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Experimental investigations on the maintenance of vibrations

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CONTENTS

Prefatory note

These investigations formed the subject of Lectures delivered at special meetings of the Indian Association for the Cultivation of Science on the 9th January, 1909, with Sir Gooroodas Banerjee presiding, and on the 9th May, 1912, with the Hon'ble Justice Sir Asutosh Mookerjee, F.R.S.E., etc. in the chair.

I. On a new form of Melde's experiment

This will be found described in my notes in *Nature* (*London*) of the 4th November, 1909, and in the *Phys. Rev.* of March, 1911 (Bulletins of the Indian Association Nos 2 and 3). When properly performed its results easily surpass in beauty and interest, those obtained with the usual arrangements in Melde's experiments. The modified form of the experiment was devised in the course of my work of 1906 at the Presidency College, Madras, when endeavouring to clear up certain anomalous observations by Mr V Appa Rao. The fine cotton or silk string which is maintained in vibration is attached to the prong of a tuning-fork which is best maintained electrically (though indeed a bowed fork is suitable enough) and is held so that it lies in a plane perpendicular to the prongs but in a direction *inclined* to their line of vibration.

Under these circumstances the motion of the prong may be resolved into two components, one parallel and the other perpendicular to the string. The latter transverse component maintains an oscillation having the same frequency as that of the fork when the tension of the string is suitably adjusted. The length of the string should be such that under the action of this force the string divides up into an even number of ventral segments, say two. The first, i.e. longitudinal component of the obligatory motion, will then generally be found to maintain simultaneously an oscillation having half the frequency of that of the fork. The success of the experiment lies in isolating the two vibrations, the frequency of one of which is double that of the other, into perpendicular planes. This is easily secured by a simple little device. The end of the string is attached to a loop of thread which is passed over the prong, instead of directly to the prong itself. The result of this mode of attachment is that the frequencies of vibration in the two planes at right angles differ slightly and this has the desired effect of keeping the two component vibrations confined to their respective planes, if the tension of the string lies anywhere within a definite range.

With the arrangements described above it is evident that the motion of each point on the string in a plane transverse to the length should be one of the

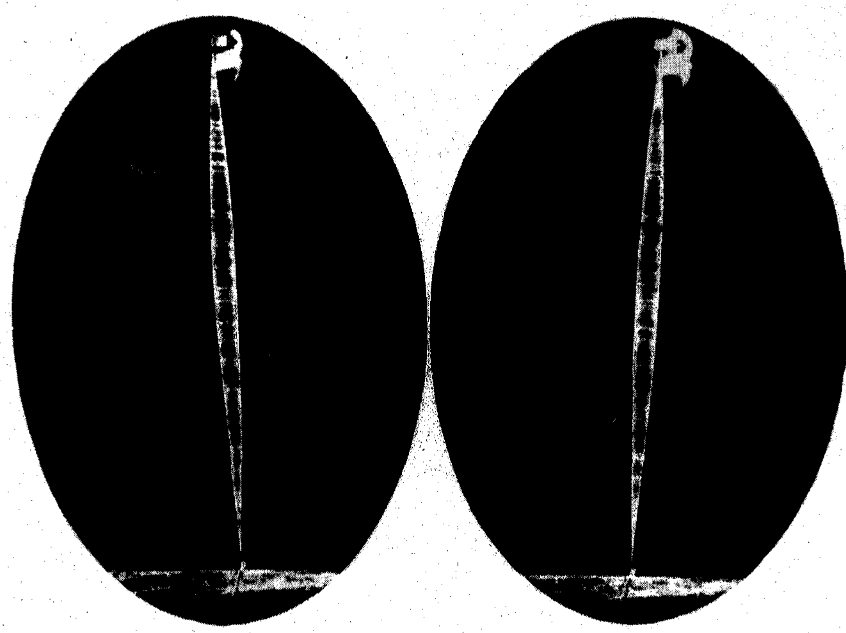


Plate I. A new form of Melde's experiment.

Lissajous figures for the interval of the octave. The shape of this figure depends on the phase-relation between the component oscillations, and this is determined by the precise value of the tension of the string. It is well to state at once that when the tension is somewhat in excess the curves are parabolic arcs. Plate I shows a stereo-photograph of a string maintained in an oscillation of this type.

We proceed to discuss the approximate theory of this case. The motion of the prong of the tuning-fork may be put equal to $\gamma \cos pt$.

The component of this transverse to the string may be put equal to $\gamma \cos pt \sin \theta$. If the distance of any point on the string from the fixed end when at rest is x, the transverse components of the maintained motion may to a first approximation be written as under

$$Y = \gamma \sin \theta \frac{R_x}{R_b} \cos (pt + E_x - E_b) \tag{1}$$

vide Lord Rayleigh's *Theory of Sound*, article 134.

$$Z = B \cos\left(\frac{pt}{2} + E\right) \sin \frac{\pi x}{b}. \tag{2}$$

If we exclude any consideration of the motion at points near the nodes of the maintained oscillation, equation (1) may be written in the simple form

$$Y = \gamma \sin \theta \sin \frac{px}{a} \cos (pt + E').$$

If $p/a = 2\pi/b$, (1) and (2) may be written in the form

$$Y = A \sin \frac{2\pi x}{b} \cos (pt + E'), \tag{3}$$

$$Z = B \sin \frac{\pi x}{b} \cos\left(\frac{pt}{2} + E\right). \tag{4}$$

It should be understood that in these equations Y and Z do *not* refer to the coordinates of any point fixed relatively to the string but to the points at which a plane transverse to its equilibrium position cuts the surface generated by the moving string. The distinction is of importance in view of the fact that each point on the string possesses a small longitudinal motion derived from that imposed by the fork and the x coordinate of any point fixed relatively to the string is therefore not itself constant.

In particular cases equations (3) and (4) may be reduced to very simple forms. Thus if $E' = 2E$ and also in the special case when both E and E' are equal to zero

$$\frac{Y}{A} \operatorname{cosec} \frac{2\pi x}{b} = \frac{2Z^2}{B^2} \operatorname{cosec}^2 \frac{\pi x}{b} - 1 \tag{5}$$

which is the equation of the surface generated by the moving string, the sections of which by planes perpendicular to the axis of x are parabolic arcs. The curvature of these parabolic arcs lies in opposite directions for values of x less and greater than $b/2$. This is exactly the type of vibration shown in the stereo-photograph (plate I). It will be noticed that in the half of the string near the tuning-fork the curvature of the parabolic arcs is in one direction and in the other half in the opposite direction.

At the mid-point there is practically no transverse motion in one of the perpendicular planes. This is also evident in the plate, but no photograph can give a really adequate idea of the beauty of the stationary form of vibration, which must be seen to be fully appreciated.

It is not difficult to make out from general considerations why and when the tension is somewhat in excess, the phase-relation between the component vibrations is such as to give us a parabolic type of vibration. It is clear that when the free period of vibration of the half-length of the string is somewhat less than that of the fork, the phase of the oscillation maintained by the transverse obligatory motion is very approximately in agreement with that of the obligatory motion itself, i.e. $E' = 0$. Under the same circumstances and generally also whenever a large amplitude of vibration is maintained by the longitudinal component of the motion of the fork, the phase of the oscillation of half-frequency is such that the displacement is very nearly a maximum when the tension is a minimum, and vice versa. This is what it would be if $E = 0$. Since E and E' are both zero, equation (5) gives us the required type of vibration.

A parabolic type of motion should also be obtained when $E' = 2E + \pi$. The equation of the surface generated by the moving string in this case may be obtained by merely writing $-Y$ for Y in equation (5)

$$\frac{Y}{A}\operatorname{cosec}\frac{2\pi x}{b} = 1 - \frac{2Z^2}{B^2}\operatorname{cosec}^2\frac{\pi x}{b}. \tag{6}$$

The sections of this surface by planes normal to the axis of x are parabolic arcs, but it will be noticed that their curvatures are in the opposite direction to those given by equation (5). A rough approximation to the case given by equation (6) is obtained when the tension of the string is somewhat in defect, i.e. the free period of the half-length of the string is more than the period of the fork, and the longitudinal component of the vibration of the fork maintains an oscillation of large amplitude.

It remains now to consider the intermediate case where $E' = 2E + \pi/2$. It is evident *á priori* that in this case the sections of the surface described by the moving string by planes normal to the axis of x should be 8 curves, and this is readily verified by experiment. The tension of the string, to obtain a motion of this type, should be roughly that at which the transverse obligatory motion maintains the most vigorous vibration, and a large motion is also maintained by the longitudinal component.

This leads me on to consider a very interesting point which was referred to above in passing. From equation (3) it appears that when $x = b/2$, $Y = 0$. This point is the 'node' of the oscillation maintained by the transverse obligatory motion. Strictly speaking Y is not zero at this point. There is a very appreciable small motion at the node, the magnitude of which is given by equation (1). This equation may by changing the origin of time be written in the more intelligible form

$$Y = \frac{\gamma \sin\theta}{R_b}\left(\sin\alpha x \cos pT + \frac{kx}{2a}\cos\alpha x \sin pT\right). \tag{7}$$

At the node the first term within the bracket is zero but the second term remains finite. It will be seen that the phase of the second term differs from that of the first by quarter of an oscillation. When the sections of the surface generated by the vibrating string at other points are 8 curves, as described in the preceding paragraph, the section of the surface at the node itself by a plane normal to the string is a parabolic arc with a fairly large radius of curvature. This is readily verifiable by experiment. I here refer to the motion in a plane transverse to the string and this is quite distinct from the curvature due to the small motion parallel to the axis of x which each point of the string (other than the fixed end) possesses in virtue of the longitudinal motion imposed by the tuning-fork.

Equations (3) and (4) represent the curves along which the string lies at any given instant. They are of course not plane curves at all (except at the epochs when either Y or Z or both are everywhere zero) and are exceedingly pretty. With the parabolic type of oscillation the peripheral curve, i.e. the position of the string at its extreme outward swing, can readily be seen. In other cases, intermittent light is required to render these curves visible. Probably the most satisfactory arrangement is to use intermittent illumination having a frequency double that of the tuning-fork which maintains the string in vibration, so that four views are obtained simultaneously and by their disposition give a much more vivid idea of the mode of motion than would be had if only one or two positions of the string were visible. For work of this kind, a stroboscopic disk with narrow radial slits and run by a Rayleigh motor synchronous with the driving tuning-fork is an extremely useful piece of apparatus. The motor purchased by the Association has thirty teeth on its armature-wheel and I have had two stroboscopic disks made with thirty and sixty slits respectively, either of which can be mounted on the motor. It is not necessary to hold the eye close to the stroboscopic disk for many purposes. If the disk is vertically held and the vibrating string is horizontal and parallel to the disk and is observed through the top row of slits, i.e. through those moving in a direction parallel to the string, the latter is seen as if divided up into a fairly large number of ventral segments. This effect is due to the fact that the string is observed through different parts of the revolving disk and it is therefore seen in successive cycles of phase along its length. We get practically a series of replicas of the string with the same amplitude of motion but of greatly diminished length, i.e.

with magnified curvature. This is specially advantageous in the present case. The vibrating string should be brightly illuminated when under observation through the stroboscopic disk.

In concluding this section I must remark that some phenomena of interest are observed when the two modes of vibration, the frequency of one of which is double that of the other, are *not* isolated in perpendicular planes. When not kept in check by some device of the kind described at the commencement of this note, the oscillation of higher frequency has a tendency to settle down into circular or elliptical motion and some very curious types of vibration are obtained by its composition with the plane vibration of half frequency. The form of these types can be varied by altering the inclination of the string and its tension. We need not however pause to discuss them in detail. Interesting as some of these modes of motion are, they sink into insignificance when compared with some of the compound types of vibration maintained by a simple harmonic force that will be illustrated in section V of this paper.

II. The small motion at the nodes of a vibrating string

I drew attention in recent publications (quoted at the commencement of the previous section) to some remarkable features of the small motion at the nodes of a vibrating string which it appears had not previously been noticed. A vivid idea of the types of motion obtaining can as I showed be had by observing under periodic illumination of approximately double the frequency of that of the oscillation. I have since succeeded in obtaining photographs under actual experimental conditions of the appearances observed.

When a stretched string is maintained in oscillation in segments by a periodic force or an obligatory motion imposed transversely at one point on it, the nodes are not of course points of absolute rest, as the energy requisite for the maintenance of the motion is transmitted through these points. Certainly the best way of observing what exactly takes place at the nodes is to use intermittent illumination, the frequency of this being nearly double that of the vibrations. We would then see *two* slowly moving positions of the string which obviously represent opposite phases of the actual motion. If the nodes were points of absolute rest, then these two positions would intersect at fixed points. One naturally expects that the 'nodes' or points of intersection actually seen under the intermittent illumination should never depart very far from the positions of the real nodes of the oscillation, i.e. the positions where the string is seen under non-intermittent illumination to divide up into segments. But this is not the case. The 'nodes' seen under the intermittent illumination travel along the string over an extraordinary range, in fact over a distance equal to the whole length of a loop. This striking effect is very readily observed, any device for securing intermittent illumination of approximately the correct frequency being sufficient for the purpose.

While visual observation is quite simple, much the most satisfactory arrangement both for ordinary daylight observation and for photographic work is the use of a stroboscopic disk with radial slits mounted on a Rayleigh motor which is actuated by the intermittent current from a self-maintaining tuning-fork interrupter and therefore runs synchronously with it. The tuning-fork interrupter, which is of frequency 60 per second, maintains the string in transverse oscillation of the same frequency. It also drives the synchronous motor on which is mounted a stroboscopic disk having just double as many apertures as the armature-wheel has teeth. The disk therefore gives two views of the fork and of the string maintained by it, which are practically stationary provided the point of observation is fixed and the motor is running satisfactorily. By changing the point of observation (which should be so chosen that the radial slits are parallel to the string and move at right angles to it) the successive stages of the motion and of the travel of the 'nodes' can be observed at leisure.

For photographic work, the stroboscopic disk is held vertically and the camera employed is brought close behind that one of the rectangular slits on the disk which is horizontal. The lens is stopped down by a plate which has a rectangular slit cut in it to correspond with those on the disk. The string and the aperture on the lens of the camera are both horizontal and by racking up the lens-front by successive small distances till it has moved through a length equal to that between contiguous apertures on the disk, a complete set of photographs can be obtained on one plate showing the successive stages of the motion of the string. Plate II reproduces a photograph obtained in this manner, the centre of the field being the position of the node of the vibrating string as seen by non-intermittent light. It shows the cycle of changes in 13 stages and was obtained by moving up the lens-front through very small distances each time. It will be seen that the point of

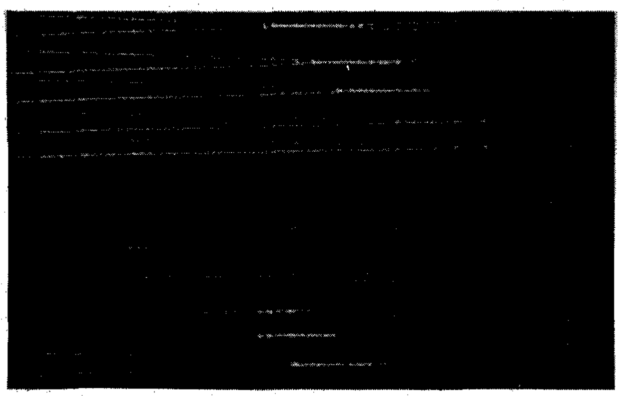


Plate II. Stroboscopic photograph of the small motion at the node of a vibrating string.

intersection or 'node' which is first in the centre moves off to one side of the field, first slowly and then more quickly, till after the lapse of a time which can be seen to be exactly half the period of the cycle it has gone well off the plate and the two positions of the string seen in the photograph are sensibly parallel. Direct observation shows that the point has moved off to a distance equal to half the length of a segment. It simultaneously appears at an equal distance on the other side and moves in from that direction first quickly and then more slowly till it reaches the centre again, and the cycle is complete.

The explanation of these phenomena is that the small motion at the node is not in the same phase as the large motion elsewhere. It is evident from the photograph that the small displacement at the node is a maximum when the large motion elsewhere is a minimum: in other words that its phase differs by exactly quarter of a period of the vibration from that of the general motion of the string. An independent method of demonstrating this was discussed incidentally in section I above. Equation (7) of that section contains in a nutshell the complete theory of the case. The sign of the phase of the small motion at any node may be found from the following rule, which is verified by observation. If the tuning-fork which imposes the obligatory transverse motion is exactly at a node, it is opposite in phase to the small motion at the next node, and in the same phase as the motion at the node next after that, and so on.

III. The amplitude and the phase of oscillations maintained by forces of double frequency

In a note published in *Nature (London)* of the 9th December, 1909, and again more fully in a communication under the title "Remarks on a paper by J S Stokes on 'Some curious phenomena observed in connection with Melde's Experiment,'" published in the *Phys. Rev.* for March, 1911 (see Bulletins 2 and 3 of the Association), I drew attention to the fact that there were considerable discrepancies between the facts of observation and the theory first published by Lord Rayleigh (*Philos. Mag.*, April 1883, August 1887 and *Theory of Sound* Art. 68*b*) as regards the maintenance of vibrations by forces of double frequency, and I also indicated the cause of these discrepancies. The phenomena observed are not only interesting in themselves but are very important in connection with the general theory of the maintenance of vibrations by a variable spring which I shall discuss in the succeeding sections.

Lord Rayleigh starts with the following as his equations of motion:

$$\ddot{u} + k\dot{u} + (n^2 - 2\alpha \sin 2pt)u = 0 \tag{1}$$

and assuming that u may be put equal to

$$A_1 \sin pt + B_1 \cos pt + A_3 \sin 3pt + B_3 \cos 3pt + A_5 \sin 5pt + \&c. \tag{2}$$

proceeds to find the conditions that must be satisfied for the assumed state of steady motion to be possible. This he does by substituting (2) for u in the left-hand side of equation (1) and equating to zero the coefficients of sin pt, cos pt, etc. The relations thus obtained are (to a first approximation)

$$\frac{B_1}{A_1}=\frac{(\alpha-kp)^{1/2}}{(\alpha+kp)^{1/2}}=\tan E \qquad (2)$$

$$(n^2-p^2)^2=\alpha^2-k^2p^2. \qquad (3)$$

By a trigonometrical transformation equation (2) may be written in the form

$$kp=\alpha\cos 2E. \qquad (4)$$

These equations show that the phase of the motion is independent of the amplitude maintained and that the latter quantity is indeterminate.

It is possible to test experimentally the phase-relation as given by equations (2) and (3). The oscillatory system used for this purpose is a stretched string which is maintained in vibration by a periodic variation of tension of double frequency imposed on it with the aid of a tuning-fork. In this case the term $-2\alpha\sin 2pt$ is proportional to the motion of the tuning-fork and u corresponds to the maintained vibration of the string. The experimental problem therefore reduces itself to a determination of the phase-relation between the vibrations of the fork and the string, the frequency of one of which is double that of the other. This can be investigated by two distinct devices.

(i) *Mechanical composition of the two motions*: This is automatically effected and needs no special experimental arrangements. For, each point on the string (except the fixed end) has two motions at right angles to each other. The first is transverse to the string and is merely that due to its general vibration. The second is longitudinal to the string and is due to the motion in that direction of the prong of the fork to which the string is attached. The resulting path of any point on the string lies in the plane of oscillation and is one of the Lissajous figures for the interval of the octave. This curve may easily be rendered conspicuous by attaching a small fragment of a silvered bead to a point on the string near the tuning-fork. This is the most convenient position, though in case the vibration of the string is in two or more ventral segments, the bead may also be attached near any one of the other nodes as well.

(ii) *Optical composition of the two motions*: This is undoubtedly the more elegant of the two. To effect this, a small mirror is attached to the extremity of the prong of the fork. A tuning-fork with a steel mirror fixed to the end of one prong (see Lord Rayleigh, *Theory of Sound*, article 39) may well be used for the purpose. One point on the stretched string is illuminated by a transverse sheet of light from a lantern or with sunlight and a cylindrical lens. When the string is set in vibration, this appears drawn out into a luminous straight line which is viewed by reflection first at a fixed mirror and then at the oscillating mirror attached to the tuning-fork. If the plane of vibration of the prongs is at right angles to that in which the string vibrates (this may be secured by a simple experimental device), the illuminated point is seen to describe a Lissajous figure which renders evident at once the phase-relation under investigation.

Working by either of these methods, it is found that the phase of the motion is *not* independent of the amplitude maintained with any given initial tension. The best way of showing this is to use a bowed fork and after starting the motion with a large amplitude to gradually allow it to die away, the Lissajous figure or 'Curve of motion' as I shall call it and the changes that occur in it being watched during the process. It is observed that the initial curve of motion and the alterations that it undergoes when the motion is gradually damped down, both depend on the initial tension of the string. With a high initial tension so that the string can be maintained in its fundamental mode of vibration only by vigorous bowing of the fork, it is found that the curve is a parabolic arc which is convex to the tuning-fork and remains as such when the motion dies away. This state of matters continues so long as the initial tension is considerably in excess of that at which the free period of vibration of the string for small amplitudes is equal to the period of the fork. As the tension is gradually reduced, it will be observed while the initial curve for large amplitudes is a parabolic arc, it becomes modified into a looped figure as the amplitude decreases, still however remaining convex. When the tension is still further reduced so that the free period of the string for small oscillations is equal to that of the fork, the curve of motion for large amplitudes is still approximately parabolic or at any rate a looped figure convex to the fork, but as the motion dies away it alters into an 8-shaped figure. The most remarkable changes are however observed with a still smaller tension. In this case very large amplitudes of motion are maintained and the initial curve of motion is still convex, but as the motion is damped away it becomes an 8-shaped figure and finally a looped figure *concave* to the fork. At this stage the motion suffers very rapid damping, and when the initial tension is below a certain value a minimum amplitude of motion of the string exists below which steady motion is not possible. In the final stage with the smallest amplitudes, the curve of motion is a parabolic arc with its *concavity* towards the fork.

To enable these observations to be satisfactorily explained, it is necessary to modify Lord Rayleigh's theory so as to take into account the variations of tension that exist in free oscillations of sensible amplitude and are proportional to the square of the motion. In my paper on 'Photographs of Vibration Curves' in the *Philos. Mag.* for May 1911 (see Bulletin No. 5 of the Indian Association) I showed experimentally that such variations of tension exist by causing them to act on a sounding-board, which was held normal to the wire and would therefore have otherwise remained appreciably at rest. The vibration curve of the sounding-board was photographed on a moving plate along with and immediately above the vibration curve of the wire itself. It was observed that the frequency of the vibrations of the sounding-board was generally double that of the oscillations of the wire. But when the equilibrium position of the wire was a catenary of small curvature and its oscillations took place in a vertical plane, the motion of the sounding-board excited by them had a component of frequency identical with their own. These and other observations proved conclusively that variations of tension existed in free oscillations of sensible amplitude which were due to the second order differences in length between the equilibrium and displaced positions of the wire or string and were in fact proportional to the square of the displacement. Taking these into account the modified equation of motion under the action of forces varying the spring may be written as

$$\ddot{u} + k\dot{u} + (n^2 - 2\alpha \sin 2pt + \beta u^2)u = 0. \tag{5}$$

Assuming that

$$u = A_1 \sin pt + B_1 \cos pt + A_3 \sin 3pt + B_3 \cos 3pt + A_5 \sin 5pt + \&c.$$

and substituting in the left-hand side of equation (5), we obtain the conditions that must be satisfied for steady motion to be possible by equating to zero the coefficients of sin pt, cos pt, etc. Neglecting the quantities A_3, B_3, etc. as too small appreciably to effect the final result, we obtain

$$\tan E = \frac{B_1}{A_1} = \frac{\alpha - kp}{n^2 - p^2 + F}. \tag{6}$$

$$(n^2 - p^2 + F)^2 = \alpha^2 - k^2 p^2 \tag{7}$$

where

$$F \text{ is equal to } \frac{3\beta}{4}(A_1^2 + B_1^2)$$

and is therefore proportional to the square of the amplitude of motion. Equation (6) may as before be written in the form

$$kp = \alpha \cos 2E. \tag{8}$$

From these equations we may draw the following inferences:

If $\alpha < kp$ no steady motion is possible. When

$$n < p \quad \text{and} \quad (p^2 - n^2)^2 > \alpha^2 - k^2 p^2$$

maintenance would evidently be impossible unless F had a certain finite minimum value. This is in accordance with the results of experiment. When the initial amplitude is less than that given by this minimum, the motion cannot be sustained and rapidly dies away. On the other hand, if the initial amplitude is equal to or greater than the required minimum, it shows a marked tendency to increase rapidly of itself up to many times the initial value. The reason for this is pretty clear from equation (7). When $(p^2 - n^2)$ is positive and greater than the right-hand side of that equation, any increase of the amplitude of motion *diminishes* the quantity on the left, with the result that a still further increase in amplitude is entailed, and it continues to increase till $(n^2 - p^2 + F)$ again becomes positive in sign and equal to $(\alpha^2 - k^2p^2)^{1/2}$ in magnitude. A somewhat similar rapid increase in the amplitude (though not of such a marked character) takes place in all cases where the initial tension is less than the theoretical value. The motion is however capable of starting from infinitely small vibrations if $(p^2 - n^2)^2$ is equal to or greater than $(\alpha^2 - k^2p^2)$. On the other hand, when the tension is sufficient or in excess no such phenomenon is observed. The increase of the motion from infinitely small amplitudes up to the value required to satisfy equation (7) is then quite gradual.

Again it is evident that for given values of α and kp, F is not a maximum when the tension is equal to the theoretical value. In other words, the maximum amplitude of motion is *not* obtained when the free period of the string for small oscillations is double that of the tuning-fork. This somewhat paradoxical result is entirely verified by observation. In fact it is clear that the amplitude maintained is largest when n is less than p and has as small a value as is consistent with steady motion in the given mode, in other words when the initial tension of the string is considerably in defect.

We now proceed to discuss the phase of the maintained motion. This is given by equations (6) or (8) above. From the former it is evident with E, i.e. the phase-difference, is always positive if $(n^2 - p^2 + F)$ is of that sign. The curve of motion is therefore convex to the fork when the tension is in excess and also when it is in defect, provided the amplitude of motion is sufficiently large. The maximum positive phase-difference is $\pi/4$ and this is attained when α is large compared with kp. It can be seen from equation (7) that a large variation of tension is required to start the motion when the initial tension is high. The curve of motion is then a parabolic arc convex to the fork and continues as such so long as the tension is in excess and the amplitude is sufficiently large. But with small amplitudes, the phase-difference though positive is less than $\pi/4$ and the curve of motion is a looped figure. When the initial tension is equal to the theoretical value and the amplitude of motion is very small $\alpha = kp$ and $E = 0$ and the curve of motion is shaped like an 8. This is in agreement with observation. But when the amplitude increase, the phase-difference again becomes finite and the curve is convex to the fork. When the initial tension of the string is in defect, the phase-difference is positive or negative according as the amplitude is large or small and the curve of motion is convex or concave under the respective circumstances. It is not at all difficult to observe any of these different cases, though in order to maintain the motion steadily with the curve in the concave position some careful adjustment of the amplitude of motion of the tuning-fork will generally be necessary. The largest negative value of the phase-difference is $-\pi/4$ and the curve is then a parabolic arc *concave* to the fork. The significance of this is that when the fork is at its extreme outward swing, the string is also at its position of maximum displacement: a paradoxical result not in accordance with the ordinary ideas of the experiment.

A glance at the Lissajous figures for the interval of the octave pictured in figure 7 of Lord Rayleigh's *Theory of Sound* will show that it is possible for two similar and similarly situated curves to represent different relative phases of motion between its components, if the moving point describes the two curves in opposite directions. In order therefore to verify the phase-relation experimentally it is necessary, in addition to observing its shape, to note the direction in which the curve of motion is described. This may be done by observation through a stroboscopic disk which is kept revolving at a speed slightly less than that at which it would give one stationary view of the vibrating string. It is then fairly easy to make out the direction in which a fragment of a silvered bead attached to a point on the string near the tuning-fork describes the curve compounded of its motions longitudinal and transverse to the string. The observed direction agrees with that indicated by theory.

There is another way of writing the equations of motion which is very useful in that it gives a clearer view of the whole case and leads us on to the subject of the next chapter. Neglecting the terms in A_3, B_3, etc. we may put $u = P\sin(pt + E)$. Equation (5) may be written as under

$$\ddot{u} + k\dot{u} + \left(n^2 + \frac{\beta P^2}{2}\right)u = \left[2\alpha \sin 2pt + \frac{\beta P^2}{2}\cos(2pt + 2E)\right]u$$

$$= \alpha P\cos(pt - E) + \frac{\beta P^3}{4}\sin(pt + E)$$

if trigonometrical functions of $3pt$ are neglected. This may be succinctly written in the form

$$\ddot{u} + k\dot{u} + N^2u = \alpha_1 P\cos(pt - E_1). \tag{9}$$

This is the ordinary form of the equation of a system subject to forced vibrations, and if Lord Rayleigh's equation, see (1) above, had been treated in the same way, we should have obtained

$$\ddot{u} + k\dot{u} + n^2u = \alpha P\cos(pt - E). \tag{10}$$

From equations (9) and (10) it is clear that a large motion might be sustained when $p = N$ or n as the case may be, and that the maintenance of vibrations by forces of double frequency is in essence only an illustration of the general principle of resonance according to which a large motion may be set up if we have equality of periods between a system and the forces acting upon it. A comparison of equations (9) and (10) shows that the introduction of the term βu^3 on the left-hand side of (5) results in a decrease in the free period of the system and also a change in the magnitude and the phase of the restoring force acting upon it. These modifications fully account for the phenomena discussed above. Equating the work done by the force represented by the right-hand side term of either of the equations (9) or (10) in any number of complete periods of the variable tension to the energy dissipated by the friction term on the left we deduce the relation

$$kp = \alpha \cos 2E$$

which is identical with that obtained, see (4) and (8) above, from the complete analysis.

IV. Vibration curves of oscillations maintained by a variable spring

In the last section I discussed the case of the maintenance of vibrations by forces of double frequency and emphasized the fact that in reality it only furnishes us with an illustration of the general principle of resonance according to which a periodic force acting on a system whose period is approximately equal to its own, may maintain a very considerable amplitude of motion, though in other cases its effect might be so small as to be of little account. In the course of the experimental work described in the preceding sections, I came across some other extremely interesting and remarkable cases of resonance which formed *apparent* exceptions to the above-stated law of approximate equality of periods. These cases I propose to discuss in the present paper. A preliminary note on this class of maintained vibrations was published by me in *Nature* (*London*) of the 9th December, 1909, and another (illustrated) in the issue of the 10th February, 1910 (see Bulletin No. 2 of the Indian Association).

The title of this section gives an indication of the character of the forces whose action we now proceed to discuss. They only alter the 'spring' or restitutional coefficient of the system and do not tend directly to displace it from the position of equilibrium. My observations showed that there were several *quite distinct* cases in which periodic forces of this character acting on a system set up a large motion. These cases may be tabulated as follows:

(1) When the period of the force is $\frac{1}{2}$ that of the system.
(2) When the period of the force is $\frac{2}{2}$ times that of the system.
(3) When the period of the force is $\frac{3}{2}$ times that of the system.
(4) When the period of the force is $\frac{4}{2}$ times that of the system.
(5) When the period of the force is $\frac{5}{2}$ times that of the system.
(6) When the period of the force is $\frac{6}{2}$ times that of the system.
(7) When the period of the force is $\frac{7}{2}$ times that of the system.
&c. &c. &c. &c.

Each of these forms a distinct type of maintained motion which can be obtained and studied separately by itself. The first is evidently identical with the case of 'double frequency' which was discussed in the preceding section.

To obtain any one of these types of motion, we adopt a procedure very similar to that by which the maintenance of vibrations by forces of double frequency is secured. Using a stretched string as our 'system' we subject it to a periodic variation of tension by attaching it to a tuning-fork whose prongs vibrate in a direction parallel to the string. The tension of the string (the length of which should be suitable) is adjusted so that its period of vibration in a given mode (for instance in its fundamental mode) bears the required ratio to the period of vibration of the tuning-fork. It will then generally be found that the equilibrium position of the string becomes unstable and it settles down into a state of permanent and (in suitable circumstances) vigorous vibration in which the number of swings (to and fro) made by it per unit of time bears the desired ratio to the frequency of the tuning-fork.

Each of these types of vibration presents some very remarkable peculiarities, a study of which enables us to explain the manner in which the maintenance is effected in a simple and intelligible way. When I first observed some of these types I proceeded to investigate them by precisely the same methods which I applied to the case of double frequency, i.e. mechanical or optical composition of the motion of the string with that of the tuning-fork. It is obvious that the methods are applicable to all these cases and in fact in some respects, e.g. for a detailed investigation of the phase of the maintained motion, they are probably superior to other methods of investigation that could be devised. It is evident that the Lissajous figure seen gives us at once the requisite information as regards the frequency and phase-relation between the motion of the string and that of the fork. For purposes of demonstration, however, the method of 'vibration curves' which I shall now proceed to discuss, yields far more striking and impressive results.

By the 'vibration curve' of an oscillation I mean of course its time-displacement diagram, or an equivalent thereof. To enable the vibration curves of the oscillation of the string and that of the tuning-fork to be recorded side by side for comparison, the following arrangement was found the most suitable. Two slits were used as sources of light. One of them was horizontal and the other which was vertical was placed immediately behind the oscillating string. Both the slits were illuminated by sunlight and had collimating lenses in front of them. The fork stood with its prongs vertical and a small silvered mirror was attached with wax to the side of one of the prongs, and this of course tilted periodically through a small angle when the fork was in vibration. The light issuing from the horizontal slit was incident in a nearly normal direction upon this mirror, and after suffering reflection at it fell upon the lens (having an aperture of $1\frac{1}{2}$ inches diameter) of a roughly constructed camera. The light issuing from the vertical slit in a direction at right angles to that from the other was deflected through 90° by reflection at a fixed mirror and also fell upon the lens of the camera. In the focal plane of the latter was placed a metal plate with a vertical slit cut in it. The images of the horizontal and vertical slits fell one immediately above the other on the slit in the plate. Of the former only a very small length, i.e. practically only a point of light, passed through to fall upon the ground-glass of the camera or its substitute, the photographic plate. Immediately below it was the narrow image of the vertical slit crossing which was seen the shadow of the string when at rest. To photograph the vibration curves the ground-glass was removed and the dark slide which held the plate was moved as uniformly as possible by hand in horizontal grooves behind the slit in the focal plane of the camera. In the positive reproductions, the vibration curve of the string appears as a dark curve on a bright ground and that of the tuning-fork vice versa. We now proceed to consider each type of maintained oscillation and its vibration curves separately.

The first type

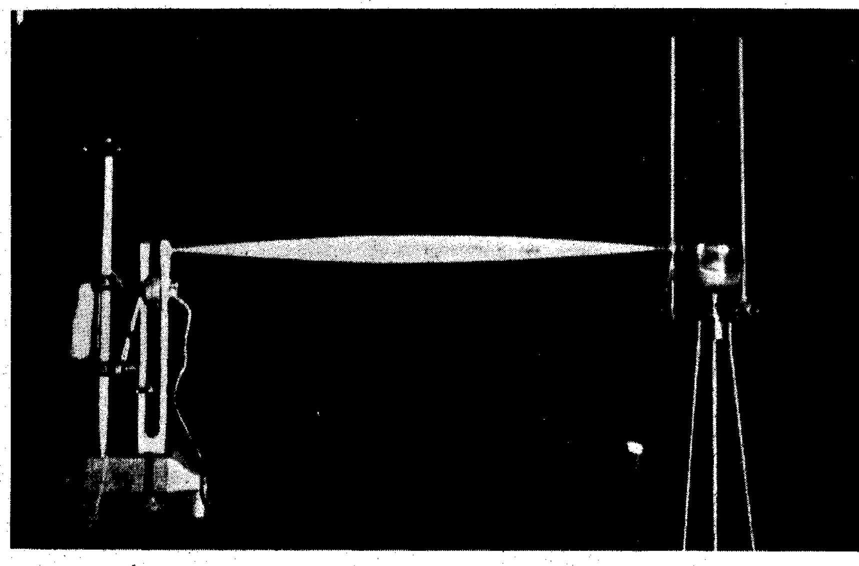
Plate III shows photographs of the well known case of the string maintained in a vibration of half the frequency of the tuning-fork, and of its vibration-curves from which it is evident at once that the tuning-fork makes two vibrations for every oscillation of the string. The photograph shows the maximum displacements of the string to have occurred almost exactly at the epochs of minimum tension, from which we may infer, since the amplitude of motion of the string was by no means very large, that the initial tension of the string was in excess of the theoretical value, *vide* section III of this Bulletin. We shall now consider in some detail.

The second type

The frequency of the oscillation of the string is in this case the same as that of the fork which varies its tension. This type is shown in figure 1, plate IV. The string vibrates in its fundamental mode, but it will be noticed that its curvature at one of the positions of maximum displacement is greater than at the other. Figure 2, plate IV and plate V, show the vibration curves of this type of oscillation, and it is clear that the frequency of the motion of the string and of that of the fork are equal. In securing the photograph shown in plate IV it was arranged that the string when at rest should exactly bisect the slit. It will be seen that its vibration curve has been displaced bodily towards one side of the slit and is thus nearer the other curve. The significance of this is that the mid-point of its oscillation is displaced to one side of the equilibrium position of the string. This is confirmed by direct observation and accounts for the greater curvature of one of the positions of maximum displacement observed in figure 1, plate IV. The transverse motion of *each* point on the string may therefore be represented by an expression of the form

$$u = P \sin(2pt + E_2) + Q \tag{1}$$

1



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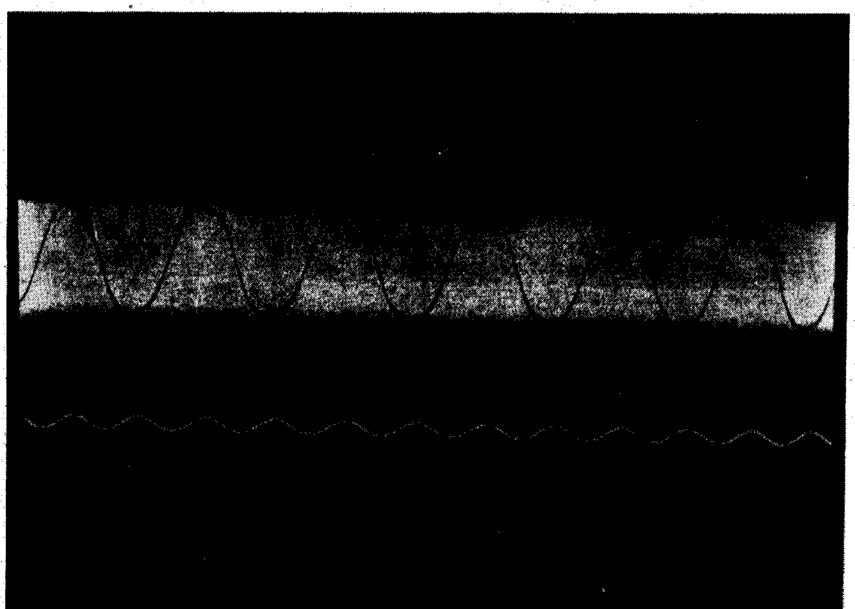
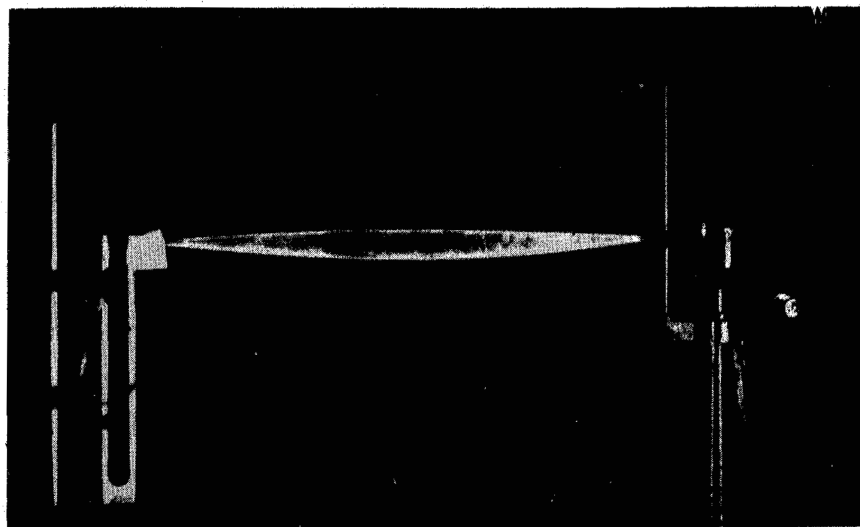


Plate III. Oscillations maintained by a variable spring of double frequency.

1



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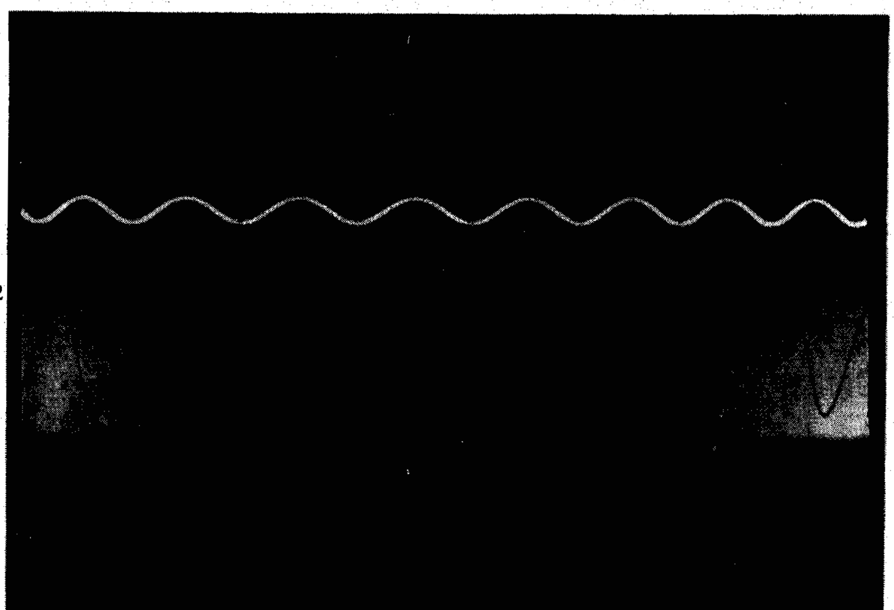

Plate IV. Maintenance of vibrations by a variable spring of equal frequency.

1

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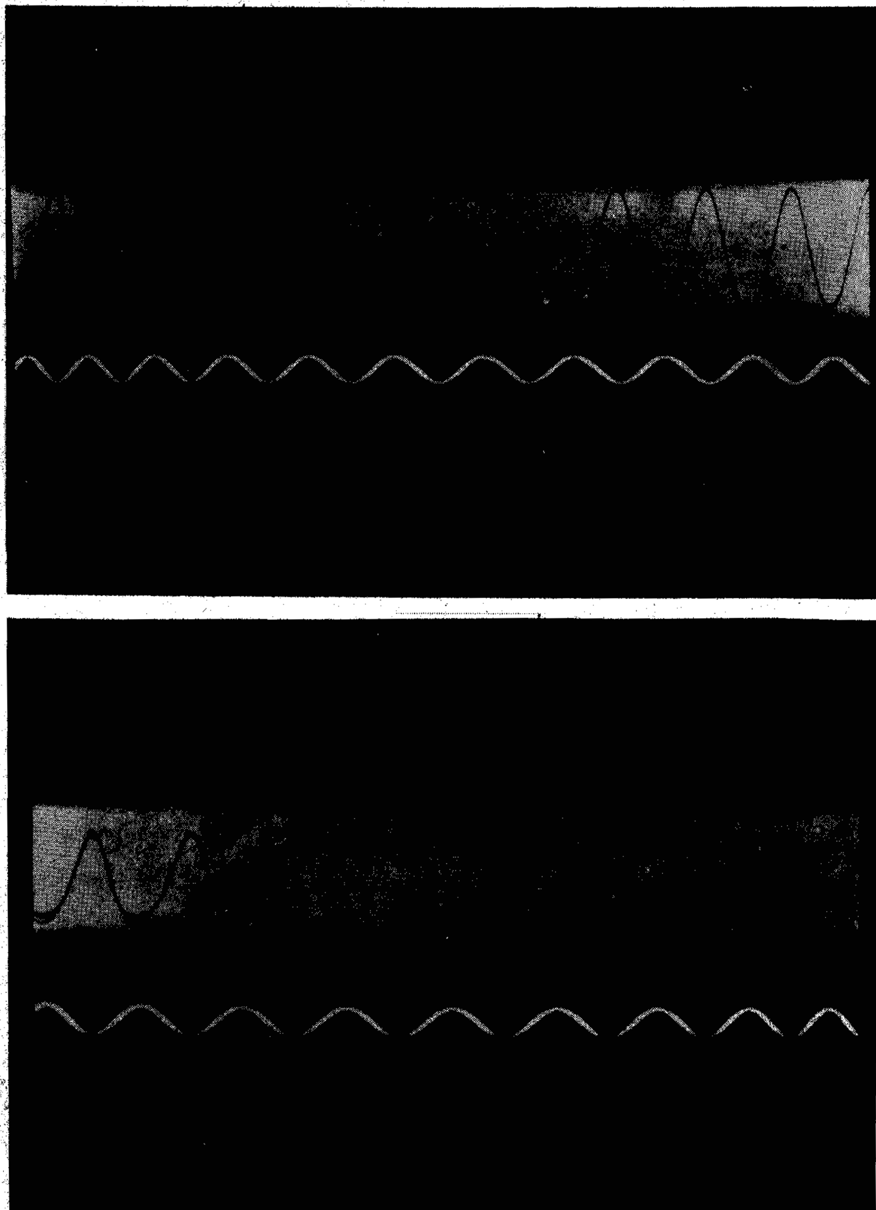


Plate V. Vibration-curves of oscillations maintained by a variable spring of equal frequency.

the ratio of the coefficients P and Q being practically the same for all points on the string.

A motion of the type represented by (1) above cannot exist if the oscillations of the string were 'free' and under constant tension, inasmuch as the restoring forces at the two positions of extreme displacement would not be equal and opposite. But we are dealing here with forced oscillations under variable tension. A reference to the vibration curves will show that the maximum displacements (on either side) of points on the string occur at epochs not very far removed from those of maximum and minimum tension. During one half of its oscillation the string is under a tension which is less than its normal value and during the other half under a tension which is correspondingly in excess. During the former half the motion being under diminished constraint swells out and increases in amplitude and during the other half the reverse is the case. The net result is that while the simple harmonic character of the motion is not generally departed from to any very considerable extent, the oscillation appears to take place about a point displaced to one side of the position of equilibrium, in the manner indicated by equation (1) above.

We are now in a position to understand in what manner the maintenance is effected in this case. We may write the equation of motion of a system having one degree of freedom and subject to a variable spring thus

$$\ddot{u} + k\dot{u} + n^2 u = 2\alpha u \sin 2pt. \tag{2}$$

Substituting $P \sin(2pt + E_2) + Q$ for u in the right-hand side of this equation, we get

$$\begin{aligned}\ddot{u} + k\dot{u} + n^2 u &= 2\alpha P \sin 2pt \sin(2pt + E_2) + 2\alpha Q \sin 2pt \\ &= 2\alpha Q \sin 2pt + \alpha P \cos E_2\end{aligned} \tag{3}$$

if we neglect trigonometrical functions of the angle $4pt$. The first term on the right represents transverse periodic forces acting on each element of the string which would maintain a large motion having the same frequency as that of the fork if n is approximately equal to $2p$. The second term stands for a system of constant forces impressed transversely at each point on the string under the action of which the mean point of the maintained motion is displaced to one side of the equilibrium position. This is just what we get. We assumed that $u = P \sin(pt + E_2) + Q$ and the importance of the term Q is sufficiently clear from what has been said above. Substituting for u in the left side of equation (3) we get the following relations:

$$\tan E_2 = \frac{2kp}{4p^2 - n^2}. \tag{4}$$

$$Q^2/P^2 = \alpha^2 \cos^2 E_2/n^4 = (n^2 - 4p^2)/2n^2. \tag{5}$$

$$n^2[(n^2 - 4p^2)^2 + 4k^2p^2] = 2\alpha^2(n^2 - 4p^2). \tag{6}$$

These equations represent the relations that must be satisfied if maintenance is to be possible. They are a fair approximation to the truth so long as the phase-difference E_2 is small. It is necessary, however, to consider the question whether the effect of terms containing trigonometrical functions of $4pt$ can be entirely ignored, particularly when according to the above formulae, Q becomes very small, which is the case when the phase-difference approaches the value $\pi/2$. We have already seen that the right-hand side of equation (3) contains such terms. We may therefore write

$$u = P\sin(2pt + E_2) + Q + R\sin(4pt + E_4) \tag{7}$$

where the ratios $P:Q:R$ are the same at all points on the string. Substituting this on the right of equation (2) we get

$$\begin{aligned}\ddot{u} + k\dot{u} + n^2u = \alpha P\cos E_2 + 2\alpha Q\sin 2pt \\ + \alpha R\cos(2pt + E_4) - \alpha P\cos(4pt + E_2).\end{aligned} \tag{8}$$

Each of the terms on the right of this equation represents a system of transverse forces, the effect of which we may consider separately. The first and the second we have already dealt with. The effect of the third depends upon its phase, i.e. upon the value of E_4. This can be found by considering the action of the component of the restoring force represented by the fourth term, which has a frequency approximately double that of the free oscillation of the string. Its effect should therefore be small and should have a phase exactly opposite to that of the force producing it. By substituting for u in equation (8), we find

$$(n^2 - 16p^2)R\sin(4pt + E_4) = -\alpha P\cos(4pt + E_2)$$

and E_4 is equal to $E_2 + \pi/2$. The third term on the right of equation (8) is therefore to $-\alpha R\sin(2pt + E_2)$ and being exactly opposite in phase to the principal part of the motion dealt with, i.e. $P\sin(2pt + E_2)$ cannot assist in maintaining it. Its effect is merely equivalent to an alteration in the free period of oscillation of the string, and the motion is maintained entirely by the force proportional to Q represented by the second term. We have then the following relations which must be satisfied for the assumed state of steady motion to be possible.

$$\tan E_2 = -\cot E_4 = \frac{2kp}{4p^2 - N^2} \tag{9}$$

$$\frac{Q^2}{P^2} = \alpha^2\cos^2 E_2/n^4 = (N^2 - 4p^2)/2n^2 \tag{10}$$

$$R/P = L/12p^2 \tag{11}$$

$$n^2[(N^2 - 4p^2)^2 + 4k^2p^2] = 2\alpha^2(N^2 - 4p^2) \tag{12}$$

where

$$N^2 = n^2 + L^2/12p^2.$$

From formulae (10) and (11) it appears that the term $R\sin(4pt + E_4)$ in the expression for the displacement is quite appreciable (having an amplitude at least $\frac{1}{3}$ that of the term Q), though it does not assist in the maintenance of the motion. Figure 2, plate V, shows this component in the motion quite clearly, The vibration-curves for all points on the string exhibit the harmonic to an equal degree. According to equations (9) to (12) above, the phase of the maintained motion is independent of the amplitude, the latter quantity being indeterminate and the adjustment of pitch must be absolutely rigorous for steady vibration. All these inferences are however subject to modification in practice. The tension of the string in free oscillations of large amplitude is not constant, generally increasing by a quantity proportional to the square of the motion, and the necessary adjustment of pitch may therefore be secured by an alteration of the amplitude of motion. With however a heavy or long string horizontally held and under moderate tension, the effect of gravity is not negligible and the law of variation of the tension with the amplitude varies with the plane of the oscillation. If this is in a vertical plane, the fact that the equilibrium position is a catenary of small curvature becomes of some importance, particularly when the vibration of the string is in its fundamental mode. The tension in free oscillations of sensible amplitude would be of the form

$$n^2 + \beta(u - a)^2$$

if they occur in a vertical plane and of the form

$$n^2 + \beta u^2$$

if in a horizontal plane.

If u is put equal to $P \sin(2pt + E_2) + Q$ it is evident that it is not open to us indifferently to alter the signs of both P and Q together and retain the conditions of the motion unchanged in the former case as would be possible in the latter. The average tension during the motion as given by the first formula would evidently be greater when Q is negative, i.e. directed downwards than when it is in the opposite direction. This appears to be the reason why as in figure 1, plate IV, the oscillation generally sets itself so that of the two extreme positions of the string the one which has the greater curvature is concave upwards.

Again

$$[n^2 + \beta(u - a)^2]$$

may when expanded be written in the form

$$n^2 + \beta[P^2/2 + (Q - a)^2] + \beta P[2(Q - a) \sin(2pt + E_2) - P/2 \cos(4pt + 2E_2)]. \tag{13}$$

Of the two periodic terms the first has the same frequency as the variation of spring imposed on the system and no doubt plays an important part in the adjustment of the phase-relation between the motions of the fork and the string. In view of the fairly complete discussion of similar effects in the case of doube frequency (section III) we need not pause to consider further detail, but proceed to discuss.

The third type of motion

This is shown in figure 1, plate VI. Figure 2, plate VII and figure 1, plate VIII, represent the vibration-curves of the string and the fork in cases coming under this class. The string makes three swings for every two vibrations of the fork, but the swings are not all of equal amplitude: This is evident from the vibration-curves and also from the appearance of the string itself in the first of the photographs. In addition to the two extreme positions, the photograph shows clearly two intermediate resting points of the string, one on each side of its equilibrium position, which mark the limits of the swings made at the epochs when the tension of the string is in excess. The two outer resting points, as can be seen from the vibration-curves, correspond almost exactly with the epochs of minimum tension at which the vibration being under diminished constraint swells out and increases in amplitude. The motion at any point of the string is capable of being very approximately represented by two terms. Thus

$$u = P\sin(3pt + E_3) + Q\sin(pt + E_1) \tag{14}$$

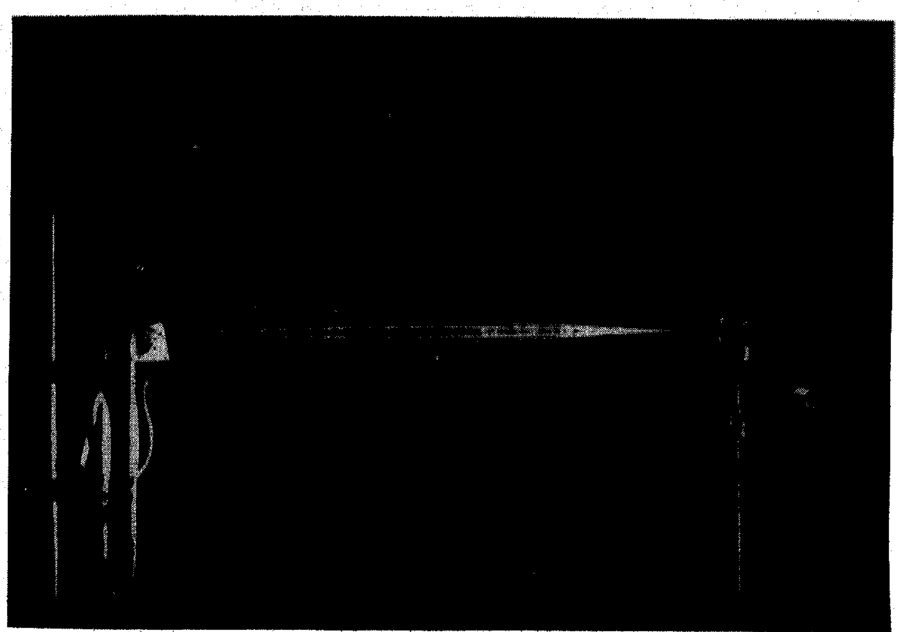
the second of which has the smaller amplitude and frequency and is brought into existence under the action of the variable tension. The ratio P/Q is the same at all points on the string, the motion of which may therefore be discussed as if it had only one degree of freedom. The frequency of the second term is less than that of the first by a quantity which is itself the frequency of the variable spring. The analogy between the motion as shown in the vibration-curves and that in the atmospheric 'beats' of two simple tones, one of which has the smaller amplitude and frequency, is fairly clear (see Helmholtz's 'Sensations of Tone,' appendix XIV). The time which is required for one swing undergoes periodic fluctuations, being greatest when the tension is least and vice versa. This corresponds to the periodic flattening and sharpening of the 'beats.'

It can be shown that the maintenance of the vibrations is effected entirely by the aid of the periodic component of lower frequency, i.e. $Q\sin(pt + E_1)$, in the expression for the steady motion under variable spring. The product of this term into the variable spring gives a transverse periodic force acting on the system, which is of the right frequency and phase for maintaining its vibrations. The equation of motion may be written

$$\begin{aligned}\ddot{u} + k\dot{u} + n^2u = \alpha[&P(\cos\overline{pt + E_3} - \cos\overline{5pt + E_3}) \\ &+ Q(\cos\overline{pt - E_1} - \cos\overline{3pt + E_1})]\end{aligned} \tag{15}$$

the fourth and last term on the right representing the force referred to above.

1



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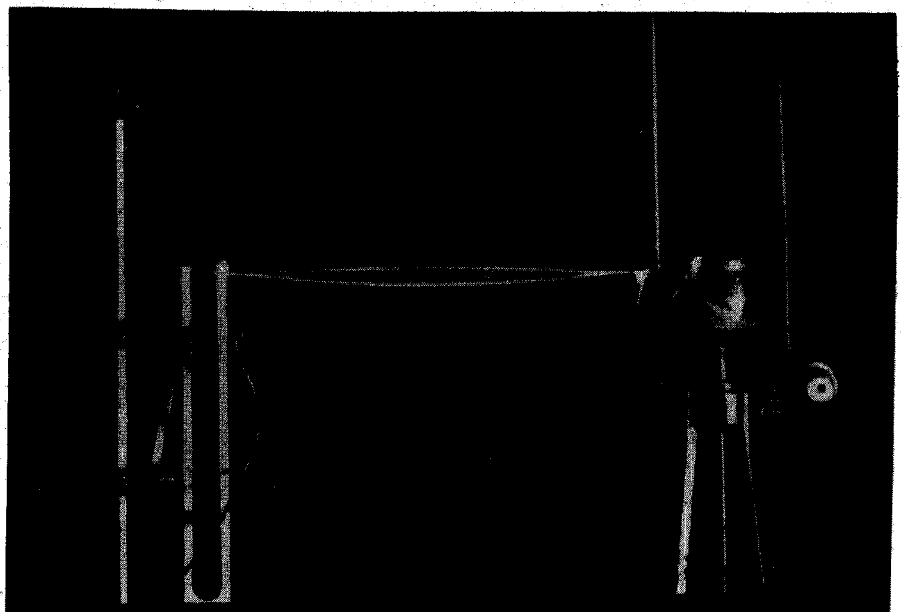

Plate VI. 1. The third type of maintenance. 2. The fourth type of maintenance.

1
2

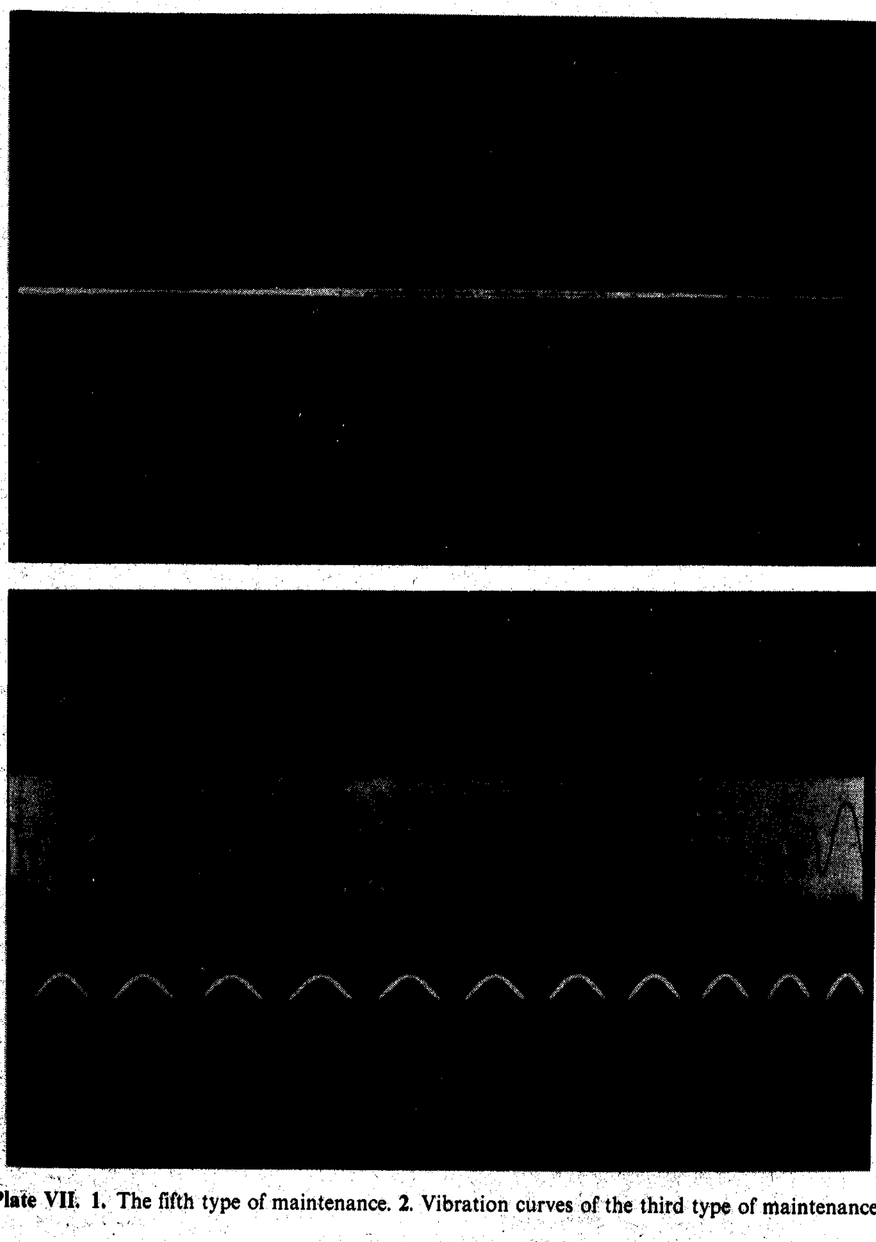

Plate VII. **1.** The fifth type of maintenance. **2.** Vibration curves of the third type of maintenance.

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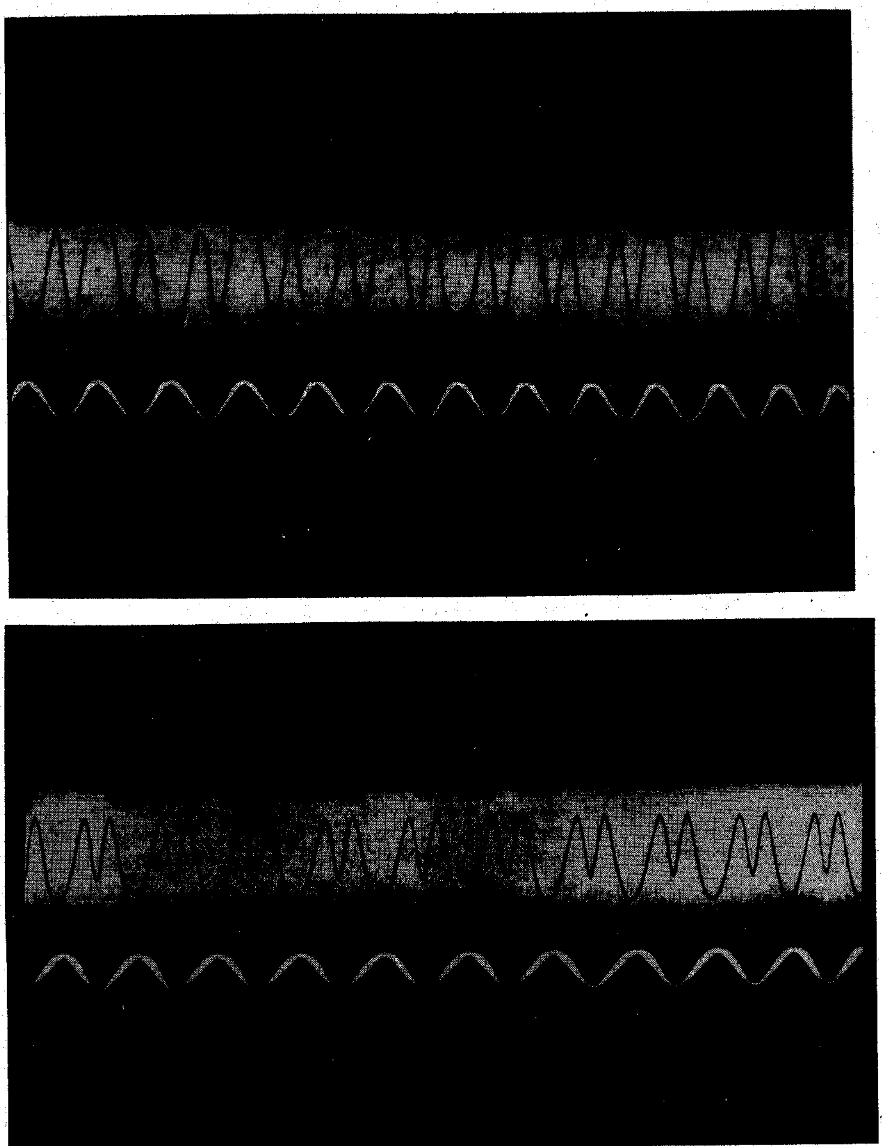

Plate VIII. Vibration curves of (**1**) the third type of maintenance and (**2**) the fourth type of maintenance.

Substituting for the terms on the left and reducing, we get four equations which give us the values of E_1, E and of the ratio Q/P and leave us in addition a relation between the 'constants' involved. The first equation is

$$3kpP = -\alpha Q \cos(E_3 - E_1) \tag{16}$$

and this expresses the relation between the energy supplied and the energy dissipated in any number of complete periods of the variable spring. The phase-difference $(E_3 - E_1)$ may be eliminated with the aid of the second relation

$$\tan(E_3 - E_1) = \frac{n^2 - 9p^2}{3kp}. \tag{17}$$

In practice, as can be seen from the vibration-curves, E_3 is nearly equal to $-\pi/4$ and E_1 nearly equal to $+\pi/4$. Cos $(E_3 - E_1)$ is therefore nearly equal to zero. It is evident from (16) that kp is very small compared with α.

The third relation is

$$\tan E_3 = \frac{(n^2 - p^2) - (\alpha + kp)\tan E_1}{(\alpha - kp) - (n^2 - p^2)\tan E_1}. \tag{18}$$

This may be simplified and written as

$$\tan E_3 = (8p^2 - \alpha \tan E_1)/(\alpha - 8p^2 \tan E_1).$$

It is of interest to note that the ratio between the amplitudes P and Q is of the same order of quantities as the ratio between the constant and variable parts of the spring. It can be readily shown that to a first approximation

$$Q = \frac{9\alpha P}{8n^2 - 9\alpha}. \tag{19}$$

Finally we get the relation between the constants involved by eliminating P, Q and $(E_3 - E_1)$ between the three equations (16), (17) and (19). Neglecting quantities of the order α^4 we get

$$n - 3p = \frac{\alpha^2}{n^2}\left(\frac{9}{16} + \frac{81}{128}\frac{\alpha}{n^2}\right) \tag{20}$$

which gives us an idea of the accuracy in adjustment of pitch that is required. In duducing this relation it is assumed that $3kp$ is of the order α^2/n^4, and this is necessary if the motion is to be maintained.

It remains to consider the effect of the force represented by the term $-\alpha P \cos(5pt + E_3)$ on the right of equation (15). For this purpose we start afresh and assume that

$$\begin{aligned} u = {} & A_1 \cos pt + A_3 \cos 3pt + A_5 \cos 5pt \\ & + B_1 \sin pt + B_3 \sin 3pt + B_5 \sin 5pt. \end{aligned} \tag{21}$$

Substituting in the equation of motion

$$\ddot{u} + k\dot{u} + n^2u = 2\alpha u \sin 2pt$$

we get the following relations

$$A_5 = \alpha B_3/16p^2, \quad B_5 = -\alpha A_3/16p^2 \tag{22}$$

$$\left.\begin{aligned} 3kpA_3 - (n^2 - 9p^2 + \alpha^2/16p^2)B_3 &= -\alpha A_1 \\ (n^2 - 9p^2 + \alpha^2/16p^2)A_2 + 3kpB_3 &= -\alpha B_1 \end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned} (\alpha + kp)A_1 - (n^2 - p^2)B_1 &= \alpha A_3 \\ (n^2 - p^2)A_1 - (\alpha - kp)B_1 &= \alpha B_3 \end{aligned}\right\} \tag{24}$$

Equations (18) and (24) are identical and by comparing equations (17) and (23) it can readily be seen that the only change is that instead of n^2 we have the very slightly larger quantity $(n^2 + \alpha^2/16p^2)$ in the latter equation. The inference is that the component in the displacement which has a frequency higher than that of the principal part of the motion does not assist in its maintenance, the effect produced by it being merely equivalent to a very small decrease in the free period of the oscillation of the string. It will be recollected that a similar result was obtained in the case of the second type of the maintenance of vibrations discussed above. From equation (22) it appears that the amplitude of the component of frequency $5p/2\pi$ in the maintained motion is less than half the amplitude of the component of frequency $p/2\pi$. If the former is represented by $R \sin(5pt + E_5)$ it is evident from what was said above that E_5 is nearly equal to $\pi/4$. We have roughly

$$u = P\sin(3pt - \pi/4) + Q\sin(pt + \pi/4) + R\sin(5pt + \pi/4)$$

when

$$pt = \pi/4, \quad u = (P + Q - R),$$

and when

$$pt = 5\pi/4, \quad u = -(P + Q - R).$$

The maximum amplitudes are therefore less than they would be if the component R did not exist. Its presence should therefore render itself evident by a flattening of the vibration-curve at the epochs of minimum tension. Some flattening of this kind, though not very marked, appears to be shown in figure 2, plate VII.

The preceding discussion gives us only the phases and the ratios of the amplitudes of the components of the maintained motion, and according to the equations the actual amplitudes are indeterminate. In practice however the equation of motion is subject to modification on account of the variation of tension in free oscillations of sensible amplitude. For simplicity we may consider a case in which the string is vertical. The effect of gravity on its transverse oscillations may then be neglected, and the equations of motion may be written in the form

$$\ddot{u} + k\dot{u} + (n^2 + \beta u^2 - 2\alpha \sin 2pt)u = 0$$

If

$$u = P\sin(3pt + E_3) + Q\sin(pt + E_1) + R\sin(5pt + E_5), \quad n^2 + \beta u^2$$
$$= n^2 + \beta(P^2 + Q^2 + R^2)/2 + \text{a large number of periodic terms.} \qquad (25)$$

As may have been expected, the average tension is increased by a large amplitude of motion and this is no doubt what secures the necessary adjustment of pitch and determines the amplitude of the maintained vibration. Of the periodic terms in (25) probably the most important are those which have a frequency equal to that of the imposed variable spring and tend directly to alter its magnitude or effectiveness. There are only three such terms, and if we neglect the others,

$$n^2 + \beta u^2 = n^2 + \beta(P^2 + Q^2 + R^2)/2 + \beta QP\cos\overline{2pt + E_3 - E_1}$$
$$- \beta Q^2\cos\overline{2pt + 2E_1} + \beta RP\cos\overline{2pt + E_5 - E_3}.$$

Putting $E_3 = -\pi/4$ and $E_1 = E_5 = \pi/4$ approximately, we have as our equation of motion

$$\ddot{u} + k\dot{u} + [N^2 - (2\alpha - Q(P + Q) + RP)\sin 2pt]u = 0.$$

From this it seems evident that when the imposed variation of tension is in excess of that just required to maintain the motion, the component of amplitude Q tends to increase at the expense of the component with amplitude R. The latter tends to become even less important than it would otherwise be, and indeed there does not appear to be any very marked indication of its existence in the vibration curves.

The fourth type of motion

This is shown as figure 2, plate VI, and its vibration curve as figure 2, plate VIII. In the former a resting point intermediate between the two extreme positions of the string is clearly visible, and it can be seen in the vibration curve that this corresponds exactly to the terminal point of the swing at the epoch of maximum tension. The string makes four swings for every two vibrations of the fork. Of these the two occurring when the tension is in excess are less in amplitude and take a shorter time than the two others made when the tension is in defect. The maintained motion consists therefore of a principal component of frequency double that of the fork, the simple harmonic character of which is modified under the action of the variable spring and which therefore appears along with a subsidiary motion of the same frequency as that of the fork. As in the previous cases discussed, the ratio between the two components is practically the same at all points of the string, and the problem may therefore be dealt with as if it related to a system with one degree of freedom only.

For a full discussion we must assume that the displacement at any instant may be represented by an expression of the form

$$u = A_2 \sin 2pt + A_4 \sin 4pt + A_6 \sin 6pt + B_0 + B_2 \cos 2pt + B_4 \cos 4pt + B_6 \cos 6pt. \tag{26}$$

The terms of frequency $4p/2\pi$ form the principal part of the maintained motion, and these and the terms of lower frequency $2p/2\pi$ are, as can be seen from the vibration curve, predominant. As in the previous cases discussed, it can be shown that the latter terms are mainly instrumental in maintaining the motion, in other words that their product into the variable spring gives a transverse periodic force of the right frequency and phase for maintaining the vibrations. The components of frequency $6p/2\pi$ have an influence on the motion of the system which is equivalent merely to a slight decrease in the period of the free vibrations of the string, and they do not otherwise assist in the maintenance of the vibrations. The constant term B_0 though small is by no means negligible, and it remains to investigate its influence in the present case. As shown in the investigation of the second type of motion, a term of this kind may be regarded as the result of a system of constant forces acting at all points on the string. To solve the equation of motion, we substitute (26) for u in the formula

$$\ddot{u} + k\dot{u} + n^2 u = 2\alpha u \sin 2pt$$

The following relations are obtained

$$A_6 = -\alpha B_4/20p^2, \quad B_6 = \alpha A_4/20p^2 \tag{27}$$

$$\left.\begin{aligned} (n^2 - 16p^2 + \alpha^2/20p^2)A_4 - 4kpB_4 &= \alpha B_2 \\ 4kpA_4 + (n^2 - 16p^2 + \alpha^2/20p^2)B_4 &= -\alpha A^2 \end{aligned}\right\} \tag{28}$$

$$n^2 B_0 = \alpha A_2 \tag{29}$$

$$\left.\begin{aligned} (n^2 - 4p^2)A_2 - 2kpB_2 &= \alpha(2B_0 - B_4) \\ 2kpA_2 + (n^2 - 4p^2)B_2 &= \alpha A_4 \end{aligned}\right\} \tag{30}$$

It is not permissible to leave out B_0 in the first of the equations (30), for, if we do so and eliminate A_2, B_2, A_4 and B_4 between the equations (28) and (30) we get an eliminant of the form $S^2 = -T^2$ which is evidently absurd. The significance of this may be understood by writing equations (28) in the form

$$4kp(A_4 + B_4)^{1/2} = -\alpha(A_2 + B_2)^{1/2} \cos(E_2 - E_4). \tag{31}$$

This formula expresses the relation that the energy dissipated by friction in a time comprising any number of complete periods of the variable spring is equal to that supplied in the same time through its agency. Now it can be seen from the vibration curve that $(E_2 - E_4)$ is very nearly equal to $\pi/2$ and $\cos(E_2 - E_4)$ is therefore very small, but still sufficient to sustain the motion. If in the first of the equations (30) we neglect B_0, the value of $(A_2 + B_2)^{1/2}$ and $(A_4 + B_4)^{1/2}$ is not very appreciably affected, but $\cos(E_2 - E_4)$ is however reduced to such an extent that it is no longer possible for equation (31) to be satisfied, in other words the motion cannot be maintained. The term B_0 is equal to $\alpha A_2/n^2$, i.e. equal to $-4\alpha^2 B_4/3n^4$ and is therefore very small, but as explained above, the maintenance of the vibrations cannot be fully explained without taking it into account.

The components A_4 and B_2 in the maintained motion are both very small. The component A_2 is approximately equal to $-4\alpha B_4/3n^2$. The ratio between the principal part of the maintained motion and the subsidiary component of lower frequency is therefore of the same order of quantities as the ratio between the constant and variable parts of the spring.

The fifth type of motion and the general case

Figure 1, plate VII and figure 1, plate IX, show this class of maintained motion and its vibration curves respectively. In this case, the frequency of the variable spring is two-fifths that of the free oscillations of the system. The forced oscillations of the system may be discussed as if it possessed one degree of freedom only, the displacement at any point on the string being given by an expression of the form

$$u = A_1 \sin pt + A_3 \sin 3pt + A_5 \sin 5pt + A_7 \sin 7pt + B_1 \cos pt + B_3 \cos 3pt + B_5 \cos 5pt + B_7 \cos 7pt \qquad (32)$$

the variable spring being as in previous cases represented by $-2\alpha \sin 2pt$, and the ratio of the constants being the same for all points on the string. The principal part of the maintained motion is $(A_5 + B_5)^{1/2} \sin(5pt + E_5)$, which is very approximately of the same frequency as the free oscillations of the system. It can be seen from the vibration curve that E_5 is approximately equal to $-3\pi/4$ and A_5 is therefore nearly equal to B_5.

The subsidiary term $(A_3 + B_3)\sin(3pt + E_3)$ in the motion is from a physical point of view of great importance. It is not at all difficult to understand in what manner it is brought into existence. The successive oscillations of the string are evidently not all executed under identical conditions. At the epoch of minimum tension the motion being under diminished constraint swells out and increases in amplitude and the contrary is the case at the epochs of maximum tension. Again at the former epochs the time taken for a swing is more than at the latter. The motion as shown in the vibration curves is very analogous to the effect of 'beats'. Taking the general case in which a variable spring $-2\alpha \sin 2pt$ acts upon a system whose free oscillations have a frequency nearly equal to $r/2$ times that of the variable spring, the frequency of the 'beats' is equal to that of the variable spring and the frequency of the subsidiary motion is *less* by that quantity than the

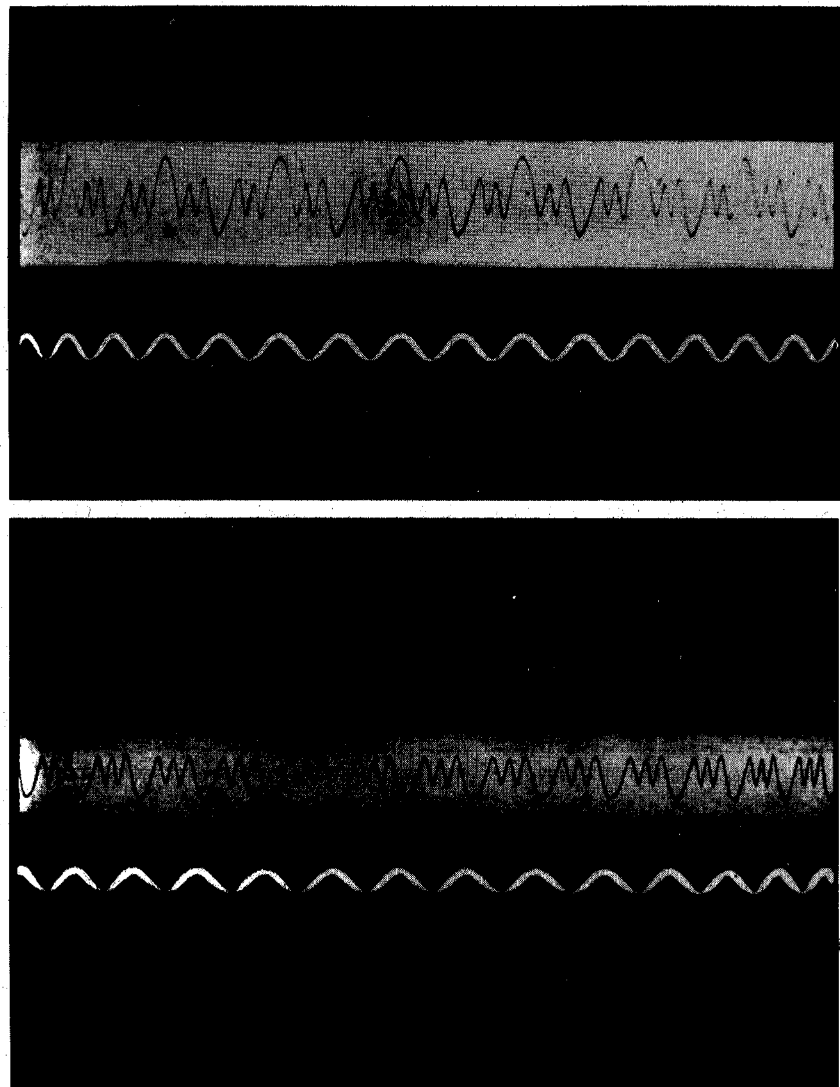

Plate IX. Vibration curves of (1) the fifth type of maintenance and (2) the sixth type of maintenance.

frequency of the principal motion, and we may therefore put

$$W = P \sin(rpt + E_r) + Q \sin(\overline{r-2}pt + E_{r-2}). \qquad (33)$$

The product of the variable spring with the displacement at any instant may be regarded as the impressed part of the restoring force. Taking the first term on the right of (33), the product $-2\alpha P \sin 2pt \sin(rpt + E_r)$ has no component of the frequency $rp|2\pi$ which is required (by the general principle of resonance) if the oscillation is to be maintained. On the other hand the product

$$-2\alpha Q \sin 2pt \sin(\overline{r-2}pt + E_{r-2})$$

does contain such a component which is equal to

$$\alpha Q \cos(rpt + E_{r-2})$$

and can maintain the motion if the other conditions are suitable. The energy dissipated in any number of complete periods of the variable spring is equal to the energy supplied during the same interval if

$$rkpP = -\alpha Q \cos(E_{r-2} - E_r). \tag{34}$$

This equation conveys the fundamental principle underlying the type of the maintenance of vibrations under discussion. In the general case Q is of the order

$$\pm \frac{\alpha P}{4(r-1)p^2},$$

if we neglect possible effects due to the variability of the tension in free oscillations of sensible amplitudes. In order to show that the value of $\cos(E_{r-2} - E_r)$ in equation (34) may be sufficiently large to ensure maintenance of the motion, it is necessary to consider the effects produced by terms of still smaller frequencies (if any) in the expression for the displacement at any instant. An illustration of this point has already been given in the case of the fourth type of motion. Such terms exist in all cases whose $r > 3$. They owe their origin to secondary and tertiary reaction between the forced oscillations and the variable spring, and though very small in magnitude play an important part in building up the requisite phase-difference between the principal motion and its immediate auxiliary of lower frequency. Thus, returning to the case of the fifth type discussed above, we cannot neglect the term

$$(A_1^2 + B_1^2) \sin(pt + E_1)$$

in the expression for the displacement, for if we do we should find that the value of $\cos(E_3 - E_5)$ is not sufficient to maintain the motion. On the other hand the terms

$$(A_7 + B_7)^{1/2} \sin(pt + E_7)$$

etc. do not play any such part in the maintenance. Their effect is merely equivalent to a slight alteration in the free period of oscillation of the string, and they are generally inconspicuous. It is hardly necessary for me to add that in each case the necessary adjustment of pitch is secured by the variation of the period of the motion with increasing amplitudes.

I have also observed the sixth and seventh and higher types of motion in the series up to the eleventh and have photographed their vibration-curves. These are shown as figure 2, plate IX and plates X and XI. The appearance of a string executing the sixth or eighth or tenth type of motion is somewhat analogous to that in the case of the fourth type, and that of the seventh or ninth or eleventh to

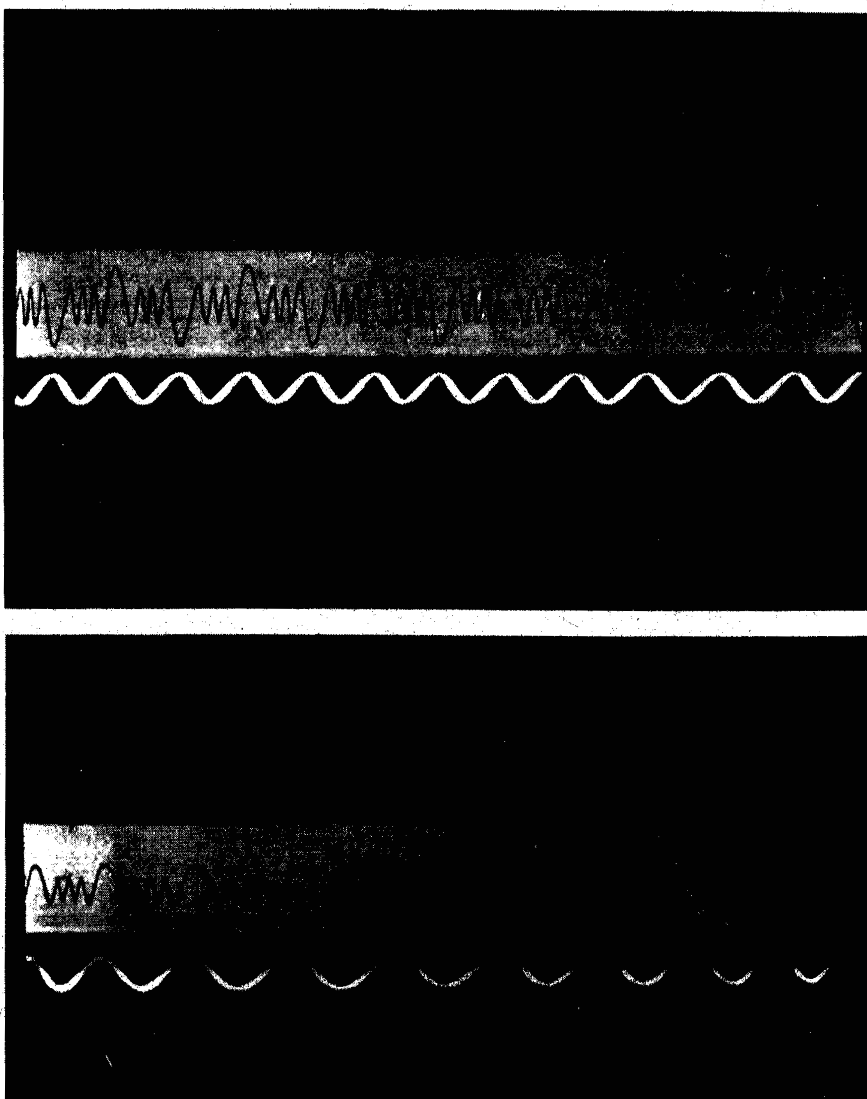
1
2

Plate X. Vibration curves of (1) the seventh type of maintenance and (2) the eighth type of maintenance.

1

2

3

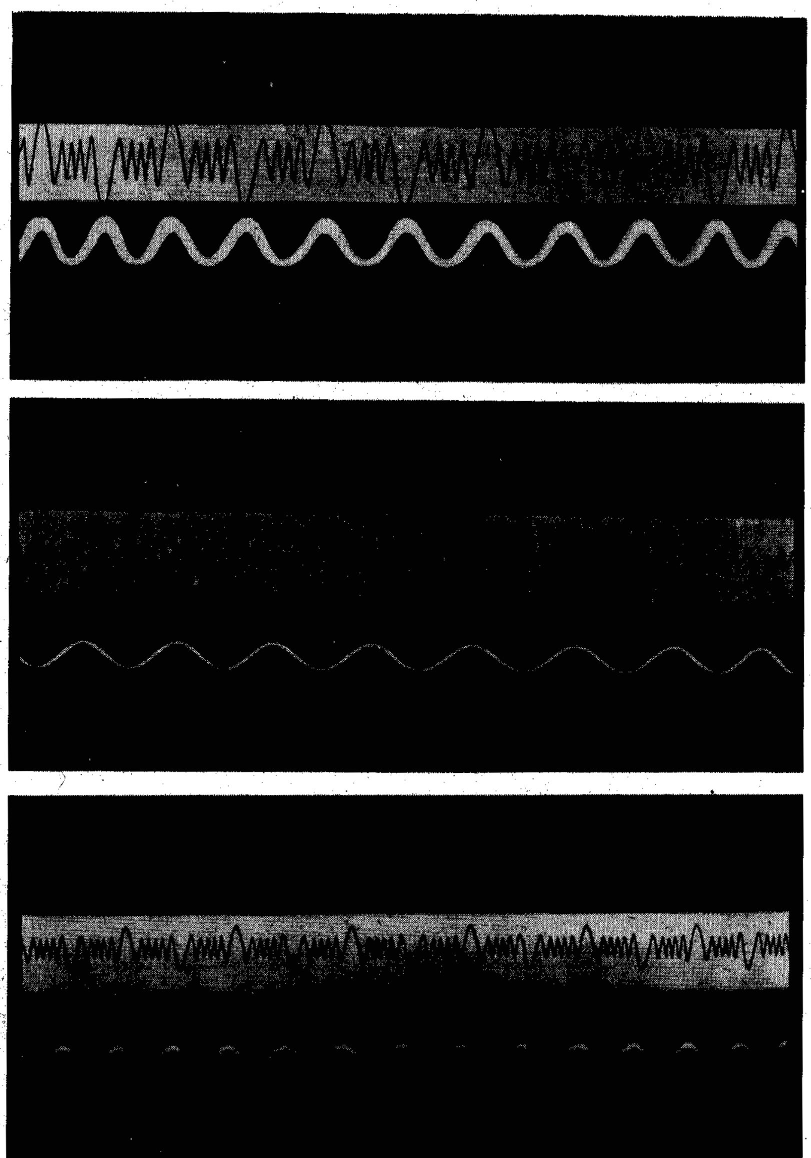

Plate XI. Vibration curves of (1) the ninth type of maintenance, (2) the tenth type of maintenance and (3) the eleventh type of maintenance.

the fifth type. The reason for this can be well understood. For the odd types are all more or less perfectly *symmetrical* and the even types are all *unsymmetrical.*

Observations with revolving mirror and with stroboscopes

That the string when maintained in any one of these types of vibration behaves as a single unit, in other words like a system having only one degree of freedom, can well be shown by observing the vibration curves at different points on the string. By illuminating any one point by a sheet of light transverse to the string and viewing the luminous line of light in a mirror kept revolving at a moderate speed, the vibration curve is seen at once. Even a mirror held in the hand which is tilted to and fro is sufficient for the purpose. Shifting the sheet of light so that it cuts the string at any other point produces no effect except to alter all the ordinates of the vibration curves in equal ratios.

Some extremely interesting phenomena are noticed when a stroboscopic disc is used in observing these types of maintained motion. A Rayleigh synchronous motor on which is mounted a blackened disc with narrow radial slits cut in it is very suitable for this purpose. As already mentioned in section I, one of the discs which I use has thirty slits on it, the armature-wheel of the motor having the same number of teeth. The electric current from the self-interrupter fork which maintains the string in vibration also runs the synchronous motor. In making the observations, the stroboscopic disc is held vertically and the string which is set horizontal and parallel to the disc is viewed through the top row of slits, i.e. those which are vertical and move in a direction parallel to the string as the disc revolves. It is advantageous to have the whole length of the string brilliantly illuminated and to let as little stray light as possible fall upon the reverse of the disc at some distance from which the observer takes his stand. A brilliant view is then obtained. I have already explained that under these circumstances we see the string in successive cycles of phase along its length, and the peculiar character of the maintained motion in these cases is brought out in an extremely remarkable way. *The string is seen in the form of a vibration curve,* which would be identical with those shown above, but for the fact that the amplitude of motion is not the same at all points of the string, being a maximum at the ventral segments and zero at the nodes.

Another point calls for remark. Using a fork with a frequency of 60 per second, the *free* oscillations of the string have a frequency of 30 in the case of the 1st type, 60 in the case of the 2nd, 90 with the 3rd, 120 with the 4th, 150 with the 5th, and so on. With the disc having 30 slits on it we get 60 views per second of any one point on the string, and with the even types of motion, i.e. the 2nd, 4th, etc. the 'vibration-curve' seen through the stroboscopic disc appears single. With the odd types, i.e. the 1st, 3rd, 5th, etc. *two* vibration-curves are seen, one of which is as nearly as can be seen the mirror-image of the other, intersecting it at points which lie or should lie upon the equilibrium position of the string. The reason why with the odd types we see the vibration-curve double is obvious enough, and I need not proceed to detail it. The double pattern brings home to the eye in an extremely vivid and convincing manner the fact that under the action of the variable spring the 'amplitude' and 'period' of the motion periodically increase and decrease after the manner of 'beats.'

An interesting variation on the experiment is made by using the disc with 60 slits. We then get 120 views per second and with the even types we get the vibration-curves double, but one of the curves is not the mirror image of the other, the motion not being symmetrical. On the other hand, with the odd types we see the vibration-curves in quadruple pattern, the third and fifth types in particular giving extremely beautiful tracery effects. It seems somewhat difficult to obtain perfectly satisfactory photographs of these phenomena on account of slight periodic alterations in the speed of the stroboscopic disc, but I am still quite hopeful.

V. The maintenance of compound vibrations by a simple harmonic force

In this and the succeeding section on 'Transitional Modes of Motion under Variable Spring' I shall consider the phenomena of the maintenance of vibrations by a variable spring of simple harmonic character acting on a system that has more than one degree of freedom. In section IV, I have shown that a variable spring acting on a system having only one normal mode of oscillation may maintain its vibrations if the frequency of the variable spring stands to that of the system in any one of the ratios 2: r where r is an integer, We know that a vibrating system of the kind here dealt with, i.e. a stretched string, has not merely one free period of oscillation, but a series of such free periods in which it divides up into one, two or more segments. Since the frequencies of oscillation which a variable spring of given frequency may maintain under suitable circumstances also form a series, it is evidently possible for more than one mode of vibration to be maintained at one and the same time, *each with its own appropriate frequency.* In other words, the variable spring may maintain a compound vibration, and as the components of this motion need not both or all be in one and the same plane of vibration of the string, we may readily obtain by a little calculation and trial, types of maintained motion in which the oscillation in one principal plane is of one frequency and in the perpendicular plane of a different frequency. Under these circumstances, the motion of a point on the string in a plane transverse to it becomes and remains the appropriate Lissajous figure, and the frequency relation between the component motions is thus rendered evident to inspection in a most striking manner.

The photographs shown in plate XII represent short sections of the string thus

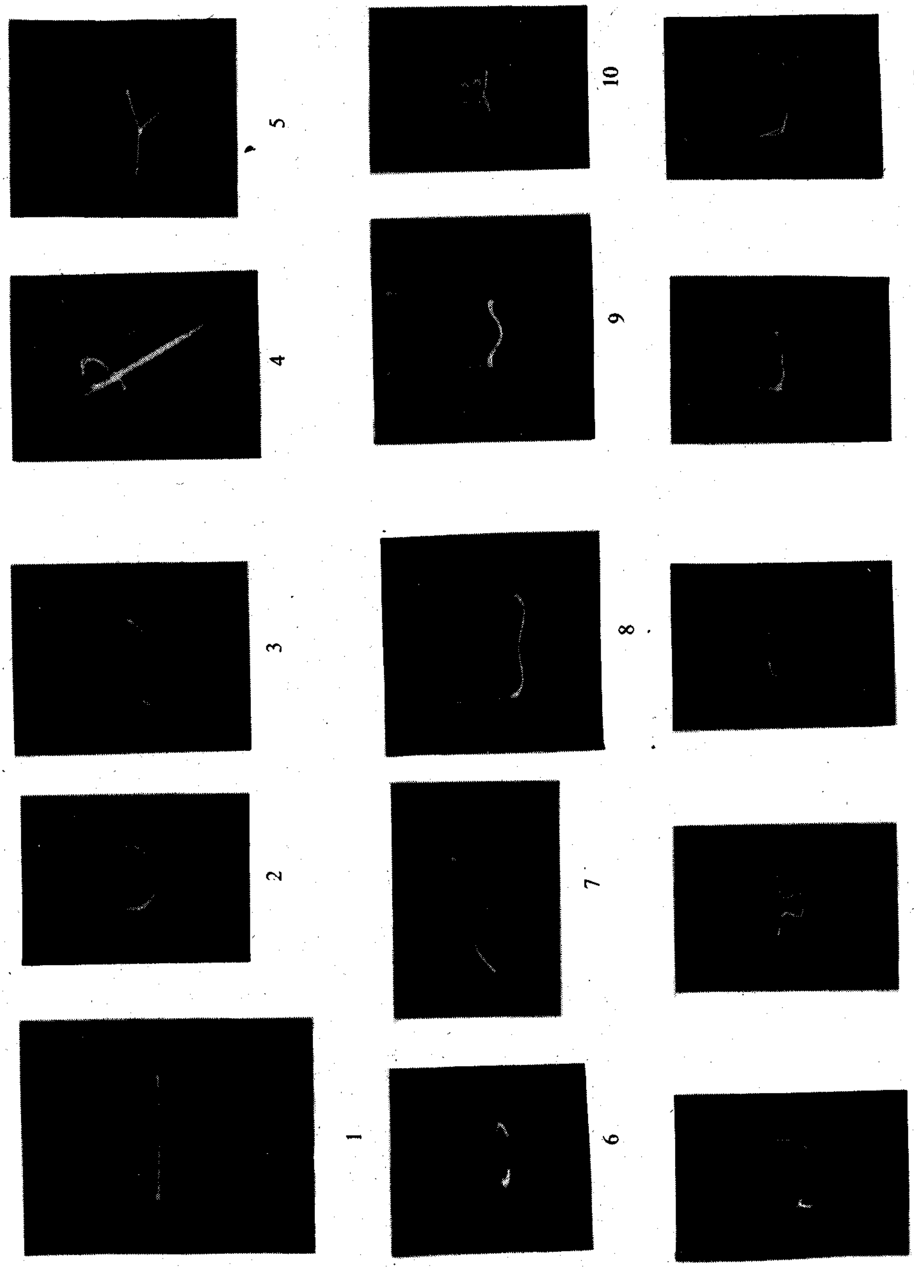

Plate XII. Compound types of vibration of a stretched string maintained by a simple harmonic variation of tension.

maintained in stationary vibration, one point in the middle of the section being brilliantly illuminated. Figure 1 shows the ordinary first type of maintenance in which the frequency of the motion is half that of the fork. Figure 2 shows a compound of the first and second types in suitable phase relation, the motion being in a parabolic arc. Figure 3 is a compound of the first and third types. Figure 4 is a compound of the second and third types of frequencies respectively equal to and half as much again as that of the fork. Figures 5 and 6 are complementary, i.e. the same mode of vibration, figure 5 showing one part of the string and figure 6 another. In these two photographs the first and third types occur in one principal plane and the second type by itself in the perpendicular plane. In figure 5, the first and third are in similar phases, but in figure 6, they are opposed, hence the very remarkable split ring effect in the latter. In figure 7 we have the first and third types again in perpendicular planes but along with the third type there is a clear addition of the second type as well. Figures 8 and 9 are complementary, and show the first type maintained in one plane and the second and fourth types together in the perpendicular plane. Figure 10 represents a compound of the second and fifth types, and shows quite clearly the characteristics of the fifth type as described in the previous section, i.e. the increase of the amplitude and period of the motion at the epoch of minimum tension and their decrease when the tension is a maximum. Figure 11 shows the first type in one plane and the second and fifth types together in a perpendicular plane. Figures 12 and 13 are complementary, i.e. show different parts of the string in the same mode of oscillation. They represent the first and fifth types together in one plane and the second by itself in the perpendicular plane. In figure 12, the first and fifth types are in the same phase and in figure 13 they are opposed. Figures 14 and 15 show the first type in one plane and the second and sixth types together in the perpendicular plane. The two latter are in different relative phases in the two photographs.

Besides the above, I have observed a very large number of permanently maintained compound modes of vibration in which two or more of the types of motion discussed in the preceding section occur in various phase-relations to each other. In the case of types of higher order than the second, the observed range of variation of phase was not however very large.

The compound modes of motion in which two or more of the types of maintained motion occur together in one plane of vibration can also be observed stroboscopically in the manner described in the preceding section. The special feature of interest in this case is that a large number of variations can be obtained and different parts of the string, sometimes even contiguous ones, show the component motions in different relative magnitudes and as seen through the stroboscopic disc in different phases. The double patterns obtained in this manner are extremely interesting and beautiful, and it is with regret that I decided not to delay the issue of this Bulletin till I had secured satisfactory photographs of some of them.

VI. Transitional modes of motion under variable spring

In discussing the maintenance of vibrations by a variable spring of double frequency, *vide* section III above, it was tacitly assumed that the motion was that of a system having one degree of freedom only, or at any rate could be treated as such to a close approximation. In other words it was taken for granted that the ordinary 'modes' of vibration under constant spring, i.e. the ratios of the displacements at any instant of different points on the system, remain unaltered. We are justified in making this assumption so long as the free periods of the system in its several normal modes of oscillation are sufficiently removed from each other. But as will be seen from what follows, it breaks down entirely when the frequencies of two natural modes of vibration between which the half frequency of the imposed variable spring lies are sufficiently close together to fall simultaneously within the range of maintenance for the given frequency. The phenomena that then result are of considerable experimental and theoretical interest and I have termed them 'Transitional Modes of Motion'. The appropriateness of this will appear as we proceed.

A variable spring of given frequency can maintain the vibrations of a system whose free period for small oscillations lies anywhere within a certain range determined by the magnitude of the imposed variation. If two of the normal modes of vibration of the system fall simultaneously within this range the steady motion, if any, that may result must evidently be of a frequency exactly half that of the variable spring imposed. It is possible to obtain the over-lapping of the ranges for two contiguous modes even if the free periods of these differ considerably, by sufficiently increasing the magnitude of the variable spring. It is experimentally observed that the maintenance of a steady state of vibration is perfectly possible under such circumstances. Thus for instance, there is absolutely no difficulty in obtaining a steady transitional mode of motion which is intermediate between the ordinary modes in which a string divides up with three and four ventral segments respectively. In the resulting vibration there is nothing that can even approximately be regarded as a 'node'. The amplitude of vibration is not however the same at all points of the string, and there are recognizable maxima and minima. It is not however easy to describe the appearance seen with much exactness and to a cursory examination the nature of the motion is by no means evident. When however we use intermittent illumination of frequency nearly equal to that fork maintaining the string in vibration, the extremely remarkable and interesting character of the motion is at once revealed. Since the frequency of the illumination is nearly but not quite double that of the motion we see simultaneously two opposite phases of the motion which undergo periodic cycles of change. At one instant the string is seen in the form of recognizably perfect sine-curves which enclose *three* ventral segments. At another phase of the motion it is seen in the form of perfect sine-curves which enclose *four* ventral segments. The periodic change from three to four segments and back again is one of the most striking and interesting phenomena that are met with in the study of the maintenance of vibrations.

When the frequency of the intermittent illumination is somewhat less than that of the fork we see the motion proceeding in the manner in which it actually takes place. One way of describing what is observed is to say that extra loops are continually being formed at the end of the string attached to the fork and continually moving off and disappearing at the fixed end. The process at one end is periodically faster and slower than that at the other, with the result that we have alternately three and four ventral segments on the visible portion of the string.

There is however another fact which is observed, i.e. the ordinates of the three-loop curve are not equal to those of the four loops, being generally larger: this is not brought out in the description given above. Perhaps a more accurate idea may be conveyed in the following manner. If we have a pair of curves whose initial positions are given by the equations

$$y = \pm P \sin\frac{3\pi x}{b} \quad \text{and} \quad z = \pm Q \sin\frac{4\pi x}{b}$$

and which continually rotate round the axis of x, the plane yz being normal to the latter, their motion as seen projected on any given plane passing through the axis of x is similar to that seen in the actual experiment with the intermittent illumination. The projected curves would be given by the equation

$$\pm u = A \sin\frac{3\pi x}{b} \sin(pt + \pi/4) + B \sin\frac{4\pi x}{b} \sin(pt + \theta). \tag{1}$$

If the axes of y and z are at right angles, $\theta = -\pi/4$, and the phase difference between the two terms would be exactly quarter of an oscillation.

From equation (1) it is clear that the phase of the resultant motion varies from point to point on the string. Working by the methods described in section III, I have observed the variation of the phase of the motion along the string and the indications of equation (1) of the present section are amply confirmed. Since A is generally larger than B, the most remarkable changes are observed on either side of the points where $x = b/3$ or $2b/3$. At some distance from these points the 'curves of motion' (*vide* section III) are parabolic arcs *convex* to the fork. As we approach nearer they become first looped figures convex to the fork and then 8-shaped curves. Nearer still, they are looped figures *concave* to the fork and finally parabolic arcs with their curvature directed towards the fork. As we recede on the other side we get the same changes in reverse order, the curves at some distance off being parabolic arcs convex to the fork. I hope later to obtain and publish photographs of these remarkable types of motion with the varying phase.

It is not difficult to see why the displacement at any point of the string is of the type given by equation (1). As already explained in the third section of this Bulletin, the maximum positive and negative phase-differences between the variable spring of double frequency and the motion maintained by it are $\pi/4$ and $-\pi/4$ respectively. When the half-frequency of the variable spring is intermediate between the frequencies of free oscillations of the system in any two given modes, we may assume that the oscillations are set up and maintained *simultaneously* in the two different modes but with the same frequency, i.e. half that of the variable spring. The two modes of oscillations are however in different phases, and the sustained vibration can well be termed a transitional mode of motion.

In concluding this Bulletin I have real pleasure in acknowledging my indebtedness of Dr Amrita Lal Sircar for his interest in the work and unfailing personal encouragement to myself, and also for his having as Hon. Secretary put the resources of the laboratory of the Association and the services of the staff unreservedly at my disposal during hours at which few institutions, if any, would remain open for work. I have also specially to mention the name of the senior demonstrator Mr Dey for having materially assisted in the early and rapid completion of the experimental work.

Appendix

Note on a stereo-optical illusion

Working with a stroboscopic disc of the pattern already described, I noticed a very curious optical illusion that seems worthy of record. The disc with 30 slits was set up vertically, and the tuning-fork which regulated its motion was placed immediately behind with its prongs vertical and facing the disc so that the observer who took his stand immediately in front of it could get a good view of the motion of both the prongs. Using both eyes for comfort, I was surprised to notice that the prongs appeared bent out of their plane, one to the front and one to the rear, and actually executed oscillations to the rear and to the front as the head was moved along the row of slits! The explanation of the phenomenon was undoubtedly that the two eyes perceived the motion of each of the prongs of the fork in two distinct phases and endeavoured to reconcile them by seeing them bent out of their plane one to the front and the other to the rear! The appearance was most realistic.