Reflection of light from a random amplifying medium with disorder in the complex refractive index: Statistics of fluctuations

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The probability distribution of the reflection coefficient for light reflected from a one-dimensional random amplifying medium with cross-correlated spatial disorder in the real and the imaginary parts of the refractive index is derived using the method of invariant imbedding. The statistics of fluctuations have been obtained for both the correlated telegraph noise and the Gaussian white-noise models for the disorder. In both cases, an enhanced backscattering (with a reflection coefficient greater than unity) results because of coherent feedback due to Anderson localization and coherent amplification in the medium. The results show that the effect of randomness in the imaginary part of the refractive index on localization and reflection is qualitatively different.

Light propagation in disordered media and the associated Anderson localization of a wave in both the active as well as the passive random media[1–4] has been studied extensively. In recent years, there has been increased interest in light propagation and lasing in active random media supported by the several experiments carried out on these systems.[5–9] However, the experimental findings of a narrowed spectral emission[5,9] and a pulse narrowing of the emission[6,8] above a well defined threshold of pumping could be explained merely as an effect of the long diffusive pathlengths in a random medium with gain and the consequent amplified spontaneous emission (ASE).[6,10] More recently, the observed supernarrowing of the emitted spectra from strongly scattering semiconducting powder[11] and from weak scatterers dispersed in high gain organic media[12] has been attributed to coherent feedback caused by recurrent multiple scattering.[13] It is, however, still debatable if the wave amplification is due to the predicted synergy between wave confinement by Anderson localization and coherent amplification.[2]

In all these studies, the active random medium is considered to scatter the propagating wave (light) due to fluctuations in the real part of the refractive index (η_r) (real potentials) while the coherent amplification is modelled by a phenomenological spatially constant imaginary part of the refractive index (η_i). However, it would be of interest to examine the effect of a spatially fluctuating imaginary part of the refractive index as well. More so, as the scattering microparticles (e.g., polystyrene microspheres TiO_2 rutile particles) used in the experiments are not active, a corresponding mismatch in the imaginary part of the refractive index is found to exist. It has been pointed out by Rubio and Kumar[14] that a mismatch in the imaginary part of the refractive index (imaginary potential) would always cause a concomitant reflection (scattering) in addition to the absorption or amplification. Mismatch in η_i alone in an amplifying medium (negative imaginary potential) with no mismatch in η_r can cause resonant enhancement of the scattering coefficients. In fact, the reflection and the transmission coefficients can even diverge as can be seen from the simple example of a single imaginary δ potential in one dimension. This would correspond to the experimental situation where the scatterers (polystyrene microspheres, say) are suspended in a fluid with the same η_r (index matching fluid) in which a laser dye is dissolved and optically pumped. The scattering caused by the fluctuations in η_i would, therefore be expected to have nontrivial effects on the wave propagation in the medium.

The transmittance across a randomly amplifying and absorbing chain was recently considered by Sen[15] numerically and was shown to decay exponentially with the increase in length of the chain, presumably due to localization. But the effects of the fluctuation in the imaginary part of the refractive index on lasing in such random media has not been studied so far. In this work, we consider the statistics of the non-self-averaging fluctuations of the reflection coefficient for light incident on a one-dimensional active random medium with spatial correlated disorder in both the imaginary part as well as the real part of the refractive index.

We consider a one-dimensional active disordered medium of length L with a random complex refractive index η ,$0 \leqslant x \leqslant L$. For simplicity, polarization effects are neglected and light is assumed to be a scalar wave. A physical realization of interest here would be an Er^{3+} doped and pumped, polarization maintaining optical fibre intentionally disordered along its length. Further, only the linear case of the gain or absorption being independent of the wave amplitude is considered and nonlinear features such as gain saturation are not considered. Here we would like to emphasize that our treatment is for the possibility of reflection ($r > 1$), i.e., for an amplifier and not an oscillator.[16] The complex wave amplitude $E(x)$ obeys the Helmhotz equation inside the medium

$$\frac{d^2E(x)}{dx^2}+k^2[1+\eta(x)]E(x)=0, \tag{1}$$

where k is the wave vector in the medium ($k^2=\omega^2/c^2\epsilon_0$) and $\eta(x)=\eta_r(x)+i[\overline{\eta_i}+\eta_i(x)]$ is the complex refractive index. Here $\eta_r(x)$ and $\eta_i(x)$ are random and $\overline{\eta_i}$ is a constant representing the average amplification or absorption in the medium according as $\overline{\eta_i}$ is negative or positive. It is well known that Eq. (1) can be transformed to give an equation for the evolution of the emergent quantity, namely, the complex amplitude reflection coefficient $R(L)=[r(L)]^{1/2}\exp[i\theta(L)]$ as a function of the sample length L, via the method of invariant imbedding[17,18] as

$$\frac{dR(L)}{dL}=2ikR(L)+\frac{ik}{2}\eta(L)[1+R(L)]^2. \tag{2}$$

Equation (2) is a stochastic differential equation and we are interested in the corresponding Fokker-Planck equation for the probability distribution $P(r,\theta;L)$ which can be readily obtained following the standard procedures. Thus, let $\Pi(r,\theta;L)$ be the density of points in the $(r,\ \theta)$ phase space. Now $\Pi(r,\theta;L)$ must satisfy the Stochastic Liouville equation,[21] and by the van Kampen lemma,[21] the probability distribution function $P(r,\theta;L)=\langle\Pi(r,\theta;L)\rangle_{\eta_r,\eta_i}$, where the angular brackets denote averaging over all the realizations of the random refractive indices η_r and η_i.

The Gaussian δ correlated (white-noise) disorder. First, let us consider the simplest case namely that of a Gaussian δ correlated (white-noise) model. In this model, η_r and η_i are assumed to have δ correlated Gaussian distributions with $\langle\eta_r(L)\rangle=0$, $\langle\eta_i(L)\rangle=0$, $\langle\eta_r(L)\eta_r(L')\rangle=\Delta_r^2\delta(L-L')$, and $\langle\eta_i(L)\eta_i(L')\rangle=\Delta_i^2\delta(L-L')$. This model would most appropriately describe the case of a continuous random medium such as a laser-dye doped gel or intralipid suspension,[19,20] where the fluctuations in η_r and η_i are uncorrelated. Using the Novikov theorem[22] to average over all configurations of η_r and η_i, we obtain in the random phase approximation (RPA) [i.e., $P(r,\theta)=P(r)/2\pi$],

$$\frac{\partial P}{\partial l}=\phi_r\mathbf{L_R}P+\phi_i\mathbf{L_I}P+2A\frac{\partial(rP)}{\partial r}, \tag{3}$$

where the linear operators $\mathbf{L_R}$ and $\mathbf{L_I}$ are given by

$$\mathbf{L_R}=\frac{1}{2}\left[r(r-1)^2\frac{\partial^2}{\partial r^2}+(5r^2-6r+1)\frac{\partial}{\partial r}+2(2r-1)\right], \tag{4}$$

$$\mathbf{L_I}=\frac{1}{2}\left[r(r^2+10r+1)\frac{\partial^2}{\partial r^2}+(5r^2+30r+1)\frac{\partial}{\partial r}+2(2r+5)\right], \tag{5}$$

and the nondimensional sample length $l=1/2\max\{\Delta_r^2,\Delta_i^2\}k^2L\equiv L/l_c$, $\phi_r=\Delta_r^2/\max\{\Delta_r^2,\Delta_i^2\}$, $\phi_i=\Delta_i^2/\max\{\Delta_r^2,\Delta_i^2\}$, and $A=2\overline{\eta_i}/\max\{\Delta_r^2,\Delta_i^2\}k\equiv l_c/l_{\mathrm{amp}}$. Here $l_{\mathrm{amp}}=(\overline{\eta_i}k)^{-1}$ is the amplification length in the medium defined by the average of the imaginary part of the refractive index and max implies the superior value of the arguments. The RPA is known to be valid in the the weak disorder limit, $kl_c\gg1$, where l_c is the localization length.[18] We point out that even if η_r and η_i were cross-correlated, the final equations do not differ in the RPA for the white-noise model [because $\langle L_1L_2P\rangle_\theta=0$ see Eqs. (9),(10)].

The asymptotic $l\to\infty$ limiting solution of Eq. (3) obtained by setting $\partial P/\partial l=0$ is given by

$$P(r;\infty)=P_0\frac{\exp(-2A/\gamma\tan^{-1}\{[(\phi_r+\phi_i)r+5\phi_i-\phi_r]/\gamma\})}{[(\phi_r+\phi_i)(1+r^2)+2(5\phi_i-\phi_r)r]},\quad \phi_r>2\phi_i,$$
$$=\frac{P_0}{(\phi_r+\phi_i)(1+r^2)+2(5\phi_i-\phi_r)r}\left[\frac{(\phi_r+\phi_i)r+5\phi_i-\phi_r-\gamma}{(\phi_r+\phi_i)r+5\phi_i-\phi_r+\gamma}\right]^{-A/\gamma},\quad \phi_r<2\phi_i, \tag{6}$$

where $\gamma=\sqrt{12\phi_i|\phi_r-2\phi_i|}$ and P_0 is a normalization constant given by $[\int_0^\infty P(r,\infty)dr]^{-1}$. The limit $l\to\infty$ implies physically $L\gg l_c$. This expression goes over straightforwardly to the result of Pradhan and Kumar[2] in the limiting case of pure real disorder ($\phi_i=0$). Thus the statistics qualitatively differ in the two regimes for an amplifying medium: (i) when the real part disorder dominates ($\phi_r>2\phi_i$) and (ii) when the imaginary part disorder dominates ($\phi_r<2\phi_i$).

We have also solved Eq. (3) numerically for finite length to investigate the approach to the asymptotic forms given by Eq. (6). In Fig. 1, the plots of $P(r,l)$ for the case of real disorder dominating ($\phi_r>2\phi_i$) for different lengths of the medium are shown. The probability distribution for the case of a pure imaginary mismatch ($\phi_r=0$), with the real part η_r being index-matched is shown in Fig. 2. The line joining the dots in both the figures corresponds to the asymptotic $P(r;\infty)$ solution. In the case of amplifying medium, the value of reflectivity ($r_{\max}$) at which $P(r;l)$ peaks increases with the average value of the amplification factor $|A|$. For the case of imaginary part disorder dominating, $P(r;l)$ has a peak at small values of the reflectivity even for moderate values of the amplification. In the case of an absorbing medium with the imaginary disorder dominating, the probability distribution has a monotonic decreasing behavior and is maximum at $r=0$. A finite probability of reflection at $r>1$ in the absorbing case and at $r<1$ in the amplifying case ($A<0$) is recognized to be a consequence of the two sidedness of the white-noise process for the complex refractive index, which allows the imaginary part of the refractive index ($\overline{\eta_i}+\eta_i$) to take on locally both positive and negative values for any given value of the average. It should be noted

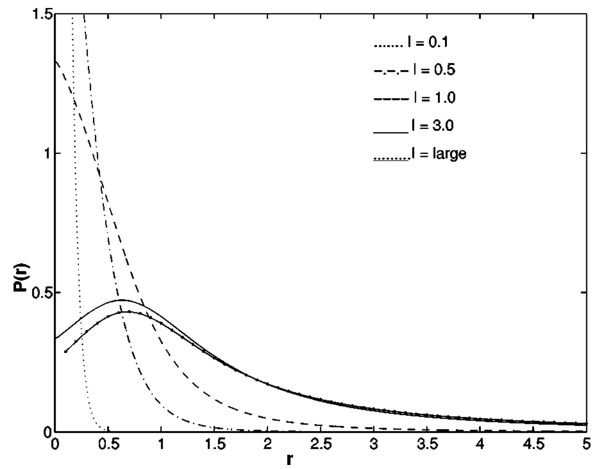

FIG. 1. The probability distribution of reflectivity $P(r;l)$ in the case of the white noise disorder given by Eq. (3) and the real disorder dominating $(\phi_r=1.0,\phi_i=0.1)$, for the different sample lengths indicated. The line joining the dots is the analtic result for $P(r;\infty)$. The amplification parameter is $A=-0.25$.

that this limiting form of $P(r,\infty)$ gives a weak logarithmic divergence for $\langle r\rangle$ (for $\phi_i\neq 0$), regardless of the sign of A for both absorption and amplification. Thus amplification has a much more drastic effect on the reflectivity than attenuation. The white-noise process allows the local fluctuations in η_i to be very large and the effect of a finite mean value $\overline{\eta_i}$ is small. It is thus a case of the fluctuations dominating over the mean. We also find that the numerical solutions saturate to the limiting forms for $l\gtrsim 1$. So most of the reflection occurs from within a localization length. This enhanced backscattering is quite different from that caused by light diffusion.[6,10] In the latter case, the distribution of optical path length, because of exponential growth of wave amplitude due to coherent amplification in one-dimension, gives $P^D(r;\infty)\sim \ln(r)^{1/2}/r$ for $r\gg 1$. This decays much slower than the $P(r;\infty)$ for $r\to\infty$, as given by Eq. (6).

Correlated telegraph disorder. In the case of the white-noise disorder, the imaginary part of the refractive index was allowed to take on both positive and negative values, i.e., the medium could be locally both amplifying or absorbing. With a view to studying purely amplifying or absorbing random media, we use the telegraph disorder model to describe the fluctuations in the refractive index. Moreover, since the gain-absorption coefficient is physically always bounded from above, the fluctuations in the imaginary part of the refractive index are better described by this dichotomic Markov process (i.e., spatial telegraph noise). Further, we recognize that in discrete random media such as microparticles suspended in a laser dye solution used in experiments, the real and the imaginary parts of the refractive index fluctuate spatially in the same manner and can, therefore, be described by the same stochastic process. A telegraph noise with a finite correlation length is most appropriate to describe such a situation. Accordingly, we will choose $\eta_r(L)=\alpha\chi(L)$ and $\eta_i(L)=\beta\chi(L)$ with an average value for the imaginary part $\overline{\eta_i}$. Here $\chi(L)$ is taken to be a dichotomic Markov process which can take on the values $\pm\chi$ such that $\langle\chi(L)\rangle=0$ and $\langle\chi(L)\chi(L')\rangle=\chi^2\exp(-\Gamma|L-L'|)$, where Γ^{-1} is the correlation length in the medium.

Now, defining as before, $P(r,\theta;L)=\langle\Pi(r,\theta;L)\rangle_\chi$ and $W(r,\theta;L)=\langle\chi(L)\Pi(r,\theta;L)\rangle_\chi$, and using the ''formulas of differentiation'' of Shapiro and Loginov[23] to average over the dichotomous configurations of $\chi(L)$, we obtain

$$\frac{\partial P}{\partial L}=-2k\frac{\partial P}{\partial\theta}+\overline{\eta_i}\mathbf{L_2}P+(\alpha\mathbf{L_1}+\beta\mathbf{L_2})W, \tag{7}$$

$$\frac{\partial W}{\partial L}=\chi^2(\alpha\mathbf{L_1}+\beta\mathbf{L_2})P-2k\frac{\partial W}{\partial\theta}+\overline{\eta_i}\mathbf{L_2}W-\Gamma W, \tag{8}$$

where the linear operators $\mathbf{L_1}$ and $\mathbf{L_2}$ are

$$\mathbf{L_1}=-k\left[\sin\theta\frac{\partial}{\partial r}\sqrt{r}(1-r)+\frac{\partial}{\partial\theta}+\frac{1}{2}\left(\sqrt{r}+\frac{1}{\sqrt{r}}\right)\frac{\partial}{\partial\theta}\cos\theta\right], \tag{9}$$

$$\mathbf{L_2}=k\left[\cos\theta\frac{\partial}{\partial r}\sqrt{r}(1+r)+2\frac{\partial}{\partial r}r+\frac{1}{2}\left(\sqrt{r}-\frac{1}{\sqrt{r}}\right)\frac{\partial}{\partial\theta}\sin\theta\right]. \tag{10}$$

We thus get a closed system of equations for $P(r,\theta,L)$ and $W(r,\theta,L)$. These equations go over correctly to the corresponding Eq. (3) in the white-noise limit obtained by taking the limit $\chi^2\to\infty$, $\Gamma\to\infty$ while keeping $\chi^2/\Gamma=\Delta^2$ constant. In this limit, the equation for $P(r,\theta;L)$ becomes autonomous, i.e., it gets decoupled from $W(r,\theta;L)$.

In the RPA and in the asymptotic limit $L\to\infty$, these equations simplify to

$$\beta\overline{\eta_i}\mathbf{L_I}P+\alpha^2\mathbf{L_R}W+\beta^2\mathbf{L_I}W=0, \tag{11}$$

$$\alpha^2\mathbf{L_R}P+\beta^2\mathbf{L_I}P+2A\frac{\partial(rP)}{\partial r}-\frac{\overline{\eta_i}\beta}{\chi^2}\mathbf{L_R}W=0, \tag{12}$$

where $\mathbf{L_R}$ and $\mathbf{L_I}$ are given by Eqs. (4) and (5) and $A=2\Gamma\overline{\eta_i}/\chi^2$. Interestingly in the case of the pure real part disorder $(\beta=0)$, the form of the telegraph noise equation for $P(r;\infty)$ is identical to that for the white-noise case, but with the coefficient $A=2\Gamma\overline{\eta_i}/k\chi^2$. Similarly, in the case of the

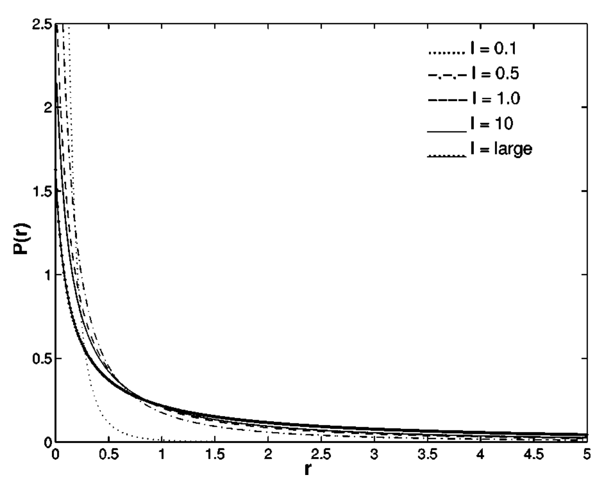

FIG. 2. The probability distribution $P(r;l)$ in the case of the white noise disorder given by Eq. (3) and a pure imaginary mismatch $(\phi_r=0)$ for different lengths of the sample. The line joining the dots is the analtic result for $P(r;\infty)$. The amplification parameter is $A=-1$.

pure imaginary part disorder ($\alpha=0$), the form of the telegraph noise equation for $P(r;\infty)$ is again identical to that for the white-noise case, but with the coefficient $A=2\Gamma\overline{\eta}_i/k(\chi^2-\overline{\eta}_i{}^2)$. However, for $\beta\chi<|\overline{\eta}_i|$, the imaginary part of the refractive index is always positive (absorbing) or negative (amplifying). Hence the solution for these two cases is also given by Eq. (6), the solutions being valid in the interval $0<r<1$ for the absorbing medium, and $1<r<\infty$ for the amplifying medium. Outside the intervals, the probability density $P(r;L)$ vanishes.

A complete solution for Eqs. (11) and (12) is obtained as

$$P(r;\infty)=P_0\left[\frac{1}{\xi_+(1+\zeta_+r+r^2)}+\frac{1}{\xi_-(1+\zeta_-r+r^2)}\right]\times\exp\{-2A[I_+(r)+I_-(r)]\},\tag{13}$$

$$I_\pm(r)=\frac{1}{\xi_\pm\sqrt{\zeta_\pm^2-4}}\ln\left|\frac{r-r_\pm^{(2)}}{r-r_\pm^{(1)}}\right|,\quad |\zeta_\pm|>2,$$

$$=\frac{1}{\xi_\pm\sqrt{\zeta_\pm^2-4}}\tan^{-1}\left(\frac{\zeta_\pm+2r}{\sqrt{\zeta_\pm^2-4}}\right),\quad |\zeta_\pm|<2,$$

where $\xi_\pm=\alpha^2+\beta^2\pm\beta\overline{\eta}_i/\chi$, $\zeta_\pm=[10(\beta^2\pm\beta\overline{\eta}_i/\chi)-2\alpha^2]/[1\pm\sqrt{\beta}+\alpha^2]$, $r_\pm^{(1)}=-1/2[\zeta_\pm+(\zeta_\pm^2-4)^{1/2}]$, $r_\pm^{(2)}=-1/2[\zeta_\pm-(\zeta_\pm^2-4)^{1/2}]$ and P_0 is a normalization coefficient. These expressions become the same as given by Eq. (6) in the white noise limit ($\chi^2\to\infty$, $\Gamma\to\infty$ and χ^2/Γ being constant).

The solutions for one-sided disorder in the imaginary part exhibit three qualitatively different behaviors corresponding to choices of the parameters α, β, and $\overline{\eta}_i$ (χ is an arbitrary constant and can be set to unity without loss of generality). First, we note that the case of $\alpha^2+\beta^2-\beta|\overline{\eta}_i|/\chi=0$, corresponds to a singular perturbation of the differential equation for $P(r;\infty)$. This condition can be interpreted as a threshold condition by noting that the localization length is given by $l_c^{-1}\sim(\alpha^2+\beta^2)$ and the effective amplification length is given by $l_{\mathrm{amp}}^{-1}\sim\beta\overline{\eta}_i$. This condition then corresponds to a matching of length scales in the problem, $l_c=l_{\mathrm{amp}}$. In the regime where the amplification dominates the localization ($\alpha^2+\beta^2-\beta|\overline{\eta}_i|<0$ or $l_c>l_{\mathrm{amp}}$), the solutions exhibit a monotonic decreasing behavior in the region of interest ($1\leqslant r<\infty$). Here the disorder in the real part (α) is small and does not affect the statistics appreciably, as can be seen from Fig. 3(a). For ($\alpha^2+\beta^2-\beta|\overline{\eta}_i|>0$ or $l_c<l_{\mathrm{amp}}$), a natural boundary arises for the solutions of the equation at $r_-^{(2)}$ which falls in the domain of physical interest ($1\leqslant r<\infty$). Now the solutions given by expression (13) are valid in the range $r_-^{(2)}\leqslant r<\infty$ with $P(r;\infty)=0$ outside. In this regime the localization dominates ($l_c<l_{\mathrm{amp}}$), if $2A/[\xi_-(\zeta_-^2-4)^{-1/2}]>1$ and we have a broad distribution with peak at $r_{\max}>r_-^{(2)}$ and $P(r_-^{(2)};\infty)=0$ [Fig. 3(b)]. The value of $r_{\max}$ is large for small disorder in the real part ($\alpha^2+\beta^2-\beta|\overline{\eta}_i|\gtrsim0$), and decreases as α increases. The behavior in this region is dominated by the disorder in the real part of the refractive index. A third qualitatively different behavior occurs for $l_c<l_{\mathrm{amp}}$ and $2A/[\xi_-(\zeta_-^2-4)^{-1/2}]<1$. Then the expression given by Eq. (13) diverges at $r_-^{(2)}$. This divergence

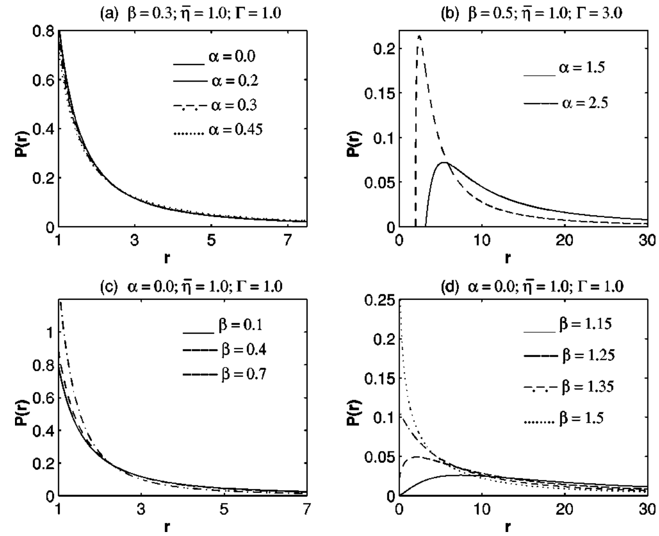

FIG. 3. The probability distribution $P(r;l)$ in the case of the correlated telegraph noise. (a) $l_c>l_{\mathrm{amp}}$ and (b) $l_c<l_{\mathrm{amp}}$ are for one-sided disorder ($\beta<|\overline{\eta}_i|$) with disorder in both the real and the imaginary parts. (c) $l_c>l_{\mathrm{amp}}$ and (d) $l_c<l_{\mathrm{amp}}$ are for two-sided disorder ($\beta>|\overline{\eta}_i|$) and pure imaginary mismatch ($\alpha=0$).

is, however, normalizable implying that $P(r;\infty)$ is peaked (in fact, sharply) at that point. This behavior can be readily understood by noting that the second condition which can be rewritten as $\overline{\eta}_i{}^2(\Gamma/k)^2<3\beta(|\overline{\eta}_i|-\beta)[\alpha^2+2\beta(|\overline{\eta}_i|-\beta)]$, is basically a condition on the correlation length ($l_{\mathrm{corr}}=\Gamma^{-1}$). This condition is satisfied for small Γ (large l_{corr}). Then the reflection is essentially from a single potential barrier and thus has a sharply defined value. It should be noted that, as $\alpha\to\infty$, $P(r;\infty)\to\delta(r-1)$, as expected.

The solutions for the case of a two-sided disorder for the imaginary part ($\beta>|\overline{\eta}_i|$) are similar to the solutions for the white noise case. It should be noted that there does not exist real $r_-^{(2)}$ which falls into the physical region of interest ($0\leqslant r<\infty$). In this case the large disorder in the imaginary part (β) causes the effects of localization to dominate. However, in all cases of amplification, for a finite A and $\alpha^2+\beta^2-\beta|\overline{\eta}_i|\neq0$, there is a universal $1/r^2$ tail for the $P(r;\infty)$. For the case of pure imaginary disorder ($\alpha=0$), we similarly see a monotonically decreasing behavior of $P(r;\infty)$ with r for one-sided disorder ($\beta<|\overline{\eta}_i|$ or $l_c>l_{\mathrm{amp}}$) [Fig. 3(c)], and a $P(r;\infty)$ with a peak for two-sided disorder ($\beta>|\overline{\eta}_i|$ or $l_c<l_{\mathrm{amp}}$) [Fig. 3(d)]. With increase in β for two-sided disorder, the peak shifts to smaller values of reflectivity as the effects of absorption show up, until for large enough β, the peaks occurs at $r=0$ and we again have a monotonically decreasing $P(r;\infty)$. It should be mentioned that all these effects are seen for the case of absorption also, with the roles of $r_-^{(1)}$ and $r_-^{(2)}$ interchanged.

Finally, it is to be noted that the domain of validity of our treatment and the results therefrom, for the super reflection from a random amplifying medium is restricted to operating conditions corresponding to below the threshold of lasing, i.e., to the parameter regime $l_c<l_{\mathrm{amp}}$. Indeed the random amplifying medium operating in the reflection mode acts as a one-sided cavity of size l_c essentially open (hence leaking) in the direction of the incident beam. (Of course, deep inside the medium, a photon injected, for example, through sponta-

neous emission will undergo indefinite amplification in an effectively closed cavity of size l_c. Such an amplified spontaneous emission will lead to large storage of photons which will eventually be limited by nonlinear effects in real systems.) As l_c approaches $l_{\rm amp}$ from below ($l_c <_\to l_{\rm amp}$), the statistical weight for the reflection coefficient moves to higher values of reflectivity as indeed can be seen in Figs. 3(b) and 3(d), and finally at $l_c > l_{\rm amp}$, we would expect the random amplifier to become a random oscillator with self-sustaining oscillations at the eigenmodes of the system. Thus one may suspect the results for $l_c > l_{\rm amp}$ [Figs. 3(a) and Fig. 3(c)] to lie outside the validity of our treatment. Indeed, it has been pointed out[16] that the time-independent wave equation (TIWE) and the associated stationary state scattering does not describe the situation above the threshold of lasing (oscillations) when the gain-length product exceeds criticality. In fact, their numerical results based on the time-dependent wave equation give a transmission which grows exponentially in time. Below, we shall clarify and interpret our results in this above-the-threshold parameter regime when $l_c > l_{\rm amp}$. To illustrate our point, we will consider a Fabry-Pérot setup treated in Ref. 16 for ease of comparison. Thus we have a gain medium of length L between the facets with reflection coefficients r and transmission coefficients t respectively placed between two distant absorbers. The reflection and transmission coefficients at the facets are related to the complex wave vector $k = k' + ik''$ ($k'' < 0$ for the case of amplification) in the medium as $r = (k - k_0)/(k + k_0) = Re^{i\phi}$ and $T = 2k/(k + k_0)$, where k_0 is the wave vector in free space outside. It can be readily shown that for a wave ($e^{-i\omega t}$) incident at the first facet at time $t = 0$, the wave amplitude outside the second face at time t is given by

$$\mathcal{T}(t) = T^2 e^{-(k''L - i\omega\tau)} e^{-i\omega t}\left[\frac{1}{1 - r^2 e^{-2(k''L - i\omega\tau)}} - \frac{(r^2 e^{-2(k''L - i\omega\tau)})^{n+1}}{1 - r^2 e^{-2(k''L - i\omega\tau)}}\right], \tag{14}$$

where $n = \mathrm{Int}[1/2(t/\tau - 1)]$, Int denotes the integer value, $\tau = L/v$ and v is the speed of propagation in the medium. It is seen that the first part on the right hand side is what we would get from a scattering treatment based on the TIWE (We have considered here the case of transmission for the ease of comparision with Ref. 16, but the case of reflection can be treated similiarly), i.e., as far as this term is concerned the expression obtained below threshold continues analytically in the expression obtained above the threshold. The second term on the right-hand side, however, is what is not contained in this analytic continuation. It, indeed, gives the exponential growth of the transmitted amplitude (intensity) as in Ref. 16. This growing oscillatory term (which may eventually get limited only by nonlinearities not considered here) essentially is a noise imposed on the relatively weak transmission noted above. Further, rewriting the second part as $\{T^2 \exp[-2(k''L - i\omega\tau] \exp(i\phi) R^{(t/\tau+1)} \exp[-k''L/\tau t] \exp[i\phi/\tau t]\}/\{1 - R^2 \exp[-2(k''L - i\omega\tau)] \exp(2i\phi)\}$, we note that this exponentially growing part is at an effective frequency ϕ/τ. Note that this frequency is nothing but the rate of change of accumulated phase shift arising from multiple reflections at the interfaces, due to the mismatch in the imaginary part of the refractive index. The growing amplitude is extremely sensitive to the change in the parameters (e.g., R, T) of the system in the limit $t \to \infty$. Indeed, in principle, it is possible to pickup the small finite part referred to above as it is synchronous with the incident wave. Thus, our results in the regime above the threshold based on the TIWE [e.g., Figs. 3(a) and 3(c)] represent just this synchronous part. This in our view gives an operational meaning to the results given by Eq. (13) in the above-the-threshold regime and shown in Figs. 3(a) and 3(c). Of course, the above is a deterministic treatment that we have chosen for the purpose of illustration. For the random case, the interpretation has to be probabilistic.

In conclusion, we have studied the statistics of super-reflection from a one-dimensional disordered system with spatial randomness both in the real and the imaginary parts of the complex refractive index. We have discussed the models of disorder qualitatively applicable to experimental systems such as intentionally disordered optical fibers with gain (En^{3+}-doped) and obtained the probability distribution function of the reflectivity for the cases of a white-noise disorder and a correlated telegraph disorder. In both cases, an enhanced reflection results because of coherent feedback due to Anderson localization and coherent amplification. In the case of white-noise disorder, the statistics are qualitatively different in the two regimes of the real part disorder dominating ($\Delta_r^2 > 2\Delta_i^2$) and the imaginary part disorder dominating ($\Delta_r^2 < 2\Delta_i^2$). In the case of telegraph disorder, we obtain three qualitatively different behaviors for $P(r;\infty)$ depending on threshold conditions involving the localization length, the amplification length and the correlation length. Thus the fluctuation in the imaginary part of the refractive index is seen to have a nontrivial and qualitatively different effect on localization and lasing from such random media. Finally, as the phenomenon considered here is concerned with the issue of statistical fluctuations (noise) in a random amplifying medium, we propose for it the acronym RAMAN (random amplifying medium and noise).

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