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On the summation of certain Fourier series involving discontinuities

C V RAMAN

In a previous communication published in the Bulletin,* I gave a brief outline of a method which I had adopted for discussing the kinematics of a bowed string in a detailed memoir on the subject which I had under preparation. Subsequently, whilst the memoir was being written, I noticed some points which were of great importance in the theory of the subject and which were not alluded to in the note published in the Bulletin of the Society. I propose now to supplement the first note by referring to the fresh results thus obtained.

The treatment adopted for discussing the motion of a bowed string in detail is based on the following two dynamical principles:

(1) The frictional force at the point of contact is a function of the relative velocity which decreases when the relative velocity is increased and becomes indeterminate when the relative velocity is zero.

(2) The force required to maintain any given harmonic component of the motion with a specified amplitude is a function of the damping coefficient and may thus be regarded as small compared with the variation of frictional force due to a finite change of relative velocity; this statement is however subject to the qualification that the point of application of the force should not coincide with a node of the given harmonic, and that the pressure with which the bow is applied is sufficiently large.

When a steady state of vibration is reached under the action of the bow, the harmonic components in the frictional force at the point of contact and the forces actually required to maintain the motion must balance one another. This can only be reconciled with the principles set out in (1) and (2) above on the assumption that during a part of the motion, the relative velocity at the point

*C V Raman, 'On some new methods in kinematical theory' 4, pages 1–4.
To the references cited in the note may be added the following:
A Harnack, *Mathematische Annalen*, Bd. 29, S. 486, 1887.
F Lindemann, *Philosophical Magazine*, March 1880.
Carslaw, *Proceedings of Edin. Math. Soc.* 1902.

of contact actually becomes zero, so that the frictional force in those stages falls below the maximum statical value.

It is also seen from principle (2) that during those stages of the motion during which the relative velocity at the point of contact is not actually zero, its value would in any case be practically a constant quantity. But this result would not hold good with the same universality as the constancy of the velocity of the bowed point during the stages in which its movement is in the same direction as that of the bow. In fact, except in special cases, the velocity in the backward movement would be rigorously constant only when the damping coefficients of all the harmonics are vanishingly small compared with the other quantities involved.

Assuming the rigorous constancy of the velocities of motion at the bowed point both in the forward and backward movements, we see that the condition $(d^2y/dt^2)=0$ is satisfied generally at the bowed point. The kinematics of the motion can then be discussed on the lines indicated in my first note. If the bowed point divides the string in an irrational ratio, all the discontinuities in the velocity-diagram of the string are equal to one another and to $(v_a - v_b)$, where v_a and v_b are the two velocities possible at the bowed point. If there are n such discontinuities on the velocity diagram of the string during the motion, the mode of vibration may be classified as belonging to the nth type. It is readily seen that the lines in the velocity-diagram are not more than $(n+1)$ in number at any instant, and that they always pass or would pass, if produced, through the $(n+1)$ nodes of the nth harmonic. These nodes are then points alternately of rest and of motion in one direction or the other.

When the bowed point divides the string in a rational ratio, the form of the velocity-diagram may be derived from that of the corresponding irrational type by the following process: Taking the form of the velocity-diagram at any specified epoch in the irrational case, we have to analyse it into its Fourier components and then effect a summation of the series of components which have a node at the bowed point in the form of a subsidiary velocity-diagram for the specified epoch. Subtracting the ordinates of this diagram from the other, we get the actual velocity-diagram in which the harmonics having a node at the bowed point are non-existent.

Let the velocity-diagram of the string at a certain epoch in the corresponding irrational type of vibration consist of parallel straight lines inclined to the x-axis at an angle a, with discontinuities d_1, d_2, d_3, etc., intervening at the points $x = c_1, c_2, c_3$, etc., respectively. Let this diagram be represented by the function $\phi(x)$. Then

$$\phi(x) = \sum_{n=1}^{n=\infty} A_n \sin\frac{n\pi x}{l},$$

where the value of A_n is determined by the equation.

$$A_n = \frac{2}{l}\int_0^l \phi(x) \quad \sin\frac{n\pi x}{l}\,\mathrm{d}x.$$

$\phi(x)$ is equal to $x \tan a$ between the limits $x = o$ and $x = c_1$. From $x = c_1$ up to $x = c_2$, $\phi(x)$ is equal $(x \tan a - d_1)$, and then changes to $(x \tan a - d_1 - d_2)$, retaining this value up to $x = c_3$, and so on. Integrating by parts, we have

$$A_n = -\frac{2}{n\pi}\left[\phi(x)\cos\frac{n\pi x}{l}\right]_0^l + \frac{2}{n\pi}\int_0^l \tan a \cos\frac{n\pi x}{l}\,\mathrm{d}x.$$

Since $\tan a$ is a constant, the second integral reduces to zero, and the equation may be written in the form

$$A_n = -\frac{2}{n\pi}\left[d_1 \cos\frac{n\pi c_1}{l} + d_2 \cos\frac{n\pi c_2}{l} + \text{etc.}\right].$$

When $n = 1$, we have

$$A_1 = -\frac{2}{\pi}\left[d_1 \cos\frac{\pi c_1}{l} + d_2 \cos\frac{\pi c_2}{l} + \text{etc.}\right].$$

When $n = s$, A_s may be written in the form

$$A_s = \frac{2}{\pi}\left[\frac{d_1}{s}\cos\frac{\pi c_1}{l/s} + \frac{d_2}{s}\cos\frac{\pi c_2}{l/s} + \text{etc.}\right].$$

The summation of the series $\sum_{n=1}^{n=\infty} A_n \sin(n\pi x)/l$ of which $A_1 \sin(\pi x)/l$ is the leading term gives us the original velocity-diagram $\phi(x)$ which consists of parallel straight lines inclined to the x-axis at an angle a and has a discontinuity d_1 at the point $x = c_1$, a discontinuity d_2 at the point c_2, and so on. From this, it follows that the series

$$\sum_{n=1}^{n=\infty} A_{ns} \sin\frac{n\pi x}{l/s}$$

of which $A_s \sin(s\pi x)/l$ is the leading term would similarly give us when summed, a diagram also consisting of straight lines inclined to the x-axis at the same angle a, the magnitude of the discontinuities in it being d_1/s, d_2/s, d_3/s, etc., and the series being periodic for increments of x by the length $2l/s$ instead of by $2l$ as with the original series. The positions of the discontinuities in the diagram thus derived are given by the abscissae obtained by subtracting from c_1, c_2, c_3, etc., the nearest multiples of the length l/s.

Subtracting the ordinates of the diagram thus derived from the ordinates $\phi(x)$ of the original diagram, the resulting figure in which the sth, $2s$th, $3s$th harmonics, etc. are all absent is seen to consist of straight lines *parallel* to the x-axis with intervening discontinuities. From the graph thus derived, the motion at any given point on the string can be drawn as a time-displacement diagram with the greatest ease and simplicity by noting the times at which the successive changes of velocity occur at the point by the passage of the discontinuities over it. In the special cases in which the motion at the bowed point is a simple two-step zig-zag, the quantities c_1, c_2, c_3, etc., are found to be merely multiples of the length l/s and the construction becomes particularly elegant.

The treatment of the cases in which the velocity at the bowed point has a continuous variation, particularly during the stages of backward motion, is naturally far more complicated. It is however of great importance in regard to the musical applications of the subject. These variations of velocity occur owing to the harmonics not being elicited in the normal strength in which they are present in any of the standard types of vibration referred to in this note. It is not within the scope of this note to consider such cases in detail.