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The diffraction of light by high frequency sound waves: Part III

Doppler effect and coherence phenomena

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1. Introduction

In part I[1] of this series of papers, a theory of the diffraction of light by high frequency sound waves was developed starting from the simple basic idea that the incident plane waves of light, after transmission through the medium traversed by the sound waves assume a corrugated form, owing to the fluctuations in the density and consequently also in the refractive index of the medium. The Fourier analysis of the emerging corrugated wave-front automatically gives the diffraction effects observed when the emergent waves are brought to focus by the lens of the observing telescope. The results deduced from the theory gave a gratifyingly satisfactory explanation of the observations of Bär[3] regarding the changes in the diffraction pattern when the supersonic intensity, the wavelength of the incident light and the length of the cell are varied.

In part II[2], we extended the theory to the case of the oblique incidence of the light on the sound waves and were successful in explaining the variations of the diffraction effects reported by Debye and Sears[4] as the angle of obliquity is varied.

In parts I and II, we deliberately ignored the variation of the refractive index with time in order to bring out the essential features of the theory without unnecessary complications. In this the third part of the paper, we proceed to take this factor also into consideration. It will be shown that light diffracted by progressive sound waves exhibits Doppler shifts of a very simple type. In the case, however, of the diffraction of light by *standing sound waves* in a medium, we get the much more interesting result that *in any even order, radiations with frequencies* $\nu \pm 2r\nu^*$ *would be present* where ν is the frequency of the incident light, ν^* is the frequency of sound in the medium and r is any integer and that *in an odd order, radiations with frequencies* $\nu \pm \overline{2r+1}\,\nu^*$ *would be present.* This implies that *any* *pair of even orders or odd orders can partly cohere* and that *an even order and an odd one are incoherent.* This latter result has already been arrived at by Bär[5] purely by his experimental investigations. The remarkable results of Bär in the field of supersonic research thus find a natural explanation in terms of our theory.

It should be, however, noted that the theory developed in the following is subject to the same limitations as those in the previous parts, viz., that the depth of the cell is not too great to permit the form of the emerging wave-front to be deduced in the simple manner indicated in part I. A more general consideration of the problem will be presented in a later communication.

2. Doppler effects due to a progressive sound wave

Let us suppose that the progressive sound wave travels in a direction parallel to the X-axis perpendicular to two faces of a rectangular vessel containing some homogeneous and isotropic medium. We use the same notation and the axes of reference as in our earlier paper. When the sound wave travels in the medium, the density of the medium and its refractive index undergo periodic fluctuations. If the sound wave is a simple one, we could assume that the variation of the refractive index at a point in the medium is given by

$$\mu(x,t) - \mu_0 = \mu \sin 2\pi(\nu^* t - x/\lambda^*) \tag{1}$$

where $\mu(x,t)$ is the refractive index of the medium at a height x from the origin at time t, μ_0 is the refractive index of the medium in its undisturbed state, μ is the *maximum variation* of the refractive index from μ_0 and ν^* and λ^* refer to the frequency and the wavelength of the sound wave in the medium.

Let the light wave be incident along the Z-axis perpendicular to two faces of the medium and the direction of the propagation of the sound wave. If the incident light wave is given by

$$\exp[2\pi i\nu t]$$

it will be

$$\exp[2\pi i\nu\{t - L\mu(x,t)/c\}]$$

when it arrives at the other face where L is the distance between the two faces.

The amplitude of the corrugated wave at a point on a distant screen parallel to the face of the medium from which light is emerging, whose join with the origin has its x-direction-cosine l depends on the evaluation of the diffraction integral

$$\int_{-p/2}^{p/2} \exp[2\pi i\{lx - \mu L \sin 2\pi(\nu^* t - x/\lambda^*)\}/\lambda]\mathrm{d}x \tag{2}$$

where p is the length of the beam along the X-axis. The real and the imaginary parts of the diffraction integral (2) are

$$\int_{-p/2}^{p/2} \{\cos ulx \cos(v \sin \overline{bx-\varepsilon}) - \sin ulx \sin(v \sin \overline{bx-\varepsilon})\} dx \tag{3}$$

and

$$\int_{-p/2}^{p/2} \{\sin ulx \cos(v \sin \overline{bx-\varepsilon}) + \cos ulx \sin(v \sin \overline{bx-\varepsilon})\} dx$$

where

$$u = 2\pi/\lambda, \quad b = 2\pi/\lambda^*, \quad v = 2\pi\mu L/\lambda \quad \text{and} \quad \varepsilon = 2\pi v^* t.$$

Putting $bx - \varepsilon$ as x' we could write the integrals* (3) as

$$\frac{2}{b}\sum_0^{\infty}{}' J_{2r}(v) \int_{-bp/2-\varepsilon}^{bp/2-\varepsilon} \cos\left(ul\frac{x'+\varepsilon}{b}\right) \cos 2rx'\, dx'$$
$$-\frac{2}{b}\sum_0^{\infty} J_{2r+1}(v) \int_{-bp/2-\varepsilon}^{bp/2-\varepsilon} \sin\left(ul\frac{x'+\varepsilon}{b}\right) \times \sin \overline{2r+1}\, x'\, dx'$$

and

$$\frac{2}{b}\sum_0^{\infty}{}' J_{2r}(v) \int_{-bp/2-\varepsilon}^{bp/2-\varepsilon} \sin\left(ul\frac{x'+\varepsilon}{b}\right) \cos 2rx'\, dx'$$
$$+\frac{2}{b}\sum_0^{\infty} J_{2r+1}(v) \int_{-bp/2-\varepsilon}^{bp/2-\varepsilon} \cos\left(ul\frac{x'+\varepsilon}{b}\right) \times \sin \overline{2r+1}\, x'\, dx'$$

or

$$\frac{1}{b}\sum_0^{\infty}{}' J_{2r}(v) \int_{-bp/2-\varepsilon}^{bp/2-\varepsilon} \left\{\cos\left(\overline{\frac{ul}{b}+2r}\, x' + \frac{ul\varepsilon}{b}\right)\right.$$
$$\left. + \cos\left(\overline{\frac{ul}{b}-2r}\, x' + \frac{ul\varepsilon}{b}\right)\right\} dx'$$
$$+\frac{1}{b}\sum_0^{\infty} J_{2r+1}(v) \int_{-bp/2-\varepsilon}^{bp/2-\varepsilon} \left\{\cos\left(\overline{\frac{ul}{b}+\overline{2r+1}}\, x' + \frac{ul\varepsilon}{b}\right)\right.$$
$$\left. - \cos\left(\overline{\frac{ul}{b}-\overline{2r+1}}\, x' + \frac{ul\varepsilon}{b}\right)\right\} dx' \tag{4a}$$

*The dash over the summation sign indicates that the coefficient of the first term has to be multiplied by half.

and

$$\frac{1}{b}\sum_0^{\infty}{}' J_{2r}(v)\int_{-bp/2-\varepsilon}^{bp/2-\varepsilon}\left\{\sin\left(\overline{\frac{ul}{b}+2r}\,x'+\frac{ul\varepsilon}{b}\right)+\sin\left(\overline{\frac{ul}{b}-2r}\,x'+\frac{ul\varepsilon}{b}\right)\right\}\mathrm{d}x'$$
$$+\frac{1}{b}\sum_0^{\infty}J_{2r+1}(v)\int_{-bp/2-\varepsilon}^{bp/2-\varepsilon}\left\{\sin\left(\overline{\frac{ul}{b}+\overline{2r+1}}\,x'+\frac{ul\varepsilon}{b}\right)\right.$$
$$\left.-\sin\left(\overline{\frac{ul}{b}-\overline{2r+1}}\,x'+\frac{ul\varepsilon}{b}\right)\right\}\mathrm{d}x'. \tag{4b}$$

Integrating and combining the real and the imaginary parts (4*a*) and (4*b*) we find that the amplitude depends on

$$p\sum_0^{\infty}{}' J_{2r}(v)\left\{\frac{\sin\{(ul+2rb)p/2\}}{(ul+2rb)p/2}e^{-i2r\varepsilon}+\frac{\sin\{(ul-2rb)p/2\}}{(ul-2rb)p/2}e^{i2r\varepsilon}\right\}$$
$$+p\sum_0^{\infty}J_{2r+1}(v)\left\{\frac{\sin\{(ul+\overline{2r+1}\,b)p/2\}}{(ul+\overline{2r+1}\,b)p/2}e^{-i\overline{2r+1}\varepsilon}\right.$$
$$\left.-\frac{\sin\{(ul-\overline{2r+1}\,b)p/2\}}{(ul-\overline{2r+1}\,b)p/2}e^{i\overline{2r+1}\varepsilon}\right\} \tag{5}$$

where $\varepsilon = 2\pi\nu^* t$. We should remember that the amplitude function has the other time factor $e^{2\pi i\nu t}$ which has been taken out as a constant from the integrand of the diffraction integral. One can see that the magnitude of each individual term of (5) attains its highest maximum when its denominator vanishes. Also, it can be seen that when any one of the terms is maximum, all the others have negligible values as the numerator of each cannot exceed unity and the denominator is some integral non-vanishing multiple of b which is sufficiently large. When

$$ul + nb = 0$$
$$\sin\theta = -\frac{n\lambda}{\lambda^*} \tag{6}$$

where n is a positive or a negative integer and θ is the angle between the direction whose x-direction-cosine is l and the Z-axis.

The wave travelling in the direction whose inclination with the incident light beam is $\sin^{-1}(-n\lambda/\lambda^*)$ is determined by

$$J_n(v)e^{2\pi i(\nu - n\nu^*)t} \tag{7}$$

having the frequency $\nu - n\nu^*$, n being a positive or negative integer; when n is negative the direction of propagation of that order has positive direction-cosines with respect to the directions of the propagation of the sound and light waves. Consequently the radiations in the different orders will be incoherent with each other. (See figure 1.)

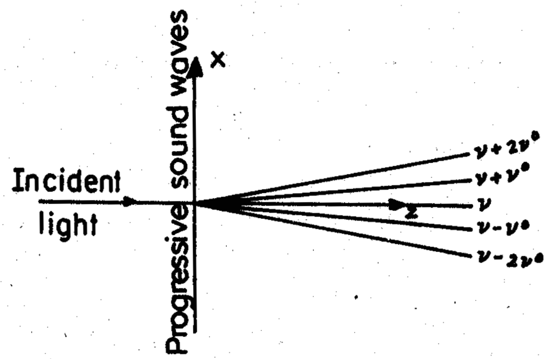

Figure 1

The relative intensity of the *m*th order to the *n*th order is given by the expression

$$J_m^2\left(\frac{2\pi\mu L}{\lambda}\right) \Big/ J_n^2\left(\frac{2\pi\mu L}{\lambda}\right)$$

identical with the one given in part I.

3. Doppler effects due to a standing sound wave

In the case of a standing wave produced by the interference of two simple waves travelling in opposite directions parallel to the X-axis, we could assume that the variations of the refractive index at a point in the medium is given by

$$\mu(x,t) - \mu_0 = -\mu \sin 2\pi\nu^* t \sin(2\pi x/\lambda^*) \qquad (8)$$

with the same notation as in the previous section. Under the same restrictions as in Part I, we find that the emerging wave-front is given by

$$\exp[2\pi i\nu\{t - L\mu(x,t)/c\}] \qquad (9)$$

The diffraction integral is then

$$\int_{-p/2}^{p/2} \exp[2\pi i\{lx + \mu L \sin\varepsilon \sin(2\pi x/\lambda^*)\}/\lambda]\,\mathrm{d}x \qquad (10)$$

where $\varepsilon = 2\pi\nu^* t$.

The real and the imaginary parts of the integral (10) are

$$\int_{-p/2}^{p/2} \{\cos ulx \cos(v' \sin bx) - \sin ulx \sin(v' \sin bx)\}\,\mathrm{d}x$$

and

$$\int_{-p/2}^{p/2} \{\sin ulx \cos(v' \sin bx) + \cos ulx \sin(v' \sin bx)\}\, dx$$

where

$$u = 2\pi/\lambda, \quad b = 2\pi/\lambda^*, \quad v' = v \sin\varepsilon = (2\pi\mu L \sin\varepsilon)/\lambda.$$

Following the same procedure as in our earlier paper, we find that the real part of the diffraction integral (10) is

$$p\sum_0^\infty{}' J_{2r}(v\sin 2\pi v^* t)\left\{\frac{\sin[(ul+2rb)p/2]}{(ul+2rb)p/2} + \frac{\sin[(ul-2rb)p/2]}{(ul-2rb)p/2}\right\}$$

$$+ p\sum_0^\infty J_{2r+1}(v\sin 2\pi v^* t)\left\{\frac{\sin[(ul+\overline{2r+1}\, b)p/2]}{(ul+\overline{2r+1}\, b)p/2}\right.$$

$$\left. - \frac{\sin[(ul-\overline{2r+1}\, b)p/2]}{(ul-\overline{2r+1}\, b)p/2}\right\}.$$

The integral corresponding to the imaginary part of the diffraction integral is zero.

Following similar arguments as in part I or in the previous section we can show that the wave travelling in the direction given by

$$ul + nb = 0$$

or

$$\sin\theta = -\frac{n\lambda}{\lambda^*}$$

is

$$\pm J_n(v\sin 2\pi v^* t)\, e^{2\pi i v t} \qquad (11)$$

multiplied by a constant usually taken out from the diffraction integral. The wave given by (11) is not a simple one but is a superposition of a number of waves given by the Fourier analysis of $J_n(v\sin 2\pi v^* t)$ and multiplied by $e^{2\pi i v t}$.

Fourier analysis of J_n $(v \sin\varepsilon)$: The well-known Neumann's addition theorem

$$J_0(\tilde{\omega}) = 2\sum_0^\infty{}' J_m(Z) J_m(z)\cos m\phi$$

where

$$\tilde{\omega} = \sqrt{(Z^2 + z^2 - 2Zz\cos\phi)}$$

has been generalised by Graf[6] as

$$J_n(\tilde{\omega})\left\{\frac{Z - ze^{-i\phi}}{Z - ze^{i\phi}}\right\}^{n/2} = \sum_{-\infty}^{+\infty} J_{n+m}(Z) J_m(z) e^{im\phi}$$

provided $|ze \pm i\phi| < Z$. If n is an integer, the inequality need not be in force. Putting

$$Z = z = v/2$$
$$\text{and } \phi = 2\varepsilon$$

we get

$$J_n(v \sin \varepsilon)e^{-in\varepsilon}(-1)^{n/2} = \sum_{-\infty}^{+\infty} J_{n+m}(v/2)J_m(v/2)e^{i2m\varepsilon}.$$

From this, changing n to $2n$ we deduce that

$$J_{2n}(v \sin \varepsilon) = (-1)^n \sum_{-\infty}^{+\infty} J_{m+2n}(v/2)J_m(v/2)e^{-i(2m+2n)\varepsilon}.$$

Putting $m = -n + r$ and after a little simplification, we get

$$J_{2n}(v \sin \varepsilon) = (-1)^n J_{-n}(v/2)J_n(v/2) + 2\sum_1^{\infty} J_{-n+r}(v/2)J_{n+r}(v/2)\cos 2r\varepsilon$$

$$= (-1)^n 2\sum_0^{\infty}{}' J_{-n+r}(v/2)J_{n+r}(v/2)\cos 2r\varepsilon$$

$$= 2\sum_0^{\infty}{}'(-1)^r J_{n-r}(v/2)J_{n+r}(v/2)\cos 2r\varepsilon.$$

Similarly we can deduce that

$$J_{2n+1}(v \sin \varepsilon) = 2\sum_0^{\infty}(-1)^r J_{n-r}(v/2)J_{n+r+1}(v/2)\sin 2r + 1\varepsilon$$

$$J_{2n}(v \cos \varepsilon) = 2\sum_0^{\infty}{}' J_{n-r}(v/2)J_{n+r}(v/2)\cos 2r\varepsilon$$

$$J_{2n+1}(v \cos \varepsilon) = 2\sum_0^{\infty} J_{n-r}(v/2)J_{n+r+1}(v/2)\cos \overline{2r+1}\,\varepsilon.$$

Returning now to the Fourier analysis of the diffraction components, the diffracted waves can be resolved into a number of simple waves, for

$$J_{2n}(v \sin 2\pi v^* t)e^{2\pi i v t}$$

$$= e^{2\pi i v t} 2\sum_0^{\infty}{}'(-1)^r J_{n-r}(v/2)J_{n+r}(v/2)\cos(2r\cdot 2\pi v^* t)$$

$$= \sum_0^{\infty}{}'(-1)^r J_{n-r}(v/2)J_{n+r}(v/2)\{e^{2\pi i(v+2rv^*)t} + e^{2\pi i(v-2rv^*)t}\}$$

and

$$J_{2n+1}(v \sin 2\pi\nu^* t)e^{2\pi i\nu t}$$

$$= \frac{1}{i}\sum_{0}^{\infty}(-1)^r J_{n-r}(v/2)J_{n+r+1}(v/2)\{e^{2\pi i(\nu+\overline{2r+1}\,\nu^*)t} - e^{2\pi i(\nu-\overline{2r+1}\,\nu^*)t}\}.$$

Thus in all even orders radiation frequencies

$$\nu \pm 2r\nu^*, \quad r \text{ a positive integer,}$$

are present. The relative intensity of the $\nu \pm 2r\nu^*$ sub-component in the $2n$th order is given by

$$J_{n-r}^2(v/2)J_{n+r}^2(v/2).$$

In all odd orders radiation frequencies

$$\nu \pm \overline{2r+1}\,\nu^*, \; r \text{ a positive integer,}$$

are present (see figure 2). The relative intensity of the $\nu \pm \overline{2r+1}\,\nu^*$ sub-component in $2n+1$th order is given by

$$J_{n-r}^2(v/2)J_{n+r+1}^2(v/2).$$

We can conclude from the above analysis that *an even order and an odd one are incoherent while any two even or any two odd orders can partly cohere.* Any two orders symmetrically situated to the 0th order are completely coherent. We have calculated the relative intensities of the various Doppler sub-components of the various orders as v ranges from 0 to 5 in steps of unity and represented them in figure 3.

We may also note that the intensity of each of the sub-components of each order depends on the amplitude of the supersonic vibration, the length of the cell and the wavelength of the incident light.

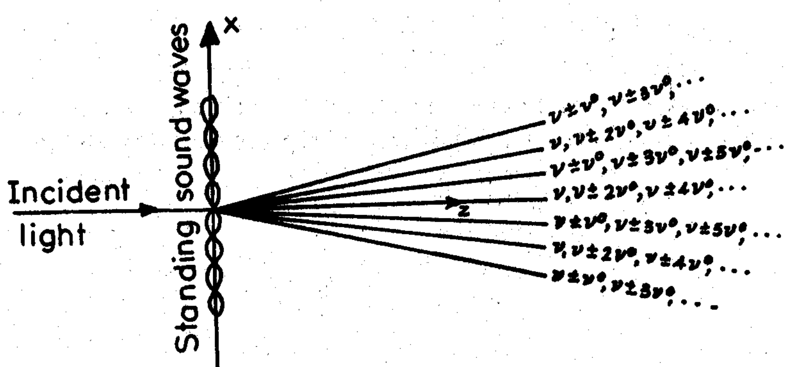

Figure 2

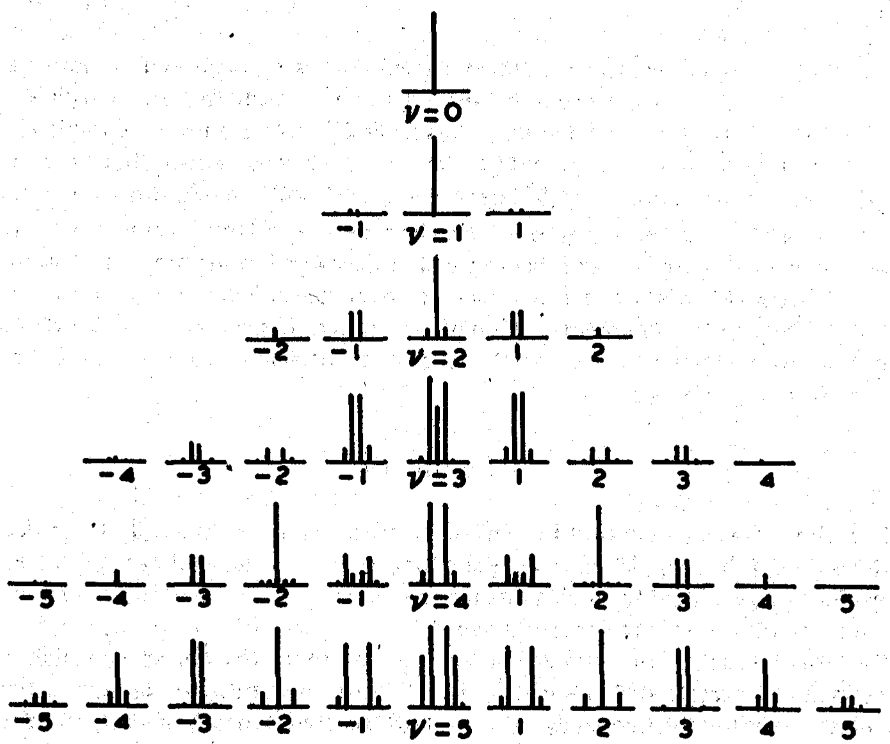

Figure 3. Relative intensities of the various sub-components of observable orders; the sub-components of an odd order standing on a base correpond to ---, $\nu - \overline{2r+1}\,\nu^*$, ---, $\nu - \nu^*$, $\nu + \nu^*$, ---, $\nu + \overline{2r+1}\,\nu^*$, --- and those of an even order standing on a base correspond to ---, $\nu - 2r\nu^*$, ---, ν, ---, $\nu + 2r\nu^*$, ---. In the figure $\nu = 5$, some lower orders are missing as their relative intensities are negligibly small.

If we ignore the spectral character of each order, then the relative intensity of the mth order to the nth order is

$$\frac{\int_0^{2\pi} J_m^2(v\sin\theta)\mathrm{d}\theta}{\int_0^{2\pi} J_n^2(v\sin\theta)\mathrm{d}\theta} \quad \text{where } v = \frac{2\pi\mu L}{\lambda}$$

which follows from Parseval's theorm.

4. Interpretation of Bär's experimental results

Bär[5] has recently investigated by an interference method the coherence of the diffraction components of light produced by a standing supersonic wave. He has found that the various orders could be classed into two groups, one comprising the even orders and the other comprising the odd orders and that any two orders of a group cohere partly while two orders from different groups are completely incoherent. These results are readily understood when we notice that an even order contains radiations with frequencies $\nu \pm 2r\nu^*$ while an odd order contains radiations with frequencies $\nu \pm \overline{2r+1}\nu^*$. The experimental results of Bär are thus fully explicable in terms of the theory we have developed in the previous section. Bär has himself remarked that the observed coherence indicates the presence of a series of frequency components in each of the diffraction spectra. It will be noticed that, according to our theory, even the zero-order spectrum includes such a series of frequency components.

5. Summary

The theory developed in part I of this series of papers has been developed in this paper to find the Doppler effects in the diffraction components of light produced by the passage of light through a medium containing (1) a progressive supersonic wave and (2) a standing supersonic wave.

(1) In the case of the former the theory shows that the nth order which is inclined at an angle $\sin^{-1}(-n\lambda/\lambda^*)$ to the direction of the propagation of the incident light has the frequency $\nu - n\nu^*$ where ν is the frequency of light, ν^* is the frequency of sound and n is a positive or negative integer and that the nth order has the relative intensity $J_n^2(2\pi\mu L/\lambda)$ where μ is the maximum variation of the refractive index, L is the distance between the faces of the cell of incidence and emergence and λ is the wavelength of light.

(2) In the case of a standing supersonic wave, the diffraction orders could be classed into two groups, one containing the even orders and the other odd orders; any even order, say $2n$, contains radiations with frequencies $\nu \pm 2r\nu^*$ where r is an integer including zero, the relative intensity of the $\nu \pm 2r\nu^*$ sub-component being $J_{n-r}^2(\pi\mu L/\lambda)\, J_{n+r}^2(\pi\mu L/\lambda)$; and odd order, say $2n+1$, contains radiations with frequencies $\nu \pm \overline{2r+1}\nu^*$, the relative intensity of the $\nu \pm \overline{2r+1}\nu^*$ sub-component being $J_{n-r}^2(\pi\mu L/\lambda)\, J_{n+r+1}^2(\pi\mu L/\lambda)$. These results satisfactorily interpret the recent results of Bär that any two odd orders or even ones partly cohere while an odd one and an even one are incoherent.

References

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