A New Method for the Absolute Determination of Frequency.

By ASHUTOSH DEY, with a Prefatory Note and an Appendix by C. V. RAMAN, M.A., Palit Professor of Physics in the Calcutta University.

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I. *Prefatory Note.*

The new method for the absolute determination of acoustical frequencies described in this paper possesses several useful features. Among these are: (*a*) the rapidity with which very accurate results may be obtained by its use, (*b*) its practical convenience and simplicity, (*c*) the inexpensiveness of the apparatus required, and (*d*) its availability for use with any of the standards of time ordinarily available in a physical laboratory, *i.e.*, a "seconds" or "half-seconds" pendulum clock, a "half-seconds" chronometer, or even merely a good stop-watch. The method is the outcome of an investigation undertaken at my suggestion by Mr. Dey, and its value has been proved in an extensive series of tests carried out by him.

The principle on which the method is based is that of the maintenance of oscillations of sub-synchronous frequency by a periodic field of force,* and is applied in practice in the following manner:—

A pendulum formed of an iron rod 0·5 cm. diameter hangs vertically from a simple wire-hook suspension. A brass bob slides on the pendulum and is capable of being fixed in any desired position. The length of the pendulum is either 35 cm. or 100 cm., so that with the bob in a suitable position,

* For some earlier observations on this class of maintained oscillation, the undermentioned references may be cited: C. V. Raman, 'Phil. Mag.,' January, 1915; C. V. Raman and Ashutosh Dey, 'Phil. Mag.,' August, 1917.

the free period of oscillation is roughly either one second or two seconds as desired. Underneath the pendulum is placed symmetrically a vertical bar-electro-magnet which has an iron core 8 cm. long, 0·9 cm. diameter, the gap between the pole and lower end of the rod when in the vertical position being quite small (1 mm. or less as required). The electro-magnet is excited by a current rendered intermittent by an electrically maintained fork. With these arrangements, the rod, when hanging vertically, does not tend to be displaced from its position of equilibrium by the magnetic forces. But if the position of the bob be suitable and the pendulum be set in motion by hand with approximately the right amplitude, its oscillations are found to be vigorously maintained with a frequency which, in practice, may be very small compared with the frequency of the fork, *but is always an exact sub-multiple of it.*

Experiment shows that the maintained oscillations of the pendulum have a frequency which may be any one of the series of fractions 1/2, 1/3, 1/4, 1/5, 1/6, etc., up to 1/140 of the frequency of the fork, and the series is capable of being extended to still smaller ratios for the possible frequency of maintenance.* The theory and certain special features of the working of the apparatus are discussed in a note which I have added at the end of the paper, but it is sufficient for its practical application to remark that theory and experiment agree in showing that the frequency of the maintained oscillation of the pendulum is an exact sub-multiple of the frequency of the fork-interrupter. Consequently, if the latter be known roughly in the first instance, it can be determined with high precision by finding the time of oscillation of the pendulum and dividing it by the appropriate number which, as remarked in the preceding footnote, is generally an *even* integer.

The rate of the pendulum which is maintained by sub-synchronous oscillation by the fork-interrupter can be found by comparison with any available standard of time. If a pendulum clock be available, all that is necessary is to put the apparatus in front of the clock and visually observe the coincidences of the two pendulums. If a "half-seconds" chronometer be the only available standard of time, the sub-synchronous pendulum can be compared with it by causing it electrically or mechanically to give an audible

* The first of these cases in which the frequency of the pendulum is one-half that of the intermittent current was noticed and described by Lord Rayleigh ('Scientific Papers,' vol. 2, p. 193, and 'Theory of Sound,' 2nd edition, vol. 1, p. 82). It should be remarked here that the cases in which the denominator of the fraction giving the frequency-ratio, is an *odd* integer, stand in a different category from those in which the denominator is an *even* integer, the maintenance being far more energetic in the latter set of cases than in the former. In fact, when the frequency of the interrupter is fairly large, maintenance is ordinarily obtained only with a frequency-ratio having an *even* integer in the denominator.

signal once in each swing and noting the coincidences with the half-second "ticks" of the chronometer by ear. If neither a clock nor a chronometer be available in the laboratory, it is possible, with the aid of an ordinary "Omega" stop-watch reading to a fifth of a second, to make absolute determinations which are sufficiently accurate to measure the changes of frequency of the fork-interrupter due to small variations in the room-temperature. All that is necessary for this purpose is a simple mechanism enabling the sub-synchronous pendulum to move the hands on a clock-dial, and thus to function as a time-keeper which can be rated at intervals against the stop-watch.*

The absolute frequency of the fork-interrupter being determined, it is obviously possible to arrange for its simultaneous comparison with a standard fork or other vibrator whose frequency is required to be ascertained. This may be done either by optical observation, as in Koenig's well-known work,† or by counting the beats with the fork-interrupter, or, if necessary, with a dependent vibrator of higher frequency electrically controlled by it, as in Lord Rayleigh's method.‡ It is obviously advantageous for this work to have a fork-interrupter, of which the frequency can be adjusted by sliding weights fixed upon it, and having also the necessary fittings for optical observation attached to one of its prongs.

In the following sections of the paper, Mr. Dey has described the working details of the method and the results of some practical tests of its accuracy.

C. V. RAMAN.

II. *Some Characteristics of Sub-synchronous Maintenance.*

In addition to the experimental details referred to in the prefatory note, it may be useful to mention a few other points to be noted in practical work. The successful working of the apparatus depends on the fact that the magnetic forces acting on the pendulum rod are powerful when it is at or near its equilibrium position, and become negligible at other points of the arc of swing, especially if the latter be large. Accordingly, if the frequency of the interrupter be more than fifty or sixty times that of the pendulum, it

* Still another way in which the sub-synchronous pendulum could be rated is, by causing it to give electric signals once in each swing, and recording these on a moving drum or tape for a few seconds at the beginning and at the end of a definite interval of time, say ten minutes, alongside of the electric signals from a standard clock or chronometer. From some trials at present being made by one of the research scholars (Mr. B. N. Banerjee), working in this laboratory, it would appear that this procedure, though somewhat more elaborate than those described in this paper, is also capable of yielding very accurate results with little labour.

† Koenig, 'Annalen der Physik,' vol. 9 (1880).

‡ Lord Rayleigh, 'Phil. Trans.,' Part I, p. 316 (1883).

is found of advantage to bevel the ends of the pole of the electro-magnet, so as to concentrate the magnetic field into a narrow region. The interrupter should operate on a mercury break, across which a condenser is shunted, so as to suppress sparking and ensure its clean and regular operation.

In order that the oscillations of the pendulum might be maintained, it is necessary that its natural frequency should lie within a certain range of values which includes the frequency of maintenance. This is secured by adjusting the position of the bob on the pendulum rod, the simplest way to do it being by actual trial, till the oscillation of the pendulum is successfully maintained. The adjustment having once been made, it is generally unnecessary to alter it so long as the same fork and pendulum are used. The manner in which the amplitude of the forced oscillation of the pendulum varies with the position of the bob in different parts of the range of maintenance is rather remarkable, and will be best understood on a reference to fig. 1, which shows the maintenance curves of a 35-cm.

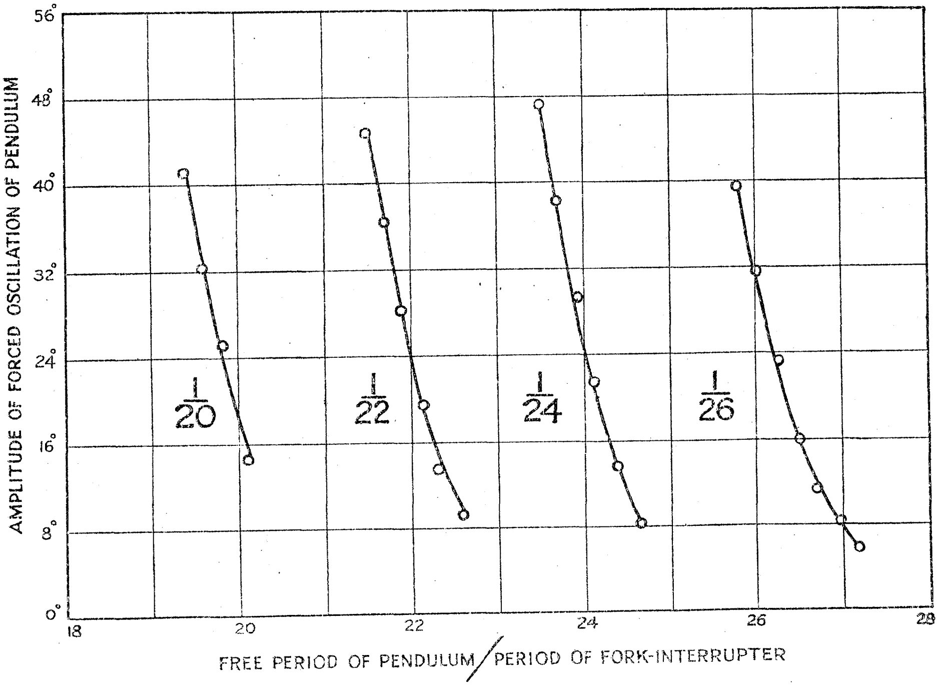

Fig. 1.

pendulum excited by a fork-interrupter of frequency roughly 24 vibrations per second. By moving up the bob of the pendulum, its free period for small oscillations can be adjusted over a considerable series of values, which, as seen in the figure, include a number of successive regions within which the motion can be maintained. With the particular fork used, the frequencies of maintenance obtained with the bob in different positions on the pendulum are respectively 1/26, 1/24, 1/22, and 1/20, of the frequency of the fork. It will be seen that, in each of these ranges, the amplitude of swing is small when the natural period of the pendulum for small oscillations is greater than the period of the maintained oscillation, but increases continually to a large value when the free period is decreased to a value below that of the maintenance, till a stage is reached at which the apparatus refuses to work.

There is little resemblance between the sequence of the phenomena shown in fig. 1 and the ordinary type of resonance curve of a simple vibrator under the action of a periodic force which shows a maximum amplitude for a particular frequency and lesser amplitudes for greater or smaller frequencies. The theoretical explanation of this feature of the working of the apparatus is dealt with in a note by Prof. C. V. Raman appended to this paper. For the present, it is sufficient to remark that, when the apparatus is in order, the ranges of maintenance are considerable, and, with the bob of the pendulum anywhere near the right position, it can be got to work almost at the first trial. Generally, when the pendulum is started, it shows "beats," the amplitude of swing varying in a cyclic manner, but these disappear fairly quickly (leaving a constant oscillation) unless the maintenance is near the lower extremity of the range where the amplitude of swing is small. In practice it is found convenient to work with the bob of the pendulum in such a position that the arc of swing is fairly large. The support from which the pendulum is suspended does not need to be particularly rigid, but it should be sufficiently firm to ensure that its movement is not irregular.

That the motion of the pendulum is of the nature of a forced oscillation whose frequency stands to that of the maintaining force in an exact numerical ratio is evident from the fact that a steady motion is sustained in the presence of dissipative forces. It is also definitely proved by the experimental results, which show (*a*) that the times of forced oscillation of the pendulum with the bob in different positions are strictly commensurable; and (*b*) that the variation of frequency of the interrupter due to changes in its temperature (even if as small as a tenth of a degree Centigrade or less) produces proportional changes in the rate of the pendulum.

III. *Comparison with Pendulum-clock by Visual Observation of Coincidences.*

For finding the time of the maintained oscillation of the pendulum, the most convenient procedure is to put it in front of the pendulum of a standard clock and to observe their coincidences visually. The oscillations compared are both of constant amplitude, and it is obviously possible to select for observation points on the two pendulum-rods which have both the same amplitude of oscillation. The method is then at its best, and it is found easy to register the instant at which the coincidence occurs to within a small fraction of the period of oscillation of either pendulum. A high degree of precision may thus be secured in measurement without unduly prolonging the interval of time included between the first and the last coincidences observed.

In the actual work, the bob of the seconds pendulum of the laboratory clock was a glass cylinder containing mercury, and this furnished a brilliant reflection of a ½-watt lamp placed at a distance from the clock case. A pair of lenses placed at a suitable distance apart, threw a moving image of this spot of light on a ground-glass screen, on which also appeared the shadow of the rod of the sub-synchronous pendulum oscillating through the same range. The coincidences could thus be seen on the screen by the observer using both his eyes, and the labour involved in making the observations was insignificantly small. The interval of time between one coincidence and say the twentieth or thirtieth coincidence following it was registered on a stop-watch.* The calculation of the frequency of the fork-interrupter from the observations was quite simple.

A few results will now be quoted to illustrate the convenience and accuracy of the method. In one series of experiments, a fork-interrupter of frequency approximately 24 vibrations per second was used, and this controlled a pendulum whose frequency was 1/46 of its own. The apparatus was started at 7 A.M. on June 29, 1918, and continued working without intermission till it was disconnected for the night at 10 P.M., that is after running without a break for 15 hours. The temperature of the fork as shown by a sensitive thermometer placed between the prongs, rose from about 28·6° C. in the forenoon to 30·1° C. at 3 P.M. and fell to 29° C. at 9.30 P.M. A series of observations of frequency were made, the routine of each observation being simply the starting of a stop-watch at a given coincidence with the pendulum of the laboratory clock and stopping it when the 14th coincidence following occurred. The time-interval (which was a little over 10 minutes) could be determined by a single observation correct to the fifth of a second. A single observation was thus sufficient to determine the frequency of the interrupter correct to about 15 parts in a million;

* If two or three good stop-watches had been available, the accuracy of the observations might have been further improved by independently observing the intervals between say, the 1st and 18th coincidences, the 2nd and 19th, the 3rd and 20th coincidences, and taking the average results.

61 such observations were made in the course of the day, the thermometer being read every few minutes. In order to eliminate the effect of the temperature-lag of the fork as far as possible, the 61 results have been divided into 12 groups, each group including observations in which the average temperature indicated did not differ by more than a tenth of a degree. The temperature readings and time-intervals for 14 coincidences have been averaged for each group and shown in the first and second columns of Table I. The third column shows the frequency of the fork-interrupter as calculated from the periods of coincidence. The fourth column shows the frequency of the fork calculated from the formula

$$\text{Frequency} = 24{\cdot}06814\,[1 - 0{\cdot}000104\,(t - 30^\circ)],$$

which was shown by a graph to give the best fit with the experimental results.

Table I.

1. Mean temperature of fork-interrupter.	2. Mean time-interval for 14 coincidences.	3. Frequency of interrupter (observed).	4. Frequency of interrupter (calculated).	5. Difference.
° C.	secs.			
28·66	601·13	24·07132	24·07147	−15
28·76	601·25	·07110	·07124	−14
28·81	601·41	·07082	·07111	−29
29·01	601·50	·07065	·07061	+ 4
29·15	601·61	·07047	·07026	+21
29·22	601·80	·07013	·07009	+ 4
29·38	602·05	·06968	·06969	− 1
29·55	602·20	·06942	·06926	+16
29·67	602·30	·06924	·06896	+28
29·72	602·35	·06915	·06884	+31
29·96	602·90	·06818	·06824	− 6
30·11	603·20	·06764	·06786	−22
Average error (irrespective of sign)			16 or 6½ parts in a million	

An interrupter of frequency approximately 60 per second was also successfully used in several series of observations. This maintained the oscillations of a pendulum having a frequency 1/110 of its own; the interval between the successive coincidences with the pendulum of the standard clock was about 22 seconds. In illustration of the results, the following series of observations made on July 1, 1918, may be quoted. The apparatus was started at 7 A.M. and continued working without intermission for 16 hours, till 11 P.M. when it was disconnected for the night. In the forenoon, 19 observations were recorded of the interval occupied by 30 consecutive coincidences, and in the afternoon and evening, 16 observations were made of the time occupied by 60 coincidences.

As usual, a thermometer placed between the prongs was read every few minutes, but as the fork was a massive one, considerable doubt must exist whether the averages of the readings during each observation really represented its mean temperature, particularly in those cases in which there was a rapid rise or fall in the reading. Table II shows the complete series of 16 observations made in the afternoon and in the evening, the mean temperature during each observation, and the frequency calculated from the results. Column 5 of the Table gives the frequency calculated from the linear formula

$$\text{Frequency} = 59{\cdot}9829\,[1-0{\cdot}000115\,(t-30^\circ)],$$

giving the best fit with the experimental results. Observations numbered 1, 7, 8, and 12 have to be rejected for the reasons mentioned in the footnotes. Ignoring these, the average of the differences shown in column 6 is seen to be about 15 parts per million of the calculated frequency. An uncertainty of a tenth of a degree Centigrade in the temperature would account for a difference of 11 parts in a million, and it seems very likely that the deviations shown in column 6 are principally due to this or other cause actually altering the frequency of the fork-interrupter in a

Table II.

1. Serial No. of observation.	2. Mean temperature of fork.	3. Time-interval of 60 coincidences.	4. Observed frequency.	5. Calculated frequency.	6. Difference.
	° C.	secs.			
1	30·20 ? *	1323·9	59·9850	59·9815 ? *	+36 ? *
2	30·08	1324·8	·9834	·9823	+11 }
3	30·10	1324·3	·9838	·9822	+16 }
4	30·85	1326·2	·9766	·9770	− 4 }
5	30·90	1326·2	·9766	·9767	− 1 }
6	30·91	1326·4	·9759	·9766	− 7 }
7	30·60 ? †	1326·5	·9754	·9787 ? †	−33 ? †
8	30·25 ? ‡	1324·5	·9830	·9811 ? ‡	+19 ? ‡
9	29·90	1324·3	·9838	·9836	+ 2 }
10	29·82	1324·4	·9834	·9841	− 7 }
11	29·63	1324·0	·9847	·9854	− 7 }
12	29·59	1322·8 ? §	·9894 ? §	·9857	+37 ? §
13	29·60	1324·2	·9842	·9856	−14 }
14	29·62	1324·2	·9842	·9855	−13 }
15	29·58	1324·3	·9838	·9858	−20 }
16	29·50	1323·5	·9868	·9864	+ 4

* Temperature rose by 0·4° C.

† Temperature fell 0·6° C.

‡ Temperature rose 0·3° C. and fell again.

§ Observation apparently an error.

progressive manner. This view is supported by the fact that consecutive observations in the series as shown bracketed together in column 6 show a very close agreement. For instance, observations 4, 5, and 6 give results for the frequency differing from their mean (corrected for temperature) by less than five parts in a million.

IV. *Method of Comparison by Ear with Half-seconds Chronometer.*

In this method the sub-synchronous pendulum used has a frequency of approximately one per second, and its rate is found by comparison with the "ticks" of the half-seconds chronometer by ear. The most satisfactory way of making this comparison has been found to be the following:—A contact-maker is fixed to the pendulum and cuts across a mercury drop placed at the lowest point of its swing, thus completing an electric circuit through a telephone receiver and giving an audible signal once in each half-oscillation. The "ticks" of the chronometer are led to the ears of the observer through the listening tubes of an ordinary stethoscope. By putting the telephone receiver in contact with the listening tubes the signals given by the pendulum can also be heard at the same time, and their periodic coincidences with the "ticks" of the chronometer can be accurately registered on a stop-watch. Ordinarily, the interval of time between one coincidence and any of the succeeding coincidences can be found to within a second, and the accuracy can be further improved by using two or more stop-watches, as suggested in a preceding foot-note, and taking the mean of the observations.

The sub-synchronous pendulum having a frequency of roughly one per second, can be successfully maintained in oscillation by fork-interrupters having any frequency up to 140 per second and even more. Observations have been made in this laboratory using fork-interrupters with frequencies ranging from 24 to 140 per second. It will suffice to quote the following determination of the frequency of a standard fork, which was made on October 2, 1918. The frequency of this fork (which was mounted on a resonance-box) was about 128, and this was compared by the method of beats with an electrically maintained fork also of about the same frequency. The latter maintained a sub-synchronous pendulum in oscillation, the frequency-ratio being 1/124.

Five successive observations were made of the time-interval (about five minutes) occupied by 15 coincidences of the "ticks" of the chronometer with the sub-synchronous pendulum. The observational data are given below:—

Time interval for 15 Coincidences.

299·2	298·5	300·2	299·0	299·2 seconds.

Temperature of Standard Fork.

30·8	30·82	30·90	30·85	30·85 degrees.

Time-interval for 16 beats of Standard Fork with the Interrupter.

51·4	51·0	51·0	52·0	51·4 seconds.
51·2	51·0	51·0	51·0	51·6 ,,

The frequency of the standard fork calculated from these observations is

127·420	127·429	127·412	127·419	127·421 seconds.

The mean of the observations is 127·420 and the average error of an individual observation is about three parts in 100,000.

V. *The Sub-synchronous Pendulum Clock.*

Instead of rating the sub-synchronous pendulum by the method of coincidences against a standard clock or chronometer as described in the preceding sections, it may itself be allowed to function as a clock, the rate being determined by comparison with any timekeeper whose accuracy can be relied upon. As the apparatus once set up will run many hours continuously without attention, the desired degree of accuracy may be obtained by sufficiently prolonging the interval of observation. Obviously the most convenient procedure is to allow the pendulum to count its own swings on a dial showing ordinary time in hours, minutes, and seconds, and to find the rate at which this gains or loses on the timekeeper used. With a fork-interrupter of frequency about 24 per second and a half-seconds pendulum with a heavy brass bob, the maintenance of sub-synchronous oscillation is very energetic, and the pendulum is easily capable of moving a rocking-lever and ratchet-wheel controlling the hands on a 12-inch clock-dial without interference with the working of the apparatus.*

As an illustration of the method, the following series of observations made on February 24, 25, and 26, 1918, may be quoted. One of the two forks having a frequency about 24 vibrations per second was used, and the sub-synchronous pendulum-clock (frequency ratio 1/24) ran continuously for 64 hours, except for a few hours each night, when the apparatus was disconnected. Each observation made was to find the number of seconds gained or lost in an "hour" by the clock, and this was determined by comparison with an "Omega" stop-watch reading to a fifth of a second; 45 such observations

* It seems hardly doubtful that with a smaller dial and a well-poised counting mechanism running on jewelled bearings, the pendulum could successfully work the hands of a clock without interfering with its maintenance by fork-interrupters of much higher frequencies up to 128 per second.

in all were made during the three days. The temperature of the fork was taken at intervals, and the stop-watch (which had a compensated balance-wheel) was insulated as far as possible against temperature variation by being kept in a covered box except when it was being actually used. The 45 observations have been divided into seven groups, in each of which the temperature of the fork was the same to within a degree Centigrade. Table III shows for each group the mean temperature of the fork, the average gain per "hour" of the sub-synchronous pendulum-clock over the stop-watch, and the frequency of the fork calculated from the observations. The fourth column shows the frequency of the fork calculated from the formula

$$\text{Frequency} = 24{\cdot}13577\,[1-0{\cdot}000123\,(t-28^\circ)],$$

giving the best fit with the observations.

Table III.

1. Mean temperature of fork.	2. Gain of sub-synchronous pendulum-clock per "hour."	3. Observed frequency of interrupter-fork.	4. Calculated frequency of interrupter-fork.	5. Difference.
° C.	secs.			
22·89	22·5	24·15094	24·15094	0
23·40	22·3	·14959	·14943	+16
24·47	21·85	·14656	·14624	+32
25·73	21·2	·14217	·14251	−34
26·74	20·8	·13946	·13951	−5
27·50	20·46	·13720	·13725	−5
28·09	20·21	·13550	·13550	0
		Average error.........	.... 13 Or 5 parts in a million.	

The sub-synchronous pendulum-clock may also be readily used to demonstrate the fact that the frequency of the maintained oscillation of the pendulum is an exact sub-multiple of the frequency of the fork. This is shown by the fact that when the bob of the pendulum is shifted, the rate remains either unaltered or else changes to some other sub-multiple of the frequency of the fork. For instance, on moving down the bob of the pendulum, referred to in Table III, a different rate of running was obtained, the frequency-ratio being 1/26 instead of 1/24. The observed frequency of the fork at 28·90° C. was found, on the average of three hours' run, to be 24·1333, which agrees with the formula given above to within one part in 100,000.

Appendix.

Note on the Theory of Sub-synchronous Maintenance.

By Prof. C. V. Raman.

The principal features of interest requiring explanation in regard to the behaviour of the sub-synchronous pendulum are: (1) The actual possibility of the maintenance of the oscillation; (2) the fact that the frequency-ratio has generally an *even* number as the denominator; and (3) the manner in which the amplitude of the maintained oscillation varies in different parts of the range of maintenance. These will now be considered in the light of dynamical theory.

The effect of the periodic field due to the electro-magnet is equivalent to a large increase in the acceleration of gravity over a small part of the arc of swing of the pendulum. The system having only one degree of freedom, its equation of motion may be written in the form

$$\ddot{\theta}+\kappa\dot{\theta}+[n^2-\alpha\theta^2+f(t)\,.\,\mathrm{F}(\theta)]\,\theta=0.$$

In this equation, the damping of the pendulum due to dissipative forces is, as usual, taken to be proportional to the angular velocity. (Strictly speaking, the law of damping would not be the same at all parts of the arc of swing, especially when the pendulum-rod is near the vertical position, owing to the Foucault currents induced in it by the electro-magnet.) The term $-\alpha\theta^2$ which appears in the coefficient of θ is necessary, in view of the large amplitudes of oscillation actually obtained in practice. The term $f(t)\,.\,\mathrm{F}(\theta)$ expresses the effect of the periodic field.

It is sufficient for our present purpose to write $f(t)$ in the form $\beta+\gamma\sin mt$, the higher harmonic components being neglected. With the experimental arrangements actually adopted, $\mathrm{F}(\theta)$ is appreciable only when θ is small, and is practically negligible elsewhere. We may now assume that the pendulum is maintained in a steady oscillation given by

$$\theta=\psi_1\sin(pt+\epsilon_1)+\psi_2\sin(2pt+\epsilon_2)+\psi_3\sin(3pt+\epsilon_3)+,\text{ etc.},$$

and the question to be determined is whether sufficient energy passes from the periodic field to the pendulum in order to sustain the motion.

The loss of energy due to dissipative forces varies as

$$\int\kappa\dot{\theta}^2dt=\tfrac{1}{2}\kappa p^2\psi_1{}^2t.$$

ψ_2, ψ_3, etc., being treated as negligible. The energy which passes from the field into the pendulum is proportional to

$$(\beta+\gamma\sin mt)\,.\,\mathrm{F}(\theta)\,.\,\theta\,.\,\frac{d\theta}{dt}dt=\tfrac{1}{2}p\psi_1{}^2\int(\beta+\gamma\sin mt)\sin 2(pt+\epsilon_1)\,.\,\mathrm{F}(\theta)dt.$$

Changing the origin of time, this may be written in the form

$$\tfrac{1}{2}p\psi_1{}^2\int[\beta+\gamma\sin m(t-\epsilon_1/p)]\sin 2pt\,.\,\mathrm{F}(\theta)dt.$$

To effect the integration, we may, as a first approximation assume that $F(\theta)$ is equal to a constant δ when θ lies between the limits $\pm\phi$, and vanishes elsewhere. The corresponding limits for the variable t are

$$t = \pm\tau + r\pi/p,$$

where $$\tau = 1/p \, . \, \sin^{-1} \phi/\psi_1.$$

From this, it is readily shown that if $m = (2s+1)p$ where s is an integer, the integral evaluated over any number of complete periods is zero. On the other hand, if $m = 2sp$, the integral is finite and increases in proportion to the time. It follows that, on the assumptions made, maintenance is *not* possible when the frequency-ratio is unity divided by an *odd* integer, while if the ratio be unity divided by an even integer, energy may pass from the field to the pendulum in quantity sufficient to maintain its motion.

The other feature requiring explanation is the manner in which the amplitude of the maintained oscillation of the pendulum changes when its free period for small oscillations is altered by moving up the bob. The sequence of phenomena within any one of the ranges of maintenance, as shown for instance in fig. 1 of the paper, is quite unlike the ordinary type of resonance of a simple vibrator. This is due, in the first place, to the fact that in present case, the amplitude of the maintained oscillation is too large for the ordinary theory of small oscillations to be applicable. Further, the field due to the electro-magnet is appreciable only when the pendulum is nearly in the vertical position. Consequently, when the arc of swing is large, the frequency of the forced oscillation does not differ sensibly from that of the free oscillation. As the bob of the pendulum is moved up, the natural frequency for small oscillations increases, but this is set off by a corresponding increase in the arc of swing, so that the free and forced periods do not differ appreciably at any stage. When the arc of spring is small, however, which is the case near the lower end of the range of maintenance, the constant part β of the field due to the electro-magnet has an appreciable effect, which is equivalent to an increase in the frequency of free oscillation. These considerations fully explain the sequence of phenomena shown in fig. 1 of the paper.

It may be remarked in conclusion, as has indeed been actually observed by Mr. Dey, that the successive ranges within which the bob must lie for maintenance to be possible may possibly overlap in certain cases. The frequency of the maintained oscillation may then assume one or another of the possible series of values according to the actual arc of swing with which the pendulum is started.