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On the mechanical theory of the vibrations of bowed strings and of musical instruments of the violin family, with experimental verification of the results—Part I

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CONTENTS

Section I—Introduction

The vibrations of stretched strings excited by bowing and their practical application in musical instruments of the violin class present many important and fascinating problems to the mathematician and to the physicist. In the present monograph which is the first part of a more complete work on the whole subject, I propose to deal with the theory of the excitation of these beautiful and characteristic types of vibration under various conditions, and of their communication to the resonator on which the string is stretched. Experimental results in confirmation of those obtained from dynamical theory are also presented.

The problem which it is proposed to consider has formed the subject of investigation by many mathematicians and physicists. A list of the works and original papers that I have consulted is given in the bibliographical appendix. The present position of the subject cannot be considered satisfactory, in view of the fact that no complete and detailed dynamical theory has been put forward which could predict and elucidate the many complicated phenomena that have already been found empirically by those who have worked in the field and that could also point the way for further research. It was this defect in the present state of knowledge of the subject that induced me to undertake the investigation. Some preliminary work had already been carried out by me on the vibrations of bowed strings and the physics of bowed instruments. Reference may be made here to three papers which may be regarded as the starting points of this investigation[1]; the exposition given in the present paper is however self-contained. In the paper on "Discontinuous Wave-Motion" by myself and another that has appeared in the *Philos. Mag.* for January 1916, it has been shown how the well-known principal mode of vibration of a bowed string discovered by Helmholtz[2] can be reproduced experimentally as a free oscillation by imposing on a stretched string a certain simple distribution of initial velocities involving a discontinuity. This experiment which was first made in September 1914, suggested my undertaking a thorough investigation of the general problem. Free use has been made of a simplified form of the theory of discontinuous wave-motion given by Harnack, Davis and others, which I have extended so as to cover cases not considered by these writers.[3] The whole subject is considered in the light of dynamical theory, and an attempt has been made to divest it of empiricism as far as possible. Emphasis is laid upon the cases which are of practical interest in music. To make the present monograph as comprehensive as possible in respect of the matters dealt with, I shall develop the theory, step by step, in detail. A summary of the treatment and of the results obtained will be found in section XIV. Many illustrative diagrams, photographic curves and numerical results will be found in the paper. Not only does the theory succeed in explaining all the known phenomena, but it has also justified itself by predicting many new relations and results which have been tested experimentally. These are also referred to in the course of the paper and in the summary.

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[1] C V Raman, M.A., May 1911. "Photographs of Vibration-Curves," *Philos. Mag.*; C V Raman, M.A., 1914. "The Motion of Bowed Strings," *Bull. Indian Assoc. Cultiv. Sci.*; See also *Sci. Abstr.* February 1915, and *Nature (London)*, August 13, 1914, page 622; C V Raman, M.A., and S Appaswamaiyar January 1916 "On Discontinuous Wave-Motion," *Philos. Mag.*

[2] *Sensations of Tone*, English Translation by Ellis.

[3] Two preliminary notes on the subject have been published by me: "On Some New Methods in Kinematical Theory," *Bull. Calcutta Math. Soc.*, Vol. IV, pages 1–4; "On the summation of certain Fourier Series involving discontinuities," *Ibid.* V pp. 5–8.

Section II—Effect of periodic force applied at a point

It is clear that the motion of a bowed string is a case of maintained vibration, and an adequate treatment of the subject is only possible if the dissipative forces to which the string is subject are taken into account. The dissipation may be due, (*a*) to the direct communication of energy to the surrounding medium from the string, or (*b*) to motion set up in the supports between which the string is stretched. The forced oscillation of a string in the presence of dissipative forces of the first kind, is readily found on the assumption that each element of the string is resisted by a force proportional to its velocity. Lord Rayleigh and others have discussed the motion that would ensue under such conditions when a periodic force is impressed at one point on the string. In practice, however, it is known that the second source of dissipation is generally of much greater importance than the first. The energy of the vibrating string is conducted through the bridges over which it is stretched to the sides of the box on which the bridges are fixed, and ultimately to the atmosphere as sound-waves.

We shall now consider the motion of a string, one end of which is supposed to the rigidly fixed at the point $x = 0$ and the other end of which ($x = l$) passes over a bridge. A periodic force $E \cos mt$ is assumed to act at the point $x = x_0$. The string may be taken to be perfectly uniform and not subject to any resistance, so that the communication of energy to the surroundings takes place only through the bridge. The equation of motion of the string is

$$\mu \frac{d^2y}{dt^2} = T_0 \frac{d^2y}{dx^2}. \tag{1}$$

The solution of the equation for values of x between 0 and x_0 may be written as

$$y = F_1 \sin px \sin mt + G_1 \sin px \cos mt \tag{2}$$

where

$$\mu m^2 = T_0 p^2. \tag{3}$$

From $x = x_0$ up to $x = l$ we may write

$$\begin{aligned} y = {} & D_2 \cos p(l-x) \sin mt + E_2 \cos p(l-x) \cos mt \\ & + F_2 \sin p(l-x) \sin mt + G_2 \sin p(l-x) \cos mt. \end{aligned} \tag{4}$$

Since y must be continuous at the point x_0,

$$F_1 \sin px_0 = F_2 \sin p(l-x_0) + D_2 \cos p(l-x_0) \tag{5}$$

$$G_1 \sin px_0 = G_2 \sin p(l-x_0) + E_2 \cos p(l-x_0). \tag{6}$$

The discontinuous change in the value of dy/dx at the point of x_0 is due to the force $E\cos mt$. From this we get the two equations

$$F_1\cos px_0+F_2\cos p(l-x_0)-D_2\sin p(l-x_0)=0$$

$$G_1\cos px_0+G_2\cos p(l-x_0)-E_2\sin p(l-x_0)=\frac{E}{pT_0}. \tag{7,8}$$

At the point $x=l$,

$$y=D_2\sin mt+E_2\cos mt$$

and this motion at the bridge must be due to its yielding under the transverse periodic components of the tension. If the equation of motion of the bridge is

$$M\frac{d^2y}{dt^2}=-T_0\frac{dy}{dx}-f^2y-g^2\frac{dy}{dt}$$

where M is the mass of the bridge and associated parts, we obtain, by substitution, the equations

$$(f^2-Mm^2)D_2=T_0pF_2+g^2mE_2 \tag{9}$$

$$(f^2-Mm^2)E_2=T_0pG_2-g^2mD_2. \tag{10}$$

From the six equations numbered (5) to (10), the six unknowns F_1, F_2, G_1, G_2, D_2 and E_2 should obviously be capable of complete determination. Putting

$$\frac{T_0p}{f^2-Mm^2}=\tan\theta \quad\text{and}\quad \frac{g^2m}{f^2-Mm^2}=\tan\phi,$$

the equations may be solved by first eliminating D_2, E_2 and then F_1, G_1. Using for brevity the expression $\tan\psi=\tan\theta\cos^2\phi$ and $\delta=\tan\theta\sin\phi\cos\phi$, the eliminant equations obtained are

$$F_2\sin(pl+\psi)+G_2\delta\cos\psi\cos pl=0$$

$$F_2\delta\cos\psi\cos pl-G_2\sin(pl+\psi)+\frac{E\cos\psi\sin px_0}{pT_0}=0.$$

Solving these two equations, we obtain

$$F_2=\frac{-E\delta\cos^2\psi\cos pl\sin px_0}{pT_0[\sin^2(pl+\psi)+\delta^2\cos^2\psi\cos^2 pl]}$$

$$G_2=\frac{-E\cos\psi\sin(pl+\psi)\sin px_0}{pT_0[\sin^2(pl+\psi)+\delta^2\cos^2\psi\cos^2 pl]}.$$

If the impressed force $E\cos mt$ is regarded as an arbitrarily determined quantity, the interpretation of the preceding result is a simple matter, provided $\tan\phi$ (which involves the damping factor g^2) is regarded as very small. The second term in the denominators is then very small relatively to the first, and the maximum of

F_2 is obtained when the first term in the denominator is zero, i.e. when

$$(pl + \psi) = \nu\pi \text{ or}$$
$$pl = \nu\pi - \theta,$$

ψ being then practically equal to θ. G_2 is found to be zero when F_2 has its maximum value.

When $\sin px_0$ is zero, it is found from equations (5) to (10) that F_2, G_2, D_2, E_2 and F_1 are all equal to zero. The significance of this is that when the point of application of the force coincides with a node of the string for the particular frequency of oscillation, the whole of the string between the bridge and the point of application remains completely at rest. Only the portion of the string between the fixed end and the point of application has any movement, this being of very small amplitude, viz. $(E/pT_0)\sin px \cos mt$. It is thus seen that a periodic force of given magnitude produces an effect which is insignificant when it is applied at a node of the resulting oscillation, and which gradually increases as the point of application is removed further and further from the node. This result has many applications, as we shall see later on.

Generally speaking, the angle θ may also be taken to be very small, the quantity $(f^2 - Mm^2)$ being either positive or negative and large compared with $T_0 p$. We then find, as may have been expected, that the vibration set up by the periodic force is a maximum when its frequency is the same as that of the free vibrations of the string of length l with both ends rigidly fixed. But the case is otherwise when $(f^2 - Mm^2)$ is small, that is, when the free periods of vibration of the string and the bridge, taken separately, are nearly equal to one another. If the two periods are nearly equal to one another, the amplitude of the vibration of the string set up by the application of a periodic force of given magnitude and of frequency equal to that of its free oscillations is considerably smaller than if the natural periods of the string and of the bridge differed appreciably. To elicit the same amplitude of vibration, therefore, a comparatively much larger force would have to be applied when the frequency of the vibration is the same as that of the free period of the bridge and associated parts. This is the explanation of the difficulty noticed in bowing a string steadily when its pitch is that of the maximum resonance of the instrument. In section XII, we shall consider the special effects observable under these conditions when the pressure of the bow is insufficient to maintain a steady vibration, and also those produced by loading the bridge.

In dealing with the motion of bowed strings, we have to consider the effect, not of a simple harmonic force, but of a system of forces whose frequencies form a harmonic series acting over a finite region of the string which may, by courtesy, be styled the "bowed point." As the bridge over which the string passes, together with its associated masses, may have several free periods of vibration, it is obvious that the formulae connecting the various harmonic components of the periodic force brought into play by the bow, with the respective components of the resulting motion, would not, in general, be of a simple character. Fortunately, however, as will be shown in the course of the paper, it is possible to build up a theory, which successfully predicts the phenomena observable under a very wide variety of conditions. The only assumptions that need be made for the present are (1), that the string is uniform and of negligible stiffness, and (2), that the yielding at the bridge is negligibly small in comparison with the motion of the string at the bowed point, or at any other point actually chosen for observation. These assumptions, which may be approximated to, in practice, as closely as desired, greatly simplify the treatment. The main result of the preceding treatment that is utilized, is that the effect of any of the harmonic components of the impressed force depends upon the point at which it is applied, vanishing when it is applied at a node, and increasing gradually as it is removed further and further from it.

The assumptions of the uniformity and flexibility of the string are made to ensure the treatment being as far as possible rigorous. Except, however, in the case of very complicated types of vibration, these assumptions are not essential, provided the frequencies of the normal modes of vibration are not so far from forming a harmonic series as to prevent the bow eliciting all the members of the series which are of importance, as components of a strictly periodic forced oscillation. Owing to this restriction and the dispersion which the waves suffer in travelling on a non-uniform string, the treatment then requires modification, as will be referred to again, later on.

Section III—The *modus operandi* of the bow

The function of the bow as normally applied is both to elicit and to maintain the vibrations of the string. The two processes are interdependent, but it is well to remember that they should not be confused with each other, inasmuch as it may well happen that the character of the motion in its initial stages is not necessarily the same or even analogous to that maintained in the final steady state. For the present, however, we need not enter into these intricacies, but may simply assume that the motion is maintained in some perfectly periodic manner by the action of the bow, and proceed to find its character. It is obvious that on the assumptions set forth in the preceding section the period of the maintained oscillation is the same as that of the free vibrations of the string.

In a well-known paper on "Maintained Vibrations" (*Philos. Mag.*, 1883) reproduced in his *Theory of Sound*, Vol. I, page 81, Lord Rayleigh has discussed the general theory of vibrations elicited by generators and has shown that the supply of energy to the system through the action of the generator in any given time, may sometimes actually exceed that lost by the system in the same time through dissipative forces. When this happens, the excited vibrations continue to increase indefinitely in amplitude, until some physical limit is reached beyond which the equations of motion originally assumed cease to apply. The motion of a bowed string is evidently a case of this kind, the physical limit beyond which the vibrations cannot increase being imposed by the finiteness of the velocity of the bow. It has been suggested (with more or less definiteness) by the previous writers on the subject, that the bowed point on the string does attain or nearly attain the velocity of the bow in its movement in one direction. As will be seen presently, the question whether the forward velocity of the bowed point is *absolutely* the same as that of the bow is one of fundamental importance in the theory of the subject, and in one of my own previous publications,[4] I have shown how this identity of velocities can be brought experimentally to a test. What I propose to do here is to discuss the dynamical principles underlying the case in some detail, and then to pursue the argument to its logical conclusions.

The magnitude of the frictional force due to the bow at any instant must obviously depend upon the pressure with which it is applied and upon the relative velocity at the point of contact. It is also clear that this relative velocity cannot oridinarily change sign during the motion, for, if it did, the entire frictional force would also change sign and the excess velocity of the bowed point would be immediately damped out. (An excellent illustration of this principle may be had by bowing a fork vigorously and then suddenly reducing the velocity of the bow. It will then be found that the amplitude of vibration of the fork also falls with practically equal suddenness.) With an efficient generator, e.g. a bow with rosined horse hair acting on the string, the frictional force exerted would be much greater when the relative-velocity is nearly but not quite zero, than when it is large. On the other hand, when the relative velocity is actually zero, the friction ceases to be a determinate function of the relative velocity. From these premises, it is clear that, when the bow is applied with sufficient pressure and not too great a velocity, the maintaining forces brought into play would be far in excess of those required to maintain the vibration of the string (the mass and damping of the latter both being small), so long as the relative velocity at the point of contact does not actually become zero during any part of the vibration. On the other hand, we know that a steady state of vibration is only possible when the energy gains and losses balance each other, i.e. when the harmonic components of the force exerted by the bow are just sufficient to maintain the motion. The only possible inference that can be drawn under the circumstances is that the bowed point does actually attain the velocity of the bow during part of its motion and ultimately throughout the fractional part or parts of the period of vibration during which it has a forward movement. During these stages, the bow merely carries forward the point of the string with which it is in contact, and it is important to notice (accordingly to the preceding argument) that the frictional force then acting on the bowed point would actually fall below the maximum statical value; by how much it would fall below this maximum, would depend on the circumstances of the case,

[4]See *Sci. Abstr.* (*Physics*), February, 1915, p. 87 (C V Raman).

viz. the magnitude of the friction during the other stages of the motion as determined by the relative velocity, and the magnitude of the forces required to sustain the motion.

When the bowed point has the velocity of the bow in all the stages of forward movement, there is necessarily a discontinuous change of velocity[5] when it starts moving backward. The preceding argument may be pressed a little further if we assume that the forces required to maintain the motion are very small compared with the variation of frictional force due to a finite change of relative velocity. (Such an assumption would, in general, be justifiable if the pressure of bowing were sufficient and the damping coefficient were sufficiently small.) It would then follow that the frictional force exerted by the bow is practically constant throughout the whole motion, and that during all the intervals in which the relative value is not zero, it has a finite constant value which is the same for all such intervals. The relative velocity changes from this value to zero and *vice versa* in a discontinuous manner.

From a consideration of dynamical principles and the relative order of magnitudes of the quantities involved, we thus arrive at the following two results: (*a*) during one or more intervals in each period of vibration, the bowed point has a forward movement which is executed with constant velocity exactly equal to that of the bow; (*b*) during the other interval or intervals, the bowed point moves backwards, also with constant velocity, this being the same for all such intervals (if there be more than one). The preliminary treatment of the vibrational modes given in the succeeding sections is mainly founded on these two results. It must be observed, however, that as the argument by which the second result was deduced, is rigorous only in the limiting case of a vanishingly small damping coefficient, this particular result, viz. the constancy of velocity in the backward movement, cannot be regarded as holding good with the same completeness and generality as the first result, i.e. the constancy of velocity in the forward movement. We are thus led by the argument to anticipate the existence of cases in which the velocity of the bowed point varies in a continuous manner, particularly in the stages in which the movement is in a direction opposite to that of the bow. This is a feature which becomes of great importance in certain cases, specially in those of musical interest, and which therefore requires to be emphasised. For the present, however, it is advantageous to consider the constancy of the velocities of the bowed point as holding good rigorously, both in the forward and backward movements. This assumption serves admirably as the basis on which the kinematical theory of the various possible modes of vibration may be discussed. We shall accordingly proceed on this basis.

We may call the two velocities possible at the bowed point, v_B and v_A respectively, v_B being the velocity of the bow, and v_A the velocity of the bowed point when it travels against the bow. The intervals of time T_1, T_3, T_5 etc. in each

[5]Equal to the velocity of the bow plus the initial speed of backward movement.

period of vibration during which the velocity is v_A and the intervals T_2, T_4, T_6 etc. in which it is v_B are obviously connected by the equations,

$$T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + + = T \text{ (the complete period of vibration).}$$

$$(T_1 + T_3 + T_5 +)v_A + (T_2 + T_4 + T_6 + +)v_B = 0.$$

As already remarked, the argument shows that in the presence of dissipative forces, the constancy of velocity in the intervals of backward movement is not by any means so generally assured as the intervals of forward movement, and a steady state of motion in which the total of the time intervals of movement in the direction opposite to that of the bow, exceeds that of the intervals of movement with the bow, is altogether out of the question. The only cases, therefore, whose kinematics need be considered in detail are those in which v_A is numerically not less than v_B.

Section IV—Simplified kinematical theory

From the general results indicating the nature of the motion at the bowed point obtained in the preceding section, we now proceed to build up a detailed kinematical theory of the motion of the bowed string. For this purpose the ordinary Fourier analysis is unsuitable, as it is neither convenient nor suggestive. I have therefore devised a simple graphical treatment which is based upon the use of the velocity-diagram of the string and appears admirably adapted for the present investigation.

The general solution of the equation of wave propagation on an infinite string not subject to damping is,

$$y = f_1(x - at) + f_2(x + at). \tag{11}$$

It is well known that this solution for the case of an infinite string can be used to represent the configuration at any instant of a vibrating string of finite length, by arranging the form of the displacement waves in such manner that the motion is periodic and satisfies the terminal condition $y = 0$ at the two ends of the string.

Similarly, the solution obtained by differentiating (1) with respect to time, viz.

$$\frac{dy}{dt} = -af_1'(x - at) + af_2'(x + at) \tag{12}$$

can be applied to represent the velocity diagram of a finite string at any instant during its vibration, if the periodicity of the motion and the terminal conditions of velocity are secured. It is obvious that solution (12) as it stands, represents the velocity waves that travel on an infinite string without change of form in the positive and negative directions respectively. In the case of a finite string of length l, the reflexions that take place at the two ends have to be taken into account and we may write

$$\frac{dy}{dt} = \theta(x - at) + \phi(x + at). \tag{13}$$

The functions $\theta(x - at)$ and $\theta(x + at)$ represent the positive and negative velocity waves which must be imagined as extending to infinity in both directions. Further they must each of them be periodic with wavelength equal to twice the length of the string, and must be so related that at the two ends of the string $x = 0$ and $x = l$, the terminal condition $dy/dt = 0$ is always satisfied. This can be secured by arranging the form of the velocity waves in much the same way as the displacement waves would be arranged to secure the terminal condition $y = 0$, i.e. the form of the positive velocity wave from $x = 0$ up to $x = l$ in its initial position is an inverted and reflected image of the negative wave from $x = l$ up to $x = 2l$, and *vice versa.*

The cases in which the positive and negative velocity waves are completely identical in their initial positions, present features of special interest. Half the initial velocity at each point on the string is then due to the positive wave, and the other half to the negative wave, and there are no initial displacements, i.e. the string is everywhere in its position of statical equilibrium. After the expiry of half a period, i.e. when the positive and negative waves have each moved through a distance equal to the length of the string, the latter is again everywhere in its position of equilibrium. This is so because one half of each wave is merely the inverted and reflected image of the other half, and the displacements resulting from the initial velocities are annulled during the same half period. During the second half period the velocity at every point on the string goes back again to its original value through exactly the same stages; in other words, the velocity is everywhere an even periodic function of the time which when plotted gives a figure with the symmetry characteristic of such functions. It is thus seen that the positive and negative velocity waves are necessarily identical in any case in which the changes of velocity at points on the string take place in a symmetrical manner with respect to time.

We now proceed to consider cases in which the velocity changes that take place at some particular point on the string, say the point $x = x_b$, can be assumed to have a specified form. Then the form of the velocity waves $\theta(x - at)$ and $\phi(x + at)$ must be such that by their movement and superposition, the known changes of velocity at the point x_b are reproduced. For example, let us assume that the string at the point x_b moves during the vibration with a succession of constant velocities, the velocity passing in a discontinuous manner from each value to the next. Then at the point x_b, d^2y/dt^2 is always zero, except at certain instants in each period of vibration when it becomes $\pm$ infinity.

Differentiating (13) with respect to time, we have

$$\frac{d^2y}{dt^2} = -a\theta'(x - at) + a\phi'(x + at). \tag{14}$$

Since at the point x_b, d^2y/dt^2 is generally zero, we must have

$$\theta'(x_b - at) = \phi'(x_b + at). \tag{15}$$

If the velocity-waves $\theta(x - at)$ and $\phi(x + at)$ are represented graphically, equation (15) may be given a geometrical significance; if any two points are taken, one on the positive wave and the other on the negative wave, the distances of which from the point x_b measured along the string are equal but in opposite directions, we should find the *slopes* of the waves at these two points to be equal. We have already seen that the positive and negative waves must satisfy certain other conditions, viz. that they are periodic with wavelength $2l$, and that, initially the form of the positive wave from $x = 0$ up to $x = l$ is the inverted and reflected image of the negative wave from $x = l$ up to $x = 2l$ and *vice versa*. It is a definite geometrical problem to find the configuration of the waves which would simultaneously satisfy these three conditions. By inspection, we get the following remarkably simple and significant solution: if the point x_b divides the string in an *irrational* ratio, the only possible form of the velocity-waves is that in which the slope is everywhere the same, in other words, they are representable by a number of straight-lines which are all parallel to one another, a discontinuity intervening wherever one straight-line leaves off and the next begins. Velocity-waves having this form also satisfy the geometrical criteria when the point x_b divides the string in a rational ratio (i.e. in the ratio of two whole numbers), but in the latter case, this is not the only form of velocity-waves *geometrically* possible. This result is not a matter for surprise, for the point x_b would then coincide with a node of one of the harmonics of the string, and the ordinary Fourier analysis of the kinematics of the vibration shows that the motion of the string as a whole is not fully determinate, even though the motion at one nodal point on the string is fully ascertained.

It is noteworthy, however, that the result stated above was obtained solely from geometrical considerations without any reference to the methods of harmonic analysis.

The utility of the preceding discussion is obvious. For, we have seen in section III that at the bowed point, generally speaking, the velocity alternates between two and only two constant values, once or oftener in each period of vibration. The condition $d^2y/dt^2 = 0$ is thus generally satisfied at the bowed point, except at the instants at which the velocity changes from one value to the other and *vice versa*. At these instants, d^2y/dt^2 becomes $\pm$ infinity. The preceding arguments are thus applicable, and it follows that when the bow is applied at some point dividing the string in a *irrational* ratio, the form of the velocity-waves is that of a number of straight-lines parallel to one another, with intervening discontinuities. It can now be seen that this is the case even when the bow is applied at a point dividing the string in any *rational* ratio, i.e. at some node on the string. For, the kinematical uncertainty in the latter case is due only to the harmonic components in the motion which have a node at the point of application of the bow, and we have established from dynamical principles that such harmonics are not excited by the bow and do not therefore exist in the motion under consideration. The quantities that determine the motion at the bowed point must therefore also determine the motion at every other point on the string whose position is known. These quantities, in the case of the bowed point, are its initial velocity and the magnitudes and positions of the discontinuities in the velocity-waves. For, the slopes of the positive and negative velocity-waves passing over the bowed point in opposite directions being equal, the velocity at that point remains unaffected except when a discontinuity passes over it, the velocity then suddenly changing by a quantity equal to the magnitude of the discontinuity: the *times* at which these changes occur are determined by the initial *positions* of the discontinuities and *vice versa.* As stated above, these quantities must also completely determine the motion at all other points on the string, and this is only possible when, between the points of discontinuity, the velocity-waves consist of straight-lines that are all parallel to one another.

It is thus seen that the problem of finding the mode of vibration of the string under the action of the bow reduces itself to one of finding the number, position and magnitudes of the discontinuities in the velocity-waves. From the mode of construction of the positive and the negative waves, it is obvious that the number of discontinuities in a wavelength of either of the two waves is the same, and is equal to the *total* number of discontinuities actually on the region of the string at any instant during the vibration. When a discontinuity travelling with the positive wave reaches the end of the string, it is reflected and returns as a discontinuity in the negative wave; moving on towards the other end, it reaches it and is again reflected and brought on to the positive wave. This process then repeats itself indefinitely. In section III it was shown that the velocity at the bowed point is alternately v_A and v_B, changing discontinuously from one value to the other, and *vice versa.* The poisitions and magnitudes of the discontinuities in the velocity-waves must be such that by their passage over the bowed point, the specified changes of velocity at that point are produced. The simplest case possible is that in which the discontinuities pass in succession over the bowed point, belonging alternately to the positive and negative waves, i.e. pass alternately over the bowed point in opposite directions. It is obvious that the discontinuities must then be all of the same magnitude and sign, i.e. $v_A - v_B$. In other words, the discontinuities are all equal in magnitude to one another and to the arithmetical sum of the two speeds possible at the bowed point, i.e. to the relative velocity of the bowed point with respect to the bow during the backward movement. Other cases that may possibly arise are the following: (*a*) two dicontinuities of the same magnitude and sign may pass simultaneously over the bowed point in opposite directions. (*b*) two discontinuities differing in magnitude or sign or both may simultaneously pass over the bowed point in opposite directions. (*c*) two or more discontinuities may pass over the bowed point in succession in the same direction instead of alternately in opposite directions. The contingency in (*c*) does not however actually arise, as it is impossible to construct positive and negative waves which would give rise to it and which would at the same time satisfy the condition that the velocity at the bowed point should alternate between two values only. Further, it is found that if the bow is applied at a point dividing the string in an irrational ratio, the contingency in (*b*) is also impossible, and the discontinuities in the velocity-waves are necessarily all equal to one another and to $v_A - v_B$. This result is of great importance in the theory of the subject.

The reason why the discontinuities are all equal to $v_A - v_B$ if the bowed point divides the string in an irrational ratio, is not very difficult to see. The result has already been demonstrated for cases in which the discontinuities pass in succession over the bowed point and never simultaneously. If two equal discontinuities pass over the point in opposite directions at the same instant, the velocity of the point is left unaffected. Further, if two discontinuities thus cross at the bowed point, they cannot again pass simultaneously over the bowed point when returning after one reflexion at the ends of the string (the distance of the bowed point from the two ends being unequal). On the return journey, the discontinuities must therefore pass the bowed point either separately or else simultaneously with certain other discontinuities. In the former case their magnitudes are necessarily equal to $v_A - v_B$. In the latter case also, a precisely similar result holds good, except when all the discontinuities of a given set pass over the bowed point in twos and twos and never singly. From very simple geometrical considerations it can be shown that the discontinuities would so all pass in twos and twos only if they were situated at regular intervals equal to an aliquot part of the wavelength, and the bow were itself applied at a point of division of the string into an equal number of aliquot parts, i.e. at a point or node dividing the string in a rational ratio.[6] We thus arrive at the following two general results regarding the form of the velocity-waves: (1) When the bow is applied at a point dividing the string in an irrational ratio, the velocity-waves consist of straight lines that are all parallel to one other with intervening discontinuities all

[6]The following simple model serves very effectively to picture the movement and successive reflexions of the discontinuities in the velocity-waves. Consider the motion of an endless cord which runs on two parallel axes between which it is stretched straight. A number of particles fixed to the cord at intervals may represent the discontinuities. If there are N particles fixed at equal intervals along the cord, the particles moving towards one axis would pass those moving the other way, at points dividing the distance between the axes into N equal parts. No particle would ever pass these points singly, i.e. by itself. A similar result would not be possible if the particles were fixed to the cord at unequal intervals or if any other point of observation were chosen. This model may be used for a lecture demonstration of the results given in sections VII to X with reference to types of vibration in which there are two, three or any larger number of discontinuities.

equal to $v_A - v_B$, and the number of such discontinuities per wavelength is the same in the positive and negative waves. (2) When the bow is applied at a point dividing the string in a rational ratio, the velocity-waves also consist of parallel straight lines with intervening discontinuities, the number of which is the same per wavelength in the positive and negative waves; the magnitude of the discontinuities is not however the same as in (1). The argument shows that in this case, the non-appearance of the harmonics having a node at the bowed point results in a number of discontinuities being present at regular intervals equal to an aliquot part of the wavelength in the positive and negative waves, viz. at intervals of $2l/s$ when the bow is applied at one of the points of division of the string into s aliquot parts.[7] The discontinuities pass in pairs (never singly) in opposite directions over the bowed point and also over the other points of rational division of the string.

From the foregoing it is seen that in any case in which the bow is applied at a point of rational division of the string, the form of the velocity waves can be derived by a very simple geometrical construction from velocity-waves of the irrational type, i.e. those in which the discontinuities are all equal to $v_A - v_B$; the construction is equivalent to the abolition or removal of those harmonics which have a node at the bowed point and leaves the resulting motion at the bowed point and at the other points of rational division unaffected. Examples of the method will be dealt with later. Its usefulness is evident from the fact that all the possible types of vibration may thus be considered as special cases of what may be termed the irrational types of vibration, the theory of which can be worked out geometrically with the greatest ease and simplicity and which we shall now proceed to discuss.

Section V—Classification of the vibrational modes

From the results given in the preceding section it is obvious that the vibrational modes in the cases in which the bow is applied at a point of irrational division of the string can be very simply classified according to the total number of discontinuities in the velocity waves. If there is one discontinuity, we may call it the first type of vibration of a bowed string. If there are two discontinuities, it may be called the second type of vibration, and so on. Generally speaking, each of these types of vibrations includes the complete series of harmonics.

We may now proceed to deduce a few results of general application, examples which will be met with in the detailed graphical discussion of individual cases to be given later in the course of the paper.

[7] s is taken to be the smallest possible number of aliquot parts.

Since the positive and negative velocity waves are representable by parallel lines separated by equal intervening discontinuities, the points at which the x-axis is cut by the parallel lines (or would be cut by them if produced) must be equidistant from one another. If there are n discontinuities per wavelength, the intercepts on the x-axis are evidently $2l/n$. If $n = 1$, the intercepts are equal to $2l$, i.e. to the wavelength, and it is obvious that in this case the positive and negative waves are necessarily of the same form (symmetrical about the x-axis) and are completely coincident twice in each period of vibration.

If the inclination of the lines to the x-axis is $\tan^{-1} c$ and there are n discontinuities per wavelength each equal to $(v_A - v_B)$, we have

$$2cl = n(v_A - v_B). \tag{16}$$

By summation of the ordinates of the positive and negative waves, the velocities at all points on the string can be ascertained and represented graphically. The velocity graph thus obtained for the string must evidently consist of parallel straight lines inclined to the x-axis at an angle $\tan^{-1} 2c$, the maximum number of such straight lines being $(n + 1)$, (there being n discontinuities on the string, some of which might be instantaneously coincident in position). Further, these $(n + 1)$ lines on the velocity diagram pass through fixed points on the x-axis situated at equal intervals. Since the two ends of the string have always zero velocity, these fixed points are in fact the $(n + 1)$ nodes of the n^{th} harmonic, and we thus obtain the result that the lines of the velocity diagram pass, or would pass if produced through some or all of the $(n + 1)$ nodes of the n^{th} harmonic, if the particular type of vibration elicited by the bow is that in which there are n equal discontinuities.

From the preceding result, we may very readily deduce an expression for the ratio v_A/v_B of the two velocities possible at the bowed point, which besides being of perfectly general application is also valid for all points on the string, the velocity at which alternates between two constant values only, once or oftener in each period of vibration. Assume first that the bow is applied at a point of irrational division of the string and the mode of vibration elicited is that in which there are n equal discontinuities. Consider the motion at a point on the string lying between the r^{th} and $(r + 1)^{th}$ nodes of the n^{th} harmonic (counting from one end) and whose distance x from that end of the string is therefore greater than $(r - 1)l/n$ and less than rl/n. The velocity of this point on the string at any instant during the vibration is given by the ordinate of the velocity diagram. As we have just seen, this velocity-diagram consists of parallel lines drawn through the successive nodes of the n^{th} harmonic at an inclination of $\tan^{-1} 2c$ to the x-axis, with intervening discontinuities. If the velocity at the particular point on the string alternates between two and only two constant values, it must be because the ordinate drawn through it intersects alternately the two lines of the velocity diagram passing through the two nodes of the n^{th} harmonic on either side of it, as a result of the movement of the discontinuities. In other words, the two velocities at the point considered are $2c(x - \overline{r-1}\, l/n)$ and $2c(x - r\, l/n)$. The ratio of these velocities is merely the ratio of the distances of the point from the two nodes, and if (for brevity) the symbol x_n is used to denote the shorter of the two distances, and ω is used to denote the total fraction of the period of vibration in which the point moves with the larger of the two velocities, we have

$$\omega = \frac{nx_n}{l}. \tag{17}$$

The algebraic difference of the two velocities at the point is $2cl/n$ and this is equal to $(v_A - v_B)$, vide equation (16). Since the result given in (17) is true for all points on the string at which the velocity alternates between two constant values once or oftener in each period of vibration, it applies also at the bowed point, x_n denoting its distance from the nearest node of the n^{th} harmonic.

The result given in (17) above is noteworthy by reason of its simplicity and perfect generality as also by reason of the simplicity and perfect generality of the reasoning from geometrical considerations by which it was deduced. The result is equally applicable in cases in which the motion at the bowed point is of the simplest possible type (one ascent followed by a descent) as well as those in which the motion is one of the so-called complicated types, a succession of several ascents and descents within the period of vibration. In deducing the result, it has been assumed that the vibration is elicited by applying the bow at a point or irrational division of the string, so that the type of motion maintained is one in which the discontinuities present in the velocity waves are all equal to $(v_A - v_B)$. Even this restriction may be removed, i.e. we may also include the cases in which the bow is applied at a point of rational division of the string, the only difference being that the result given in (17) would then be applicable only at the bowed point and at some or all of the other nodes of the principal member of the missing series of harmonics, and not at any other point on the string. For, as already referred to in the preceding section, any type of vibration elicited by applying the bow at a nodal point on the string can be considered as one of the modes of vibration of the 'irrational' type with the series of harmonics having a node at the bowed point dropped out. The process leaves the motion at the bowed point and at the other nodes of the principal member of the missing series of harmonics, unaffected.

When $n = 1$, equation (17) reduces to the well known relation discovered by Helmholtz, i.e. the ratio of the velocities of ascent and descent at the point considered is the same as the ratio of its distances from the two ends of the string. Krigar-Menzel and Raps found in their experimental work that Helmholtz's relation was satisfied when the bow was applied in the normal manner at any point very close to an end of the string or else very exactly at one of the nodal points distant $l/2$, $l/3$, $l/4$, $l/5$, $l/6$ or $l/7$ from the end. The value of ω for other points of application of the bow was also measured by Krigar-Menzel and Raps, and they state as the result of these measurements that no general algebraic relation connecting the value of ω at the bowed point with its position on the string could be found even when the motion at the bowed point was of the simplest possible type representable by a two-step zig-zag. Their deliberate conclusion on this question was that, except when the bow was applied close to the end of the string or at one of the nodes of some fairly important harmonic, the value of ω was to be regarded as a purely empirical quantity depending on the experimental conditions. It is obvious that if the value of ω is thus regarded as an arbitrary quantity determinable by experiment, no complete theoretical discussion of the kinematics of the string is possible, and in fact Krigar-Menzel and Raps did not attempt any such complete discussion. While on the experimental side their paper was a notable contribution to the subject, the treatment given by them on the theoretical side was thus obviously defective and incomplete. The general kinematical analysis set out in the present paper shows that the value of ω in all cases (i.e. both for rational and irrational points of bowing) should satisfy the relation given in equation (17), n being given by the appropriate integral value, 1, 2, 3, 4 or 5, etc. The failure of Krigar-Menzel and Raps to discover this general algebraic relation, or rather this series of relations connecting the value of ω at the bowed point with its position on the string, must be attributed to their having adopted an almost exclusively empirical method of treatment. If, instead of relying solely on the result of the measurements which were necessarily subject to experimental error in some degree, they had investigated in detail the kinematics of some of the simpler types of vibration other than those known through the work of Helmholtz, e.g. that obtained by applying the bow at a point close to but not coincident with the centre of the string, the functional relation connecting the value of ω at the bowed point with its position on the string could have been looked for with a greater chance of success. That such a functional relation exists must indeed have been evident from the fact that the characteristic vibration-curves in such cases also are perfectly reproducible, time after time, with strings of any length, diameter or material.

The failure to establish a proper scheme of classification of the vibrational modes and to find the general form of the functional relation connecting ω at the bowed point with its position anywhere on the string was also one of the fundamental defects in the paper by A. Stephenson cited in the bibliography. In this paper (published in 1911) only the work of Helmholtz is referred to, and a perusal of it shows that Stephenson was unacquainted with the work of Krigar-Menzel and Raps published in 1891, and that he was, indeed, unaware of many facts which anyone who has experimented with a bow and monochord could readily observe for himself. It is not a matter for surprise therefore that, though Stephenson's paper is noteworthy as an attempt to treat the motion of a bowed string as a case of maintained vibration, it takes us little beyond the work of Helmholtz. Stephenson also failed to realise that the Fourier analysis is obviously incapable of giving any useful indication of what would happen if the bow is applied at a point of irrational division of the string, i.e. at a point not exactly coinciding with any nodal point of importance, and it is precisely such indication that is required to explain the phenomena observed in experiment.

We may now pass on to consider the kinematics of the irrational types of vibration more in detail. If there are n equal discontinuities, the velocity-diagram of the string consists of not more than $(n + 1)$ parallel lines passing through the $(n + 1)$ nodes of the n^{th} harmonic. As the discontinuities move one way or the other, the lines of the velocity-diagram increase in length or else shorten and sometimes vanish altogether, and given the form of the velocity-diagram at any epoch of the vibration, it is quite an easy matter to find its form at any subsequent epoch or to trace directly the succession of velocity-changes at any point on the string and thus to determine the form of the vibration curves. If the positive and negative velocity-waves are of identical form, it is obviously convenient to commence with the epoch at which they are completely coincident and the string everywhere passes through its position of statical equilibrium. At that epoch, the discontinuities are everywhere situate in pairs along the string, the odd discontinuity, if any being at one end of the string, and the lines of the velocity-diagram pass through the *alternate* nodes of the n^{th} harmonic. In the subsequent motion, the discontinuities situate along the string separate and move off in opposite directions, the odd discontinuity, if any, situate at the end of the string moving straight off towards the other end. After half a period, the positive and negative waves again coincide and the velocity-changes at every point on the string are gone through in the reverse order, as already described in section II of the paper. It should be remarked that the positive and negative waves are necessarily of the same form when the motion at any one point on the string is representable by a simple two step zig-zag, or by any other curve possessing a similar type of symmetry. As normally applied, the bow excites the vibrations of the string from an initial state in which the latter is everywhere in its position of equilibrium. The tendency is thus, in a large majority of cases, to set up vibrations having this characteristic type of symmetry.

Section VI—The first type of vibration

Of the possible types of vibration set up by the application of the bow at a point of irrational division, the first type with only one discontinuity on the velocity-diagram is the simplest and most important. In this case, as already remarked, the positive and negative velocity-waves are necessarily of the same form, and at the instant at which they are coincident, the velocity-diagram is a straight line passing through one end of the string ($x = 0$), with a discontinuity at the other end ($x = l$). As this discontinuity moves in along the string, the velocity-diagram consists of parallel lines passing through its two ends, and the velocities at any point before and after its passage are respectively proportional to the distances from the two ends. When the discontinuity reaches the end $x = l$, it is reflected and the velocity-diagram then passes back through the same stages to its original form.

Figure 1 (first column) shows the successive velocity-diagrams at intervals of one-twelfth of an oscillation.

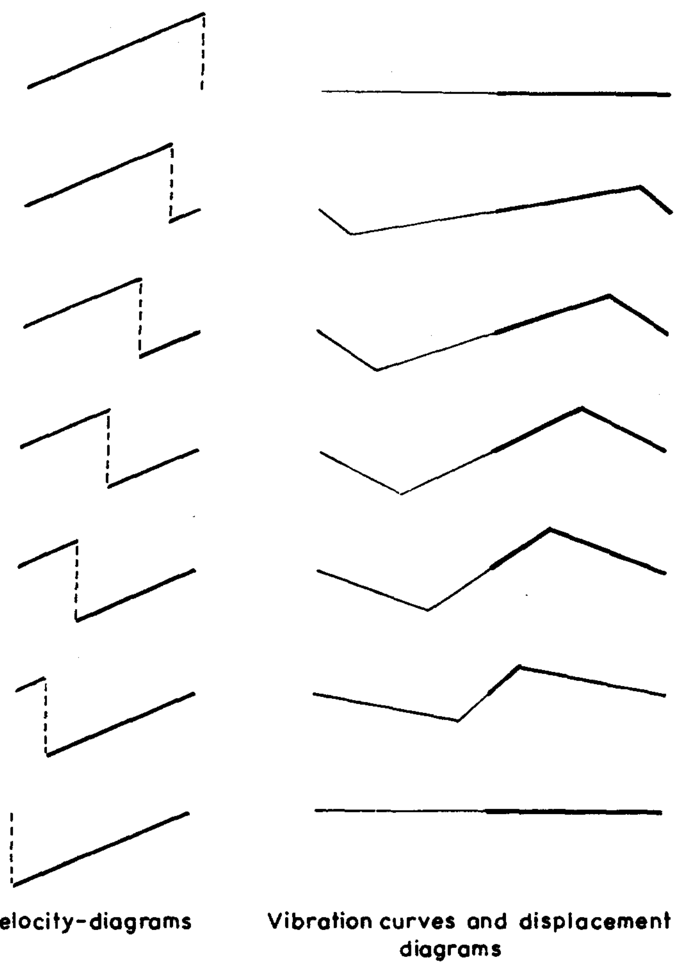

Figure 1. First type of vibration.

Figure 1 (second column with the heavy and thin lines taken separately) also shows the displacements of the string from its initial position at similar equal intervals throughout the complete period of vibration. These configuration-diagrams, as we may call them, evidently consist of two straight lines passing through the ends of the string and meeting at the point up to which the discontinuity in the velocity-diagram has travelled at any instant. (It should be remarked here that the displacements are always measured from the position of equilibrium of the string under the steady frictional force exerted by the bow, and not from the position of equilibrium attained when the bow is removed.) The second column of figure 1, with the heavy and thin lines taken together represents on the same scale of ordinates, the vibration curves for the complete period, of points situated on the string at successive equal intervals of one-sixth of its length, commencing from one end. When extended on either side for a number of complete periods, the vibration-curves are seen to consist of simple two-step zig-zags, the fraction ω of the complete period during which any point moves with the larger of the two velocities being given by the relation

$$\omega = x_1/l, \tag{18}$$

where x is the distance of the point from the nearer of the two ends of the string.

The correspondence noticed above between the configuration of the string as a whole and the vibration-curves of individual points on it is not peculiar to the present case, but may be established with generality for any possible type of vibration of a stretched string in which the positive and negative velocity-waves are of the same form. A geometrical proof is very readily given by noticing that the displacement at any point is the time-integral of the velocity and is therefore representable by the area enclosed by two ordinates drawn at equal distances on either side of the point under consideration, to intersect the velocity-wave. The following is an analytical proof. In such cases we have

$$y = \sum_1^{\infty} a_n \sin\frac{n\pi x}{l} \sin\frac{2\pi nt}{T}. \tag{19}$$

If x is regarded as constant and t as a variable, equation (19) represents the form of the vibration-curves. On the other hand, if t is regarded as constant and x as the variable, the equation gives us the configuration of the string. By taking only half the complete period into consideration, i.e. from $t = 0$ up to $t = T/2$ or from $t = T/2$ up to $t = T$, we can get identical geometrical representations for the motion at individual points on the string, and for the configuration of the string as a whole, provided that the times for the latter and the positions for the former are so chosen that $2t/T = x/l$.

Two other important consequences of equation (19) may also be noticed here. If two points are taken on the string, one on either side of the centre at equal distances from it, the form of their vibration-curves are the mirror-images of one another with respect to the centre of the string. The second consequence is that the vibration-curve at a point very close to the end of the string from $t = 0$ to $t = T/2$ or from $t = T/2$ up to $t = T$ is, in the limit, of the same form as the *velocity-diagram* of the string at time $t = 0$ or $t = T/2$ as the case may be. This may be regarded as a particular case of the correspondence of form noticed in the preceding para obtained by putting the chosen values of t and x very small. For, when t is very small but not actually zero, the small displacement at any point is proportional to the initial velocity at the point.

Section VII—The second type of vibration

We now pass on to consider the case in which there are two discontinuities in the velocity-diagram.

At some epoch or other of the vibration, the two discontinuities must necessarily coincide, and we thus see that in the second type of vibration also, the positive and negative velocity-waves are necessarily of the same form. The particular point on the string at which the discontinuities cross, remains, however, at our disposal. If this point is the centre of the string, the discontinuities would again cross at that point after half a period, and it is obvious that the string would vibrate in two segments, the frequency of vibration being twice that of the fundamental, and the vibration-curves would everywhere be simple two-step zig-zags. If, however, the discontinuities cross at a point distant $l/2 + b$ from one end, their second crossing after the expiry of a half-period would be at a point distant $l/2 - b$ from the same end, and the frequency of the vibration would be that of the fundamental. Figure 2, first column, shows the velocity-diagrams for this case. Initially, the velocity-diagram consists of two parallel lines passing through the two ends of the string and separated by the two coincident discontinuities. When these move off in opposite directions, the third line that forms on the velocity-diagram and extends both ways, passes through the centre of the string. During the greater part of each half-period, therefore, the centre of the string remains at rest,[8] displaced from the position of equilibrium first to one side and then to the other, the movement from one position to the other being executed with great velocity (equal to $v_A - v_B$). The two stationary positions of the centre of the string should therefore appear very brillianty visible on a dark ground. The vibration-curves of any desired points on the string can be very readily set down by inspection of the velocity-diagrams or otherwise. They are shown in the second column of figure 2. At the centre, the motion is of the type already described.[9] The

[8]This phenomenon of which, it is believed, the simplest explanation is here given, and the analogous appearances at the respective nodes in the case of the third, fourth and higher types of vibrations were observed and figured so long ago as 1800 by Dr Thomas Young in the *Philos. Trans.* for that year. Young obtained them by applying the bow at points close to but not coincident with the points of aliquot division of the string. He appears to have fully understood the fact that the types of vibration thus set up were totally different from those obtained by applying the bow at the points of aliquot division, and not merely modifications thereof. His explanation of the difference in terms of the impulses which he assumed the bow to send out and of their interference at the bowed point, was remarkably near the truth.

[9]Three typical vibration-curves of the type here described appear among the photographs published by Krigar–Menzel and Raps, but these authors failed to observe the essential kinematical relation given in (20) necessary to connect them together.

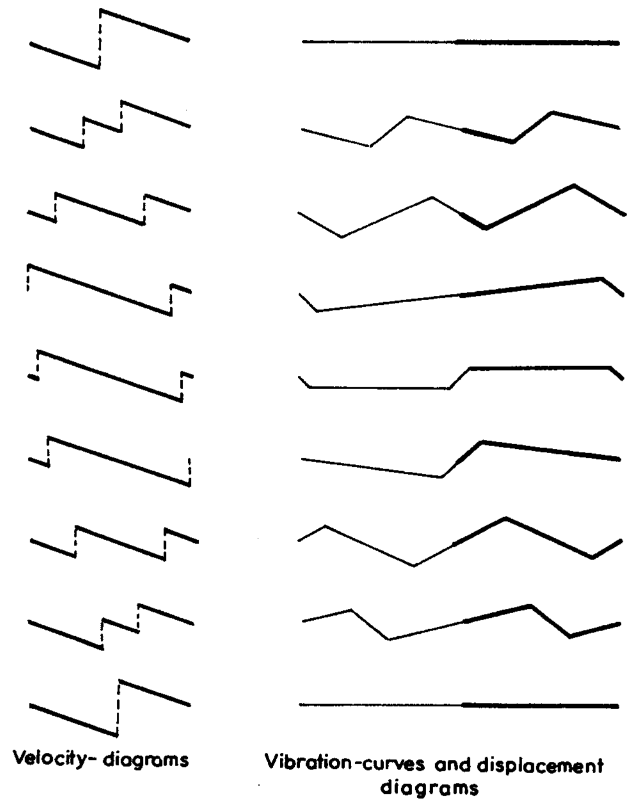

Figure 2. Second type of vibration.

nature of the movement at points intermediate between the limits $x = l/2 \pm b$ is not shown, but this the reader can find for himself from the velocity-diagram. At the points $x = l/2 \pm b$, the vibration-curve is a simple two-step zig-zag, the ratio ω of the time of movement with the larger velocity to the total period of vibration being given by the simple relation

$$\omega = 2b/l. \tag{20}$$

Outside the limits $x = l/2 \pm b$ the motion is representable by a four-step zig-zag, the velocity alternating twice between two constant values, ω being given by the relation

$$\omega = 2x_2/l, \tag{20a}$$

where x_2 is the distance of the point of observation from the centre of the string or from the end, whichever is less.

A motion of the type here described would, normally speaking, be excited if the bow were applied at one or other of the two points $x = l/2 \pm b$. It is obvious, however, that it might also be excited, if the bow were applied outside these limits, the motion at the bowed point then being a four-step zig-zag. In either case, the maximum value of b is $l/4$, since this would make ω in (20) equal to $\frac{1}{2}$.

As already explained in section V, the relation $\omega = 2x_2/l$ would hold good at the bowed point even if this divided the string in a rational ratio, as the dropping out of the harmonics having a node at the bowed point would leave the motion there unaffected. It is instructive to compare the values of ω for various points on the string, for the first and second types of vibration. These are shown in table I. The distance of the point from the end of the string is given as a fraction of the total length. For convenience, the ratio of two large numbers prime to one another may be used as practically equivalent to an irrational ratio. If the motion

Table I

Position of bowed point	$\frac{1}{2}$	$\frac{10}{21}$	$\frac{29}{61}$	$\frac{17}{36}$	$\frac{7}{15}$	$\frac{5}{11}$	$\frac{9}{20}$	$\frac{4}{9}$	$\frac{3}{7}$	$\frac{2}{5}$	$\frac{4}{11}$	$\frac{1}{3}$	$\frac{2}{7}$	$\frac{3}{11}$	$\frac{1}{4}$
Value of ω for the first type	$\frac{1}{2}$	$\frac{10}{21}$	$\frac{29}{61}$	$\frac{17}{36}$	$\frac{7}{15}$	$\frac{5}{11}$	$\frac{9}{20}$	$\frac{4}{9}$	$\frac{3}{7}$	$\frac{2}{5}$	$\frac{4}{11}$	$\frac{1}{3}$	$\frac{2}{7}$	$\frac{3}{11}$	$\frac{1}{4}$
Value of ω for the second type	$\frac{1}{\infty}$	$\frac{1}{21}$	$\frac{3}{61}$	$\frac{2}{36}$	$\frac{1}{15}$	$\frac{1}{11}$	$\frac{2}{20}$	$\frac{1}{9}$	$\frac{1}{7}$	$\frac{1}{5}$	$\frac{3}{11}$	$\frac{1}{3}$	$\frac{3}{7}$	$\frac{5}{11}$	$\frac{1}{2}$

at the bowed point is of the simplest possible type, a two-step zig-zag, it is seen from the third row of figures in table I that one of the sides of the zig-zag becomes very steep if the bow is applied close to the centre of the string. It becomes less and less steep as the bow is removed farther and farther from the centre of the string, but it continues steeper than in the vibration-curve for the first type until the point $l/3$ is reached when ω becomes identical for the first and second types of vibration; the significance of this is that in the second type of vibration the octave is the dominant harmonic and far more powerful than the fundamental when the point of application of the bow is close to the centre of the string, but the difference becomes less and less marked as the bow is removed farther from the centre; when the bow is applied at a distance $l/3$ from the end of the string, the octave and the fundamental are present in the same proportion in the second type of vibration as in the first, and, in fact, the two types of vibration then become identical owing to the dropping out of all harmonics having a node at $l/3$. Similar relations are met with in the theory of the third and higher types of vibration, and a fuller discussion of this identity of vibration types in certain cases will be given when dealing in detail with the theory for rational points of bowing. Between $l/3$ and $l/4$, the value of ω is actually less for the second type than for the first, and the octave becomes feebler and feebler as the bow is removed from the centre, till at $l/4$, it vanishes altogether, ω being equal to $\frac{1}{2}$.

Section VIII—The third type of vibration

When there are three equal discontinuities on the velocity-diagram, the positive and negative velocity-waves are not necessarily of the same form. If at an epoch at which two of the discontinuities are coincident in position, the third is at one of the ends of the string, then, obviously, the positive and negative waves are of the same form and actually coincident at that instant; the vibration at every point on the string is then of the symmetrical type. At the epoch referred to, the velocity-diagram would consist of only two parallel lines, one of which passes through an end of the string, and the other through the point of trisection farthest from it. In the subsequent motion, the number of lines in the velocity-diagram would, in general, be four, and at certain instants, three or two. On the other hand, in the unsymmetrical cases, the number of lines is never less than three and is generally four. Apart from this, the form of the vibration-curve at any point on the string can be found in precisely the same way in the unsymmetrical cases as in the symmetrical ones, i.e. by considering the changes of velocity due to the movement of the discontinuities.

As the symmetrical cases are much the more important, we shall now consider them a little more fully. Figures 3 and 4 illustrate the mode of vibration, the initial coincident position of the two discontinuities lying outside the two points of trisection of the string in figure 3 and between them in figure 4. If this position is at $x = 2l/3 \pm 2b$, the two discontinuities again coincide after a half period at the point $x = l/3 \mp 2b$. It can be seen that at the points $l/3 + b$ and $2l/3 - b$ in figure 3 and at the points $l/3 - b$ and $2l/3 + b$ in figure 4, the vibration-curve is a simple two-step zig-zag, these being also points at which discontinuities cross. At the points $l/3 - 2b$ and $2l/3 + 2b$ in figure 3, and at the points $l/3 + 2b$ and $2l/3 - 2b$ in figure 4, the vibration-curve is seen to be a four-step zig-zag. Except in the region on either side of the points of trisection between the limits $l/3 - 2b < x < l/3 + b$ and $2l/3 - b < x < 2l/3 + 2b$ in figure 3 and the limits $l/3 - b < x < l/3 + 2b$ and $2l/3 - 2b < x < 2l/3 + b$ in figure 4, the velocity at any point on the string alternates between two constant values thrice in each vibration-period. The fraction ω of the complete period during which the larger velocity subsists is given by the relation

$$\omega = 3x_3/l \tag{21}$$

where x_3 is the distance of the point of observation from either point of trisection or from the end of the string whichever is the *least*. The types of vibration shown in figures 3 and 4 may be regarded as set up by application of the bow at a point distant b from one of the points of trisection, and lying between them in respect of figure 3, and outside of them in figure 4. The motion at the bowed point is then of the simplest possible type (representable by a two-step zig-zag), the value of ω there being given by the relation

$$\omega = 3b/l. \tag{21a}$$

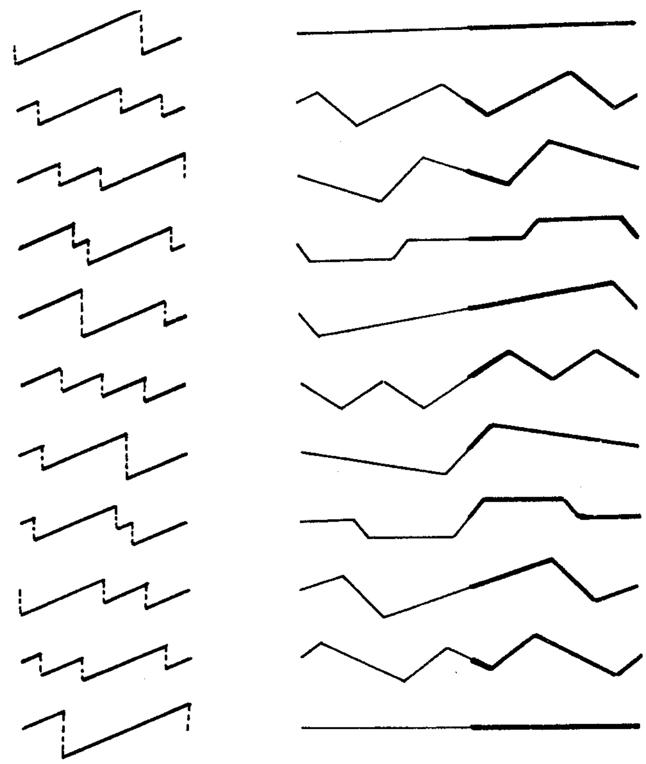

Figure 3. The third type of vibration (negative).

The maximum value of b is $l/6$ as ω is then equal to $\frac{1}{2}$.

The values of ω as given by equation (21a) for various values of b within the limit stated above, are shown in table II. The distance of the bowed point from one end of the string is shown as a fraction of the total length l.

Table II

Position of bowed point	$\frac{1}{2}$	$\frac{5}{11}$	$\frac{4}{9}$	$\frac{3}{7}$	$\frac{2}{5}$	$\frac{5}{13}$	$\frac{3}{8}$	$\frac{4}{11}$	$\frac{5}{14}$	$\frac{23}{65}$	$\frac{6}{17}$
Value of ω for the third type	$\frac{1}{2}$	$\frac{4}{11}$	$\frac{3}{9}$	$\frac{2}{7}$	$\frac{1}{5}$	$\frac{2}{13}$	$\frac{1}{8}$	$\frac{1}{11}$	$\frac{1}{14}$	$\frac{4}{65}$	$\frac{1}{17}$
Position of bowed point	$\frac{1}{3}$	$\frac{5}{16}$	$\frac{17}{55}$	$\frac{3}{10}$	$\frac{5}{17}$	$\frac{2}{7}$	$\frac{3}{11}$	$\frac{1}{4}$	$\frac{2}{9}$	$\frac{1}{5}$	$\frac{1}{6}$
Values of ω for the third type	$\frac{1}{\infty}$	$\frac{1}{16}$	$\frac{4}{55}$	$\frac{1}{10}$	$\frac{2}{17}$	$\frac{1}{7}$	$\frac{2}{11}$	$\frac{1}{4}$	$\frac{3}{9}$	$\frac{2}{5}$	$\frac{1}{2}$

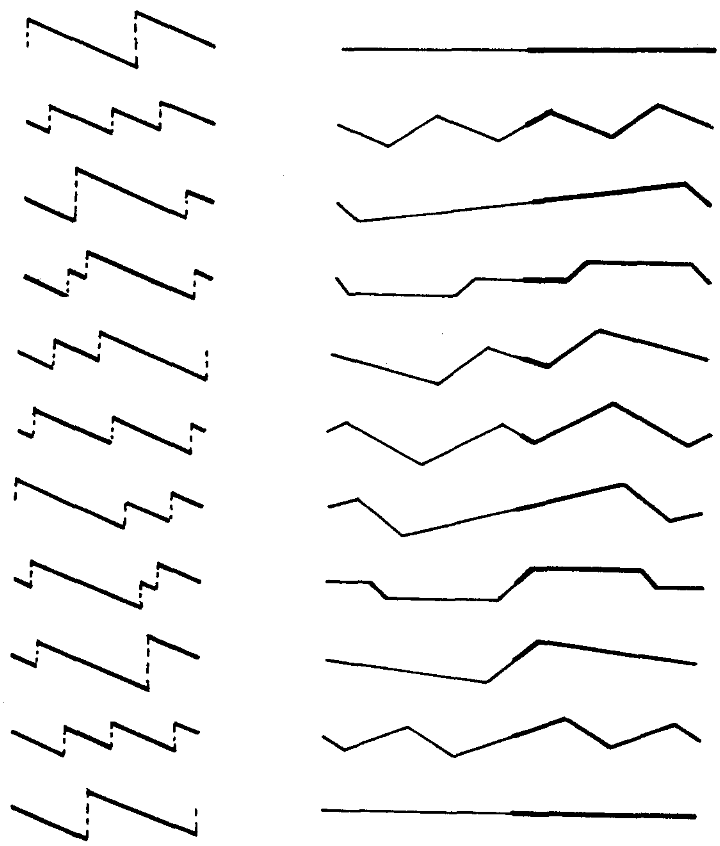

Figure 4. The third type of vibration (positive).

From table II it will be seen that ω is very small at the bowed point when this is situate close to the point of trisection. One side of the zig-zag vibration-curve at the bowed point is then much steeper than the other and, as can be seen from figures 3 and 4, the third harmonic is dominant in the resulting motion, being in fact far more powerful than even the fundamental. As the bow is removed from the node, ω becomes larger and the amplitude of the third harmonic falls off rapidly, and vanishes completely when the bow is applied at the points $x = 4l/9$ or $2l/9$, the value of ω being then exactly $\frac{1}{3}$. A distance of $l/9$ either way from any of the nodes of the third harmonic may thus in a sense be regarded as the range of application of the bow for the excitation of the third type of vibration. On comparing the values of ω in table II with those for the first and second types shown in table I, it will be noticed that at the point $2l/5$ the second and third types both give the ratio $\frac{1}{5}$ and that at the point $l/4$ the first and third types both give $\omega = \frac{1}{4}$. It is instructive to note the gradual changes in the form of the vibration-curves for the third type at the centre of the string and at $l/5$ as the point at which the bow is applied approaches $l/4$ and $2l/5$ respectively.

Figure 5(*a*), vibration-curve of the 9th type, bowed at $5l/9 - l/135$ and observed at $l/15$.

Figure 5(*b*), (*c*), (*d*), (*e*), vibration curves at the bowed point for the 2nd, 4th, 6th and 8th types respectively, when the bow is applied close to the centre of the string.

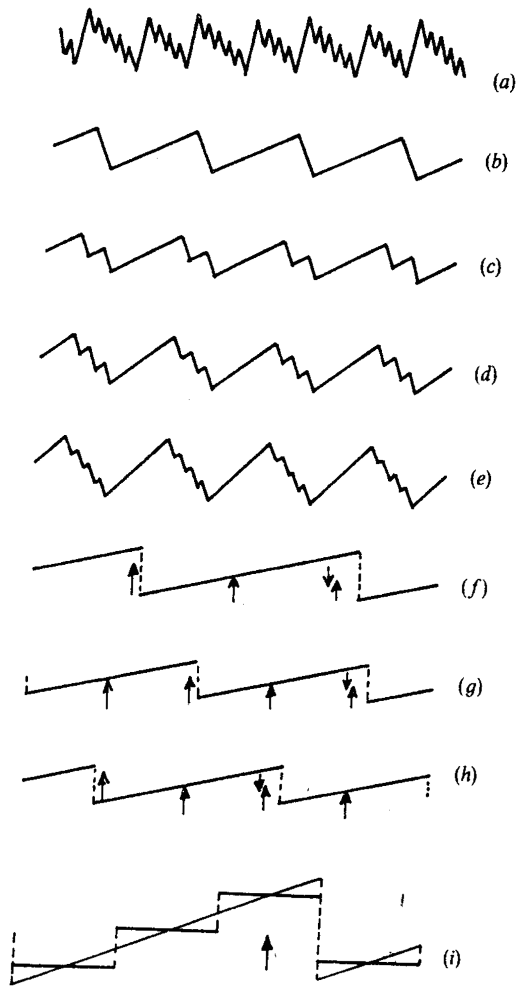

Figure 5. (For explanation see text).

Figure 5(f), velocity-diagram of the 4th type: (g) and (h), velocity diagrams of the 5th type: (i), velocity-diagram of 3rd type, bow applied at $5l/8$.

Section IX—The fourth and higher types of vibration

The cases in which there are four or more equal discontinuities moving on the velocity-diagram of the string are also capable of investigation with facility by the graphical method. Certain general laws are readily established of which we have already seen instances in the second and third types of vibration. Certain important and very remarkable differences are also noted between the cases in which n (the number of discontinuities) is a prime integer, e.g. 5, 7 or 11, and the cases in which n is not prime, e.g. 4, 6, 8, 9 or 10. One result that holds good for all values of n is that the motion at every point on the string lying outside certain limits is representable by a vibration curve consisting of $2n$ straight lines, alternate ones being parallel to one another. In other words, outside certain limits the velocity at every point on the string alternates between two constant values, n times in each complete period, the fractional part of the period ω during which the larger velocity subsists being given by the relation $\omega = nx_n/l$ already found (equation 17).

The regions within which the velocity may have more than two values in each period, lie on either side of the $(n-1)$ intermediate nodes of the n^{th} harmonic and are bounded by points at which discontinuities moving in opposite directions cross one another. The vibration curve of a point at which one such crossing takes place consists of $(2n-2)$ straight lines in each complete period; if two crossings occur at one point in each period, its vibration curve consists of $(2n-4)$ straight lines and so on. If $(n-1)$ crossings take place at a point, the vibration curve at that point is of the simplest possible form, i.e. a two-step zig-zag.

We now proceed to find the conditions that must be satisfied if the motion at some specified point on the string is to be of the simplest possible type.

Choosing as origin of time, the instant at which the positive and negative waves are completely coincident, it is easy to find the positions on the string at which the coincident discontinuities in the waves should lie, if the motion at some specified point on the string is to be representable by a simple two-step zig-zag. Take first, the case in which $n = 4$ and let the x-coordinate of the specified point be $3l/4 - b$. The velocity-diagram at time $t = 0$ consists of three parallel lines passing through the centre and the two ends of the string respectively. Let the coordinates at time $t = 0$ of the two pairs of coincident discontinuities be $l/4 + c$ and $3l/4 + d$ respectively, see figure 5(f). In this diagram the positions of the nodes and of the specified point are shown by arrow-heads. In the subsequent movement, the discontinuities would pass over the point referred to, after times corresponding to the following distances of travel: $(b + d)$, $(l/2 - c - b)$, $(l/2 - d + b)$, $(l + c - b)$, $(l - c + b)$, $(3l/2 + d - b)$, $(3l/2 + c + b)$ and $(2l - b - d)$. It is seen that by putting

$c = b$ and $d = 3b$, the alternate distances (excluding the first and the last) become equal to one another. The condition for the vibration curve at the point $3l/4 - b$ being a simple two-step zig-zag is thus that the initial positions of the coincident discontinuities should be $(l/4 + b)$ and $(3l/4 + 3b)$. We may then mark off the following scheme of points on the string:

$$0 \quad \frac{l}{4} \quad \frac{l}{2} \quad \frac{3l}{4} \quad l$$

$$\left(\frac{l}{4} - 3b\right), \left(\frac{l}{4} + b\right), \left(\frac{l}{2} - 2b\right), \left(\frac{l}{2} + 2b\right), \left(\frac{3l}{4} - b\right), \left(\frac{3l}{4} + 3b\right)$$

6-step 2-step 4-step 4-step 2-step 6-step

zig-zags.

The nature of the motion at the points marked off is indicated below them in the scheme. If the point at which the motion is of the simplest possible type is $(3l/4 + b)$, then we get instead the following scheme:

$$0 \quad \frac{l}{4} \quad \frac{l}{2} \quad \frac{3l}{4} \quad l$$

$$\left(\frac{l}{4} - b\right), \left(\frac{l}{4} + 3b\right), \left(\frac{l}{2} - 2b\right), \left(\frac{l}{2} + 2b\right), \left(\frac{3l}{4} - 3b\right), \left(\frac{3l}{4} + b\right),$$

2-step 6-step 4-step 4-step 6-step 2-step

zig-zags.

It will be noticed that in both schemes, the region on either side of the central node is bounded by points at which the motion is representable by a four-step zig-zag. It is not possible, by assigning arbitrary values to c and d or in any other way, to secure that the region round the central node should be bounded by points at which the vibration-curve is a two-step zig-zag, so long as we are dealing with the fourth type of vibration. This result has a very important significance. If the fourth type of vibration is excited by applying the bow near either of the two nodes $l/4$ or $3l/4$, the motion at the bowed point might be (and, in fact, generally would be) of the simplest possible type, i.e. having a two-step zig-zag as the vibration-curve. On the other hand, if the bow is applied near the centre of the string (which is also a node of the fourth harmonic), the fourth type of vibration, if elicited, would not give a motion of the simplest possible type at the bowed point. Nor, for that matter, would the sixth, eighth or tenth types, if elicited. For the second type of vibration, a simple two-step zig-zag vibration-curve for the bowed point is possible. But for the fourth type, if elicited by bowing near the centre of the string, the minimum number of lines in the vibration-curve at the bowed point for each complete period is *four*; for the sixth type, it is six, for the eighth type, it is eight, and so on.

Figures 5(*b*), (*c*), (*d*) and (*e*) illustrate the preceding remarks. They were drawn from the initial velocity-diagrams of the respective cases, which were disposed so as to give a vibration-curve of the minimum complexity admissible at the bowed point.

Kinematical analysis by the method described above leads to analogous results in every case in which n is not a prime integer and the vibration is elicited by application of the bow near a point which is a node of the n^{th} harmonic but is also a node of some harmonic of lower frequency. For instance, the sixth type of vibration, if elicited by bowing near a point of trisection, would not give a vibration-curve at the bowed point of a simpler type than a four-step zig-zag; the ninth type under similar circumstances would give a vibration-curve at the bowed point with not less than six lines per period; the twelfth type would not give less than eight lines per period, and so on. Near the point of quadrisection of a string, the eighth type would give a vibration-curve at the bowed point with at least four lines per period and so on. As a final example we may mention the case in which the bow is applied very close to one end of the string. Only the first type ($n = 1$) would then give a simple two-step zig-zag at the bowed point. For other values of n, i.e. 2, 3, etc. the vibration-curve at the bowed point would consist of 4, 6, or more lines as the case may be, in each complete period.

When n is a prime number, e.g. 5, 7, or 11, the corresponding type of vibration may be elicited by applying the bow on either side of any one of the intermediate nodes of the n^{th} harmonic on the string, with a simple two-step zig-zag as an admissible vibration-curve for the bowed point. The initial positions of the coincident discontinuities in the velocity-diagram differ very considerably, however, according to the particular node selected, and the character of the vibration-curves elsewhere than at the bowed point is correspondingly different in the respective cases. On the other hand when n is not a prime, a two-step zig-zag would be possible as the vibration-curve at the bowed point, only if this is at the boundary of the region on either side of a node of the n^{th} harmonic which is not also a node of some harmonic of lower frequency; e.g. $l/6$ or $5l/6$ for the sixth type, $l/8$, $3l/8$, $5l/8$ or $7l/8$ for the eighth type and $l/9$, $2l/9$, $4l/9$, $5l/9$, $7l/9$ or $8l/9$ for the ninth type. Near the other nodes, the motion at the bowed point for these types is necessarily of a less simple type, as we have already seen.

We shall now consider one after another the fifth and higher types of vibration confining our attention to those cases in which the motion at the bowed point has the simplest character possible under the circumstances, that is, a two, or four, or a six-step zig-zag, etc. as the case may be. It is obvious, from considerations of symmetry, that it is sufficient to consider the cases in which the bow is applied at some point between the centre and one end of the string, that is between $x = l/2$ and $x = l$. For any given value of n, as we have seen, the velocity-diagram consists of parallel lines passing through the nodes of the n^{th} harmonic, and it follows that at these nodes the state is alternately one of rest and of motion in one direction or the other. These positions of rest at intervals during the vibration at the respective nodes are readily visible to the eye on inspecting the string, appearing the brighter, the longer the intervals of rest. If two such positions coincide, the line of rest seen is twice as bright, and so on. For instance, in the fourth type of vibration already discussed, three lines of rest are visible at the centre of the string, and four lines at the points $x = l/4$ and $x = 3l/4$.

The fifth type of vibration

This may be elicited by applying the bow at the points

$$\left(\frac{4l}{5}+b\right), \left(\frac{4l}{5}-b\right), \left(\frac{3l}{5}+b\right) \text{ or } \left(\frac{3l}{5}-b\right)$$

the motion at the bowed point being in each case a simple two-step zig-zag. In the first case the initial positions of the discontinuities in the velocity-diagram are

$$0, \left(\frac{2l}{5}-2b\right) \text{ and } \left(\frac{4l}{5}-4b\right).$$

By writing b for $-b$ in the above, the initial position of the discontinuities in the second case is found at once. The initial velocity-diagram for this case is shown in figure 5(g). In the third case the initial positions of the discontinuities are

$$\left(\frac{l}{5}+2b\right), \left(\frac{3l}{5}-4b\right) \text{ and } l.$$

The solution for the fourth case is obtained by writing $-b$ for b in the preceding. The initial velocity-diagram for the fourth case is shown in figure 5(h).

On working out the form of the vibration-curves from the velocity-diagrams, it is found that in all these cases, five lines of rest should be visible at each of the four nodes $l/5$, $2l/5$, $3l/5$ and $4l/5$. Except in the limited regions lying on either side of each of these nodes, the vibration-curve at any point on the string is a ten-step zig-zag in which the alternate lines are all parallel to one other. The boundaries of the regions about the nodes and the character of the motion at the limiting points are indicated in table III.

Table III
(Fifth type of vibration)

$l/5 - b)$: 2-step	$(l/5 - 4b)$: 8-step	$(l/5 - 3b)$: 6-step	$(l/5 - 2b)$: 4-step
$(l/5 + 4b)$: 8-step	$(l/5 + b)$: 2-step	$(l/5 + 2b)$: 4-step	$(l/5 + 3b)$: 6-step
$(2l/5 - 2b)$: 4-step	$(2l/5 - 3b)$: 6-step	$(2l/5 - b)$: 2-step	$(2l/5 - 4b)$: 8-step
$(2l/5 + 3b)$: 6-step	$(2l/5 + 2b)$: 4-step	$(2l/5 + 4b)$: 8-step	$(2l/5 + b)$: 2-step
$(3l/5 - 3b)$: 6-step	$(3l/5 - 2b)$: 4-step	$(3l/5 - 4b)$: 8-step	$(3l/5 - b)$: 2-step
$(3l/5 + 2b)$: 4-step	$(3l/5 + 3b)$: 6-step	$(3l/5 + b)$: 2-step	$(3l/5 + 4b)$: 8-step)
$(4l/5 - 4b)$: 8-step	$(4l/5 - b)$: 2-step	$(4l/5 - 2b)$: 4-step	$(4l/5 - 3b)$: 6-step
$(4l/5 + b)$: 2-step	$(4l/5 + 4b)$: 8-step	$(4l/5 + 3b)$: 6-step	$(4l/5 + 2b)$: 4-step

zig-zags.

It will be seen that the distance between the two points on either side of each node as shown in the table, is in all cases equal to $5b$.

The sixth type of vibration

By applying the bow at either of the two points $5l/6 \pm b$ the type of vibration with six equal discontinuities on the velocity-diagram may be elicited with a two-step zig-zag as the vibration-curve at the bowed point.

It is obvious that for this to be possible, the discontinuities should cross at the bowed point five times in each period of vibration, and the initial positions of the discontinuities are thus found to be $l/5 \mp b$, $3l/6 \mp 3b$ and $5l/6 \mp 5b$, the upper minus signs being taken for the first case and the plus signs for the second. The following are the eight points on the string at which crossings of discontinuities take place during the vibration and the character of the vibration-curve determined by the number of such crossings is indicated below the respective points:

First case

$$\left(\frac{l}{6} - b\right), \left(\frac{l}{6} + 5b\right), \left(\frac{2l}{6} - 2b\right), \left(\frac{2l}{6} + 4b\right), \left(\frac{3l}{6} - 3b\right)$$

2-step 10-step 4-step 8-step 6-step

$$\left(\frac{3l}{6} + 3b\right), \left(\frac{4l}{6} - 4b\right), \left(\frac{4l}{6} + 2b\right), \left(\frac{5l}{6} - 5b\right) \text{ and } \left(\frac{5l}{6} + b\right)$$

6-step 8-step 4-step 10-step 2-step

zig-zags.

Second case

$$\left(\frac{l}{6}-5b\right),\left(\frac{l}{6}+b\right),\left(\frac{2l}{6}-4b\right),\left(\frac{2l}{6}+2b\right),\left(\frac{3l}{6}-3b\right)$$

10-step 2-step 8-step 4-step 6-step

$$\left(\frac{3l}{6}+3b\right),\left(\frac{4l}{6}-2b\right),\left(\frac{4l}{6}+4b\right),\left(\frac{5l}{6}-b\right) \text{ and } \left(\frac{5l}{6}+5b\right)$$

6-step 4-step 8-step 2-step 10-step

zig-zags.

It will be noticed that the regions on either side of the nodes are all of the same length $6b$. (With the fifth type of vibration, this length was found to be $5b$, with the fourth type $4b$, with the third $3b$, and with the second type $2b$, the motion at the bowed point being in each case a simple two-step zig-zag. The generalization of the result is obvious).

From the vibration-curves, it is found that six lines of rest should be visible at each of the nodes $l/6$ and $5l/6$ in the case discussed above, five lines of rest at each of the nodes $2l/6$ and $4l/6$, and four lines of rest at the central node $3l/6$.

Passing on to the cases in which the bow is applied in the neighbourhood of the node $4l/6$, it is found that the maximum number of crossings in each period possible at the bowed point is four. (If we attempt to find the position of the discontinuities on the assumption that there are five crossings, the resulting equations are inconsistent with each other). A four-step zig-zag thus represents the least complicated motion possible at the bowed point in this case. We have then three pairs of coincident discontinuities in the initial velocity-diagram of the string, and as there are two crossings of discontinuities at the bowed point in each half period, we get two simple algebraic equations connecting the position of the bowed point with the initial positions of the discontinuities. Thus if the position of the bowed point is at $4l/6+b$, and the initial positions of the three pairs of discontinuities are $l/6+c$, $3l/6+d$ and $5l/6+e$, we obtain the two equations $(e-b)=(b-d)$, and $(e+b)=(c-b)$. It is obviously not possible from these two equations alone to determine the three quantities c, d and e, uniquely in terms of the known quantity b. While c, d and e are thus in a sense arbitrary quantities, in practice owing to the flexibility of the dynamical conditions under which the motion is maintained (this will be referred to again later on), the vibration would tend to settle down into a type in which b, c, d and e are all integral multiples of one and the same quantity, these multiples being as small as is consistent with the kinematical conditions referred to above and the dynamical conditions of maintenance in the presence of dissipative forces. For instance, by putting $c=5b$, $d=-b$, and $e=3b$, we get a solution which satisfies the two kinematical equations and gives a characteristic type of vibration.

The case in which the sixth type is elicited by applying the bow in the neighbourhood of the node $3l/6$, i.e. near the centre of the string, may be similarly dealt with. In this case, the maximum number of crossings of discontinuities at the bowed point in each period is three and the simplest type of motion at the bowed point is a six-step zig-zag. The initial position of one pair of discontinuities then necessarily coincides with the bowed point. To find the initial positions of the other two pairs, we have only one equation which is given by the crossing that takes place within each half-period. The case has therefore to be dealt with in a manner analogous to that described in the preceding. Thus if the position of the bowed point is at $3l/6 + b$, the discontinuities may initially be taken to be at the positions $(l/6 - 3b)$, $(3l/6 + b)$ and $(5l/6 + 5b)$.

The seventh type of vibration

For this type, we have to consider the bowed point as being at one or other of the six positions $(6l/7 \pm b)$, $(5l/7 \pm b)$, and $(4l/7 \pm b)$. Each of these cases gives a mode of vibration characteristically different from the others, though at the bowed point the vibration-curve is the same in all the cases, a two-step zig-zag. When the bow is applied at $6l/7 + b$, the discontinuities are initially at the positions 0, $(2l/7 - 2b)$, $(4l/7 - 4b)$ and $(6l/7 - 6b)$. In the resulting vibration, if b be small, the seventh harmonic is very strong. By writing $-b$ for b in these quantities, we get the solution for the second case in which the bow is applied at $(6l/7 - b)$. Both the sixth and the seventh harmonics are then powerful in the mode of vibration set up. If the position of the bowed point is shifted to the point $(5l/7 + b)$ the initial positions of the discontinuities are found to be $(l/7 - 4b)$, $(3l/7 + 2b)$, $(5l/7 - 6b)$ and l respectively. In the resulting vibration the fourth and seventh harmonics are dominant. By writing $-b$ for b as before, the solution for the bowed point $(5l/7 - b)$ is obtained. The third and seventh harmonics are then dominant. When the bow is applied at the point $4l/7 + b)$, the initial positions of the discontinuities are 0, $(2l/7 + 4b)$, $(4l/7 - 6b)$ and $(6l/7 - 2b)$. It can be shown that the second, fifth and seventh harmonics are then dominant. The solution for the case in which the bow is applied at $(4l/7 - b)$ may be obtained as above by changing the sign. The second and seventh harmonics are found to be dominant in the resulting vibration.

The eighth, ninth and higher types may be readily discussed in accordance with the general principles already outlined. The form of the vibration-curve for the ninth type of vibration elicited by bowing at the point $(5l/9 - l/135)$ and calculated for the point of observation distant $l/15$ from one end of the string is shown in figure 5(*a*). It will be seen from the general form of the vibration-curve that the second and ninth harmonics are dominant in the resulting motion.