# ON SOME APPLICATIONS OF HERTZ'S THEORY OF IMPACT.

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SYNOPSIS.

*Statement of the Theory.*—One of the premises on which the mathematical theory of the collision of elastic solids given by Hertz is based is that the strains produced in the immediate neighborhood of the region of contact are determined by the pressure subsisting at any instant between the bodies, and are practically the same as under statical conditions. This premise is valid even when the impinging bodies do not move as rigid bodies, and the impact results in part of the translational kinetic energy being transformed into energy of elastic wave-motion in the substance of the solids. *Hertz's theory of impact* with suitable modifications may accordingly be applied under a very wide variety of conditions. In the present paper, an attempt is made to discuss the problem of *the transverse impact of a solid sphere on an infinitely extended elastic plate of finite thickness* and to calculate the theoretical *coefficient of restitution* which is a function of the elastic constants and densities of the materials, the diameter of the sphere and the thickness of the plate, and of the velocity of impact. As the result of the impact, annular waves of flexure are set up in the plate, and the sum of the kinetic and potential energies of the wave-motion may be determined in terms of the magnitude and duration of the impulse on certain simplifying assumptions. The calculation results in a *simple formula for the coefficient of restitution.*

*Experiments.*—A series of *experiments* carried out in the author's laboratory by Mr. A. Venkatasubbaraman has furnished a quantitative confirmation of the formula within the limits of its applicability, that is for plates not thinner than about half the diameter of the spheres. For plates much thinner than this, theory and experiment agree in indicating a zero coefficient of restitution. The formula indicates that the coefficient of restitution should increase and approach unity for greatly diminished velocities of impact, and this is also confirmed in experiment. The paper concludes with indications of some *further applications and extensions* of Hertz's theory of impact.

## 1. INTRODUCTION.

As is well known, Hertz[1] developed a solution of the problem of the collision of curved elastic solids on the following premises: (1) the elastic state of the two bodies near the point of impact during the whole duration of impact is very nearly the same as the state of equilibrium which would be produced by the total pressure subsisting at any instant between the bodies, supposing it to act for a long time; (2) it is further assumed that the time of impact is large compared with the time taken by elastic waves to traverse the impinging bodies from end to end,

[1] Hertz's Miscellaneous Papers, English edition, page 146, and Love's Theory of Elasticity, page 195 (1906 edition).

which consequently move practically as rigid bodies except in the immediate neighborhood of the region of contact. From these premises, it follows at once that the energy of the colliding bodies remains as translational energy after the impact, a deduction which is closely borne out in experiment, provided the impinging bodies are of appropriate shape, *e.g.*, solid spheres, and the stresses set up do not transcend the limits of perfect elastic recovery.[1] The first of the two premises on which Hertz's theory is based is of very general validity, in as much as it depends for its truth on the consideration that the elastic deformations near the region of contact are determined mainly by the pressure subsisting between the bodies at the instant, and being of a local character and relatively large, are un-influenced by any changes in the elastic deformations that might be developed elsewhere as the result of movements of the bodies. The second premise of Hertz is however purely an assumption[2] which is of comparatively restricted validity and may even fail completely. Indeed it is often the case that the colliding bodies *cannot* even approximately be considered to move as rigid bodies during and after the collision, and a considerable proportion of the energy is transformed into the energy of elastic wave-motion set up in the substance of the solids. Hertz remarked in his paper, though he did not fully develop the idea, that even in such cases, the first hypothesis (which remains valid) taken together with the equations of wave-motion in an elastic solid might enable the course of the impact to be traced. This suggestion of Hertz does not appear to have been generally followed up, though indeed in one case, that of the longitudinal impact of rods with rounded ends, its utility has been established.[3] It is proposed in the present paper to consider the application of the method suggested by Hertz to the problem of the *transverse* impact of a sphere or other solid of limited dimensions upon an infinitely extended elastic plate of finite thickness. It will be shown how the proportion of energy of impact transformed to energy of elastic wave-motion may be approximately calculated, in other words how the coefficient of restitution of the impinging body may be theoretically determined.

## 2. On the Nature of the Wave-Motion set up by Impact.

The effect of an impulse of short duration applied at a point on the plane face of an infinite mass of elastic solid has been investigated by Lamb,[4] who found that the main shock produced by the impulse travels

[1] C. V. Raman, Physical Review, Dec., 1918, page 442.

[2] Except in the case of *extremely small* velocities of impact when it is true irrespective of the shapes of the impinging bodies.

[3] J. E. Sears, Trans. Camb. Phil. Soc., Vol. XXI (1908), page 49.

[4] Phil. Trans. Roy. Soc., A, Vol. 203 (1904), pages 1–42.

along the surface of the solid as a solitary wave (with one maximum and one minimum, both in the horizontal and vertical displacements), with its time-scale constant, and its amplitude decreasing in accordance with the usual law of annular divergence, so that its total energy remains undiminished. The velocity of this solitary wave is that of the free Rayleigh waves on the surface of the solid which is somewhat less than that of the equivoluminal waves in an unlimited medium. As the depth to which the Rayleigh waves penetrate is comparable with their wave-length, it may be readily shown that for the case of a sphere impinging on the plane face of an infinite solid, the fraction of the translational energy transformed to energy of elastic wave-motion is extremely small. When, however, the impact takes place on an unlimited elastic plate or bar of *finite* thickness, this is no longer true, and a considerable proportion of the energy (in some cases, the whole of it) may be so transformed. The theory of wave-motion on a cylindrical rod of infinite length has been discussed by Pochhammer in a well-known memoir,[1] and for the case of an infinite elastic plate of finite thickness by Lamb[2] in a recent paper. Lamb finds that the types of wave-motion possible in an elastic plate may be divided into two classes, (*a*) the symmetrical modes, and (*b*) the asymmetrical modes. The former class travel with very high velocities ranging from a maximum equal to the highest value possible in an unlimited medium to a minimum equal to that of the Rayleigh surface-waves. The asymmetrical modes of wave-motion have relatively smaller velocities ranging from a maximum equal to that of the Rayleigh waves to a minimum value which tends to zero for very long flexural waves. *Prima facie*, it is clear that the modes of wave-motion excited by transverse impact would be chiefly of the asymmetrical class, and that, even of the latter, those chiefly concerned in taking up the momentum of the blow would be the relatively slowly-moving waves. Which particular type of wave-motion preponderates would obviously depend on the duration of impact and the manner in which the pressure exerted by the impinging body varies during impact.

An approximate calculation of the energy taken up by the wave-motion excited in the plate may be founded on the simplifying assumption (which may be justified at least as a first approximation) that the disturbance set up by the impact travels outward in the plate with the velocity of flexural waves having a half-period equal to the duration of impact. Let $2f$, and $2b$ be the thickness of the plate, and the diameter of the sphere respectively, $\rho_1$, $\rho_2$ their densities, $q_1$, $q_2$, their Young's

[1] Crelle, Vol. 81, page 324, see also Love's Elasticity, 1906, page 275.

[2] Proc. Roy. Soc., Vol. 93 (A series), 1917, page 114.

Moduli, $\sigma_1$, $\sigma_2$ the values of Poisson's ratio, and $\tau$ the duration of impact. The velocity $V$ of long flexural waves of wave-length $\lambda$ in the plate is given by the formula

$$V^2 = \frac{4}{3} \cdot \pi^2 \cdot \frac{f^2}{\lambda^2} \cdot \frac{q_1}{\rho_1(1 - \sigma_1^2)}. \tag{1}$$

On the foregoing assumption, the radius $a$ of the circle on the plate over which the disturbance has spread at the termination of the impact is given by the relation $a = V\tau = \lambda/2$. Accordingly, we have

$$a^2 = \pi\tau f \sqrt{q_1/3\rho_1(1 - \sigma_1^2)}. \tag{2}$$

The next step is to find the kinetic and potential energies of the wave-motion in the plate. The kinetic energy may be determined if we know the transverse velocity of the plate at each point over the circle of radius $a$ covered by the wave. The problem is one of two-dimensional wave-propagation analogous to that treated by Lamb,[1] who has discussed the configuration of the annular solitary wave diverging from a point at which a local pressure is applied, rising from zero to a maximum and falling again to zero. Lamb has given a sketch in the paper cited, showing the form of the wave when it has moved out to a considerable distance from the origin, from which it is seen that the wave consists of two parts: a rising part in which the transverse velocity of a point over which the wave passes increases quickly from zero to a maximum positive velocity and decreases again to zero, and a falling part which consists of an infinitely extended 'tail' in which the transverse velocity after reaching a certain maximum *negative* value (which is numerically less than in the first part) gradually drops down to zero again. The first part of the wave in Lamb's diagram passes over any specified point in about 5/8 of the duration of the original impulse. The form of the wave when near the origin just after the impulse has ceased would, of course, differ in details from that described above, but may be approximately represented as in Fig. 1 in which the ordinates represent transverse velocities, and the abscissæ are the radial distances from the point of impact. The direction of the impact is shown in the figure by an arrow.

### 3. Calculation of the Coefficient of Restitution.

The kinetic energy contained in the wave is given by

$$E = \int_0^a \tfrac{1}{2} \cdot 2f\rho_1 \cdot 2\pi r v^2 dr, \tag{3}$$

where $v$ is the transverse velocity at any point as given by the graph

[1] Proc. Lond. Math. Soc., Vol. 35, page 141 (1902).

in Fig. 1. We may, without appreciable error, denote the sum of the potential and kinetic energies in the wave by twice the integral in (3), that is by $2E$. The impulse given by the impinging body to the plate is given by

$$I = \int_c^a 2f\rho_1 \cdot 2\pi r v dv, \tag{4}$$

the integration in which is to be carried out having due regard to the

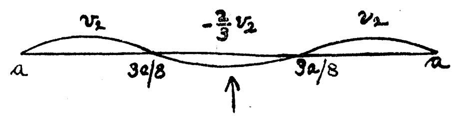


Fig. 1.

sign of $v$. If $M$ be the mass of the impinging body, $v_1$ its velocity before impact, and $e$ the coefficient of restitution, we have on the assumption that the energy is fully conserved, the two relations

$$\left.\begin{aligned} \tfrac{1}{2}Mv_1^2(1 - e^2) &= 2E \\ Mv_1(1 + e) &= I \end{aligned}\right\}. \tag{5}$$

On evaluating the integrals in (3) and (4), and substituting the same in (5), we get the value of the coefficient of restitution, $e$. The necessary integrations are readily carried out by taking the graph of the transverse velocity in Fig. (1) to be made up of arcs of sine-curves. The formula finally obtained is

$$e = \frac{f\rho_1 a^2 - 0.56M}{f\rho_1 a^2 + 0.56M}. \tag{6}$$

The distribution of transverse velocity shown in Fig. (1) is, as explained above, based on Lamb's investigation of two dimensional wave-propagation, and has thus theoretical justification. It also appears to be that

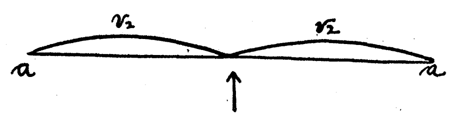


Fig. 2($a$).

most closely agreeing with facts. Nevertheless, it is of interest to see how far other assumed distributions of velocity would modify the formula for $e$ given in (6). If the distribution of transverse velocity were that shown in Fig. 2($a$), the formula for $e$ is found to be

$$e = \frac{f\rho_1 a^2 - 0.39M}{f\rho_1 a^2 + 0.39M}. \tag{7}$$

For the distribution shown in Fig. 2(*b*), the formula is

$$e = \frac{f\rho_1 a^2 - 0.44M}{f\rho_1 a^2 + 0.44M}. \tag{8}$$

To make use of the formula given in (6), we have to substitute in it the value of $a$ as given by (2), and to enable this to be done, we have to ascertain the duration of the impulse $\tau$. It is obvious that to a first approximation, this may be taken to be the same as that given by Hertz's theory of impact, the mass of the impinging body being $M$, and

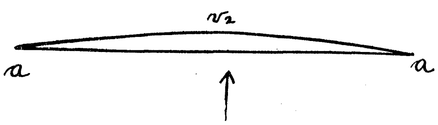


Fig. 2(*b*).

the mass of the plate being taken to be infinitely great. If there is any deviation from this in actual practice, such deviation should be sensible only when the velocity of impact is very large or when the thickness of the plate is much smaller than the diameter of the impinging sphere. The duration of impact on Hertz's theory is given by the equation

$$\tau = 2.94\alpha/v_1, \tag{9}$$

where

$$\alpha = \left[\frac{15}{16}v_1^2\left(\frac{1-\sigma_1^2}{q_1} + \frac{1-\sigma_2^2}{q_2}\right)M\right]^{2/5} b^{-1/5} \tag{10}$$

and $b$ is the radius of the impinging sphere. These values have to be substituted in equations (2) and (6) above.

## 4. Comparison with Experiment, and Some Concluding Remarks.

The correctness of the formula for the coefficient of restitution developed in (6) above has been tested in the author's laboratory in a very careful series of experiments carried out by Mr. A. Venkatasubbaraman to whom his best thanks are due. The impacts observed were those of polished hard steel spheres impinging on horizontally held glass plates of sufficient size to permit of the application of the theory, these materials being chosen as for moderate velocities of impact they very nearly satisfy the condition that the system is of the conservative type. As remarked above, the calculations assume that the duration of impact is the same as that given by Hertz's formulæ which would be practically correct, provided the velocity of impact is not very great and that the thickness of the plate is not very small. Mr. Venkatasubbaraman's experiments give results for the coefficient of restitution under these conditions closely agreeing with those found from (6).

TABLE I.

*Coefficient of Restitution, e.*

Velocity of impact = 234 cms. per second.

| Thickness of Plate in Centimeters. | Diameter of Spheres in Centimeters. | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0.791. | | 0.714. | | 0.637. | | 0.555. | | 0.396. | | 0.314. | | 0.237. | |
| | Calc. *e.* | Obsd. *e.* | Calc. *e.* | Obsd. *e.* | Calc. *e.* | Obsd. *e.* | Calc. *e.* | Obsd. *e.* | Calc. *e.* | Obsd. *e.* | Calc. *e.* | Obsd. *e.* | Calc. *e.* | Obsd. *e.* |
| 2.53....... | 0.98 | 0.95 | 0.99 | 0.96 | 0.99 | 0.97 | 0.99 | 0.97 | 0.99 | 0.98 | 0.99 | 0.98 | 0.99 | 0.98 |
| 1.93....... | 0.97 | 0.95 | 0.98 | 0.96 | 0.98 | 0.96 | 0.99 | 0.97 | 0.99 | 0.97 | 0.99 | 0.98 | 0.99 | 0.98 |
| 1.29....... | 0.93 | 0.91 | 0.94 | 0.93 | 0.96 | 0.93 | 0.97 | 0.94 | 0.98 | 0.95 | 0.99 | 0.97 | 0.99 | 0.98 |
| 1.01....... | 0.89 | 0.87 | 0.91 | 0.89 | 0.93 | 0.91 | 0.95 | 0.93 | 0.97 | 0.94 | 0.98 | 0.96 | 0.99 | 0.97 |
| 0.71....... | 0.79 | 0.77 | 0.83 | 0.80 | 0.86 | 0.83 | 0.89 | 0.86 | 0.94 | 0.90 | 0.96 | 0.95 | 0.98 | 0.96 |
| 0.49....... | 0.61 | 0.60 | 0.67 | 0.66 | 0.73 | 0.71 | 0.79 | 0.77 | 0.89 | 0.86 | 0.93 | 0.91 | 0.96 | 0.94 |
| 0.35....... | 0.36 | 0.38 | 0.44 | 0.45 | 0.53 | 0.53 | 0.62 | 0.61 | 0.79 | 0.77 | 0.86 | 0.84 | 0.92 | 0.89 |
| 0.225...... | 0 | 0 | 0.03 | 0.13 | 0.15 | 0.23 | 0.28 | 0.34 | 0.55 | 0.55 | 0.69 | 0.68 | 0.81 | 0.79 |
| 0.150...... | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.12 | 0.21 | 0.28 | 0.42 | 0.45 | 0.62 | 0.63 |
| 0.105...... | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.08 | 0.09 | 0.24 | 0.36 | 0.43 |

The following facts emerge on an examination of the figures shown in the table of results. For the thicker plates, the experimental values for $e$ are smaller by two or three per cent. than the theoretical values. This is evidently due to various minor causes of dissipation of energy not being taken into account in the theoretical treatment. For moderate thicknesses of plate, the calculated and observed coefficients of restitution agree well. Theory and experiment also agree in the case of very thin plates in giving a zero coefficient of restitution. In other words, in such cases, the sphere on impact with the plate remains in contact with it. But in certain intermediate cases, where the thickness of the plate is less than about half the diameter of the impinging sphere, but not so small as to give a zero coefficient of restitution, the observed values of $e$ are somewhat larger than the calculated values. These cases are shown enclosed in heavy lines in the columns of the table. It has already been remarked that in such cases, the assumption made that the duration of the impulse is given by Hertz's formula for impact with an infinite mass of solid would cease to be valid, and the discrepancy between the observed and calculated figures may possibly be due to this. An alternative explanation would be that in such cases, the configuration of the annular wave set up in the plate may slightly differ from that shown graphically in Fig. 1, and may approach more closely to that shown in Fig. 2($a$). It is obvious then that the formula (6) would have to be modified for such cases by slightly decreasing the numerical constant 0.56 appearing in it to 0.50 or 0.45 for the thinnest plates. As a matter of fact, if this is done, the discrepancy between the observed and calculated values of $e$ in these cases disappears.

Another noteworthy result indicated by formulas (2) and (6) and which is confirmed by some observations made by the writer is that the coefficient of restitution for impact on an elastic plate should depend on the velocity of impact, and should approach unity for very small values of this velocity. This is a consequence of the fact that the duration of impact as given by Hertz's formula varies inversely as the one fifth power of the velocity, and the radial distance $a$ covered by the annular wave diverging from the origin should therefore increase with decreasing velocity of impact. Fuller quantitative data showing the relation between the coefficient of restitution and the velocity of impact will be obtained and presented in due course in a further communication.

It may be remarked that much higher velocities of impact or larger spheres than those indicated in the table cannot be used, for the reason that the impact in such cases ceases to be of the conservative type, and results in internal fractures of a local character and of peculiar geometrical form in the glass plates. The character of these fractures bears an interesting relation to the distribution of stress in impact as given by Hertz's mathematical theory, and will be more fully discussed in a later paper.

The method used in the present paper can of course be applied also to the problem of transverse impact on an elastic bar, which is of a somewhat simpler character owing to the wave-propagation being in a single dimension instead of in two dimensions as in an elastic plate. The nature of the wave-motion set up by impact in this case is also capable of somewhat stricter analytical treatment if, following Boussinesq, we use Fourier's well known integral for the transverse vibration of an infinitely long bar to find the motion resulting from the initial impulse. In both cases also, it would be of interest to attempt a theoretical treatment (with experimental verification) of the manner in which the duration of impact varies with the thickness of the bar or plate, and also a direct experimental determination of the form of the wave at the instant at which the impact ceases.

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