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DETECTION OF PARITY OF A BINARY STAR IN TRIPLE CORRELATION SPECKLE INTERFEROMETRY. I. SIGNAL-TO-NOISE RATIO

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ABSTRACT

In a recent paper, Karbelkar and Nityananda rectified, in the wave limit (high flux), an earlier overestimate of signal-to-noise ratio (SNR) for the bispectrum of a speckle image. The relevance of this work to the opposite case of low photon levels is pointed out. For the specific case of binary stars, we present SNR calculations for the detection of the parity (the side of the brighter component) of the binary. Parity, defined as a special case of the focal plane triple correlation, is shown to have significantly poorer limiting faintness than that found in the autocorrelation case. The simplified nature of the object and speckle model allow a rather complete study of the noise properties of triple correlation analysis, which should be useful in more complex situations.

Subject headings: instruments — interferometry

I. INTRODUCTION

It is well known that atmospheric "seeing" limits the resolution (smallest detectable angular feature) in long exposure images formed by large telescopes. The incoming plane wave due to a point source encounters random refractive index inhomogeneities associated with the temperature-induced density fluctuations in Earth's turbulent atmosphere. By the time starlight reaches the entrance pupil of a telescope, it shows random variations of both phase and amplitude. Typically, the field decorrelation length is about 10 cm in the optical region. An immediate consequence is that the instantaneous image of a point source has a spread of about 1″ (corresponding to the 10 cm decorrelation length) and contains many bright spots, the so-called speckles, with nearly diffraction-limited size. These atmospheric corrugations of wave front change with time (typically 10 ms), resulting in a time-dependent speckle pattern. Although at any instant the finest features (speckles) in the image have diffraction-limited size, on long exposures one records only the envelope "seeing disk," which is typically 1″ wide. The goal of speckle interferometry, pioneered by Labeyrie (1970), is to recover diffraction-limited stellar images from (perhaps a large number of) short-exposure images. Although the system response (the focal plane image of a point source) is quite random, both spatially and temporally, two nearby point sources will produce almost the same image (in the high flux limit). This so-called isoplanatic patch, within which two point sources produce almost the same image, is typically 10″ in size. Wave fronts emanating from two stars not within an isoplanatic patch encounter significantly different turbulent regions of the atmosphere.

The stochastic nature of the system response requires the use of statistical methods of image reconstruction. Labeyrie (1970) proposed and demonstrated successfully (Gezari, Labeyrie, and Stachnik 1972) the use of the second-order correlation (the power spectrum),

$$\langle I_u I_{-u} \rangle = \langle R_u R_{-u} \rangle S_u S_{-u} , \tag{1}$$

where I_u, R_u, and S_u are Fourier components of the observed focal plane intensity $I(x)$, the system response (telescope + atmosphere) $R(x)$, and the source structure, $S(x)$, respectively, and angle brackets denote the average over atmospheric fluctuations. The power spectrum, however, does not contain the phases of the Fourier components S_u, which are necessary for unambiguous reconstruction of the source structure $S(x)$. In the phase recovery scheme proposed by Weigelt (1977) and discussed in detail by Lohmann, Weigelt, and Wirnitzer (1983), the bispectrum

$$\langle I_u I_v I_{-u-v} \rangle = \langle R_u R_v R_{-u-v} \rangle S_u S_v S_{-u-v} \tag{2}$$

is obtained as an intermediate step. The signal-to-noise ratio (SNR) for recovering the individual phases of the S_u's contains two factors: (1) the SNR for the bispectrum and (2) a factor representing improvement due to the redundantly stored phase information in the bispectrum. Wirnitzer (1985) has calculated the SNR for the bispectrum for general light levels. For one frame of data, his results can be summarized as follows:

$$\mathrm{SNR}_{\mathrm{Bispectrum}} \begin{cases} \sim 1 & \text{high flux} \quad \mathcal{N} > 1 , & (3) \\ \sim \mathcal{N}^{3/2} & \text{low flux} \quad \mathcal{N} < 1 , & (4) \end{cases}$$

where $\mathcal{N}$ is the average photon count per speckle in an exposure. Recently, Karbelkar and Nityananda (1987; hereafter KN 1987) showed that in the wave limit (high flux), this calculation overestimated the SNR for the bispectrum by a factor of the order $N_S^{1/2}$ (with N_S the number of speckles):

$$\mathrm{SNR}_{\mathrm{Bispectrum}} \sim N_S^{-1/2} \quad \text{high flux } \mathcal{N} > 1 . \tag{5}$$

In § II, we present a brief summary of the KN paper and the SNR estimates for the bispectrum that are valid at low photon levels.

The signal-to-noise properties of the bispectrum analysis of image reconstruction need to be understood, and the present paper is a step in this direction. In § III, we present the SNR for the parity of a binary. A binary is a simple situation where parity (the side of the brighter component) is left undetermined by autocorrelation methods. Parity is known to binary star observers as "quadrant ambiguity." These focal plane calculations for the specific case of a binary have the advantage that the intermediate step of obtaining the bispectrum is bypassed: there is no redundancy of parity information in the focal plane triple correlation values. The detailed calculations that are valid for general light levels are given in the Appendix. Our calculations use an idealized approximation to the point spread function. This model for the point spread function is described in § III. The model is physically motivated and is shown to reproduce well-known results for the autocorrelation analysis. The simple nature of the source and a simple model for speckle imaging tame (to some extent) the complexities of the general bispectrum analysis and allow an analysis which brings out the physical origin of various effects.

II. KN RESULTS AND THEIR RELEVANCE FOR LOW LIGHT LEVELS

Karbelkar and Nityananda (KN 1987) idealize the point source response $R(x)$ (telescope + atmosphere) by a sum of delta functions: $R(x) = R_0 \sum_{i=1}^{N_S} \delta(X - X_i)$ representing $N_S \sim 100\ D^2$ (where D is the diameter in meters) constant intensity speckles whose positions, represented by X_i's, are statistically independent of each other. In the frequency domain, the response function $R_u = R_0 \sum_{j=1}^{N_S} \exp(iuX_j)$ (here $i = (-1)^{1/2}$) is then just a sum of N_S uncorrelated complex numbers whose average is zero because of the assumption of sufficient randomness of speckle positions. In this picture, the triple product $R_u R_v R_{-u-v}$ (which is the bispectrum transfer function) can be shown to contain N_S constant real positive terms of value R_0^3 and $N_S^3 - N_S$ uncorrelated complex terms with zero mean. The N_S constant terms give the signal, while $N_S^3 - N_S \sim N_S^3 (N_S > 1)$ complex terms contribute to the noise. This simple calculation (more rigorous calculations, taking into account variations in speckle size and intensity, do not change these scalings) gives

$$\mathrm{SNR}_{\mathrm{Bispectrum}} \sim N_S^{-1/2} \quad \text{high flux } \mathcal{N} > 1 \; . \tag{5}$$

Typically, $N_S^{1/2} \sim 50$. This is to be compared with the SNR of the order unity obtained by Wirnitzer (1985) in the wave limit.

At low light levels (faintness more than about 13 m when a speckle receives less than single photon per 10 ms exposure with a 100 Å bandwidth), one must also consider noise resulting from the Poisson fluctuations in the number of photons detected. This involves two steps. (1) Since photon statistics introduces bias terms that are dominant at low light levels, one starts with an unbiased estimator for the bispectrum which, when averaged over the photon statistics alone (at a fixed intensity distribution determined by wave theory), gives the classical bispectrum for that realization of the atmosphere. Note that given this construction, the average, even for low light levels, is going to be the same as in the wave limit. (2) Calculation of the variance of such an unbiased estimator considering both the photon and atmospheric noise. Wirnitzer (1985), starting with the correct unbiased estimator, gets the right leading term N_0^3 (Wirnitzer 1985, eq. [A8]) for the variance at low light levels, where $N_0 = N_S \mathcal{N}$ is the total photon count per exposure. However, as seen above, the atmospheric fluctuations have not been propertly taken into account while evaluating the average of the bispectrum. The previous overestimate in step (1) continues to exist, even in the low-flux limit. The correct result for the SNR of the bispectrum, including the effect of the atmospheric noise, is:

$$\mathrm{SNR}_{\mathrm{Bispectrum}} \begin{cases} \sim N_S^{-1/2} & \text{high flux} \quad \mathcal{N} > 1 \; , \quad (5) \\ \sim N_S^{-1/2} \mathcal{N}^{3/2} & \text{low flux} \quad \mathcal{N} < 1 \; , \quad (6) \end{cases}$$

where $\mathcal{N}$ is the average number of photons detected in one speckle per exposure. Since the system response R_u contains N_S statistically independent correlation areas in the u-plane, the bispectrum $R_u R_v R_{-u-v}$ contains $N_S^2/4$ statistically independent regions. If N_S phase values are reconstructed from $N_S^2/4$ bispectrum values, then the phase error is reduced by a factor of $N_S^{1/2}$ (Wirnitzer 1985). Taking this improvement factor with our estimate, equation (7), for the bispectrum, we obtain

$$\mathrm{SNR}_{\mathrm{Phase}} \begin{cases} \sim 1 & \text{high flux} \quad \mathcal{N} > 1 \; , \quad (7) \\ \sim \mathcal{N}^{3/2} & \text{low flux} \quad \mathcal{N} < 1 \; , \quad (8) \end{cases}$$

Note that as long as the photon count per speckle is less than 1, the SNR for the phase reconstruction is lower than the SNR for the power spectrum (Dainty and Greenaway 1979). At the recent NOAO–ESO conference on high-resolution imaging by interferometry, it became clear that several groups have reached this conclusion independently: Ayers, Dainty and Northcott (1988), Hofmann (1988), Karbelkar (1988), and Nakajima (1988).

III. SNR FOR THE PARITY DETECTION

a) *Model of the Point Spread Function*

The triple correlation is a statistics of third order in intensity. The noise on the triple correlation contains terms of third, fourth, fifth and sixth order in the intensities. The intensities themselves are second-order quantities, considering the fields as the basic quantities. A rigorous calculation should therefore involve sixth-, eighth-, tenth-, and twelfth-order integrals in the fields. A simpler model is welcome. Results based on field correlations will be presented elsewhere. In the following calculations, we make reasonable approximations about the point spread function. First of all, we take the seeing disk to be uniform. Second, we divide the focal plane into pixels with the diffraction-limited size. Since the telescope aperture acts as a filter for spatial frequencies, we expect the intensities over a pixel to be correlated. We therefore approximate the intensity correlations in the focal plane as follows. Intensity over any pixel is regarded as uniform, and intensities over different pixels are uncorrelated. Thus, in our model, the point source response is completely specified by intensities, represented by μ_i, at the ith pixel. The μ_i values are statistically independent and have the same distribution for all i within the seeing disk. This approximation neglects effects arising from the edge of the seeing disk where the intensity gradually falls to zero. We assume further that the distribution of intensity at any pixel is the Rayleigh (exponential) distribution. We also consider other distributions in § III*e*, but as the following argument shows, the Rayleigh distribution for the μ_i values is the natural choice. Any point in the focal plane receives complex fields from roughly N_S different correlation patches in the pupil plane. The result of the addition of a large number of complex fields is a complex number whose real and imaginary parts have a Gaussian distribution because of the central limit theorem. The intensity, which is the modulus of the resultant field, is therefore distributed according to the Rayleigh statistics (Rayleigh 1899):

$$P(\mu)d\mu = (d\mu/\langle\mu\rangle) \exp(-\mu/\langle\mu\rangle) \; ,$$

or, equivalently,

$$\langle\mu^m\rangle = m!\langle\mu\rangle^m \; , \tag{9}$$

where m is a nonnegative integer. As we move away from a point in the focal plane, the intensity will begin to decorrelate with that at the first point. Therefore, the intensity within a speckle-sized pixel will show variations, and the statistics may deviate from the Rayleigh statistics which holds for intensity at one point. Since the focal plane intensity correlation length is of the order of the pixel size, we expect the statistics to be close to that of Rayleigh. Deviations from this statistic can be checked by more detailed calculations dealing with field correlations. This statistical model allows us to deal with the intensities themselves, thus reducing the order of correlations we have to deal with. We assume also that the fluctuation in the number of photons detected in a pixel is a Poisson distribution with the instantaneous intensity as the mean. There are correlations of the Hanbury-Brown Twiss type, but it is easy to see that these are negligible for speckle interferometric observations. Our averaging procedure therefore involves two steps. In the first step, we compute the Poisson average for a given focal plane intensity distribution. In the second step, we perform the classical averaging (over the atmospheric realizations.) We denote the Poisson average by an overbar, and we denote the classical average by angle brackets. Later, in § III*g* we show that the above model for the system response correctly reproduces the known results for the autocorrelation of a binary. Our calculations (Karbelkar 1989) based on field correlations show that for binaries close to the diffraction limit, the present approximations are justified. In particular, the edge-effects arising from the finite size of the seeing disk are negligible.

b) Parity

In this section, we present SNR estimates for the parity of a binary system. These focal plane calculations have the advantage that the intermediate step of obtaining the bispectrum is bypassed. It should be noted that the information about the parity (the side of the brighter component) is stored in N_S Fourier components, and therefore parity should have a higher SNR than that for the phase of the individual Fourier components. We expect

$$\mathrm{SNR}_{\mathrm{Parity}} \sim N_S^{1/2}\, \mathrm{SNR}_{\mathrm{Phase}} . \qquad (10)$$

The situation is quite similar to the second-order statistics in the case of a binary. The focal plane correlation has a nonzero value only at the binary spacing. This has a higher SNR (Dainty 1974) than the frequency domain correlation (Dainty and Greenaway 1979) at a single u value. Of course, the full SNR can be recovered by combining the power at different u values:

$$\mathrm{SNR}_{\mathrm{Autocorrelation}} \sim N_S^{1/2} \mathrm{SNR}_{\mathrm{Power\ spectrum}} . \qquad (11)$$

In our model for the focal plane image of a point source, intensities over different pixels are uncorrelated. However, another point source within the isoplanatic patch will give an exactly similar but shifted intensity pattern. In the high-flux limit, the speckle patterns resulting from the two stars have the same relative intensity as the true binary. In this paper, we use the word "speckle" for contribution to pixel intensity from a single star. Therefore all pixels, except those near the edge of the seeing disk, receive two speckles: one from each of the two stars. We are mainly concerned with cases where binary separation is smaller than the seeing disk, and therefore we neglect any edge effects. More specifically, we denote the speckle intensity from star 1 (on the left) at the pixel i by μ_i. This pixel also receives a speckle from star 2. The intensity of this speckle is the same (appropriately scaled) as the intensity of speckle from star 1 at the $(i-b)$th pixel, where b is the binary separation. The speckle at the ith pixel from star 2 has the intensity $\nu_{i-b} = (\mathscr{N}_2/\mathscr{N}_1)\mu_{i-b}$, where $\mathscr{N}_2/\mathscr{N}_1$ is the true ratio of the intensity of star 2 to that of star 1. If we denote the total intensity at the ith pixel by $\bar{n}_i$ then

$$\bar{n}_i = \mu_i + \nu_{i-b} = \mu_i + \frac{\mathscr{N}_2}{\mathscr{N}_1}\, \mu_{i-b} . \qquad (12)$$

Since the μ_i values are statistically independent, two $\bar{n}_i$ values, say $\bar{n}_i$ and $\bar{n}_j$, are also statistically independent unless either $i = j$ or $j = i + b$. The correlation between $\bar{n}_i$ and $\bar{n}_{i+b}$ is caused by a pair of speckles common to these pixels. This pair of speckles has the same relative intensity as the binary. It can be easily checked that in our model (which neglects edge or gradient effects), the general focal plane triple correlation,

$$\langle T_{kj} \rangle = \sum_i \langle \bar{n}_i \bar{n}_{i+k} \bar{n}_{i+j} \rangle , \qquad (13)$$

is symmetric in the average source strengths $\mathscr{N}_1$ and $\mathscr{N}_2$, except for the following six cases for which T_{kj} is asymmetric in $\mathscr{N}_1$ and $\mathscr{N}_2$:

$$\text{i) } k = b,\ j = 0 \quad \text{ii) } k = 0,\ j = b \quad \text{iii) } k = -b,\ j = -b , \qquad (14)$$

$$\text{iv) } k = b,\ j = b \quad \text{v) } k = -b,\ j = 0 \quad \text{vi) } k = 0,\ j = -b . \qquad (15)$$

It can be seen from the definition that the three triple correlations in equation (14) are identical; the same is true of the three triple correlations in equation (15). Note that this is true without the average denoted by angle brackets and an overbar ($\langle\bar{\ }\rangle$); therefore, these three values are identical in all realizations. There is no gain in SNR by a factor of $3^{1/2}$ when combining these three values. Out of the two sets of triplets, we choose only two statistically independent terms:

$$T_{ob} = \sum_i \bar{n}_i^2\, \bar{n}_{i+b} ; \quad T_{b,b} = \sum_i \bar{n}_i \bar{n}_{i+b}^2 .$$

It is possible to combine these two terms in order to get a single parity statistic that is antisymmetric in $\mathscr{N}_1$ and $\mathscr{N}_2$. Thus, the only third-order correlation that contains the parity information is

$$\langle \bar{P} \rangle = \sum_i \langle (\bar{n}_i^2\, \bar{n}_{i+b} - \bar{n}_i \bar{n}_{i+b}^2) \rangle , \qquad (16)$$

where n_i is the number of photons in the ith pixel, b is binary separation, the overbar denotes average over the assumed Poisson statistics obeyed by the photons for a given intensity $\bar{n}_i$, and angle brackets denote average over the atmospheric noise.

Note that the expression in equation (16) is unbiased under Poisson fluctuations; i.e., the unbiased estimator of the parity statistics is

$$P = \sum_i (n_i^2\, n_{i+b} - n_i n_{i+b}^2) . \qquad (17)$$

We can write this as

$$P = \sum_i p_i ; \quad p_i = n_i^2\, n_{i+b} - n_i n_{i+b}^2 , \qquad (18)$$

where p_i is the contribution from the ith pair of pixels consisting of the ith and $(i+b)$th pixel.

c) SNR for the Parity Detection at Low Light Levels

SNR values for this parity statistic for general light levels are presented in § III*f*. The algebraic complexity makes the general

TABLE 1

TRUNCATED POISSON DISTRIBUTION

n	$P(n)$
0...........	$1-\bar{n}+\frac{1}{2}\bar{n}^2$
1...........	$\bar{n}-\bar{n}^2$
2...........	$\frac{1}{2}\bar{n}^2$

TABLE 3

PROBABILITY DISTRIBUTION OF PARITY

Parity	Probability
2.........	$\frac{1}{2}\bar{n}^2\bar{n}_1$
−2.........	$\frac{1}{2}\bar{n}\bar{n}_1^2$
0.........	Remainder

case less physically transparent. Here we treat the case of low light levels for which the algebra is simpler and the physical origin of various contributions to the SNR is clearer. At low flux, the number of pairs of pixels giving nonzero values of parity is small compared to the total number of possible pairs. It will be shown later that their overlap contributes to the fourth and higher order terms in the variance, and not to the lowest third-order terms. Since we approximate the seeing disk by a uniform disk, and since the overlap in the different pairs of pixels does not contribute in the third order, it is sufficient to consider a representative pair of pixels with the binary separation. The second simplification is that such a representative pair of pixels must not receive all three photons in one pixel. As Table 2 shows, one of these pixels must receive one photon and the other must receive two. It is enough, therefore, to use the truncated Poisson distribution shown in Table 1, which gives the probability $P(n)$ of detecting n photons when the mean is $\bar{n}$. The SNR resulting from one such pair must then be multiplied by $N_S^{1/2}$ to find the SNR for parity statistics.

Consider, then, a pair of pixels with binary separation, with intensity $\bar{n}$ and $\bar{n}_1$ for one atmospheric realization. We suppress the subscript "i" and denote $\bar{n}_b$ (which is actually $\bar{n}_{i+b}$) by $\bar{n}_1$. In all meaningful correlations, b is the basic displacement. Later, we shall denote $\bar{n}_{i+mb}$ by $\bar{n}_m$ for simplicity. The parity statistics for this representative pair of pixels becomes $p = n^2 n_1 - n n_1^2$, which takes values with the probabilities shown in Tables 2 and 3, respectively. It is clear from Table 3 that the Poisson average and variance (up to third order) of the parity statistics for one pair is

$$\bar{p} = \bar{n}^2\bar{n}_1 - \bar{n}\bar{n}_1^2 , \quad (19a)$$

$$\overline{p^2} - \bar{p}^2 \sim \overline{p^2} = 2(\bar{n}^2\bar{n}_1 + \bar{n}\bar{n}_1^2) . \quad (19b)$$

One should have subtracted $\bar{p}^2$, but this is of sixth order in the flux per speckle and neglected here. However, the atmospheric average remains to be calculated. The pixel on the left contains a speckle from star 1 with intensity μ, and the pixel on the right contains a corresponding speckle from star 2 with intensity $\nu = (\mathcal{N}_2/\mathcal{N}_1)\mu$. In addition to this correlated pair, the pixel on the left will contain a speckle from star 2 with intensity of, say, ν_-, and the pixel on the right will contain a speckle from star 1 with intensity μ_+. The quantities with different subscripts are uncorrelated with each other. After calculating, the atmospheric average we obtain for the per pair parity average

$$\langle\bar{p}\rangle = (\langle\mu^3\rangle - 3\langle\mu^2\rangle\langle\mu\rangle + 2\langle\mu\rangle^3)\,\frac{\mathcal{N}_2(\mathcal{N}_1 - \mathcal{N}_2)}{\mathcal{N}_1^2} . \quad (20)$$

Note that equation (20) does not make any assumptions about the distribution of μ and allows us to discuss other statistics as well. We assume (see § III*a*) the intensities of the speckles to be distributed according to the Rayleigh statistics. We discuss other interesting cases in § III*e*). The average and the variance of the per pair parity are:

$$\langle\bar{p}\rangle = 2\mathcal{N}_1\mathcal{N}_2(\mathcal{N}_1 - \mathcal{N}_2) , \quad (21)$$

$$\langle\overline{p^2}\rangle - \langle\bar{p}\rangle^2 = 4(2\mathcal{N}_1^3 + 7\mathcal{N}_1^2\mathcal{N}_2 + 7\mathcal{N}_1\mathcal{N}_2^2 + 2\mathcal{N}_2^3) . \quad (22)$$

This gives our estimate for the low-flux SNR for the parity,

$$\mathrm{SNR}_{\mathrm{Parity}} = \frac{M^{1/2}q^{3/2}\mathcal{N}_S^{1/2}N_1\mathcal{N}_2(\mathcal{N}_1 - \mathcal{N}_2)}{(2\mathcal{N}_1^3 + 7\mathcal{N}_1^2\mathcal{N}_2 + 7\mathcal{N}_1\mathcal{N}_2^2 + 2\mathcal{N}_2^3)^{1/2}}$$
$$\mathcal{N}_1\mathcal{N}_2 < 1 , \quad (23)$$

where M frames of data are used and q is the detector efficiency (optics + quantum). Note that for one realization, this is consistent with our calculation in the frequency domain (eqs. [7] and [8]) and expectation (eq. [10]) relating the SNR for parity to the SNR for phase (take $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_1 - \mathcal{N}_2 \sim \mathcal{N}$). Preliminary results of the low-flux SNR estimates were presented at the NOAO–ESO conference on high-resolution imaging by interferometry (Karbelkar 1988).

d) Parity Detection in the Presence of Sky Background

In this section, we consider the effect of a uniform sky background of K photons on the average per pixel per exposure. In the previous section, it was seen that only events registering one photon in one pixel and two in the other contribute to the parity information. In the absence of sky background noise, these photons come from the binary and therefore contain information about the parity of the binary. Sky background can mimic parity events: as an extreme example, all three photons may come from the background. Assuming uniform background, such spurious events will have their negatives, so on an average there is no signal resulting from the background. Of course, there will be additional noise caused by such spurious events. To calculate the fluctuations in parity caused by photon noise, we convolve the Poisson fluctuations in the photons from the binary (as before) with the fluctuations in the photons from the background. Since the sum of the Poisson fluctuations is again a Poisson fluctuation with the sum of the means, the calculation of probabilities in Table 3 continues to hold with

$$\bar{n}_i = \mu_i + \nu_{i-b} + K , \quad (24)$$

instead of equation (12), which is true for no background.

TABLE 2

PARITY VALUES AS A FUNCTION OF n AND n_1

n	n_1 = 0	1	2
0...........	0	0	0
1...........	0	0	−2
2...........	0	2	0

Assuming that the speckles have a Rayleigh distribution as before, we obtain for one pair of pixels:

$$\langle \bar{p} \rangle = 2\mathcal{N}_1 \mathcal{N}_2(\mathcal{N}_1 - \mathcal{N}_2) , \quad (25)$$

$$\sigma^2 = \langle \overline{p^2} \rangle - \langle \bar{p} \rangle^2 = 8\mathcal{N}_1^3 + 14\mathcal{N}_1^2 \mathcal{N}_2 + 14\mathcal{N}_1 \mathcal{N}_2^2 + 8\mathcal{N}_2^3 + 8K(\mathcal{N}_1^2 + 3\mathcal{N}_1 \mathcal{N}_2 + \mathcal{N}_2^2) + 12K^2(\mathcal{N}_1 + \mathcal{N}_2) + 4K^3 . \quad (26)$$

Stationarity for the statistics of the μ_i values was used in the derivation of equation (26). We must subtract the square of the averaged parity, but this is of sixth order and will be taken care of in § III*f*, where all orders are considered. Combining equations (25) and (26), we find the SNR for the parity in the presence of sky background.

e) Effects of Various Intensity Distributions for Pixels

We check now to ensure that the general formula (eq. [20]) gives correct results in other limiting cases of the statistics for the pixel intensities. Consider a long exposure image. The intensities on the pixels are positive. However, as a result of a large number of speckles, one may find, to the zeroth order, a constant intensity, as in the case of long exposure images. To higher order, we could approximate the intensities as a Gaussian distribution around the mean because of the central limit theorem. The variance of this Gaussian component has to be much smaller than the mean, so that the unphysical negative intensities predicted by the approximation have negligible probability. Such a distribution is represented by $\mu = 1 + x$, where x is a zero mean Gaussian with $\langle x^2 \rangle \ll 1$. For this distribution,

$$\langle \mu \rangle = 1; \quad \langle \mu^2 \rangle = 1 + \langle x^2 \rangle; \quad \langle \mu^3 \rangle = 1 + 3\langle x^2 \rangle ,$$

and there is no parity signal. The Gaussian distribution may arise in another way. Consider a source more complex than a binary, in which every pixel gets a speckle from every component. The result of such a large number of independent Rayleigh distributions is a Gaussian distribution. As is well known for Gaussian distributions, the bispectrum is identically zero. Note that the expression in parentheses in equation (20) is just the third moment about the mean. Whatever the statistics of the pixel intensities are, the third cumulant must be nonvanishing for parity detection.

f) SNR for General Light Levels

The variance for the parity statistics contains terms in the fourth, fifth, and the sixth order, in addition to those in the third order considered already. We summarize the results here, while the details are left to the Appendix. Starting with the unbiased parity statistic

$$P = \sum_i (n_i^2 n_{i+b} - n_i n_{i+b}^2) ,$$

where n_i is the number of photons recorded in the ith pixel in a realization of the atmospheric noise. We calculate the Poisson average for the square of the parity first:

$$\overline{p^2} = [\sum_i (\bar{n}_i^2 \bar{n}_{i+b} - \bar{n}_i \bar{n}_{i+b}^2)]^2 + \sum_i (2\bar{n}_i^2 \bar{n}_{i+b} + 2\bar{n}_i \bar{n}_{i+b}^2 + 4\bar{n}_i^3 \bar{n}_{i+b} + 4\bar{n}_i \bar{n}_{i+b}^3 - 4\bar{n}_i^2 \bar{n}_{i+b}^2 - 4\bar{n}_i \bar{n}_{i+b}^2 \bar{n}_{i+2b} + \bar{n}_i^4 \bar{n}_{i+b} + \bar{n}_i \bar{n}_{i+b}^4 + 4\bar{n}_i^2 \bar{n}_{i+b}^2 \bar{n}_{i+2b} + 4\bar{n}_i \bar{n}_{i+b}^2 \bar{n}_{i+2b}^2 - 2\bar{n}_i^2 \bar{n}_{i+b} \bar{n}_{i+2b}^2 - 8\bar{n}_i \bar{n}_{i+b}^3 \bar{n}_{i+2b}) . \quad (27)$$

Note that the sixth-order term is just the classical variance, and the Poisson contribution exists only in the lower orders.

Here we give an interpretation of the Poisson contribution to the variance. We note that the square of parity statistics (in the form of eq. [18]) is

$$\sum_{i,j} p_i p_j .$$

Since n_i is an independent Poisson variable for a realization of intensity distribution $\bar{n}_i$, in the first stage of (Poisson) averaging p_i and p_j are independent unless $i = j$ or $i = j \pm b$. The Poisson average of the above expression is

$$\sum_{i,j} \overline{p_i p_j} = (\sum_i \bar{p}_i)^2 + \sum_i (\overline{p_i^2} - \bar{p}_i^2) + 2 \sum_i (\overline{p_i p_{i+b}} - \bar{p}_i \bar{p}_{i+b}) .$$

The first term is the square of the classical parity statistics. The second term is a variance term of the type discussed before while considering the low flux noise. The third term includes the effects of overlapping pairs. For example, consider events where three pixels with interpixel separation equal to the binary register one, two, and one photons, respectively. The first pair of pixels contributes -2, while the middle and the extreme right pixels contribute $+2$ to the parity. Such an event has a net parity of zero. However, in the previous calculations, the two pairs were taken to contribute independently to the parity, so their contributions to the variance are summed up. The event under consideration has zero parity and should not have contributed to the variance. Such excess counting must be corrected for. The probability of such an event is in the fourth order and contributes $-4\bar{n}_i \bar{n}_{i+b}^2 \bar{n}_{i+2b}$ in the fourth order. Just as the third-order Poisson events were tabulated in Tables 2 and 3, the higher order Poisson contributions can be worked out, and we can convince ourselves that the result is the same as given above (eq. [27]). The atmospheric noise needs to be averaged, and when this is done (see Appendix), we get for the variance the explicitly positive definite form:

$$\langle \overline{p^2} \rangle - \langle \bar{p} \rangle^2 = 4N_S[2\mathcal{N}_1^3 + 7\mathcal{N}_1^2 \mathcal{N}_2 + 7\mathcal{N}_1 \mathcal{N}_2^2 + 2\mathcal{N}_2^3] + 4N_s[6(\mathcal{N}_1^2 - \mathcal{N}_2^2)^2 + 20\mathcal{N}_1 \mathcal{N}_2(\mathcal{N}_1^2 + \mathcal{N}_2^2) + 2\mathcal{N}_1^2 \mathcal{N}_2^2] + 8N_s[3\mathcal{N}_1^5 + 5\mathcal{N}_1^4 \mathcal{N}_2 + 2\mathcal{N}_1^3 \mathcal{N}_2^2 + 2\mathcal{N}_1^2 \mathcal{N}_2^3 + 5\mathcal{N}_1 \mathcal{N}_2^4 + 3\mathcal{N}_2^5] + N_S[8(\mathcal{N}_1 - \mathcal{N}_2)^6 + 124\mathcal{N}_1^2 \mathcal{N}_2^2(\mathcal{N}_1 - \mathcal{N}_2)^2 + 32\mathcal{N}_1 \mathcal{N}_2(\mathcal{N}_1^2 - \mathcal{N}_2^2)^2 + 8\mathcal{N}_1^3 \mathcal{N}_2^3] . \quad (28)$$

g) SNR for Autocorrelation (Low Flux)

The SNR for autocorrelation of a binary is well known in the literature. Here we show that the SNR estimate for the autocorrelation, based on our assumptions and simplifications, agrees with the known result. As before, assuming a uniform seeing disk, we treat the N_S terms in the general autocorrelation,

$$\sum_i n_i n_{i+x} , \quad (29)$$

equivalent at low flux levels, and we consider only one representative pair:

$$a_x = n_i n_{i+x} . \quad (30)$$

The pair of pixels registers n_i and n_{i+x} photons, respectively, with average intensity $\bar{n}$ and $\bar{n}_1$ in one realization of atmo-

TABLE 4

AUTOCORRELATION VALUES

	n_1	
n	0	1
0...........	0	0
1...........	0	1

spheric noise. Note that for $x \neq 0$, the statistics is unbiased under Poisson statistics. The autocorrelation gets its leading contribution (low flux) in the second order in the flux per pixel, and this enables us to truncate the Poisson distribution after first order: the values of the autocorrelation and their probabilities are given in Tables 4 and 5, respectively. It is then clear that the Poisson average and variance are

$$\bar{a}_x = \bar{n}\bar{n}_x \, , \quad \overline{a_x^2} = \bar{n}\bar{n}_x \, . \tag{31}$$

Consider the case where x equals the binary separation. Then, from equation (24), which gives intensities in the presence of sky background,

$$\langle \bar{a}_b \rangle = \mathcal{N}_1^2 + 3\mathcal{N}_1\mathcal{N}_2 + \mathcal{N}_2^2 + 2K(\mathcal{N}_1 + \mathcal{N}_2) + K^2 \, . \tag{32}$$

However, one should actually be able to measure the slight bump in $\langle \bar{a}_x \rangle$ at $x = b$ relative to its neighbors, where $\langle \bar{a}_x \rangle$ takes the value, say $\langle a_{DC} \rangle$. For $x \neq 0, x \neq \pm b$, we have

$$\langle \bar{a}_{DC} \rangle = \mathcal{N}_1^2 + 2\mathcal{N}_1\mathcal{N}_2 + \mathcal{N}_2^2 + 2K(\mathcal{N}_1 + \mathcal{N}_2) + K^2 \, , \tag{33}$$

$$\mathrm{SNR}_{\mathrm{Autocorrelation}} = \frac{M^{1/2} q \mathcal{N}_1 \mathcal{N}_2 N_S^{1/2}}{[\mathcal{N}_1^2 + 3\mathcal{N}_1\mathcal{N}_2 + \mathcal{N}_2^2 + 2K(\mathcal{N}_1 + \mathcal{N}_2) + K^2]^{1/2}} \, . \tag{34}$$

We should calculate the variance in $a_b - a_{DC}$; however, we can determine $\langle \bar{a} \rangle_{DC}$ from many separations x which give independent a_{DC} values, and we therefore consider it almost noise free when compared to a_b. The scaling with M, q, and $\mathcal{N}$ agrees with previous results from Dainty (1974), who considers a binary with $\mathcal{N}_1 = \mathcal{N}_2$.

IV. CONCLUSION

For concreteness, we consider the specific case of a 4 m telescope with optical bandwidth 100 Å and an exposure time of 10 ms. In this case, $N_S = 1600$, and the per speckle photon count of unity corresponds to a 12.25 mag star. In the high-flux limit, the sixth-order terms dominate, and the SNR is a function of the relative strength r of the two components:

$$r = \mathcal{N}_2/\mathcal{N}_1 = 10^{-0.4(m_2 - m_1)} \quad \mathcal{N}_1 > \mathcal{N}_2 > 1 \, , \tag{35}$$

$$\mathrm{SNR}_{\mathrm{Parity}} = M^{1/2} q^{3/2} N_S^{1/2} r(1 - r)/(2 - 4r + 61r^2 - 116r^3 + 61r^4 - 4r^5 + 2r^6)^{1/2} \, , \tag{36}$$

where m_1 and m_2 are the magnitudes of the two stars, and we have taken detector efficiency (optics + quantum) $q = 0.2$. This high-flux SNR is given in Table 6 as a function of the magnitude difference $\Delta m = m_2 - m_1$. Calculations based on all orders (exact in our model) show that this gives an SNR accurate to a few percent if the brighter component is brighter than 7 and the fainter component is brighter than about 13. We note from the table that for bright binaries, parity cannot be detected with SNR > 3 if the magnitude difference is greater than 6, although it may be possible to see the binary nature in the autocorrelation. Table 7 gives the limiting magnitude of the fainter component, for a given magnitude of the brighter component, for which parity and autocorrelation can be detected with SNR > 3. For the chosen observation parameters, sky background noise is unimportant and makes no difference in the limiting magnitudes. The limiting magnitudes given in Table 7 for the autocorrelation case are based on our calculations, which are outlined in § III*g*. In this case, keeping the 21 mag sky background in mind, the SNR calculations were terminated at 20.5 mag. We conclude from Table 7 that parity detection has a significantly lower SNR than the autocorrelation. We then raise the following legitimate question. From autocorrelation, which has a much higher SNR, we know $\mathcal{N}_1$ and $\mathcal{N}_2$ and therefore the magnitude $|\mathcal{N}_1^2\mathcal{N}_2 - \mathcal{N}_1\mathcal{N}_2^2|$ of the parity quite well. It is the sign of parity which is unknown. So the relevant statistical question is whether to assign probability distribution to the signs when the observed parity value is given. This question can be answered only if we know how the parity is distributed around its mean (which has to be consistent with the modulus of parity obtained from autocorrelation). Knowing a distribution means knowing all the moments of the variable (the parity). The complexity involved in evaluating the second moment of the parity statistics (despite a simple model) indicates the near-impossibility of carrying out such a task by analytical methods. Two

TABLE 5

PROBABILITY DISTRIBUTION FOR AUTOCORRELATION

Autocorrelation	Probability
1.....................	$\bar{n}\bar{n}_1$
0.....................	$1 - \bar{n}\bar{n}_1$

TABLE 6

HIGH-FLUX SNR AS A FUNCTION OF MAGNITUDE DIFFERENCE

Δm	SNR	Δm	SNR
0.0......	0	3.5......	31
0.2......	112	4.0......	20
0.5......	130	4.5......	13
1.0......	130	5.0......	8
1.5......	118	5.5......	5
2.0......	96	6.0......	3
2.5......	70	6.5......	2
3.0......	48	7.2......	1

TABLE 7

COMPARISON OF LIMITING MAGNITUDES FOR THE PARITY AND THE AUTOCORRELATION OF A BINARY

Magnitude of the Brighter Component	Magnitude of the Limiting Fainter Component: Parity	Autocorrelation
13.0.........................	17.2	20.5
14.0.........................	17.0	20.5
15.0.........................	16.0	20.5
16.0.........................	...	20.5
17.0.........................	...	20.5
18.0.........................	...	20.5
19.0.........................	...	20.5
20.0.........................	...	20.0

extreme cases are, however, trivial. For large values of SNR, the sign of parity is well defined. On the other hand, for poor SNR, the observed value might just as well have come from any one of the two signs. We take SNR = 3 as the case where one sign has significantly greater probability than the other. We note also that we have considered an ideal situation; in reality, there could be other sources of noise, and present-day speckle work does not attain these theoretical limits. The limits themselves are still of interest.

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APPENDIX

SNR FOR PARITY DETECTION AT GENERAL LIGHT LEVELS

Here we give the details of the SNR calculations for the parity statistics

$$P = \sum_i (n_i^2 n_{i+b} - n_i n_{i+b}^2) , \qquad b \neq 0 , \tag{A1}$$

introduced before. The average of this is given by equation (25). The square of the parity statistics is

$$P^2 = \sum_{i,j} n_i^2 n_{i+b} n_j^2 n_{j+b} + \sum_{i,j} n_i n_{i+b}^2 n_j n_{j+b}^2 - 2 \sum_{i,j} n_i^2 n_{i+b} n_j n_{j+b}^2 . \tag{A2}$$

I. POISSON FLUCTUATIONS

Since the Poisson fluctuations in different pixels are uncorrelated for a given intensity distribution, correlations come only when any two of the subscripts are equal. Since we restrict ourselves to $b \neq 0$, no three subscripts in any of the three terms in equation (A2) can be equal. So while taking the Poisson average of terms in equation (A2), we can partition the summation $\sum_{i,j}$ into four sums: (1) j distinct from i, $i + b$, and $i - b$; (2) $j = i$; (3) $j = i + b$; (4) $j = i - b$. These partitions are mutually exclusive ($b \neq 0$) and cover all the terms implied by the original summation without any restriction over i and j. When i is distinct from $j, j \pm b$, the Poisson average of a term takes simpler form. For example,

$$\overline{n_i^2 n_{i+b} n_j^2 n_{j+b}} = \overline{n_i^2}\, \bar{n}_{i+b} \overline{n_j^2}\, \bar{n}_{j+b} .$$

We can relax the restriction on j and pretend that the Poisson average can always be split like this, although this is not true for partitions other than (1). For other partitions, we must first write the correct result implied by the partition and then subtract from it the above (pretend) uncorrelated average. For example,

$$\sum_{i,j} \overline{n_i^2 n_{i+b} n_j^2 n_{j+b}} = \sum_{i,j} \overline{n_i^2}\, \bar{n}_{i+b} \overline{n_j^2}\, \bar{n}_{j+b} + \sum_i \left(\overline{n_i^4}\, \overline{n_{i+b}^2} - \overline{n_i^2}^2\, \bar{n}_{i+b}^2\right) + 2 \sum_i \overline{n_i^2}\, \bar{n}_{i+2b}\left(\overline{n_{i+b}^3} - \overline{n_{i+b}^2}\, \bar{n}_{i+b}\right) . \tag{A3}$$

Using the well-known results

$$\bar{n}_i = \bar{n}_i , \qquad \overline{n_i^2} = \bar{n}_i^2 + \bar{n}_i , \qquad \overline{n_i^3} = \bar{n}_i^3 + 3\bar{n}_i^2 + \bar{n}_i , \qquad \overline{n_i^4} = \bar{n}_i^4 + 6\bar{n}_i^3 + 7\bar{n}_i^2 + \bar{n}_i , \tag{A4}$$

for the Poisson distribution for n_i with given mean $\bar{n}_i$, we can write the averages in terms of the given intensities. A compact notation is useful. Since except for the first method of partitioning, other partitions involve only one summation over i, we drop the explicit $\sum_i$ in such terms. We also suppress the subscript "i" and denote $\bar{n}_i$ by $\bar{n}$, $\bar{n}_{i+b}$ by $\bar{n}_1$, $\bar{n}_{i+2b}$ by $\bar{n}_2$, etc. Also, since i is a dummy index, we have

$$\bar{n}\bar{n}_1^2 \bar{n}_2 = \bar{n}_{-1}\bar{n}^2\bar{n}_1 .$$

With this compact notation, we have

$$\begin{aligned} \sum_{i,j} \overline{n_i^2 n_{i+b} n_j^2 n_{j+b}} = {} & \sum_{i,j} (\bar{n}_i^2 + \bar{n}_i)\bar{n}_{i+b}(\bar{n}_j^2 + \bar{n}_j)\bar{n}_{j+b} + \bar{n}^4\bar{n}_1 + 4\bar{n}^3\bar{n}_1^2 + 4\bar{n}^2\bar{n}_1^2\bar{n}_2 \\ & + 2\bar{n}^2\bar{n}_1\bar{n}_2 + 6\bar{n}^3\bar{n}_1 + 6\bar{n}^2\bar{n}_1^2 + 4\bar{n}\bar{n}_1^2\bar{n}_2 + 7\bar{n}^2\bar{n}_1 + \bar{n}\bar{n}_1^2 + 2\bar{n}\bar{n}_1\bar{n}_2 + \bar{n}\bar{n}_1 , \end{aligned} \tag{A5}$$

$$\begin{aligned} \sum_{i,j} \overline{n_i n_{i+b}^2 n_j n_{j+b}^2} = {} & \sum_{i,j} \bar{n}_i(\bar{n}_{i+b}^2 + \bar{n}_{i+b})\bar{n}_j(\bar{n}_{j+b}^2 + \bar{n}_{j+b}) + \bar{n}\bar{n}_1^4 + 4\bar{n}^2\bar{n}_1^3 + 4\bar{n}\bar{n}_1^2\bar{n}_2^2 \\ & + 2\bar{n}\bar{n}_1\bar{n}_2^2 + 6\bar{n}\bar{n}_1^3 + 6\bar{n}^2\bar{n}_1^2 + 4\bar{n}\bar{n}_1^2\bar{n}_2 + 7\bar{n}\bar{n}_1^2 + \bar{n}^2\bar{n}_1 + 2\bar{n}\bar{n}_1\bar{n}_2 + \bar{n}\bar{n}_1 , \end{aligned} \tag{A6}$$

$$\begin{aligned} -2\sum_{i,j} \overline{n_i^2 n_{i+b} n_j n_{j+b}^2} = {} & -2\sum_{i,j} (\bar{n}_i^2 + \bar{n}_i)\bar{n}_{i+b}\bar{n}_j(\bar{n}_{j+b}^2 + \bar{n}_{j+b}) - 4\bar{n}^3\bar{n}_1^2 - 4\bar{n}^2\bar{n}_1^3 - 2\bar{n}^2\bar{n}_1\bar{n}_2^2 \\ & -8\bar{n}\bar{n}_1^3\bar{n}_2 - 16\bar{n}^2\bar{n}_1^2 - 2\bar{n}\bar{n}_1^3 - 2\bar{n}^3\bar{n}_1 - 2\bar{n}^2\bar{n}_1\bar{n}_2 - 2\bar{n}\bar{n}_1\bar{n}_2^2 \\ & -12\bar{n}\bar{n}_1^2\bar{n}_2 - 6\bar{n}^2\bar{n}_1 - 6\bar{n}\bar{n}_1^2 - 4\bar{n}\bar{n}_1\bar{n}_2 - 2\bar{n}\bar{n}_1 . \end{aligned} \tag{A7}$$

This gives us the Poisson average of the square of the parity statistics

$$\begin{aligned} \overline{P^2} = {} & \left[\sum_i (\bar{n}_i^2\bar{n}_{i+b} - \bar{n}_i\bar{n}_{i+b}^2)\right]^2 \\ & + \bar{n}^4\bar{n}_1 + \bar{n}\bar{n}_1^4 + 4\bar{n}^2\bar{n}_1^2\bar{n}_2 + 4\bar{n}\bar{n}_1^2\bar{n}_2^2 - 2\bar{n}^2\bar{n}_1\bar{n}_2^2 - 8\bar{n}\bar{n}_1^3\bar{n}_2 \\ & + 4\bar{n}^3\bar{n}_1 + 4\bar{n}\bar{n}_1^3 - 4\bar{n}^2\bar{n}_1^2 - 4\bar{n}\bar{n}_1^2\bar{n}_2 + 2\bar{n}^2\bar{n}_1 + 2\bar{n}\bar{n}_1^2 . \end{aligned} \tag{A8}$$

II. RAYLEIGH AVERAGE UP TO THE FIFTH ORDER

Note that the third-, forth-, and fifth-order terms in equation (A8) involve only one summation and are easier to average over the assumed Rayleigh distribution for individual speckles. We assume, as before, that the seeing disk has a uniform profile. The average of these sums can then be replaced by N_S times the average for one term. The sixth-order terms involve double summations and need different handling, similar to the Poisson double summation. We therefore consider three pixels within the seeing disk separated by the binary separation with contributing speckles $\bar{n}$, $\bar{n}_1$, and $\bar{n}_2$ (compact notation) given by equation (12). The quantities with different subscripts are uncorrelated. The results of Rayleigh averages of correlated variables can be summarized as follows:

$$\langle \mu_i^{m_1} \nu_i^{m_2} \rangle = (m_1 + m_2)! \mathcal{N}_1^{m_1} \mathcal{N}_2^{m_2} , \tag{A9}$$

where angle brackets denote the Rayleigh average. The following Rayleigh averages are needed:

$$\begin{aligned}
\langle \bar{n}^2 \bar{n}_1 \rangle &= 2(\mathcal{N}_1^3 + 4\mathcal{N}_1^2 \mathcal{N}_2 + 3\mathcal{N}_1 \mathcal{N}_2^2 + \mathcal{N}_2^3); \qquad \langle \bar{n}^3 \bar{n}_1 \rangle = 6\mathcal{N}_1^4 + 30\mathcal{N}_1^3 \mathcal{N}_2 + 24\mathcal{N}_1^2 \mathcal{N}_2^2 + 18\mathcal{N}_1 \mathcal{N}_2^3 + 6\mathcal{N}_2^4 , \\
\langle \bar{n}^2 \bar{n}_1^2 \rangle &= 4\mathcal{N}_1^4 + 16\mathcal{N}_1^3 \mathcal{N}_2 + 36\mathcal{N}_1^2 \mathcal{N}_2^2 + 16\mathcal{N}_1 \mathcal{N}_2^3 + 4\mathcal{N}_2^4 , \\
\langle \bar{n}\bar{n}_1^2 \bar{n}_2 \rangle &= 2\mathcal{N}_1^4 + 12\mathcal{N}_1^3 \mathcal{N}_2 + 22\mathcal{N}_1^2 \mathcal{N}_2^2 + 12\mathcal{N}_1 \mathcal{N}_2^3 + 2\mathcal{N}_2^4 , \\
\langle \bar{n}^4 \bar{n}_1 \rangle &= 24\mathcal{N}_1^5 + 144\mathcal{N}_1^4 \mathcal{N}_2 + 120\mathcal{N}_1^3 \mathcal{N}_2^2 + 96\mathcal{N}_1^2 \mathcal{N}_2^3 + 72\mathcal{N}_1 \mathcal{N}_2^4 + 24\mathcal{N}_2^5 , \\
\langle \bar{n}^2 \bar{n}_1^2 \bar{n}_2 \rangle &= 4\mathcal{N}_1^5 + 28\mathcal{N}_1^4 \mathcal{N}_2 + 72\mathcal{N}_1^3 \mathcal{N}_2^2 + 68\mathcal{N}_1^2 \mathcal{N}_2^3 + 24\mathcal{N}_1 \mathcal{N}_2^4 + 4\mathcal{N}_2^5 , \\
\langle \bar{n}^2 \bar{n}_1 \bar{n}_2^2 \rangle &= 4\mathcal{N}_1^5 + 24\mathcal{N}_1^4 \mathcal{N}_2 + 44\mathcal{N}_1^3 \mathcal{N}_2^2 + 44\mathcal{N}_1^2 \mathcal{N}_2^3 + 24\mathcal{N}_1 \mathcal{N}_2^4 + 4\mathcal{N}_2^5 , \\
\langle \bar{n}\bar{n}_1^3 \bar{n}_2 \rangle &= 6\mathcal{N}_1^5 + 42\mathcal{N}_1^4 \mathcal{N}_2 + 84\mathcal{N}_1^3 \mathcal{N}_2^2 + 84\mathcal{N}_1^2 \mathcal{N}_2^3 + 42\mathcal{N}_1 \mathcal{N}_2^4 + 6\mathcal{N}_2^5 ,
\end{aligned} \tag{A10}$$

where $\langle \bar{n}\bar{n}_1^2 \rangle$, $\langle \bar{n}\bar{n}_1^3 \rangle$, $\langle \bar{n}\bar{n}_1^4 \rangle$, and $\langle \bar{n}\bar{n}_1^2 \bar{n}_2^2 \rangle$ are obtained from $\langle \bar{n}^2 \bar{n}_1 \rangle$, $\langle \bar{n}^3 \bar{n}_1 \rangle$, $\langle \bar{n}^4 \bar{n}_1 \rangle$, and $\langle \bar{n}^2 \bar{n}_1^2 \bar{n}_2 \rangle$, respectively, by interchanging $\mathcal{N}_1$ and $\mathcal{N}_2$.

III. RAYLEIGH AVERAGE OF THE CLASSICAL SIXTH-ORDER TERMS

The sixth-order terms are the classical terms in the sense that even in the absence of photon noise, these terms would survive. This is the reason why there are no terms in the sixth order with single summation: the photon noise merges into the wave noise. The double summation in the sixth-order terms has similarities to the double summation $\sum_{i,j}$ in the Poisson case. In the Poisson case, the Poisson fluctuations in different pixels were independent, so the subscripts on the n variables has to be equal in order for correlations to arise. This meant that $j = i$ or $j = i \pm b$. In the Rayleigh case under consideration, the correlations in the intensities also arise if two pixels are separated by the binary separation. This is because such pixels have one pair of speckles with the true intensity ratio for the binary. Thus, the subscripts on the $\bar{n}$ variables must either be equal or differ by b, the binary separation. This means that $j = i$, $j = i \pm b$, or $j = i \pm 2b$ give correlation. Barring these five possibilities, the Rayleigh average can be split. This, of course, includes the above five cases for which such splitting is not possible. So when we write these five cases as they should be written, we subtract from them the terms with split averages, much as we did before for the Poisson statistics. The sixth-order terms can be expanded as (compact notation for single summation)

$$\begin{aligned}
\left\langle \left[\sum_i (n_i^2 \bar{n}_{i+b} - \bar{n}_i \bar{n}_{i+b}^2) \right]^2 \right\rangle = {} & \left[\left\langle \sum_i (\bar{n}_i^2 \bar{n}_{i+b} - \bar{n}_i \bar{n}_{i+b}^2) \right\rangle \right]^2 \\
& + \langle \bar{n}^4 \bar{n}_1^2 \rangle + \langle \bar{n}^2 \bar{n}_1^4 \rangle + 2\langle \bar{n}^2 \bar{n}_1^3 \bar{n}_2 \rangle + 2\langle \bar{n}\bar{n}_1^3 \bar{n}_2^2 \rangle + 2\langle \bar{n}^2 \bar{n}_1 \bar{n}_2^2 \bar{n}_3 \rangle \\
& + 2\langle \bar{n}\bar{n}_1^2 \bar{n}_2 \bar{n}_3^2 \rangle - 5\langle \bar{n}^2 \bar{n}_1 \rangle^2 - 5\langle \bar{n}\bar{n}_1^2 \rangle^2 + 10\langle \bar{n}^2 \bar{n}_1 \rangle \langle \bar{n}\bar{n}_1^2 \rangle \\
& - 2\langle \bar{n}^3 \bar{n}_1^3 \rangle - 2\langle \bar{n}^2 \bar{n}_1^2 \bar{n}_2^2 \rangle - 2\langle \bar{n}^2 \bar{n}_1 \bar{n}_2 \bar{n}_3^2 \rangle - 2\langle \bar{n}\bar{n}_1^4 \bar{n}_2 \rangle - 2\langle \bar{n}\bar{n}_1^2 \bar{n}_2^2 \bar{n}_3 \rangle .
\end{aligned} \tag{A11}$$

As before, we consider only a representative term whenever a single summation occurs. However, since pixel intensities are correlated if the pixel separation equals the binary separation, we need to consider four consecutive pixels (instead of three in the Poisson case with the binary separations. The following averages are needed:

$$\begin{aligned}
\langle \bar{n}^4 \bar{n}_1^2 \rangle &= 48\mathcal{N}_1^6 + 288\mathcal{N}_1^5 \mathcal{N}_2 + 960\mathcal{N}_1^4 \mathcal{N}_2^2 + 672\mathcal{N}_1^3 \mathcal{N}_2^3 + 432\mathcal{N}_1^2 \mathcal{N}_2^4 + 192\mathcal{N}_1 \mathcal{N}_2^5 + 48\mathcal{N}_2^6 , \\
\langle \bar{n}^2 \bar{n}_1^4 \rangle &= 48\mathcal{N}_1^6 + 192\mathcal{N}_1^5 \mathcal{N}_2 + 432\mathcal{N}_1^4 \mathcal{N}_2^2 + 672\mathcal{N}_1^3 \mathcal{N}_2^3 + 960\mathcal{N}_1^2 \mathcal{N}_2^4 + 288\mathcal{N}_1 \mathcal{N}_2^5 + 48\mathcal{N}_2^6 , \\
\langle \bar{n}^2 \bar{n}_1^3 \bar{n}_2 \rangle &= 12\mathcal{N}_1^6 + 96\mathcal{N}_1^5 \mathcal{N}_2 + 264\mathcal{N}_1^4 \mathcal{N}_2^2 + 432\mathcal{N}_1^3 \mathcal{N}_2^3 + 288\mathcal{N}_1^2 \mathcal{N}_2^4 + 84\mathcal{N}_1 \mathcal{N}_2^5 + 12\mathcal{N}_2^6 , \\
\langle \bar{n}\bar{n}_1^3 \bar{n}_2^2 \rangle &= 12\mathcal{N}_1^6 + 84\mathcal{N}_1^5 \mathcal{N}_2 + 288\mathcal{N}_1^4 \mathcal{N}_2^2 + 432\mathcal{N}_1^3 \mathcal{N}_2^3 + 264\mathcal{N}_1^2 \mathcal{N}_2^4 + 96\mathcal{N}_1 \mathcal{N}_2^5 + 12\mathcal{N}_2^6 , \\
\langle \bar{n}^2 \bar{n}_1 \bar{n}_2^2 \bar{n}_3 \rangle &= 4\mathcal{N}_1^6 + 36\mathcal{N}_1^5 \mathcal{N}_2 + 108\mathcal{N}_1^4 \mathcal{N}_2^2 + 132\mathcal{N}_1^3 \mathcal{N}_2^3 + 92\mathcal{N}_1^2 \mathcal{N}_2^4 + 32\mathcal{N}_1 \mathcal{N}_2^5 + 4\mathcal{N}_2^6 , \\
\langle \bar{n}\bar{n}_1^2 \bar{n}_2 \bar{n}_3^2 \rangle &= 4\mathcal{N}_1^6 + 32\mathcal{N}_1^5 \mathcal{N}_2 + 92\mathcal{N}_1^4 \mathcal{N}_2^2 + 132\mathcal{N}_1^3 \mathcal{N}_2^3 + 108\mathcal{N}_1^2 \mathcal{N}_2^4 + 36\mathcal{N}_1 \mathcal{N}_2^5 + 4\mathcal{N}_2^6 , \\
\langle \bar{n}^2 \bar{n}_1 \bar{n}_2 \bar{n}_3^2 \rangle &= 4\mathcal{N}_1^6 + 32\mathcal{N}_1^5 \mathcal{N}_2 + 84\mathcal{N}_1^4 \mathcal{N}_2^2 + 120\mathcal{N}_1^3 \mathcal{N}_2^3 + 84\mathcal{N}_1^2 \mathcal{N}_2^4 + 32\mathcal{N}_1 \mathcal{N}_2^5 + 4\mathcal{N}_2^6 , \\
\langle \bar{n}\bar{n}_1^2 \bar{n}_2^2 \bar{n}_3 \rangle &= 4\mathcal{N}_1^6 + 36\mathcal{N}_1^5 \mathcal{N}_2 + 128\mathcal{N}_1^4 \mathcal{N}_2^2 + 200\mathcal{N}_1^3 \mathcal{N}_2^3 + 128\mathcal{N}_1^2 \mathcal{N}_2^4 + 36\mathcal{N}_1 \mathcal{N}_2^5 + 4\mathcal{N}_2^6 , \\
\langle \bar{n}\bar{n}_1^4 \bar{n}_2 \rangle &= 24\mathcal{N}_1^6 + 192\mathcal{N}_1^5 \mathcal{N}_2 + 408\mathcal{N}_1^4 \mathcal{N}_2^2 + 432\mathcal{N}_1^3 \mathcal{N}_2^3 + 408\mathcal{N}_1^2 \mathcal{N}_2^4 + 192\mathcal{N}_1 \mathcal{N}_2^5 + 24\mathcal{N}_2^6 , \\
\langle \bar{n}^3 \bar{n}_1^3 \rangle &= 36\mathcal{N}_1^6 + 180\mathcal{N}_1^5 \mathcal{N}_2 + 504\mathcal{N}_1^4 \mathcal{N}_2^2 + 1044\mathcal{N}_1^3 \mathcal{N}_2^3 + 504\mathcal{N}_1^2 \mathcal{N}_2^4 + 180\mathcal{N}_1 \mathcal{N}_2^5 + 36\mathcal{N}_2^6 , \\
\langle \bar{n}^2 \bar{n}_1^2 \bar{n}_2^2 \rangle &= 8\mathcal{N}_1^6 + 56\mathcal{N}_1^5 \mathcal{N}_2 + 192\mathcal{N}_1^4 \mathcal{N}_2^2 + 256\mathcal{N}_1^3 \mathcal{N}_2^3 + 192\mathcal{N}_1^2 \mathcal{N}_2^4 + 56\mathcal{N}_1 \mathcal{N}_2^5 + 8\mathcal{N}_2^6 .
\end{aligned} \tag{A12}$$

Using these averages and noting that the remaining double sum in equation (A11) is just the square of the averaged (both Poisson and Rayleigh) parity we get the variance for the parity (which can be put in the explicitly positive form of eq. [28]):

$$\begin{aligned}\langle \overline{p^2} \rangle - \langle \bar{p} \rangle^2 = {} & 4N_S(2\mathcal{N}_1^3 + 7\mathcal{N}_1^2\mathcal{N}_2 + 7\mathcal{N}_1\mathcal{N}_2^2 + 2\mathcal{N}_2^3) \\ & + 4N_S(6\mathcal{N}_1^4 + 20\mathcal{N}_1^3\mathcal{N}_2 - 10\mathcal{N}_1^2\mathcal{N}_2^2 + 20\mathcal{N}_1\mathcal{N}_2^3 + 6\mathcal{N}_2^4) \\ & + 8N_S(3\mathcal{N}_1^5 + 5\mathcal{N}_1^4\mathcal{N}_2 + 2\mathcal{N}_1^3\mathcal{N}_2^2 + 2\mathcal{N}_1^2\mathcal{N}_2^3 + 5\mathcal{N}_1\mathcal{N}_2^4 + 3\mathcal{N}_2^5) \\ & + 4N_S(2\mathcal{N}_1^6 - 4\mathcal{N}_1^5\mathcal{N}_2 + 61\mathcal{N}_1^4\mathcal{N}_2^2 - 116\mathcal{N}_1^3\mathcal{N}_2^3 + 61\mathcal{N}_1^2\mathcal{N}_2^4 - 4\mathcal{N}_1\mathcal{N}_2^5 + 2\mathcal{N}_2^6) \qquad \text{(A13)}\end{aligned}$$

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