaré invariant *interacting* field theories, there exist observables corresponding to spatially smeared Cauchy data. We prefer to formulate our investigation of free field theory in terms of

canonical coordinates and momenta for a couple of reasons: (1) this is the formulation used in [5], whose results we are trying to extend; (2) in canonical quantum gravity, for which this work is intended as a humble model, one formulates the quantization problem in terms of “observables” constructed from operator representatives of (functions of) Cauchy data for the field equations.

Our investigation proceeds as follows. Using the standard Fock space representation of a free scalar field on a two-dimensional flat spacetime we consider Heisenberg picture field operators (operator-valued distributions) associated with arbitrary (curved) spacelike slices. We ask whether the evolution of field operators from one such slice to another, as dictated by the field equations, is unitarily implemented on the Fock space. This issue, although formulated in the context of slice-dependent operators in the Heisenberg picture, is intimately connected with the existence of the Schrödinger picture. In the Schrödinger picture, field operators are slice-independent and are associated with some fixed initial slice of the foliation. The dynamics are encoded in the slice-dependent state vectors which, presumably, satisfy a functional Schrödinger equation, usually associated with the names Tomonaga and Schwinger [6,7]; see also the book of Dirac [8]. Given a foliation, if there exists a one-parameter family of unitary transformations which implement the operator evolution from slice to slice of the foliation, then the Schrödinger picture is defined as the unitary image of the Heisenberg picture. In this paper we show that such unitary transformations exist for a free, massless scalar field propagating on a flat spacetime with manifold structure $R \times S^1$, and we investigate properties of the Schrödinger picture quantum states. We thus largely complete the quantization program initiated in [5] by rigorously constructing the physical quantum states in the Schrödinger picture. In so doing, we derive the anomaly potential, proposed in [5], which appears in the quantum constraint equations as a c -number quantum correction. With a rigorous construction of the physical states in hand, it is now possible to investigate in detail various diffeomorphism invariance-related issues in quantum field theory. In this paper we answer the question: to what extent are the physical states of the parametrized quantum field theory actually invariant under spatial diffeomorphisms? This invariance is usually assumed in approaches to canonical quantization of diffeomorphism invariant field theories, but at least for the two-dimensional models such as considered here, spatial diffeomorphism invariance is called into question by the quantum corrections which appear in the constraints.

Let us emphasize what we are *not* doing in this paper. We are not considering the effect of classical gravitational fields on quantum matter fields, which is the subject of quantum field theory in curved spacetime. We are not considering different quantization schemes in flat spacetime. The complex structure and Fock space that we use are the standard ones associated with the timelike Killing vector field of the Minkowski metric and are fixed once and for all. So, for example, in this paper we do not (explicitly) consider slice-dependent complex structures and Fock spaces. As mentioned before, the simplest definition of slice-dependent state

is as the unitary image of a Heisenberg picture state. We do not discuss how to measure/prepare such a state. We hope to return to this question in a future work. Finally, we do not investigate the feasibility or existence of other definitions of slice dependent states.

The outline of the paper is as follows. In Sec. II we summarize the classical theory of a free scalar field on $R \times S^1$, and we remind the reader of the standard Fock space quantization of the theory in the Heisenberg picture. We provide the relation to the framework of parametrized field theory and its Dirac quantization as constructed in [5]. Finally, we demonstrate the existence of the unitary transformation which dictates evolution of operators from one time slice to another. In Sec. III, we construct the Schrödinger picture for the theory and give an explicit construction of the Schrödinger picture states on an arbitrary time slice as unitary images of the Heisenberg states. We show that the Schrödinger picture states satisfy a functional Schrödinger equation which includes an embedding-dependent quantum correction relative to the classical equation. This c -number correction is related to the ‘‘anomaly potential’’ of [5]. Section IV is devoted to the issue of spatial diffeomorphism invariance of the solutions to the functional Schrödinger equations. There we relate the factor ordering of the spatial projection of the Schrödinger equation to a version of the Schwarzian derivative due to Segal [4]. This leads to an interpretation of the spatially covariant ‘‘gauge’’ choice advocated by Kuchař for the anomaly potential. With this result in hand we are able to show that the functional Schrödinger equation implies spatial diffeomorphism invariance of physical states in the Schrödinger representation. In Sec. V we briefly consider generalizations of our results to massive free fields and to spacetimes with topology R^2 . We also indicate the fundamental difficulties inherent in generalizing our results to higher spacetime dimensions.

Notation. Classical fields are distinguished from their quantum counterparts by adopting bold face type for the former [e.g., $\phi(x)$ is the quantum counterpart of the classical field $\boldsymbol{\phi}(x)$]. Inertial coordinates on $R \times S^1$ are $T \in (-\infty, \infty)$ and $X \in [0, 2\pi]$, with respect to which the line element is

$$ds^2 = -dT^2 + dX^2. \quad (1)$$

We denote by $T^\pm := T \pm X$ the advanced and retarded null coordinates. Derivatives with respect to T^\pm are denoted with the subscripts ‘‘ \pm ’’ (e.g., $\phi_{,+} = \partial\phi/\partial T^+$). On a generic spacelike foliation we denote the spatial coordinate on a leaf of the foliation by $x \in [0, 2\pi]$. Spatial derivatives (with respect to x) are denoted with the subscript ‘‘ x ’’ [e.g., $f_{,x}(x) = df(x)/dx$]. Leaves of the foliation are labeled by the parameter t . We define a foliation by specifying the parametric equations

$$T^\alpha = T^\alpha(t, x), \quad (2)$$

where the superscript α labels coordinates on $R \times S^1$, e.g., $T^\alpha = (T, X)$ or $T^\alpha = (T^+, T^-)$, and

$$T^+_{,x}(t, x) > 0, \quad T^-_{,x}(t, x) < 0, \quad (3)$$

$$T^\pm(t, 2\pi) = T^\pm(t, 0) \pm 2\pi. \quad (4)$$

A particular spacelike slice is determined by an *embedding*:

$$T^\alpha = T^\alpha(x), \quad (5)$$

which can be identified with a leaf $t = t_0$ of a foliation via

$$T^\alpha(x) = T^\alpha(t_0, x).$$

II. THE HEISENBERG PICTURE FOR A FREE MASSLESS SCALAR FIELD ON $R \times S^1$

A. The classical theory

The massless scalar field on $R \times S^1$ satisfies the wave equation

$$\square \boldsymbol{\phi} = 0, \quad (6)$$

$$\Rightarrow \boldsymbol{\phi}(T^+, T^-) = \boldsymbol{\phi}^+(T^+) + \boldsymbol{\phi}^-(T^-). \quad (7)$$

We expand the scalar field in modes as

$$\boldsymbol{\phi}^\pm = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{2} (\mathbf{q} + \mathbf{p}T^\pm) + \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k}} \mathbf{a}_{(\pm)k} e^{-ikT^\pm} + \frac{1}{\sqrt{k}} \mathbf{a}_{(\pm)k}^* e^{ikT^\pm} \right) \right]. \quad (8)$$

The real numbers \mathbf{q}, \mathbf{p} will be referred to as the zero modes of the field. The complex numbers $\mathbf{a}_{(+k)}, \mathbf{a}_{(-k)}$ and their complex conjugates $\mathbf{a}_{(+k)}^*, \mathbf{a}_{(-k)}^*$ are the familiar Fourier mode coefficients (note that $k > 0$).

The field can be restricted to an embedding (i.e., a leaf of a foliation) $T^\alpha = T^\alpha(x)$, which results in the definition

$$\boldsymbol{\phi}(x) := \boldsymbol{\phi}(T^\alpha(x)) = \boldsymbol{\phi}^+(T^+(x)) + \boldsymbol{\phi}^-(T^-(x)). \quad (9)$$

Given an embedding $T^\alpha(x)$, we also define

$$\boldsymbol{\pi}(x) := \sqrt{\gamma} n^\alpha \nabla_\alpha \boldsymbol{\phi} \Big|_{T^\alpha = T^\alpha(x)}, \quad (10)$$

where $\sqrt{\gamma}$ is the determinant of the 1-metric induced on the spatial slice and n^α is the future-pointing unit normal to the slice. Thus $\boldsymbol{\pi}(x)$ is the field momentum associated with the given embedding. A simple computation shows that

$$\boldsymbol{\pi}(x) = T^+_{,x}(x) \boldsymbol{\phi}_{,+}(T^+(x)) - T^-_{,x}(x) \boldsymbol{\phi}_{,-}(T^-(x)). \quad (11)$$

The slice-dependent fields $(\boldsymbol{\phi}(x), \boldsymbol{\pi}(x))$ are Cauchy data for Eq. (6) and provide a canonical coordinate chart on the phase space of solutions of the wave equation. The wave equation can be used to determine the evolution of the fields $(\boldsymbol{\phi}(x), \boldsymbol{\pi}(x))$ from one arbitrary slice to another. This evolution is encoded in the following functional evolution equations:

$$\frac{\delta\boldsymbol{\phi}(x)}{\delta T^\pm(x')} = \pm \frac{\boldsymbol{\pi}(x) \pm \boldsymbol{\phi}_{,x}(x)}{2T^\pm(x)} \delta(x, x'), \quad (12)$$

$$\frac{\delta\boldsymbol{\pi}(x)}{\delta T^\pm(x')} = \frac{\boldsymbol{\pi}(x') \pm \boldsymbol{\phi}_{,x}(x')}{2T^\pm(x')} \frac{\partial \delta(x, x')}{\partial x}. \quad (13)$$

In the context of a particular foliation, $T^\alpha = T^\alpha(t, x)$, Eqs. (12), (13) give the infinitesimal change of $(\boldsymbol{\phi}(x), \boldsymbol{\pi}(x))$ corresponding to evolution from the slice $T^\alpha(x, t)$ to the slice $T^\alpha(x, t+dt)$ via

$$\frac{\partial \boldsymbol{\phi}(x, t)}{\partial t} = \int_0^{2\pi} \frac{\partial T^\alpha(x', t)}{\partial t} \frac{\delta \boldsymbol{\phi}(x, t)}{\delta T^\alpha(x', t)} dx', \quad (14)$$

$$\frac{\partial \boldsymbol{\pi}(x, t)}{\partial t} = \int_0^{2\pi} \frac{\partial T^\alpha(x', t)}{\partial t} \frac{\delta \boldsymbol{\pi}(x, t)}{\delta T^\alpha(x', t)} dx'. \quad (15)$$

This time evolution is a one-parameter family of canonical transformations which we would like to carry over into unitary transformations in the quantum theory. In particular, we shall deal with dynamical evolution along an arbitrary foliation connecting a fixed initial slice $T_0^\alpha(x)$ to a slice $T^\alpha(x)$. Data on $T_0^\alpha(x)$ will be denoted by $(\boldsymbol{\phi}_0(x), \boldsymbol{\pi}_0(x))$. For simplicity, we restrict attention to the case where the initial slice of our foliation is flat, and corresponds to $T=0$ with arc-length parametrization. Thus

$$T_0^+(x) = -T_0^-(x) = x, \quad (16)$$

and $(\boldsymbol{\phi}_0(x), \boldsymbol{\pi}_0(x))$ are Eqs. (9), (11) evaluated on $T_0^\alpha(x)$. Equations (12), (13) with initial data $(\boldsymbol{\phi}_0(x), \boldsymbol{\pi}_0(x))$ on the initial slice given by Eq. (16) can be solved to give a unique solution to Eq. (6).

B. Quantum theory: The Hilbert space

We now consider the operators $q, p, a_{(\pm)k}, a_{(\pm)k}^\dagger$ corresponding to the classical quantities $\mathbf{q}, \mathbf{p}, \mathbf{a}_{(\pm)k}, \mathbf{a}_{(\pm)k}^*$. We recall the standard Hilbert space construction [5] on which the only nontrivial commutation relations are

$$[q, p] = i\mathcal{I}, \quad (17)$$

$$[a_{(\pm)k}, a_{(\pm)l}^\dagger] = \delta_{kl}\mathcal{I}, \quad (18)$$

where \mathcal{I} is the identity. The Hilbert space \mathcal{H} of the theory is a product of three Hilbert spaces:

$$\mathcal{H} = \mathcal{F}^{(+)} \otimes \mathcal{F}^{(-)} \otimes \mathcal{L}^2(R), \quad (19)$$

where $\mathcal{F}^{(\pm)}$ are the standard Fock spaces on which the $a_{(\pm)k}^\dagger, a_{(\pm)k}$ operators are represented as creation and annihilation operators. $\mathcal{L}^2(R)$ is the representation space for the zero mode operators (q, p) .

To illustrate our notation and conventions we recall the standard construction of the Fock space associated with the ‘‘+’’ operators. The vacuum state $|(+); 0\rangle \in \mathcal{F}^{(+)}$ is such that

$$a_{(+k)}|(+); 0\rangle = 0 \quad \forall k. \quad (20)$$

The normalized N -particle states are generated from $|(+); 0\rangle$ by the action of the creation operators so that

$$|(+); n_{k_1} \dots n_{k_m}\rangle := \frac{(a_{(+k_1)}^\dagger)^{n_{k_1}}}{\sqrt{n_{k_1}!}} \dots \frac{(a_{(+k_m)}^\dagger)^{n_{k_m}}}{\sqrt{n_{k_m}!}} |0\rangle, \quad (21)$$

$$\sum_{i=1}^m n_{k_i} = N.$$

The vectors $|(+); n_{k_1}, \dots, n_{k_m}\rangle \forall m, \forall \{k_i, n_{k_i}, i=1, \dots, m\}$ with $|(+); 0\rangle$ form an orthonormal basis for $\mathcal{F}^{(+)}$. The action of $a_{(+k)}$ on any state in this basis is obtained from Eqs. (18), (20), (21).

The operators $a_{(-)k}, a_{(-)k}^\dagger$ are represented in an identical manner on $\mathcal{F}^{(-)}$, while q, p are densely defined on $\mathcal{L}^2(R)$ in the usual way. For our purposes, we find the momentum representation convenient: $p\psi(\mathbf{p}) = \mathbf{p}\psi(\mathbf{p})$ and $q\psi(\mathbf{p}) = i(d\psi/d\mathbf{p})$.

We identify the operator-valued distributions corresponding to Eqs. (9), (11) by replacing $\mathbf{p}, \mathbf{q}, \mathbf{a}_{(\pm)k}, \mathbf{a}_{(\pm)k}^*$ in these expressions with the operators $q, p, a_{(\pm)k}, a_{(\pm)k}^\dagger$. Since the classical evolution equations are linear, the operator valued distributions $\boldsymbol{\phi}(x)$ and $\boldsymbol{\pi}(x)$ satisfy the corresponding evolution equations for operators in the Heisenberg picture. In Sec. II D we will show that the corresponding dynamical evolution is unitarily implemented.

C. Relation to parametrized field theory and its Dirac quantization

It is a simple matter to check that the quantum system described above is the same as that arising in the Heisenberg picture constraint quantization of parametrized field theory developed in [5]. The only differences lie in our notation and different normalizations for the quantities $(\mathbf{a}_{(\pm)k}, \mathbf{a}_{(\pm)k}^*)$ and their quantum counterparts. We briefly summarize the treatment of [5] in our slightly different notation and conventions.

The phase space of a parametrized, free, massless, scalar field on the Minkowskian cylinder consists of the embedding fields $T^\alpha(x)$, and their conjugate momenta $P_\alpha(x)$,¹ along with the scalar field $\boldsymbol{\phi}(x)$ and its conjugate momentum $\boldsymbol{\pi}(x)$. Corresponding to the diffeomorphism invariance of the parametrized theory, there are two constraints

$$C_\pm = P_\pm \pm \frac{(\boldsymbol{\pi}(x) \pm \boldsymbol{\phi}_{,x}(x))^2}{4T^\pm_{,x}(x)} \approx 0, \quad (22)$$

¹The notation for the classical embedding coordinates and their conjugate momenta is an exception to our convention of denoting classical quantities by bold face type. This is to minimize confusion with the notation of [5] in which bold face type does not have the same meaning as in this paper.

which completely fix the embedding momenta in terms of the remaining fields. These constraints are first class (they have strongly vanishing Poisson brackets) and indicate that the embeddings can be viewed as ‘‘pure gauge.’’ The phase space variables can be mapped via an embedding-dependent canonical transformation to a new set of phase space coordinates $(\mathbf{P}_\pm(x), T^\pm(x), \mathbf{p}, \mathbf{q}, \mathbf{a}_{(\pm)k}, \mathbf{a}_{(\pm)k}^*)$ via Eqs. (8)–(11) [5]. The transformation leaves the embedding fields unchanged, while the new embedding momenta are the constraint functions:

$$\mathbf{P}_\pm(x) := C_\pm \approx 0. \quad (23)$$

This transformation hinges upon the fact that the constraint functions C_α satisfy an Abelian Poisson algebra. In these ‘‘Heisenberg’’ variables, the constraints are therefore simply the vanishing of the embedding momenta.

Based upon the Heisenberg variables just described, Kuchař implements the Dirac constraint quantization of the parametrized field theory in the Heisenberg picture as follows. In the quantum theory the operators $q, p, a_{(\pm)k}, a_{(\pm)k}^\dagger$ are represented as in Sec. II B. The embedding fields act by multiplication and the embedding momenta act by functional differentiation. The quantum constraints,

$$\mathbf{P}_\alpha |\Psi\rangle = \frac{1}{i} \frac{\delta}{\delta T^\alpha} |\Psi\rangle = 0, \quad (24)$$

then imply that the physical states are time independent, that is, independent of the embedding. The physical states can thus be identified with the embedding-independent Fock states of Sec. II B. Thus, constraint quantization based upon the canonical variables $(\mathbf{P}_\pm(x), T^\pm(x), \mathbf{p}, \mathbf{q}, \mathbf{a}_{(\pm)k}, \mathbf{a}_{(\pm)k}^*)$, corresponds exactly to the canonical quantum theory in the Heisenberg picture outlined in Sec. II B.

From the point of view of Dirac quantization of parametrized field theory, our primary goal in this paper is to recover the quantum theory in the Schrödinger picture. In particular, we aim to obtain physical states satisfying quantum constraints of the form

$$\hat{C}_\pm |\Psi\rangle = 0, \quad (25)$$

where \hat{C}_\pm is a quantum version of the classical constraint function (22).

D. Unitarity of time evolution

For each embedding, the quantum fields $(\phi(x), \pi(x))$ generate a *-algebra of observables via their canonical commutation relations [3]. In this section we show that the observable algebras associated with different, arbitrary time slices are unitarily equivalent. We do this by comparing $(\phi(x), \pi(x))$ and $(\phi_0(x), \pi_0(x))$ and building up the unitary transformation relating these operator-valued distributions on each of $\mathcal{F}^{(+)}$, $\mathcal{F}^{(-)}$ and $\mathcal{L}^2(R)$. To this end, expand the fields $(\phi(x), \pi(x))$ and $(\phi_0(x), \pi_0(x))$ in Fourier series:

$$\begin{aligned} \phi_0(x) = \frac{1}{\sqrt{2\pi}} \left(q + \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} [(a_{(+)k} + a_{(-)k}^\dagger) e^{-ikx} \right. \\ \left. + (a_{(+)k}^\dagger + a_{(-)k}) e^{ikx}] \right), \end{aligned} \quad (26)$$

$$\begin{aligned} \pi_0(x) = \frac{1}{\sqrt{2\pi}} \left(p - \frac{i}{\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} [(a_{(+)k} - a_{(-)k}^\dagger) e^{-ikx} \right. \\ \left. - (a_{(+)k}^\dagger - a_{(-)k}) e^{ikx}] \right), \end{aligned} \quad (27)$$

$$\begin{aligned} \phi(x) = \frac{1}{\sqrt{2\pi}} \left(q[T] + \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} [(a_{(+)k}[T] \right. \\ \left. + a_{(-)k}^\dagger[T]) e^{-ikx} + (a_{(+)k}^\dagger[T] \right. \\ \left. + a_{(-)k}[T]) e^{ikx}] \right), \end{aligned} \quad (28)$$

$$\begin{aligned} \pi(x) = \frac{1}{\sqrt{2\pi}} \left(p[T] - \frac{i}{\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} [(a_{(+)k}[T] \right. \\ \left. - a_{(-)k}^\dagger[T]) e^{-ikx} - (a_{(+)k}^\dagger[T] \right. \\ \left. - a_{(-)k}[T]) e^{ikx}] \right), \end{aligned} \quad (29)$$

where

$$\begin{aligned} a_{(\pm)k}[T] = \frac{1}{2\pi\sqrt{k}} \int_0^{2\pi} e^{\pm ikx} T_{,x}^\pm \\ \times \left[\pm \frac{ip}{\sqrt{2}} \pm \sum_{n=1}^{\infty} \sqrt{n} (a_{(\pm)n} e^{-inT^\pm(x)} \right. \\ \left. - a_{(\pm)n}^\dagger e^{inT^\pm(x)}) \right] dx, \end{aligned} \quad (30)$$

$$\begin{aligned} q[T] = q + \frac{1}{2\pi} p \int_0^{2\pi} T(x) dx \\ + \frac{1}{2\pi} \int_0^{2\pi} dx \left(\frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} [a_{(+)n} e^{-inT^+(x)} \right. \\ \left. + a_{(+)n}^\dagger e^{inT^+(x)} + a_{(-)n} e^{-inT^-(x)} \right. \\ \left. + a_{(-)n}^\dagger e^{inT^-(x)}] \right) dx, \end{aligned} \quad (31)$$

$$p[T] = p. \quad (32)$$

It is straightforward to verify at a purely algebraic level (that is, ignoring issues of domain), that the commutation

relations between the variables (28), (29) are independent of the embedding fields $T^\pm(x)$. In other words, $(q[T], p[T], a_{(\pm)k}[T], a_{(\pm)k}^\dagger[T])$ have the non-vanishing commutators given in Eqs. (17), (18). The transformation

$$(q[T], p[T], a_{(\pm)k}[T], a_{(\pm)k}^\dagger[T]) \leftrightarrow (q, p, a_{(\pm)k}, a_{(\pm)k}^\dagger) \quad (33)$$

is a symplectic transformation which is a quantum analog of the canonical transformation mentioned in Sec. II C. We now want to see that there is an embedding-dependent unitary transformation $U = U[T]$ on \mathcal{H} such that

$$q[T] = U^\dagger q U, \quad p[T] = U^\dagger p U, \quad a_{(\pm)k}[T] = U^\dagger a_{(\pm)k} U. \quad (34)$$

The basic theory of the unitary implementability on Fock space of symplectic transformations on the vector space of solutions to linear field equations is due to Shale [10]; see also [11]. Because of the existence of the zero modes, we find it convenient to first decompose the symplectic transformation (33) into two successive symplectic transformations, and then check that each transformation is unitarily implementable. To this end, we view the transformation (33) as being defined by the composition of the symplectic transformation

(I)

$$(q, p, a_{(\pm)k}, a_{(\pm)k}^\dagger) \rightarrow (q, p, c_{(\pm)k}[T], c_{(\pm)k}^\dagger[T]), \quad (35)$$

where

$$c_{(\pm)k}[T] = \pm \frac{1}{2\pi\sqrt{k}} \int_0^{2\pi} e^{\pm ikx} T_{,x}^\pm \times \left[\sum_{n=1}^{\infty} \sqrt{n} (a_{(\pm)n} e^{-inT^\pm(x)} - a_{(\pm)n}^\dagger e^{inT^\pm(x)}) \right] dx, \quad (36)$$

followed by the symplectic transformation

(II)

$$(q, p, c_{(\pm)k}[T], c_{(\pm)k}^\dagger[T]) \rightarrow (q[T], p, a_{(\pm)k}[T], a_{(\pm)k}^\dagger[T]), \quad (37)$$

where

$$a_{(\pm)k}[T] = c_{(\pm)k}[T] \pm \frac{ip}{\sqrt{2}} \frac{1}{2\pi\sqrt{k}} \times \int_0^{2\pi} e^{\pm ikx} T_{,x}^\pm dx, \quad (38)$$

and $q[T]$ is defined in Eq. (31).

Because $T^+(x)$ and $T^-(x)$ each define diffeomorphisms of the circle [see Eqs. (3), (4)], the transformation (I) involves two copies of the ‘‘metaplectic representation’’ of the group $\text{Diff}(S^1)$, which is discussed in [4]. It follows that the

transformation (I), for each sign $+$ and $-$, arises as a unitary transformation $U_1^{(\pm)}[T]$ on $\mathcal{F}^{(\pm)}$ (and the identity on the zero mode sector of the Hilbert space):

$$U_1^{(\pm)\dagger} q U_1^{(\pm)} = q \quad (39)$$

$$U_1^{(\pm)\dagger} p U_1^{(\pm)} = p \quad (40)$$

$$U_1^{(\pm)\dagger} a_{(\pm)k} U_1^{(\pm)} = c_{(\pm)k}[T]. \quad (41)$$

The gist of the proof involves showing that the Bogolubov coefficients

$$B_{mn}^{(\pm)}[T] = \mp \frac{1}{2\pi} \sqrt{\frac{n}{m}} \times \int_0^{2\pi} e^{\pm imx} T_{,x}^\pm(x) e^{inT^\pm(x)} dx, \quad (42)$$

are Hilbert-Schmidt type, i.e., satisfy

$$\sum_{m,n=1}^{\infty} |B_{mn}^{(\pm)}|^2 < \infty. \quad (43)$$

This latter result is guaranteed if the embedding is taken to be sufficiently smooth (see the Appendix).

Next, it is straightforward to check that both

$$Z_n^{(\pm)} := \frac{1}{2\pi\sqrt{n}} \int_0^{2\pi} e^{\mp inx} T_{,x}^\pm dx \quad (44)$$

and

$$\zeta_{(\pm)n} := \frac{1}{2\pi\sqrt{n}} \int_0^{2\pi} e^{inT^\pm(x)} dx \quad (45)$$

are rapidly decreasing functions of n , that is, as $n \rightarrow \infty$, $|Z_n^{(\pm)}|$ and $|\zeta_n^{(\pm)}|$ vanish faster than any power of $1/n$. For details, see the Appendix. From this it follows that $U_{\text{II}}[T]$, defined as

$$U_{\text{II}}[T] = \exp \left\{ -i \left[\frac{p^2}{4\pi} \int_0^{2\pi} T(x) dx - \left(\frac{p}{\sqrt{2}} \sum_{n=1}^{\infty} [c_{(+n)} Z_n^{(+)} + c_{(+n)}^\dagger Z_n^{(+)*} - c_{(-n)} Z_n^{(-)} - c_{(-n)}^\dagger Z_n^{(-)*}] \right) \right] \right\} \\ = \exp \left\{ -i \left[\frac{p^2}{4\pi} \int_0^{2\pi} T(x) dx + \left(\frac{p}{\sqrt{2}} \sum_{n=1}^{\infty} [a_{(+n)} \zeta_n^{(+)*} + a_{(+n)}^\dagger \zeta_n^{(+)} + a_{(-n)} \zeta_n^{(-)*} + a_{(-n)}^\dagger \zeta_n^{(-)}] \right) \right] \right\}, \quad (46)$$

is a unitary operator on the Hilbert space \mathcal{H} . U_{II} implements the transformation **(II)**:

$$U_{\text{II}}^\dagger q U_{\text{II}} = q[T] \quad (47)$$

$$U_{\text{II}}^\dagger p U_{\text{II}} = p \quad (48)$$

$$U_{\text{II}}^\dagger c_{(\pm)k}[T] U_{\text{II}} = a_{(\pm)k}[T]. \quad (49)$$

The combined transformation $U[T] = U_{\text{I}}^{(+)} U_{\text{I}}^{(-)} U_{\text{II}}$ is the unitary map implementing dynamical evolution from the initial spacelike embedding $T_0^\pm(x) = \pm x$ to the final spacelike embedding $T^\alpha(x) = (T^+(x), T^-(x))$.

III. THE SCHRÖDINGER PICTURE

A. Schrödinger picture image of the Fock basis

A vector in the Hilbert space for the quantum field theory is any normalizable superposition of the Fock basis vectors (see Sec. II B). In the Heisenberg picture of dynamics, any such vector can represent the state vector $|\Psi\rangle_{\text{H}}$ of the system for all time. Dynamical results depend upon specification of an embedding, and are expressed in terms of expectation values of observables built from the embedding-dependent operator-valued distributions $(\phi(x), \pi(x))$ defined in Sec. II D. In the Schrödinger picture, dynamical evolution is encoded in embedding-dependent state vectors $|\Psi[T]\rangle_{\text{S}}$ according to the unitary mapping

$$|\Psi[T]\rangle_{\text{S}} = U[T] |\Psi\rangle_{\text{H}}, \quad (50)$$

and dynamical results are expressed in terms of operator observables constructed from $(\phi_0(x), \pi_0(x))$.

In the last section we showed that $U[T]$ exists; here we explicitly define this operator by giving its action on the Fock basis of Sec. II B. To begin, we express the Fock ground state (Heisenberg vacuum state) as

$$|0, \psi\rangle = \psi(\mathbf{p}) \otimes |(+); 0\rangle \otimes |(-); 0\rangle, \quad (51)$$

where $\psi \in \mathcal{L}^2(R)$. The Schrödinger picture image of this state is denoted by $|0, \psi; T\rangle$:

$$|0, \psi; T\rangle = U[T] |0, \psi\rangle. \quad (52)$$

We note that

$$|0, \psi; T_0\rangle = |0, \psi\rangle. \quad (53)$$

To evaluate $|0, \psi; T\rangle$ it is convenient to decompose U as

$$U = V_{\text{II}} U_{\text{I}}, \quad (54)$$

where V_{II} is the unitary operator

$$V_{\text{II}} := U_{\text{I}} U_{\text{II}} U_{\text{I}}^{-1}, \quad (55)$$

and

$$U_{\text{I}} = U_{\text{I}}^{(+)} U_{\text{I}}^{(-)}. \quad (56)$$

Using Eq. (46) and Eqs. (39)–(41),

$$V_{\text{II}}[T] = \exp \left\{ -i \left[\frac{p^2}{4\pi} \int_0^{2\pi} T(x) dx - \left(\frac{p}{\sqrt{2}} \sum_{n=1}^{\infty} [a_{(+)n} Z_n^{(+)} + a_{(+)n}^\dagger Z_n^{(+)*} - a_{(-)n} Z_n^{(-)} - a_{(-)n}^\dagger Z_n^{(-)*}] \right) \right] \right\}, \quad (57)$$

Our strategy is to first evaluate $U_{\text{I}}|0, \psi\rangle$ and then compute the action of V_{II} on the resulting state. The vector $U_{\text{I}}|0, \psi\rangle$ can be computed from the observation that it is annihilated by

$$d_{(\pm)k} := U_{\text{I}} a_{(\pm)k} U_{\text{I}}^\dagger \quad (58)$$

$$= \sum_{n=1}^{\infty} (\alpha_{(\pm)kn} a_{(\pm)n} + \beta_{(\pm)kn} a_{(\pm)n}^\dagger), \quad (59)$$

where

$$\alpha_{(\pm)kn} = \frac{1}{2\pi} \sqrt{\frac{n}{k}} \int_0^{2\pi} e^{ikT^\pm(x)} e^{\mp inx} dx \quad (60)$$

$$\beta_{(\pm)kn} = -\frac{1}{2\pi} \sqrt{\frac{n}{k}} \int_0^{2\pi} e^{ikT^\pm(x)} e^{\pm inx} dx. \quad (61)$$

Let us note some important properties of these Bogolubov coefficients (see [4] for a more rigorous treatment of most of these results). First, note that the operators $d_{(\pm)n}$ can be obtained from Eq. (36) using the inverse diffeomorphisms $(T^\pm)^{-1}$:

$$d_{(\pm)n} = c_{(\pm)n} [(T^\pm)^{-1}]. \quad (62)$$

The coefficients $\alpha_{(\pm)mn}$ and $\beta_{(\pm)mn}$ satisfy the relations

$$\sum_{k=1}^{\infty} (\alpha_{(\pm)ik} \alpha_{(\pm)jk}^* - \beta_{(\pm)ik} \beta_{(\pm)jk}^*) = \delta_{ij}, \quad (63)$$

$$\sum_{k=1}^{\infty} (\alpha_{(\pm)ik} \beta_{(\pm)jk} - \beta_{(\pm)ik} \alpha_{(\pm)jk}) = 0, \quad (64)$$

which are equivalent to saying that the transformation **(I)** of Sec. II D is symplectic. The coefficients $\beta_{(\pm)mn}$ are Hilbert-Schmidt

$$\sum_{m,n=1}^{\infty} |\beta_{(\pm)mn}|^2 < \infty; \quad (65)$$

this result is equivalent to Eq. (43). The infinite arrays $\alpha_{(\pm)mn}$ admit inverses $\alpha_{(\pm)mn}^{-1}$ which can be written as

$$\alpha_{(\pm)mn}^{-1} = \alpha_{(\pm)nm}^* - \sum_{k=1}^{\infty} \gamma_{(\pm)mk} \beta_{(\pm)nk}^*, \quad (66)$$

where we have defined the Hilbert-Schmidt operators

$$\gamma_{(\pm)mn} = \sum_{k=1}^{\infty} \alpha_{(\pm)mk}^{-1} \beta_{(\pm)kn}. \quad (67)$$

It is straightforward to verify that, for any embedding-dependent function of p , $N(p, T)$,

$$U_1|0, \psi\rangle = N(p, T) \exp \left\{ -\frac{1}{2} \sum_{k, l=1}^{\infty} [(\gamma_{(+)}{}_{kl} a_{(+)}^{\dagger} a_{(+)}^{\dagger})_{kl} + \gamma_{(-)}{}_{kl} a_{(-)}^{\dagger} a_{(-)}^{\dagger}] \right\} |0, \psi\rangle \quad (68)$$

is annihilated by $d_{(\pm)k}$ for all k (see [12] for some properties of such a state). Since U_1 is trivial on the zero mode sector, (39), (40), $N(p, T)$ must be independent of p . Thus

$$N(p, T) = N(T), \quad (69)$$

and $N(T)$ is determined, up to an embedding-dependent phase factor, by normalization to be

$$N(T) = e^{i\Lambda(T)} \det(1 - \gamma_{(+)}^* \gamma_{(+)})^{1/4} \times \det(1 - \gamma_{(-)}^* \gamma_{(-)})^{1/4}, \quad (70)$$

where $\Lambda(T)$ is an arbitrary real-valued function of the embedding and we have used a matrix notation in which $\gamma_{(\pm)}$ denotes the symmetric matrix $\gamma_{(\pm)mn}$. $N(T)$ is well-defined thanks to the fact that γ is Hilbert-Schmidt type.

It is now straightforward to compute the action of V_{II} (57) on (68) to be

$$|0, \psi, T\rangle = M(p, T) \exp \left\{ \sum_{k=1}^{\infty} \left[\frac{-ip}{\sqrt{2}} (\xi_{(+)}{}_{k} a_{(+)}^{\dagger})_{k} + \xi_{(-)}{}_{k} a_{(-)}^{\dagger})_{k} - \frac{1}{2} \sum_{k, l=1}^{\infty} (\gamma_{(+)}{}_{kl} a_{(+)}^{\dagger})_{kl} + \gamma_{(-)}{}_{kl} a_{(-)}^{\dagger})_{kl} \right] \right\} |0, \psi\rangle, \quad (71)$$

where

$$M(p, T) = \exp \left\{ -i \left[\frac{p^2}{4\pi} \int_0^{2\pi} T(x) dx \right] \right\} \times \exp \left\{ \frac{p^2}{4} \sum_{k=1}^{\infty} [\xi_{(+)}{}_{k} Z_k^{(+)} - \xi_{(-)}{}_{k} Z_k^{(-)}] \right\} N(T) \quad (72)$$

with $N(T)$ defined by Eq. (70) and

$$\xi_{(\pm)k} := \sum_{l=1}^{\infty} \alpha_{(\pm)kl}^{-1} \zeta_{(\pm)l}. \quad (73)$$

Note that the various sums and products in the expressions above converge because γ is Hilbert-Schmidt type and ξ, Z are rapidly decreasing.

The vector $|0, \psi, T\rangle$ serves as the vacuum (or ‘‘cyclic’’) vector for the Fock representation associated with the annihilation and creation operators $b_{(\pm)k}$ and $b_{(\pm)k}^{\dagger}$ where

$$b_{(\pm)k} := U a_{(\pm)k} U^{\dagger} \quad (74)$$

$$= i \zeta_{(\pm)k} \frac{p}{\sqrt{2}} + \sum_{n=1}^{\infty} (\alpha_{(\pm)kn} a_{(\pm)n} + \beta_{(\pm)kn} a_{(\pm)n}^{\dagger}). \quad (75)$$

This Fock space representation of the algebra of creation and annihilation operators and zero modes is unitarily equivalent to the representation on \mathcal{H} we used originally. By repeatedly applying the creation operators $b_{(\pm)k}^{\dagger}$ to $|0, \psi, T\rangle$, and allowing ψ to range over an orthonormal basis for $\mathcal{L}^2(R)$, we obtain an orthonormal basis $\{|e_i(T)\rangle\}$ for the Hilbert space \mathcal{H} . This basis is just the Schrödinger picture unitary image of the original orthonormal basis of states used in the Heisenberg picture. From the point of view of the parametrized field theory description of [5] and Sec. II C, the embedding-independent Fock states are the ‘‘physical states’’ of the Dirac quantization based upon the Heisenberg variables. The physical states of the Dirac quantization in the Schrödinger picture are obtained as the unitary image of the Heisenberg physical states. The (pure) physical states in the Schrödinger picture are thus obtained by taking finite-norm superpositions of the basis $\{|e_i(T)\rangle\}$ for \mathcal{H} that we described above. The Dirac quantization of the parametrized field theory of [5] in the Schrödinger picture is thereby completed. However, we would still like to see explicitly how the physical states satisfy the quantum constraints in the Schrödinger picture. This is our next topic.

B. Functional Schrödinger equation

The Schrödinger picture states constructed in the last subsection are determined by a choice of embedding. In this subsection we consider the change induced in these states by a variation of the embedding. In particular, we derive a functional Schrödinger equation that describes the evolution of the state vector from one slice to another of an arbitrary spacelike foliation. This functional Schrödinger equation is the quantum constraint equation arising in the Dirac quantization of parametrized field theory in the Schrödinger picture.

To begin, we consider the embedding dependence of the Schrödinger vacuum state given in Eqs. (71), (70), (72). We want to consider the change induced in this state by an infinitesimal change in the embedding $T^{\alpha}(x)$. With this result in hand, it is straightforward to compute the corresponding results for the basis $\{|e_i(T)\rangle\}$. Evidently, we need to compute the functional derivatives of $\xi_{(\pm)k}$, $\gamma_{(\pm)mn}$, and $Z_k^{(\pm)}$ with respect to $T^{\alpha}(x)$. To display the results of the computation it is convenient to present the Fourier modes of the functional derivatives. We define

$$\delta_{(\pm)n} = \int_0^{2\pi} e^{inT^\pm(x)} \frac{\delta}{\delta T^\pm(x)} dx. \quad (76)$$

Direct computation yields

$$\delta_{(\pm)n} \mathcal{Y}_{(\mp)lm} = 0, \quad (77)$$

$$\delta_{(\pm)n} \xi_{(\mp)k} = 0, \quad (78)$$

$$\delta_{(\pm)n} Z_{(\mp)k} = 0, \quad (79)$$

$$\delta_{(\pm)n} \mathcal{Y}_{(\pm)lm} = 0 \quad \text{for } n \geq 0, \quad (80)$$

$$\begin{aligned} \delta_{(\pm)n} \mathcal{Y}_{(\pm)lm} = & -i \sum_{j=1}^{|n|-1} \sqrt{j|n+j|} \alpha_{(\pm)lj}^{-1} \left[\alpha_{(\pm)|n+j|m}^* \right. \\ & \left. - \sum_{q=1}^{\infty} \beta_{(\pm)|n+j|q}^* \mathcal{Y}_{(\pm)qm} \right] \quad \text{for } n < 0, \end{aligned} \quad (81)$$

$$\delta_{(\pm)n} \xi_{(\pm)k} = 0 \quad \text{for } n \geq 0, \quad (82)$$

$$\begin{aligned} \delta_{(\pm)n} \xi_{(\pm)k} = & i \sqrt{|n|} \alpha_{(\pm)k|n|}^{-1} \\ & + i \sum_{j=1}^{|n|-1} \sqrt{j|n+j|} \alpha_{(\pm)kj}^{-1} \left[\xi_{(\pm)|n+j|}^* \right. \\ & \left. + \sum_{q=1}^{\infty} \beta_{(\pm)|n+j|q}^* \xi_{(\pm)q} \right] \quad \text{for } n < 0, \end{aligned} \quad (83)$$

$$\delta_{(\pm)n} Z_{(\pm)k} = \pm i \sqrt{n} \alpha_{(\pm)nk} \quad \text{for } n > 0, \quad (84)$$

$$\delta_{(\pm)n} Z_{(\pm)k} = 0 \quad \text{for } n = 0, \quad (85)$$

$$\delta_{(\pm)n} Z_{(\pm)k} = \mp i \sqrt{|n|} \beta_{(\pm)|n|k}^* \quad \text{for } n < 0. \quad (86)$$

It is now a simple matter to apply $\delta_{(\pm)n}$ to the state $|0, \psi, T\rangle$ as written in Eqs. (71), (70), (72). The result is a sum of four terms acting on $|0, \psi, T\rangle$:

$$\delta_{(\pm)n} |0, \psi, T\rangle = \{ \mathcal{P}_{(\pm)n} + \mathcal{Q}_{(\pm)n} + \mathcal{R}_{(\pm)n} + \mathcal{S}_{(\pm)n} \} |0, \psi, T\rangle, \quad (87)$$

where $\mathcal{P}_{(\pm)n}$ is a term proportional to the identity \mathcal{I} arising from the derivative of $N(T)$,

$$\mathcal{P}_{(\pm)n} = \delta_{(\pm)n} (\log N(T)) \mathcal{I}; \quad (88)$$

$\mathcal{Q}_{(\pm)n}$ is quadratic in p ,

$$\begin{aligned} \mathcal{Q}_{(\pm)n} = & \frac{p^2}{4} \left\{ -i \left(\frac{1}{2\pi} \int_0^{2\pi} e^{inT^\pm(x)} dx \right) \right. \\ & + \sum_{k=1}^{\infty} \left[\delta_{(\pm)n} \xi_{(+k)} Z_{(+k)} + \xi_{(+k)} \delta_{(\pm)n} Z_{(+k)} \right. \\ & \left. \left. - \delta_{(\pm)n} \xi_{(-k)} Z_{(-k)} - \xi_{(-k)} \delta_{(\pm)n} Z_{(-k)} \right] \right\}; \end{aligned} \quad (89)$$

$\mathcal{R}_{(\pm)n}$ is bilinear in p and a^\dagger ,

$$\mathcal{R}_{(\pm)n} = -\frac{ip}{\sqrt{2}} \sum_{k=1}^{\infty} (\delta_{(\pm)n} \xi_{(+k)} a_{(+k)}^\dagger + \delta_{(\pm)n} \xi_{(-k)} a_{(-k)}^\dagger); \quad (90)$$

and $\mathcal{S}_{(\pm)n}$ is quadratic in a^\dagger ,

$$\begin{aligned} \mathcal{S}_{(\pm)n} = & -\frac{1}{2} \sum_{k,l=1}^{\infty} (\delta_{(\pm)n} \mathcal{Y}_{(+k)l} a_{(+k)}^\dagger a_{(+l)}^\dagger \\ & + \delta_{(\pm)n} \mathcal{Y}_{(-k)l} a_{(-k)}^\dagger a_{(-l)}^\dagger). \end{aligned} \quad (91)$$

The explicit forms of these terms can be obtained by substituting Eqs. (77)–(86). In particular, it follows immediately that, for $n \geq 0$,

$$\mathcal{Q}_{(\pm)n} = -i \frac{p^2}{4} \delta_{n0}, \quad (92)$$

$$\mathcal{R}_{(\pm)n} = 0, \quad (93)$$

$$\mathcal{S}_{(\pm)n} = 0. \quad (94)$$

We now want to compare these results with the action on $|0, \psi, T\rangle$ of the Schrödinger picture Hamiltonian. We therefore digress for a moment to define this Hamiltonian.

The classical dynamical evolution equations (12)–(15) are generated by the Hamiltonian

$$\begin{aligned} H = & \int_0^{2\pi} \frac{1}{4} \left\{ \frac{\partial T^+(x,t)}{\partial t} (T_{,x}^+(x,t))^{-1} [\boldsymbol{\pi}(x) + \boldsymbol{\phi}_{,x}(x)]^2 \right. \\ & \left. - \frac{\partial T^-(x,t)}{\partial t} (T_{,x}^-(x,t))^{-1} [\boldsymbol{\pi}(x) - \boldsymbol{\phi}_{,x}(x)]^2 \right\} dx. \end{aligned} \quad (95)$$

Quantum mechanically, the Hamiltonian (95) can be made well-defined (i.e., densely defined, self-adjoint) for any choice of $T^\alpha(x,t)$ by normal-ordering with respect to the creation and annihilation operators and (a^\dagger, a) . (This feature does not seem to generalize to higher-dimensional models, see Sec. V). In this way the normal-ordered Hamiltonian, denoted by $:H:$, generates the Heisenberg equations of motion

$$i \frac{\partial \phi(x)}{\partial t} = [\phi(x), :H:], \quad (96)$$

$$i \frac{\partial \pi(x)}{\partial t} = [\pi(x), :H:] \quad (97)$$

associated with an arbitrary spacelike foliation $T^\alpha(x,t)$. Because the foliation is arbitrary, the Heisenberg equations shown above are equivalent to a set of functional Heisenberg equations,

$$i \frac{\delta \phi(x)}{\delta T^\pm(x')} = [\phi(x), \mathcal{H}_\pm(x')] \quad (98)$$

$$i \frac{\delta \pi(x)}{\delta T^\pm(x')} = [\pi(x), \mathcal{H}_\pm(x')], \quad (99)$$

where

$$\mathcal{H}_\pm(x) = \pm \frac{:(\pi(x) \pm \phi_{,x}(x))^2:}{4T_{,x}^\pm(x)}. \quad (100)$$

It is important to keep in mind that normal ordering is essentially a renormalization prescription that discards an infinity. It is still possible to renormalize by a finite amount. This possibility corresponds to the freedom to add multiples of the identity operator to the Hamiltonian without disturbing the Heisenberg equations of motion. As we shall see, this finite renormalization is needed to define dynamical evolution of the state vector along an arbitrary foliation.

Recalling the time evolution operator $U[T]$, and the usual correspondence between the Schrödinger picture and the Heisenberg picture, it follows that the time evolution of state vectors is (up to the possible addition of multiples of the identity) controlled by the Schrödinger Hamiltonian,

$$H_S := U[T]:H:U^\dagger[T], \quad (101)$$

and Schrödinger Hamiltonian densities,

$$\mathcal{H}_{S\pm}(x) := U[T]\mathcal{H}_\pm(x)U^\dagger[T]. \quad (102)$$

From the definition (75) of the operators $b_{(\pm)k}$ and $b_{(\pm)k}^\dagger$, it is straightforward to verify that H_S and $\mathcal{H}_{S\pm}$ are the same functions of $b_{(\pm)k}$ and $b_{(\pm)k}^\dagger$ that $:H:$ and \mathcal{H}_\pm are functions of $a_{(\pm)k}$ and $a_{(\pm)k}^\dagger$. In particular, the Schrödinger Hamiltonians H_S and $\mathcal{H}_{S\pm}$ are normal-ordered in the b , b^\dagger operators.

We now return to our derivation of the functional Schrödinger equation satisfied by $|0, \psi, T\rangle$. To this end, we consider the action of the operators $\mathcal{H}_{S\pm}(x)$ on $|0, \psi, T\rangle$. Again, we introduce Fourier modes:

$$h_{(\pm)n} = \int_0^{2\pi} e^{inT^\pm(x)} \mathcal{H}_{S\pm}(x) dx. \quad (103)$$

These Fourier modes are Virasoro operators (familiar from string theory) built from the b , b^\dagger operators:

$$h_{(\pm)0} = \frac{p^2}{4} + \sum_{k=1}^{\infty} k (b_{(\pm)k}^\dagger b_{(\pm)k}), \quad (104)$$

and, for $n > 0$,

$$\begin{aligned} h_{(\pm)n} = & -i \sqrt{\frac{n}{2}} p b_{(\pm)n} + \sum_{k=1}^{\infty} \sqrt{k(k+n)} b_{(\pm)k}^\dagger b_{(\pm)(k+n)} \\ & - \frac{1}{2} \sum_{k=1}^{n-1} \sqrt{k(n-k)} b_{(\pm)k} b_{(\pm)(n-k)}, \end{aligned} \quad (105)$$

$$h_{(\pm)-n} = h_{(\pm)n}^\dagger. \quad (106)$$

We now compute the action of $h_{(\pm)n}$ on $|0, \psi, T\rangle$ in order to compare with Eq. (87). To begin we note that, because this state is the vacuum associated with the $(b_{(\pm)n}, b_{(\pm)n}^\dagger)$ operators, we have

$$h_{(\pm)n}|0, \psi, T\rangle = \delta_{n0} \frac{p^2}{4} |0, \psi, T\rangle \quad n \geq 0. \quad (107)$$

Using Eqs. (77)–(83), (87)–(91) we see that

$$\left[\frac{1}{i} \delta_{(\pm)n} + h_{(\pm)n} + i(\delta_{(\pm)n} \log N(T)) \mathcal{I} \right] |0, \psi, T\rangle = 0, \quad n \geq 0. \quad (108)$$

Thus, up to addition of a multiple of the identity to the Schrödinger Hamiltonian, we have obtained the expected functional Schrödinger equation for $n \geq 0$.

In order to compute the action of $h_{(\pm)-n} = h_{(\pm)n}^\dagger$ on $|0, \psi, T\rangle$ we expand the $(b_{(\pm)n}, b_{(\pm)n}^\dagger)$ operators in terms of the $(a_{(\pm)n}, a_{(\pm)n}^\dagger)$ operators using the Bogolubov transformation (75) and apply the resulting operator to $|0, \psi, T\rangle$. At this point it is convenient to take note of the identities

$$\alpha_{(\pm)kl}^{-1} = \alpha_{(\pm)lk}^* - \sum_{r=1}^{\infty} \beta_{(\pm)lr}^* \gamma_{(\pm)rk}, \quad (109)$$

$$\sum_{k=1}^{\infty} \alpha_{(\pm)kl}^{-1} Z_k = \mp \left(\zeta_{(\pm)l}^* + \sum_{k=1}^{\infty} \beta_{(\pm)lk}^* \xi_{(\pm)k} \right). \quad (110)$$

We get four types of terms:

$$h_{(\pm)n}^\dagger |0, \psi, T\rangle = (P_{(\pm)n} + Q_{(\pm)n} + R_{(\pm)n} + S_{(\pm)n}) |0, \psi, T\rangle. \quad (111)$$

Here $P_{(\pm)n}$ is proportional to the identity \mathcal{I} ,

$$P_{(\pm)n} = -\frac{1}{2} \sum_{j=1}^{n-1} \sum_{r=1}^{\infty} \sqrt{j(n-j)} \beta_{(\pm)jr}^* \alpha_{(\pm)r, n-j}^{-1} \mathcal{I}, \quad (112)$$

$Q_{(\pm)n}$ is quadratic in p ,

$$\begin{aligned} Q_{(\pm)n} = & \frac{p^2}{2} \left\{ \sqrt{n} \zeta_{(\pm)n}^* + \sqrt{n} \sum_{l=1}^{\infty} \beta_{(\pm)nl}^* \xi_{(\pm)l} \right. \\ & + \sum_{k=1}^{n-1} \sqrt{k(n-k)} \left[\frac{1}{2} \zeta_{(\pm)k}^* \sum_{l=1}^{\infty} \beta_{(\pm)(n-k, l)}^* \xi_{(\pm)l} \right. \\ & + \frac{1}{2} \zeta_{(\pm)(n-k)}^* \sum_{l=1}^{\infty} \beta_{(\pm)k, l}^* \xi_{(\pm)l} + \frac{1}{2} \zeta_{(\pm)k}^* \zeta_{(\pm)(n-k)}^* \\ & \left. \left. + \frac{1}{2} \sum_{l, m=1}^{\infty} \beta_{(\pm)kl}^* \beta_{(\pm)(n-k, m)}^* \xi_{(\pm)l} \xi_{(\pm)m} \right] \right\}. \end{aligned} \quad (113)$$

$R_{(\pm)n}$ is bilinear in p and $a_{(\pm)n}^\dagger$:

$$\begin{aligned}
 R_{(\pm)n} &= i \sqrt{\frac{n}{2}} p \sum_{j=1}^{\infty} \left(\alpha_{(\pm)nj}^* - \sum_{r=1}^{\infty} \beta_{(\pm)nr}^* \gamma_{(\pm)rj} \right) a_{(\pm)j}^\dagger \\
 &+ \frac{i}{\sqrt{2}} p \sum_{j=1}^{\infty} \sum_{k=1}^{n-1} \sqrt{k(n-k)} a_{(\pm)j}^\dagger \alpha_{(\pm)jk}^{-1} \\
 &\times \left\{ \zeta_{(\pm)n-k}^* + \sum_{l=1}^{\infty} \beta_{(\pm)n-k,l}^* \xi_{(\pm)l} \right\}. \quad (114)
 \end{aligned}$$

Finally, $S_{(\pm)n}$ is quadratic in $a_{(\pm)k}^\dagger$:

$$\begin{aligned}
 S_{(\pm)n} &= -\frac{1}{2} \sum_{l,m=1}^{\infty} \sum_{k=1}^{n-1} \sqrt{k(n-k)} \left[\alpha_{(\pm)kl}^* \alpha_{(\pm)n-k,m}^* \right. \\
 &- \sum_{r=1}^{\infty} \beta_{(\pm)kr}^* \alpha_{(\pm)n-k,l}^* \gamma_{(\pm)rm} \\
 &- \sum_{r=1}^{\infty} \beta_{(\pm)n-k,r}^* \alpha_{(\pm)kl}^* \gamma_{(\pm)rm} \\
 &\left. + \sum_{r,s=1}^{\infty} \beta_{(\pm)kr}^* \beta_{(\pm)n-k,s}^* \gamma_{(\pm)rl} \gamma_{(\pm)sm} \right] a_{(\pm)l}^\dagger a_{(\pm)m}^\dagger. \quad (115)
 \end{aligned}$$

We now compare $Q_{(\pm)n}$, $R_{(\pm)n}$, $S_{(\pm)n}$ with $Q_{(\pm)n}$, $\mathcal{R}_{(\pm)n}$, $S_{(\pm)n}$; we find that

$$Q_{(\pm)n} = i Q_{(\pm)-|n|} \quad (116)$$

$$R_{(\pm)n} = i \mathcal{R}_{(\pm)-|n|} \quad (117)$$

$$S_{(\pm)n} = i S_{(\pm)-|n|}. \quad (118)$$

Combining our results, we have for all n

$$\left[\frac{1}{i} \delta_{(\pm)n} + h_{(\pm)n} + \mathcal{A}_{(\pm)n} \mathcal{I} \right] |0, \psi, T\rangle = 0, \quad (119)$$

where

$$\mathcal{A}_{(\pm)n} = i(\delta_{(\pm)n} \log N(T)), \quad \text{when } n \geq 0, \quad (120)$$

$$\begin{aligned}
 &= i(\delta_{(\pm)n} \log N(T)) \\
 &+ \frac{1}{2} \sum_{j=1}^{n-1} \sum_{r=1}^{\infty} \sqrt{j|n+j|} \beta_{(\pm)jr}^* \alpha_{(\pm)r,|n|-j}^{-1} \\
 &\quad \text{when } n < 0. \quad (121)
 \end{aligned}$$

This equation is equivalent to

$$\left[\frac{1}{i} \frac{\delta}{\delta T^\alpha(x)} + \mathcal{H}_{S_\alpha(x)} + \mathcal{A}_\alpha(x) \mathcal{I} \right] |0, \psi, T\rangle = 0, \quad (122)$$

where

$$\mathcal{A}_\pm(x) = \frac{1}{2\pi} T_{,x}^\pm(x) \sum_{n=-\infty}^{\infty} e^{-inT^\pm(x)} \mathcal{A}_{(\pm)n}. \quad (123)$$

The presence of the c -number contribution \mathcal{A}_α to the Schrödinger picture image of the normal-ordered Heisenberg Hamiltonian was proposed by Kuchař in [5]. Its presence is needed to ensure the integrability of Eq. (122) given the appearance of an anomaly (Schwinger terms) in the algebra of the operators $\mathcal{H}_\alpha(x)$. As such, following Kuchař, we refer to \mathcal{A}_α as the ‘‘anomaly potential.’’ The form of \mathcal{A}_α as a functional of embeddings is not uniquely determined because of the freedom to specify $\Lambda[T]$ in Eq. (70). The results of [5] imply that the phase $\Lambda[T]$ can be chosen to put the anomaly potential into the following local, spatially covariant form [16]:

$$\mathcal{A}_\pm = \frac{1}{24\pi} \left[\mp \frac{1}{2} (T_{,x}^\pm)^{-1} + ((T_{,x}^\pm)^{-1} K_x)_{,x} \right], \quad (124)$$

where

$$K_x = \frac{1}{2} \left[\frac{T_{,xx}^-}{T_{,x}^-} - \frac{T_{,xx}^+}{T_{,x}^+} \right] \quad (125)$$

is the mean extrinsic curvature of the embedding multiplied by the square root of the determinant of the metric induced on the embedded circle.

Having derived the functional Schrödinger equation satisfied by the Schrödinger image of the Heisenberg vacuum state, it now is easy to see that the basis $\{|e_i(T)\rangle\}$ described in Sec. III A also satisfies the same equation. This follows from the fact that the operators p , $b_{(\pm)k}$, $b_{(\pm)k}^\dagger$, $k = 1, 2, \dots$ satisfy

$$\left[p, \frac{1}{i} \frac{\delta}{\delta T^\alpha(x)} + \mathcal{H}_{S_\alpha(x)} + \mathcal{A}_\alpha(x) \mathcal{I} \right] = 0, \quad (126)$$

and

$$\begin{aligned}
 &\left[b_{(\pm)k}, \frac{1}{i} \frac{\delta}{\delta T^\alpha(x)} + \mathcal{H}_{S_\alpha(x)} + \mathcal{A}_\alpha(x) \mathcal{I} \right] = 0 \\
 &= \left[b_{(\pm)k}^\dagger, \frac{1}{i} \frac{\delta}{\delta T^\alpha(x)} + \mathcal{H}_{S_\alpha(x)} + \mathcal{A}_\alpha(x) \mathcal{I} \right]. \quad (127)
 \end{aligned}$$

The states $\{|e_i(T)\rangle\}$ thus define a basis of solutions to the functional Schrödinger equation.

Finally, we emphasize that the functional Schrödinger equation (122) can be viewed as the quantum constraint in the Dirac quantization of parametrized field theory in the Schrödinger picture. As predicted in [5], the factor ordering of this constraint is quite non-trivially related to that of normal ordering in the (a^\dagger, a) operators. Note also that the operators (p, b^\dagger, b) used to build the physical states are ‘‘Dirac observables;’’ as shown in Eqs. (126) and (127) they commute with quantum constraint operators.

IV. SPATIAL DIFFEOMORPHISMS

In the quantum theory of generally covariant systems one often partitions the constraint equations of the theory into dynamical constraints (the ‘‘super-Hamiltonian constraint,’’ the ‘‘Wheeler-DeWitt equation’’) and gauge constraints (the

“super-momentum constraint,” the “diffeomorphism constraint”). The physical states constructed in Sec. III A satisfy the functional Schrödinger equation (122), which governs the propagation of the state vector from hypersurface to hypersurface in spacetime. As described in Sec. II C this equation can be interpreted as representing a quantization of the constraints which arise in the Hamiltonian description of a parametrized field theory. If Eq. (122) is projected along the normal to the embedding $T^\alpha(x)$ then we obtain an analog of the Wheeler-DeWitt equation, which governs the change of the state as time is pushed forward along the normal to the embedding. If we project this equation tangentially to the embedding $T^\alpha(x)$, then we get

$$\left[\frac{1}{i} T_{,x}^\alpha \frac{\delta}{\delta T^\alpha} + \mathcal{H}_{(S)x} + \mathcal{A}_x \right] |\Psi(T)\rangle = 0, \quad (128)$$

where

$$\mathcal{H}_{(S)x} = T_{,x}^\alpha \mathcal{H}_{(S)\alpha}, \quad (129)$$

and

$$\mathcal{A}_x = T_{,x}^\alpha \mathcal{A}_\alpha. \quad (130)$$

Normally, this gauge constraint is viewed as enforcing some kind of spatial diffeomorphism invariance of the state vector. Indeed, the analog of this equation in canonical quantum gravity is usually interpreted as saying that wave functions in the metric representation depend only upon diffeomorphism equivalence classes of the spatial metric [13]. Alternatively, in the loop representation of canonical quantum gravity, the analog of Eq. (128) is interpreted as saying that wave functions only depend upon diffeomorphism equivalence classes of closed curves (knots, links, etc.) [14,15]. Here we would like to relate Eq. (128) to the action of spatial diffeomorphisms in quantum parametrized field theories. In particular, we would like to see how/if one can maintain the interpretation of Eq. (128) as enforcing spatial diffeomorphism invariance at the quantum level. The issue is not trivial given the factor ordering used to define $\mathcal{H}_{(S)x}$ and, in particular, given the c -number term \mathcal{A}_x which appears in Eq. (128).

We will present two results. First we show that the phase freedom ($\Lambda[T]$ in Eq. (70)) can be used to cast Eq. (128) into the form

$$\left[\frac{1}{i} T_{,x}^\alpha \frac{\delta}{\delta T^\alpha} + h_x \right] |\Psi(T)\rangle = 0, \quad (131)$$

where

$$h_x =: \pi_0(\phi_0)_{,x} :, \quad (132)$$

is a particular ordering of the Schrödinger picture momentum density for the field, and the field operators $\phi_0(x)$ and $\pi_0(x)$ are defined in Eqs. (26), (27). By definition, the operator h_x is normal ordered in the (a^\dagger, a) creation and annihilation operators. Second, we show Eq. (131) can be interpreted as indicating that the physical states constructed in

Sec. III A are invariant under an action of the group of (spatial) diffeomorphisms of the circle.

To begin, we note that $\mathcal{H}_{(S)x}$ is, up to operator ordering, the Schrödinger momentum density in Eq. (132). As a consequence, the difference between $\mathcal{H}_{(S)x}(x)$ and $h_x(x)$ is a “ c -number” functional of the embeddings, $\sigma[T](x)$:

$$\mathcal{H}_{(S)x} = h_x + \sigma \mathcal{I}. \quad (133)$$

A direct computation of this c -number is straightforward but not immediately enlightening. We compute $\sigma[T](x)$ indirectly as follows. Because of Eq. (133), the variation of $\mathcal{H}_{(S)x}$ with respect to the embedding $T^\alpha(x)$ is a multiple of the identity which is related to $\sigma[T]$ via

$$\frac{\delta \mathcal{H}_{(S)x}[T](x)}{\delta T^\alpha(y)} = \frac{\delta \sigma[T](x)}{\delta T^\alpha(y)} \mathcal{I}. \quad (134)$$

We take the expectation value of this operator relation in the Schrödinger vacuum state $|0, \psi, T\rangle$. Using the Schrödinger equation (122) we can put the expectation value in the form

$$\begin{aligned} \frac{\delta \sigma[T](x)}{\delta T^\alpha(y)} &= iT_{,x}^\beta(x) \langle 0, \psi | [\mathcal{H}_\beta(x), \mathcal{H}_\alpha(y)] | 0, \psi \rangle \\ &\quad - i \frac{\delta}{\delta T^\alpha(y)} \langle 0, \psi | T_{,x}^\beta(x) \mathcal{H}_\beta(x) | 0, \psi \rangle. \end{aligned} \quad (135)$$

The right-hand side of Eq. (135) can be evaluated using results of Kuchař [5]. As usual, we will compute in null coordinates; we have [16]

$$\begin{aligned} \frac{\delta \sigma[T](x)}{\delta T^\pm(y)} &= \pm \frac{1}{24\pi} T_{,x}^\pm(x) \{ \delta_{,x}(x, y) \\ &\quad + \partial_x [(T_{,x}^\pm(x))^{-1} \partial_x ((T_{,x}^\pm(x))^{-1} \\ &\quad \times \delta_{,x}(x, y))] \}. \end{aligned} \quad (136)$$

It is a straightforward exercise to solve the functional differential equation (136); we get

$$\begin{aligned} \sigma[T] &= \frac{1}{24\pi} \left[\frac{1}{2} (T_{,x}^+)^2 - \frac{3}{2} (T_{,x}^+)^{-2} (T_{,xx}^+)^2 + (T_{,x}^+)^{-1} T_{,xxx}^+ \right. \\ &\quad \left. - \frac{1}{2} (T_{,x}^-)^2 + \frac{3}{2} (T_{,x}^-)^{-2} (T_{,xx}^-)^2 - (T_{,x}^-)^{-1} T_{,xxx}^- \right], \end{aligned} \quad (137)$$

where we have eliminated an integration constant by taking into account the boundary condition that $\sigma[T] = 0$ when $T^\alpha(x) = T_0^\alpha(x)$.

As mentioned in Sec. II D, the dynamical evolution of field operators arises via two copies of the metaplectic representation of the group of diffeomorphisms of the circle. As noted in [4], this representation is closely related to a version of the Schwarzian derivative. The Schwarzian derivative defined in [4] is a non-linear third-order differential operator

mapping diffeomorphisms of the circle into functions on the circle. It is defined on diffeomorphisms $f: S^1 \rightarrow S^1$ via

$$S(f) = \frac{1}{12} \frac{f'''}{f'} - \frac{1}{8} \left(\frac{f''}{f'} \right)^2 + \frac{1}{24} [(f')^2 - 1]. \quad (138)$$

The difference between the two different orderings of the Schrödinger momentum densities can therefore be expressed in terms of the Schwarzian derivative as

$$\sigma[T] = \frac{1}{2\pi} [S(T^+) - S(T^-)]. \quad (139)$$

From the result (137), it is now easy to show that, for an appropriate choice of $\Lambda[T]$ in Eq. (70), we can turn Eq. (128) into Eq. (131), i.e.,

$$A_x[T] + \sigma[T] = 0. \quad (140)$$

Indeed, the local, spatially covariant choice of ‘‘gauge’’ advocated by Kuchař in [5] leads precisely to Eq. (140). This is easily verified using Eq. (124), and then using the relation between the extrinsic curvature and the embeddings (125). We thus get an interpretation of Kuchař’s covariant choice of gauge: In this gauge the anomaly potential exactly compensates for the difference in factor ordering between the Schrödinger momentum density $\mathcal{H}_{(S),x}$ appearing in Eq. (122) and the naive Schrödinger momentum density (132).

Given an appropriate choice of phase $\Lambda[T]$ in Eq. (70), we can assume that the spatial projection of the functional Schrödinger equation takes the form (131). We now show that this equation implies spatial diffeomorphism invariance of the Schrödinger picture physical states. Although this could be demonstrated directly in the Fock representation we have been using for the non-zero modes of the field, we will instead place our discussion in the Schrödinger coordinate representation since that is the representation one usually has in mind in such discussions. We now digress to describe this representation.

The Schrödinger representation we shall use is a natural extension to infinitely many degrees of freedom of an analogous representation for the harmonic oscillator. Because of the absence of an infinite-dimensional generalization of the usual translationally invariant Lebesgue measure, we use a Gaussian measure $d\mu$ to define the Hilbert space inner product [2,18]. So, the Hilbert space \mathcal{H} of states is defined as a space of functionals $\Psi = \Psi[Q]$ of a scalar field $Q(x)$ on a circle. We assume that the scalar field lies in the function space which is the topological dual to the space of smooth functions on the circle. Thus $Q(x) \in \mathcal{S}'$, the space of distributions on the circle (see, e.g., [17]). It is convenient to work with the Fourier modes of $Q(x)$. We have

$$Q(x) = \sum_{n=-\infty}^{\infty} Q_n e^{-inx}, \quad (141)$$

and, since $Q(x)$ is real,

$$Q_n = Q_{-n}^*. \quad (142)$$

The scalar product (\cdot, \cdot) on \mathcal{H} is that associated with the Gaussian measure $d\mu[Q]$ on the space of fields $Q(x)$ with covariance $(1/\pi)(-d^2/dx^2)^{-1/2}$ for the non-zero modes of $Q(x)$. The zero mode Q_0 gets the standard translationally invariant measure dQ_0 . So, for example, if we consider wave functions depending upon a finite number of modes, say, $\{Q_n, |n| \leq N\}$, we have

$$(\Psi, \Phi) = \int \Psi^*[Q] \Phi[Q] dQ_0 \times \prod_{n=-N}^N \star \frac{|2n|^{1/2}}{\pi^{1/2}} e^{-|n|Q_n Q_{-n}} dQ_n. \quad (143)$$

Here the star on the product symbol indicates one should omit $n=0$. The Hilbert space inner product based upon the Gaussian measure $d\mu[Q]$ arises formally as the limit of Eq. (143) as $N \rightarrow \infty$.

Because we use the measure $d\mu[Q]$, the wave functions $\Psi[Q]$ cannot be quite interpreted as probability amplitudes in the traditional way. Note, for example, that the Fock vacuum $|0, \psi\rangle$ in this representation is simply given by the wave function $\Psi[Q] = \psi(Q_0)$, where $\psi \in \mathcal{L}^2(\mathbb{R})$. In general, if the wave function is given by $\Psi = \Psi[Q]$, the probability $\mathcal{P}[Q]$ for measuring the field $\phi(x)$ and obtaining a value (in an infinitesimal neighborhood of) $Q(x)$ is given by

$$\mathcal{P}[Q] = \Psi^*[Q] \Psi[Q] d\mu[Q]. \quad (144)$$

Inclusion of the Gaussian measure in Eq. (144) is essential for the probability interpretation of the wave functions.

Keeping in mind that the Heisenberg picture fields on the initial slice $X_0^\alpha(x)$, namely $(\phi_0(x), \pi_0(x))$, are the Schrödinger picture fields, we expand these operators as

$$\phi_0(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \phi_n e^{-inx}, \quad (145)$$

$$\pi_0(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \pi_n e^{inx}. \quad (146)$$

The Fourier representatives (ϕ_n, π_n) of the Schrödinger picture operators $(\phi_0(x), \pi_0(x))$ are to satisfy the commutation relations

$$[\phi_n, \pi_m] = i \delta_{n,m}, \quad (147)$$

and the Hermiticity requirements

$$\phi_n^\dagger = \phi_{-n} \quad \text{and} \quad \pi_n^\dagger = \pi_{-n}. \quad (148)$$

The basic operators (ϕ_n, π_n) are represented on wave functions as

$$\phi_n \Psi[Q] = Q_n \Psi[Q], \quad (149)$$

$$\pi_n \Psi[Q] = \frac{1}{i} \left(\frac{\partial \Psi[Q]}{\partial Q_n} - |n| Q_{-n} \Psi[Q] \right). \quad (150)$$

The creation and annihilation operators are represented as

$$a_{(\pm)n}\Psi[Q]=\frac{1}{\sqrt{2n}}\frac{\partial\Psi}{\partial Q_{\mp n}} \quad (151)$$

$$a_{(\pm)n}^\dagger\Psi[Q]=-\frac{1}{\sqrt{2n}}\frac{\partial\Psi}{\partial Q_{\pm n}}+\sqrt{2n}Q_{\mp n}\Psi. \quad (152)$$

The Schrödinger representation described here is unitarily equivalent to the Fock representation [2,18].

It is now a simple matter to express the Schrödinger momentum density (132) as a differential operator-valued distribution on a suitable dense domain of functions $\Psi[Q]$. We get

$$\begin{aligned} h_x(x)\Psi[Q]&=-\frac{1}{2\pi}\sum_{n,m=-\infty}^{\infty}e^{i(n-m)x}mQ_m\frac{\partial\Psi[Q]}{\partial Q_n} \\ &-\frac{1}{2\pi}\sum_{n=1}^{\infty}\sum_{m=1}^n m(n-m)[e^{inx}Q_{-(n-m)}Q_{-m} \\ &-e^{-inx}Q_{n-m}Q_m]\Psi[Q]. \end{aligned} \quad (153)$$

We now consider an action of the spatial diffeomorphism group $\text{Diff}(S^1)$ on state vectors in this representation. Let $f:S^1\rightarrow S^1$ be a diffeomorphism of the circle. In coordinates, f is represented by a smooth map $x\rightarrow f(x)$ with a smooth inverse, satisfying

$$f(2\pi)=f(0)+2\pi. \quad (154)$$

We consider the usual pull-back action of spatial diffeomorphisms on the field $Q(x)$:

$$Q(x)\rightarrow(f^*Q)(x):=Q(f(x)). \quad (155)$$

This action induces an action of $\text{Diff}(S^1)$ on the Fourier modes:

$$Q_n\rightarrow(f^*Q)_n:=\sum_{m=-\infty}^{\infty}\Xi_{nm}Q_m, \quad (156)$$

where

$$\Xi_{nm}=\frac{1}{2\pi}\int_0^{2\pi}e^{inx}e^{-imf(x)}dx. \quad (157)$$

In order to interpret Eq. (131) we need the infinitesimal form of this action. Consider a 1-parameter family f_λ of spatial diffeomorphisms and define

$$V(x):=\left(\frac{df_\lambda(x)}{d\lambda}\right)_{\lambda=0}=\sum_{n=-\infty}^{\infty}V_n e^{-inx}, \quad (158)$$

$$\delta Q_n:=\left(\frac{d(f_\lambda^*Q)_n}{d\lambda}\right)_{\lambda=0}. \quad (159)$$

It is easy to see that

$$\delta Q_n=-i\sum_{m=-\infty}^{\infty}mV_{n-m}Q_m. \quad (160)$$

Let us now define an operator δ_V which provides the infinitesimal action of a one parameter family of spatial diffeomorphisms f_λ generated by V on functionals $\Psi[Q]$:

$$\delta_V\Psi[Q]=\left(\frac{d\Psi[f_\lambda^*Q]}{d\lambda}\right)_{\lambda=0}. \quad (161)$$

If we also define

$$h_x(V)=\int_0^{2\pi}h_x(x)V(x)dx, \quad (162)$$

then, using Eq. (153), it is easily verified that

$$h_x(V)\Psi[Q]=\frac{1}{i}\delta_V\Psi[Q]+F[Q]\Psi[Q], \quad (163)$$

where

$$\begin{aligned} F[Q]&=-\sum_{n=1}^{\infty}\sum_{m=1}^n m(n-m)[V_nQ_{-(n-m)}Q_{-m} \\ &-V_{-n}Q_{n-m}Q_m]. \end{aligned} \quad (164)$$

We remark that the infinite sum in $F[Q]$ converges for sufficiently smooth $V(x)$. From Eq. (163) we see that $h_x(V)$ would generate the action of spatial diffeomorphisms on wave functions $\Psi[Q]$ if not for the presence of the term $F[Q]$. This extra term simply reflects the presence of the Gaussian measure in Eq. (143). The role of $F[Q]$ is to guarantee that $h_x(V)$ generates the action of spatial diffeomorphisms on the probability (144). Indeed, we have the identity

$$\delta_V\mathcal{P}=\{[ih_x(V)\Psi]^*\Psi+\Psi^*[ih_x(V)\Psi]\}d\mu. \quad (165)$$

Next we recall that a functional $\Phi[T]$ of the embeddings changes under an infinitesimal spatial diffeomorphism via

$$\left(\frac{d\Phi[T^\alpha\circ f_\lambda]}{d\lambda}\right)_{\lambda=0}=\int_0^{2\pi}\frac{\delta\Phi}{\delta T^\alpha(x)}T_{,x}^\alpha(x)V(x)dx. \quad (166)$$

Because of Eq. (165), the spatial projection of the functional Schrödinger equation, given in Eq. (131), then implies that the probabilities occurring on a given embedding are invariant under orientation preserving spatial diffeomorphisms. More precisely, associated with a physical state vector, such as Eq. (71), there is a wave function

$$\Psi=\Psi[Q,T] \quad (167)$$

which defines the probability $\mathcal{P}[Q,T]$ for a measurement of the field $\phi(x)$ on the circle embedded as $T^\alpha=T^\alpha(x)$ to result in $Q(x)$:

$$\mathcal{P}[Q,T]=\Psi^*[Q,T]\Psi[Q,T]d\mu[Q]. \quad (168)$$

The probability $\mathcal{P}[Q,T]$ is spatially diffeomorphism invariant: If $f:S^1\rightarrow S^1$ is an orientation-preserving diffeomorphism, then

$$\mathcal{P}[Q, T] = \mathcal{P}[f^*Q, T \circ f]. \quad (169)$$

The result (169) is checked as follows. Because any two orientation preserving diffeomorphisms of the circle can be connected by a one parameter family of such diffeomorphisms, it suffices to consider a one parameter family of diffeomorphisms in Eq. (169) and check that

$$\left(\frac{d\mathcal{P}[f_\lambda^*Q, T \circ f_\lambda]}{d\lambda} \right)_{\lambda=0} = 0. \quad (170)$$

Using Eqs. (166), (165), and (131), Eq. (170) follows.

We note that while Eq. (131) depends upon the choice of phase $\Lambda[T]$, the result (169) is independent of such a choice of phase. This is, of course, due to the fact that the phase factor does not contribute to the probability. Viewing the state of a quantum system as the totality of probability distributions for the outcome of any and all measurements made on an ensemble of identically prepared systems, we thus conclude that the functional Schrödinger equation (122) enforces spatial diffeomorphism invariance of states in the Schrödinger representation of the Schrödinger picture.

Physically speaking, there is little else to discuss regarding the role of spatial diffeomorphisms in the space of Schrödinger picture physical states. Mathematically, there are a few other interesting issues. In particular, while the probabilities are spatially diffeomorphism invariant in the sense of Eq. (169), in the present representation neither the measure $d\mu[Q]$ nor the wave functions $\Psi[Q, T]$ satisfying Eq. (122) are separately invariant under the spatial diffeomorphism transformation

$$(Q, T) \rightarrow (f^*Q, T \circ f). \quad (171)$$

This is because the representation we are working in is designed to render the initial field operators (the Schrödinger picture field operators) diagonal and keep in a simple form the representation of the (a^\dagger, a) creation and annihilation operators as well as the representation of the Fock vacuum $|0, \psi\rangle$. From the point of view of the parametrized field theory of [5], this representation is tailored to the Heisenberg picture quantization in which physical states are embedding independent and the action of spatial diffeomorphisms is trivial on the field variables:

$$(Q, T) \rightarrow (Q, T \circ f). \quad (172)$$

Presumably, there exists a representation in which the wave functionals and measure are separately invariant under the action of spatial diffeomorphisms that naturally arise in the Schrödinger picture quantization of parametrized field theory [5]:

$$(Q, T) \rightarrow (f^*Q, T \circ f). \quad (173)$$

We will explore this representation of the quantum field theory elsewhere.

V. GENERALIZATIONS

There are a number of ways one might try to generalize the results presented in the previous sections. Here we briefly

discuss partial results pertaining to such generalizations; details will appear elsewhere. The generalizations that we consider include: inclusion of nonzero mass, massive and massless fields on flat spacetimes diffeomorphic to $R \times R$, and higher-dimensional generalizations of these models.

We begin by presenting a generic form for the Bogolubov coefficient relevant for a discussion of unitary implementability of dynamical evolution along an arbitrary foliation. We consider a free scalar field ϕ propagating on a flat $(n+1)$ -dimensional spacetime M . We assume that $M \approx R \times \Sigma$, where either $\Sigma = R^n$ or $\Sigma = \mathbf{T}^n$ (\mathbf{T}^n is the n -torus). We assume ϕ satisfies the Klein-Gordon equation

$$(\square - m^2)\phi = 0. \quad (174)$$

Let T^α and x^i denote inertial coordinates on M and arbitrary coordinates on Σ , respectively. An embedding $T: \Sigma \rightarrow M$ of a Cauchy surface is represented by $n+1$ functions of n variables:

$$T^\alpha = T^\alpha(x). \quad (175)$$

The induced metric and future pointing unit normal of a slice embedded by $T^\alpha(x)$ are denoted by γ_{ij} and n^α , respectively. Creation and annihilation operators $(a_{\mathbf{p}}^\dagger, a_{\mathbf{p}})$, are labeled by the wave vector \mathbf{p} for plane waves. This vector takes on discrete or continuous values when $\Sigma = \mathbf{T}^n$ or $\Sigma = R^n$. Dynamical evolution from an initial slice $T_0^\alpha(x)$ to a final slice $T^\alpha(x)$ can be viewed as a symplectic transformation on the space of solutions to Eq. (174). Consequently, there is a corresponding Bogolubov transformation of the creation and annihilation operators. If we choose the initial embedding to be flat with Cartesian coordinates, $T^\alpha(x) = (0, x^i)$, the mixing between creation and annihilation operators is controlled by the coefficients:

$$\beta_{\mathbf{k}, \mathbf{p}} = \frac{1}{\sqrt{\omega(k)\omega(p)}} \int (\sqrt{\gamma} n^\alpha k_\alpha + \omega(\mathbf{p})) \times e^{-i(\mathbf{p} \cdot \mathbf{x} + k_\alpha T^\alpha(x))} d^n x. \quad (176)$$

Here $\omega(k) = \sqrt{|\mathbf{k}|^2 + m^2}$ and $k_\alpha = (-\omega(k), \mathbf{k})$. We have dropped an irrelevant overall numerical factor in Eq. (176).

The Bogolubov coefficients (176) define an operator β on the one particle Hilbert space that underlies the Fock space. Unitary implementability of dynamical evolution from $T_0^\alpha(x)$ to $T^\alpha(x)$ requires β to be Hilbert-Schmidt. We have seen that this is so when $\Sigma = S^1$ and $m=0$ (there we had to also take account of zero modes). With compact spatial sections, the Hilbert-Schmidt condition only involves the ultraviolet behavior of β , and one therefore expects that, for $\Sigma = S^1$, β is Hilbert-Schmidt even when $m \neq 0$. This is indeed the case. We can prove that dynamical evolution along arbitrary spacelike foliations is unitarily implemented when $M = R \times S^1$ for any value of the mass m . When $M = R \times R$ the massless case is rather similar to the case studied in detail in the previous sections. In particular, we can show that the ultraviolet behavior of β does not spoil the Hilbert-Schmidt property provided the embeddings are asymptotically flat. However, one encounters an infrared divergence if one uses the usual Schwartz space as the space of test functions. We

expect that this case can nevertheless be handled with an appropriate choice of test functions for operator valued distributions representing the scalar field [2]. Likewise, we expect the operator β for a massive field on $M=R \times R$ to be well-behaved in the infrared and ultraviolet for evolution involving asymptotically flat spacelike slices. Consequently, we conjecture that our results for a massless, free, scalar field on $R \times S^1$ generalize to any free field on a flat two-dimensional spacetime. In particular, we expect that dynamical evolution along arbitrary spacelike foliations is unitarily implemented for free fields on flat spacetimes $M=R \times S^1$ and along asymptotically flat spacelike foliations of $M=R \times R$.

The situation in higher dimensions is not nearly so simple as it is for two-dimensional spacetimes. It is possible to obtain unitary evolution on the Fock space for free fields in higher dimensions if one restricts attention to special classes of foliations. For example, dynamical evolution along a foliation obtained by dragging an arbitrary spacelike slice along the integral curves of a Killing vector field can be shown to be unitarily implementable. However, using the stationary phase approximation, we have estimated Eq. (176) for the case $\Sigma=\mathbf{T}^m$ and found that β is *not* Hilbert-Schmidt for a generic embedding $T^\alpha(x)$. This means that dynamical evolution along arbitrary spacelike foliations is *not* unitarily implemented in the usual Poincare-invariant Fock representation for free fields on flat spacetime. A related difficulty is that the smeared energy-momentum densities do not have the particle number eigenstates (e.g., the Fock vacuum) in their domain (this point has already been noted in [19]). This fact would explain the divergent Schwinger terms that are encountered when computing the algebra of energy-momentum tensors [20]. We remark that an analogous situation arises in current algebra [21].

It is an interesting open question to find a Hilbert space quantization of free fields on flat spacetime of dimension greater than two which yields the correct physical results for dynamical evolution along foliations by flat slices and which also allows for dynamical evolution along more general foliations. In particular, the standard apparatus of Hilbert space and unitary time evolution does not seem adequate to deal with quantization of parametrized field theory models of quantum gravity in spacetime dimensions greater than two. It is well-known that analogous difficulties arise in the construction of quantum field theories in curved spacetime, where generically there are no preferred foliations available for the purposes of canonical quantization. In this case progress can be made by using algebraic methods of quantization (see, e.g., [22]), and it is likely that such methods can be fruitfully applied to the class of problems we are considering here. Thus, even in the simplest context of free fields in flat spacetime, our results suggest that one is forced to abandon “traditional” approaches to quantization of generally covariant theories in favor of the more flexible algebraic (or other) approaches.

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APPENDIX

In this appendix we show that the matrix $B_{mn}^{(+)}$ satisfies the Hilbert-Schmidt condition (43).

Since $T^+(x)$ is a diffeomorphism of the circle, it can be used as a coordinate. Put $T^+(x) - T^+(0) = \theta$ and define χ to be the inverse function to θ , that is, $\chi(\theta) := x$. Then

$$B_{mn}^{(+)} = -\frac{1}{2\pi} \sqrt{\frac{n}{m}} e^{inT^+(0)} \times \int_0^{2\pi} e^{im\chi(\theta) + in\theta} d\theta. \quad (\text{A1})$$

For any $t \in [0, 1]$, the function

$$\chi_t(\theta) := t\chi(\theta) + (1-t)\theta \quad (\text{A2})$$

is also a diffeomorphism. With $t = m/(m+n)$,

$$B_{mn}^{(+)} = -\frac{1}{2\pi} \sqrt{\frac{n}{m}} e^{inT^+(0)} \times \int_0^{2\pi} e^{i(m+n)\chi_t(\theta)} d\theta. \quad (\text{A3})$$

Put $\chi_t(\theta) = y$ and denote the inverse function to χ_t as φ_t . Then

$$B_{mn}^{(+)} = -\frac{1}{2\pi} \sqrt{\frac{n}{m}} e^{inT^+(0)} \times \int_0^{2\pi} e^{i(m+n)y} \frac{d\varphi_t}{dy} dy. \quad (\text{A4})$$

On integrating by parts k times,

$$B_{mn}^{(+)} = -\frac{i^k}{2\pi} \sqrt{\frac{n}{m}} (m+n)^{-k} e^{inT^+(0)} \times \int_0^{2\pi} e^{i(m+n)y} \frac{d^{k+1}\varphi_t}{dy^{k+1}} dy, \quad (\text{A5})$$

which gives the estimate

$$|B_{mn}^{(+)}| \leq (n+m)^{-k} \sqrt{\frac{n}{m}} \times \sup \left\{ \left| \frac{d^{k+1}\varphi_t}{dy^{k+1}} \right| : 0 \leq y \leq 2\pi, 0 \leq t \leq 1 \right\}. \quad (\text{A6})$$

(Note that for sufficiently smooth embeddings $\sup\{|d^{k+1}\varphi_t/dy^{k+1}|\}$ exists.) Clearly Eq. (A6) suffices to show that $B_{mn}^{(+)}$ is Hilbert-Schmidt type.

Similar considerations, involving appropriate integrations by parts, suffice to show that $B_{mn}^{(-)}$, $\alpha_{(\pm)mn}$, and $\beta_{(\pm)mn}$ are Hilbert-Schmidt type and that $Z_n^{(\pm)}$ and $\zeta_{(\pm)n}$ are rapidly decreasing in n .

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