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ON THE MAINTENANCE OF COMBINATIONAL VIBRATIONS BY TWO SIMPLE HARMONIC FORCES.

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Introductory.

IN a previous publication in this Review[1] I dealt with my experimental investigations on a new class of forced oscillations maintained by periodic variation of spring, of which the well-known phenomenon of the maintenance of vibrations by forces of double frequency (dealt with by Faraday, Melde and Lord Rayleigh) is a specific case. Working with the same apparatus as is used for one of the forms of Melde's experiment, I showed that a simple harmonic force acting longitudinally upon a stretched string could maintain its vibrations when the frequency of the free oscillations of the string in any given mode, is sufficiently nearly equal to any integral multiple of half the frequency of the fork. To illustrate and explain the manner in which such maintenance is effected, a series of photographs were published with the paper, showing simultaneous vibration-curves of the exciting tuning-fork and the maintained motion of the string, in which the natural frequencies of the latter were various integral multiples of half the frequency of the former. From these photographs and the mathematical discussion, it became clear that a very important part in the maintenance of the motion is played by certain subsidiary components introduced into it under the action of the variable spring. The principal component of the maintained vibration together with the subsidiary motions thus introduced could, it was shown, be arranged in the form of a Fourier series, the difference of frequency between the successive terms being that of the variable spring itself.

The successful investigation of this class of resonance-vibrations suggested further experiments with systems subjected simultaneously

[1] "Some Remarkable Cases of Resonance," Physical Review, Dec., 1912.

to *two* simple harmonic forces of differing frequencies varying the spring. These have been productive of some very interesting results and will form the subject of the present paper.

EXPERIMENTAL ARRANGEMENTS.

As mentioned above, the idea underlying the investigation now to be described was to subject a system, having a frequency of vibration that could be adjusted to any desired value over a wide range, *e. g.*, a stretched string, simultaneously to two simple harmonic forces of known frequencies varying its spring, and then to observe and record the various cases in which the state of equilibrium which usually obtains becomes unstable and the system settles down into vigorous vibration.

The experimental method adopted was extremely simple. Two electrically-maintained turning-forks were used. These stood on the table at some distance apart with their prongs vertical, in one plane, and directions of vibration parallel. A fine silk or cotton string, one or two metres in length, was stretched horizontally between the two forks, its extremities being attached to one prong of each fork (*i. e.*, to those nearest to each other). The tension of the string when the forks were at rest could be readily adjusted by merely sliding one fork slightly towards or away from the other along the table. Since the prongs of the fork are vertical and the string is parallel to their direction of vibration, we have as the result when the forks are excited, that the tension of the string is periodically varied by the vibrations of both simultaneously.

Of course, with the arrangements described neither of the forks, whether acting by itself or conjointly with the other, tends *directly* to displace the string from its position of equilibrium. They vary the tension of the string, but the latter remains undisturbed so far as transverse movement is concerned, except when the initial tension, *i. e.*, the frequency of free oscillation of the string, is adjusted so as to coincide more or less accurately with certain values which we may for convenience term "resonance frequencies," leaving the justification of this phraseology to be dealt with later.

Certain of the resonance-frequencies should obviously be multiples of half the frequency of one or the other of the forks by itself. For, each of the forks acting alone can maintain a vigorous vibration in a number of cases as shown in the paper referred to above, and this vibration is excited and maintained under suitable conditions even in the presence of the other periodic force varying the spring. To put it mathematically, if the frequency of free oscillation of the string in any given mode is sufficiently nearly equal to either $\frac{1}{2}rN_1$ or $\frac{1}{2}sN_2$, where N_1, N_2 are the frequencies of the forks, r, s, being any positive integers, we would get resonance-vibrations as already shown. That certain of the resonances observed are of this class, can readily be verified by stopping the fork which does not play a part in the maintenance, when the vibration of the other fork continues to sustain the motion of the string.

Besides the resonances of the kind described in the preceding paragraph, the observer is surprised and delighted to find, even at a first trial of the experiment, a large number of other cases of vigorous maintenance which have evidently to be ascribed to the joint action of the two forks on the string. Their variety and number is extraordinary, and these, together with the way in which they come rapidly following one another particularly at the higher frequencies, remind the observer, by a vivid analogy, of the lines in a complicated spectrum-series. It is readily guessed at once that these are cases of "combinational" resonance in which the frequency of the principal term in the maintained motion is related *jointly* to the frequencies of both the forks. This fact is readily verified by experimental investigation as described below, and the results obtained can be stated with generality thus. Under suitable conditions the equilibrium of the system becomes unstable and a vigorous motion is maintained if the frequency of free vibration in any given mode is sufficiently nearly equal to $\frac{1}{2}rN_1 \pm \frac{1}{2}sN_2$, where r and s are *positive* integers. The degree of accuracy of adjustment necessary for maintenance increases as r and s increase. Where the positive sign applies we have "summational" resonances. With the negative sign we have "differential" resonances. The frequency of the maintained motion is *exactly* equal to $\frac{1}{2}rN_1 \pm \frac{1}{2}sN_2$, where r and s have the values assigned.

Of course N_1 and N_2, which are the frequencies of the forks, do not in general stand in any simple arithmetical ratio, and the cases of "combinational" resonance described in this paper could in almost all cases be recognized and distinguished from "simple" resonance due to either of the forks acting alone by a peculiar appearance of "flicker" due to the presence of *small* components of very low frequencies in the motion. Even if this method failed, there is the alternative test of stopping either of the two forks when a "combinational" is instantly extinguished, whereas a "simple" resonance is only abolished by stopping *one* of the two forks, and not *either*. One very characteristic feature which was noticed in the experiments was that while resonance-vibrations of the *summational* class were obtained with great ease up to fairly high orders and vigorously maintained, *differential* vibrations were not nearly so readily maintained, and it was found necessary, in order to realize them to arrange matters so that none of the other resonances due to the forks, simple or summational, lay in the neighborhood of the one sought for and could therefore extinguish it, the former being maintained by preference. The result noticed above is an inversion of the ordinary experience in acoustical work with combinational tones in which it is found that differentials are generally stronger and easier to demonstrate than summationals. The theoretical explanation of the effect will be discussed later in this paper.

Photographic Record of Combinational Vibrations.

In order to demonstrate that the frequency of maintenance is that given by the combinational formula referred to above, the method of vibration-curves was used. Arrangements were made to obtain simultaneous photographic records of the vibration of the two forks and of the maintained oscillation of the string. These records incidentally throw light on the *modus operandi* of the maintenance. The disposition of the apparatus employed is shown in Fig. 1.

T_1 and T_2 are the two forks which stand with prongs vertical on the table. The string is stretched horizontally between the inner prongs of the forks as shown.

To enable its plane of vibration to be brought into the vertical at pleasure, the following very simple device is adopted. Each end of the string is attached to a loop of thread which is passed over the prong of the fork, instead of directly to the prong itself. The result of this mode of attachment is that the frequencies of vibration in the horizontal and vertical planes differ slightly, and this has the desired effect of keeping the vibration confined to the vertical if the tension of the string is suitably adjusted in each case. Immediately in front of the string is placed a camera, the plate-carrier of which has been removed, and which carries instead a square sheet fitted with a narrow vertical slit S as nearly as possible contiguous to the string. The light from an electric arc emerges from the nozzle of the lantern L; and is then divided into three parts.

1. One part passes first through the vertical slit S, then through the lens of the camera carrying it, and after suffering reflexion at the fixed mirror M passes on to the lens of the moving-plate camera DD (to be described below).

2. The second part is deflected by the mirror M_1, and after passing through a narrow horizontal slit S_1 suffers reflexion at the surface of a small plane mirror M_2 attached to the prong of the fork T_1 and is finally deflected by the mirror M_3 to the lens of the camera DD.

3. The third part after deflexion by the mirror M_4 passes through the

horizontal slit S_2, and thence after reflexion at a plane mirror M_5 attached to the prong of the fork T_2 passes on to the camera *DD*.

By suitably adjusting the position of the slits S_1 and S_2 and the distance between the slit *S* and the lens of the camera carrying it, it was found possible without the use of any additional collimating lenses to obtain

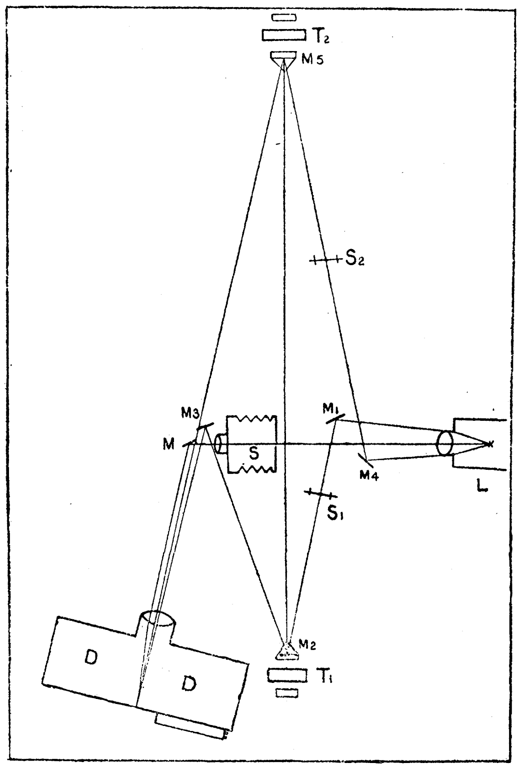


Fig. 1.

Apparatus for the production and photographic record of combinational vibrations.

in the focal plane of the camera *DD* real images of the vertical slit *S* and of the horizontal slits S_2 and S_1, the images of the latter being formed respectively above and below the images of the vertical slit *S*. A narrow vertical slit placed in the focal plane of the camera immediately in front of the ground-glass or its substitute, the photographic plate, cuts off everything except two brilliant spots of light, one for each fork, and between them the image of the illuminated vertical slit crossing which is seen the shadow of the string when at rest. The plate-holder can be moved (by hand) in horizontal grooves behind the slit in the focal plane, and when the forks are set in vibration we can thus obtain photographic records of simultaneous vibration-curves of the string and forks.

By counting the number of swings shown in each of the three curves appearing in a record, the relation between the frequencies of the forks and that of the maintained motion of the string can be observed and tested with the exact combinational formula $\frac{1}{2}(rN_1 \pm sN_2)$. Illustrations of this method will be given later.

Isolation of Individual Types.

As already mentioned above, the vibrations of the summational class are obtained with great ease. Taking only the first four types of summationals, *i. e.*, with $r = 1$ or 2 or 3 or 4 and $s = 1$ or 2 or 3 or 4, we have 4×4, *i. e.*, 16 distinct types possible, or if we include the cases in which either r or s is zero, *i. e.*, those of resonance due to each of the forks acting by itself, 24 distinct types. In practice summationals of types even higher than the fourth are distinguishable. For each of these various frequencies of vibration, the string may divide up into one, two, three or more ventral segments, according to circumstances. We have therefore a very large number of cases in which resonance is possible, and it is a matter for considerable surprise that it should be at all possible (not to speak of its being quite easy) to isolate in experiment any one of these manifold modes and frequencies of vibration and obtain distinctive photographic records of the same. The explanation of this result is very instructive. It rests upon the following facts: for each of these summationals there is a limited and fairly well-defined range of frequency within which the natural frequency of the system should lie if maintenance is to be possible. If the natural frequency of the system lies within this range, the vibration is vigorously maintained. If outside it, the summational does not put in an appearance. The range becomes narrower and narrower as we go higher up the scale and is smallest in the very region where the summationals are relatively speaking most numerous. This effectually prevents their crowding in unduly upon each other.

The facts mentioned in the foregoing paragraph sufficiently explain the successful isolation of the several vibrational types. After a little practice it will be found easy to arrange that any given member of the series of summationals (if not of too high an order) is obtained and vigorously maintained. The necessary guide to the proper adjustment of tension is to be had by noticing the tensions at which the "simple" resonances in various modes due to either of the forks acting alone occur, and by drawing up a table of frequencies of the summational vibrations it is a simple matter to get the right tension for any one of them.

In practical work, it will be found a useful device (besides adjusting the tension of the string to correspond with the frequency required), to regulate the amplitude of vibration of the forks in a suitable manner. This is readily done, if the forks are electrically maintained by altering the driving current or the position of the contact-maker. The formula to be borne in mind is, if r is large and s is small, to work the N_1 fork vigorously and the N_2 fork with quite a small amplitude of vibration: vice-versa if r is small and s is large. If r and s are to be nearly equal, the amplitudes are to be roughly commensurate with the values of r and s. This regulation of the amplitude ensures the desired summational being obtained without fail, and unaccompanied by other modes of vibration.

Vibration-Curves of Summationals.

Figs. 3 to 13 exhibit the photographic records for all the nine summationals comprised within the first three types, for one summational of the fourth type and one of the fifth, *i. e.*, eleven photographs in all. The two forks used had frequencies of 60 and 23.7 respectively per second. In the reproductions their vibration-curves appear white on a dark ground and that of the string dark on a bright background. The two former appear one on each side of the latter. It is obvious from an inspection of the vibration-curves of the string that in each case the principal part of the maintained motion is accompanied by subsidiary components. These components are introduced by the alteration of the character of the maintained motion due to the imposed variable spring, and it will be seen from the theoretical discussion to follow that they act as vehicles for the supply of energy to the system. In a few cases their periodicity is fairly evident to inspection.

Fig. 3.—This is the first and most important summational, the frequency of maintenance being equal to the sum of half the frequencies of the forks. This is readily shown by counting the swings shown on each curve. Thus—

Summational $\frac{1}{2}N_1 + \frac{1}{2}N_2$.

	Fork N_1.	Fork N_2.	String.
Number of swings	17.50	6.90	12.23
Calculated frequency	60.0	23.70	41.95
Observed frequency	60.0	23.70	41.93

An inspection of the vibration-curve of the string clearly shows the periodic flattening and sharpening of the maintained motion under the joint action of the components of the variable spring.

Fig. 4.—This shows the next higher frequency of maintenance, summational $\frac{1}{2}N_1 + N_2$, frequency 53.7 approximately.

Fig. 5 shows the summational $\frac{1}{2}N_1 + \frac{3}{2}N_2$, frequency 65.6.

Fig. 6 shows the summational $N_1 + \frac{1}{2}N_2$, frequency 71.9.

Fig. 7 shows the summational $N_1 + N_2$, frequency 83.7.

Fig. 8 shows the summational $N_1 + \frac{3}{2}N_2$, frequency 95.6.

Fig. 9 shows the summational $\frac{3}{2}N_1+\frac{1}{2}N_2$, frequency 101.9.

Fig. 10 shows the summational $\frac{3}{2}N_1 + N_2$ frequency 113.7.

Fig. 11 shows the summational $\frac{3}{2}N_1 + \frac{3}{2}N_2$, frequency 125.6.

Fig. 12 shows one of the summationals of the fourth type, frequency $2N_1 + \frac{1}{2}N_2$, *i. e.*, 131.9.

Fig. 13 shows one of the summationals of the fifth type, frequency $2N_1 + \frac{5}{2}N_2$, *i. e.*, 179.3.

Vibration-Curves of Differentials.

Using the two forks of frequencies 60 and 23.7 it was not found possible to obtain any cases of differential resonances, as these lay in the region in which the primaries and summationals were present and were strongly maintained in preference. After some trial, however, using forks of adjustable frequencies with which the frequencies of possible differentials lay far removed from that of the stronger resonances due to either of the forks alone or their summationals, I succeeded in isolating two cases of differential resonance. Fig. 14 represents the differential of the first type, frequency $\frac{1}{2}N_1$ and $\frac{1}{2}N_2$, being respectively 128 and 23.7. Fig. 15 represents a differential of the second type, frequency $N_1 - N_2$ being 92.3, the frequencies of the two forks used, N_1 and N_2, being respectively 128 and 35.7.

It will be seen that in both of these cases, the frequency of the differential is such that it cannot be readily confused with that of any resonances due to either of the forks acting alone or to their summationals.

General Theory of Combinational Maintenance.

The equation of motion of a simple oscillatory system having one degree of freedom when subject to two periodic forces varying its spring may be written as

$$\ddot{U} + k\dot{U} + n^2U = 2U[\alpha_1 \sin 2p_1t + \alpha_2 \sin 2p_2t + \beta_1 \cos 2p_1t + \beta_2 \cos 2p_2t] \quad (1)$$

$$= U[2\gamma_1 \sin (2p_1t + E_1) + 2\gamma_2 \sin (2p_2t + E_2)]. \quad (2)$$

It may also be written in the form

$$\ddot{U} + k\dot{U} + (n^2 - 2\gamma_1 \sin \overline{2p_1t + E_1} - 2\gamma_2 \sin \overline{2p_2t + E_2})U = 0. \quad (3)$$

It is instructive to compare the equation of motion as it is written in forms (1) and (2) above, with that of an asymmetrical system subject to double forcing considered by Helmholtz. The latter may be written as

$$\ddot{U} + k\dot{U} + (n^2 - \gamma U)U = 2\gamma_1 \sin (2p_1t + E_1) + 2\gamma_2 \sin (2p_2t + E_2) \quad (4)$$

or as

$$\ddot{U} + k\dot{U} + n^2U = \gamma U^2 + 2\gamma_1 \sin (2p_1t + E_1) + 2\gamma_2 \sin (2p_2t + E_2). \quad (5)$$

Equations (2) and (5) are analogous in so far as the term on the right-hand side which represents the "disturbing force" acting on the system is in both cases a function of two variables, *i. e.*, the time, and the configuration of the system: the form of the function is however different in the two cases.

Equations (3) and (4) are analogous in so far as that the coefficient of the third term on the left which represents the "spring" of the system is in both cases a variable. But here the analogy ends, for in one case the variable part of the spring is an independent function of the time, in the other case it is a function of the configuration only.

In any case, however, there is abundant material to suggest that the maintenance of a series of combinational vibrations should be possible under the joint action of two periodic forces of different frequencies varying the spring, and this is fully verified by the results of the experiments described above.

Discussion of Diagram of Periodicities.

The exact process by which the maintenance of the combinational vibrations is effected in these experiments is best understood by analogy with the simpler case of oscillations under variable spring of only one periodicity, and by reference to Fig. 2. In the theory of oscillations maintained by a simple variable spring referred to above, it was shown that when resonance was secured by adjusting the natural frequency of the system to any multiple of half the frequency of the impressed force, the principal part of the maintained motion and the subsidiary components of smaller amplitudes introduced under the action of the variable spring could be arranged in the form of a Fourier series. For example, in the case of the fourth type of maintenance, the equation of motion is

$$\ddot{U} + k\dot{U} + n^2U = 2\alpha U \sin 2pt,$$

and n being nearly equal to $4p$, we have as the solution

$$U = A_2 \sin 2pt + \boldsymbol{A_4 \sin 4pt} + \text{etc.}$$
$$+ B_0 + B_2 \cos 2pt + \boldsymbol{B_4 \cos 4pt} + \text{etc.}$$

Similarly in the case of the third type of maintenance where n is nearly equal $3p$ we have

$$U = A_1 \sin pt + \boldsymbol{A_3 \sin 3pt} + \text{etc.}$$
$$+ B_1 \cos pt + \boldsymbol{B_3 \cos 3pt} + \text{etc.}$$

It will thus be seen that in each case, the subsidiary components introduced under the action of the variable spring proceed by successive differences of $2pt$. The components having smaller frequencies than that of the system were the vehicles for the supply of energy required for the maintenance of the oscillations: those having higher frequencies play no such part, but are equivalent merely to a small alteration in the natural frequency of the system. The components of smaller frequencies are none of them negligible so far as the explanation of the maintenance is concerned: those of higher frequencies can generally (though not always) be safely ignored.

In Fig. 2 each of the points marked corresponds to a frequency of the system at which (to a close approximation) resonance is possible under the sole or joint action of the two components of variable spring in the

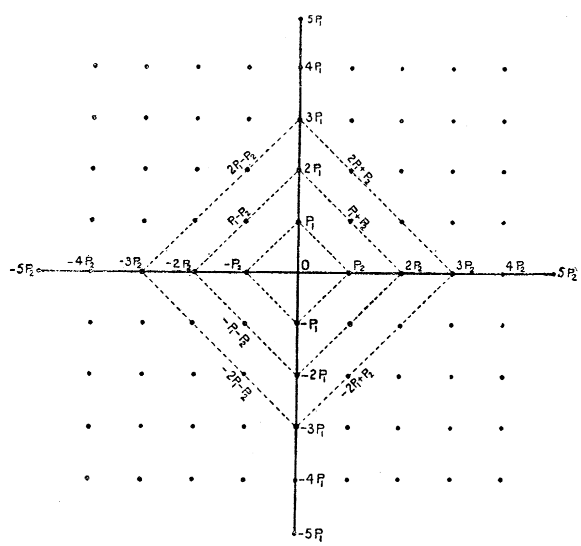


Fig. 2.

Representation of periodicities of combinational maintenance.

experiments described above. The diagram also enables us to see at a glance the periodicity of the subsidiary components in the motion in each such case. For, starting at the resonance-point for the case, we move by successive steps of $2p_1$ and $2p_2$ respectively parallel to the two axes, and each point so arrived at represents a component in the motion induced under the action of the variable spring. Only about one fourth of the total number of points in the diagram can be arrived at in this way from a given starting-point. Such of the components as are represented by points lying outside the diagonal square drawn through the resonance-point (as in the specimens shown by dotted lines in the diagram), may be neglected in considering the maintenance of the motion. For example, when considering the case $n = (p_1 + p_2)$ nearly, only the components which are trigonometrical functions of $(p_1 + p_2)t$ and $(p_1 - p_2)t$ need be taken into account to arrive at an approximate solution. It will be seen that alternative paths starting from the same point lead to periodicities having the same value, with or without the sign reversed; and it is this fact and the negligibility of points lying outside the diagonal square in each case that enables a definite solution of the equation of motion to be obtained easily by the method of approximations.

The diagram of periodicities has other uses. It enables us to obtain at a glance an indication of the circumstances under which maintenance can be successfully effected in each case, and to arrange the experimental conditions accordingly. For instance, in the case of motion under one of the components of variable spring alone, we know that the adjustment of the natural frequency of the system with reference to that of the impressed force should be more and more accurate as we proceed outwards along the axes of the diagram and meet the successive resonance-points. In the case of the point p_1 the adjustment should be accurate to the order α, in the case of the point $2p$ to the order α^2, and so on, α being the coefficient of the variable spring (supposed small). The diagram suggests that a similar increase in the accuracy of adjustment would be found necessary as we proceed outwards from the origin in any other direction and meet resonance-points situated on successive diagonals (shown as dotted lines). This indication is amply confirmed both by experiment and by the detailed mathematical treatment. For instance it will be shown below that a degree of adjustment accurate to the order α^2 would be necessary for the point $p_1 + p_2$ which lies midway between $2p_1$ and $2p_2$.

The position of any resonance-point on the diagram is also found to indicate approximately the amplitude of vibration of the two tuning-forks required for successful maintenance.

Theory of Summationals of the First Type.

We now proceed to consider the detailed theory of the simplest summational type in which the frequency of maintenance is equal to the sum of half the frequencies of the forks. As already explained, the whole of the string behaves as one unit, in other words, we may write down the equation of motion as that of a system having only one degree of freedom. The equation is

$$\ddot{U} + k\dot{U} + n^2 U = 2U[\alpha_1 \sin 2p_1 t + \alpha_2 \sin 2p_2 t + \beta_1 \cos 2p_1 t + \beta_2 \cos 2p_2 t]. \quad (1)$$

Since by assumption n is nearly equal to $p_1 + p_2$, we may commence building up a solution by putting

$$U = A_1 \sin (p_1 + p_2)t + B_1 \cos (p_1 + p_2)t + \text{etc.} + \text{etc.} \quad (6)$$

On substituting the first two terms on the right of (6) for U in the right-hand side of equation (1) and expanding the terms in it, we find that none of them is a sine or cosine of $(p_1 + p_2)t$, *i. e.*, represents a force which is competent to excite resonance, if we regard (1) as the ordinary equation of forced oscillations. It is obvious, therefore, that to solve the equation even approximately and explain the maintenance of the motion, we have to take a second subsidiary pair of terms in the expression for U. The frequency of these terms is ascertained at once by reference to the diagram of periodicities given above, and we may then write

$$U = A_1 \sin (p_1 + p_2)t + B_1 \cos (p_1 + p_2)t + A_2 \sin (p_1 - p_2)t + B_2 \cos (p_1 - p_2)t + \text{etc.} \quad (7)$$

The terms in A_2 and B_2 are of course small compared with those in A_1B_1, yet are sufficiently large to make their presence felt in the vibration-curve pictured previously. They are introduced by the action of *either* of the two periodic components of the variable spring on the fundamental motion, and in their turn maintain the latter by making the requisite supply of energy to the system possible. This can be shown by writing out the equation of the work supplied to the system by the variable spring and that dissipated by frictional forces in an equal time. In the present case it can be shown in a simpler way by merely equating separately the terms of various periodicities on either side of equation (1) after substituting the value of U given by (7).

We have thus:

(writing $n^2 - (p_1 + p_2)^2 = \theta_1$ and $n^2 - (p_1 - p_2)^2 = \vartheta_2$, $k(p_1 + p_2) = \phi_1$ and $k(p_1 - p_2) = \phi_2$ for brevity,

$$\begin{aligned}
\theta_1 A_1 - \phi_1 B_1 &= A_2(\beta_2 - \beta_1) + B_2(\alpha_1 + \alpha_2),\\
\phi_1 A_1 + \theta_1 B_1 &= A_2(\alpha_1 - \alpha_2) + B_2(\beta_1 + \beta_2),\\
\theta_2 A_2 - \phi_2 B_2 &= A_1(\beta_2 - \beta_1) + B_1(\alpha_1 - \alpha_2),\\
\phi_2 A_2 + \theta_2 B_2 &= A_1(\alpha_1 + \alpha_2) + B_1(\beta_1 + \beta_2), \quad (8)
\end{aligned}$$

It will be seen that these four equations were derived by retaining only terms containing trigonometrical functions of $(p_1 + p_2)t$ and $(p_1 - p_2)t$ and neglecting all others. Before considering the effect, if any, of the neglected terms, it is well to discuss the physical significance of the equations. They give us the three ratios A_1, B_1, A_2, B_2 in terms of the known quantities θ_1, θ_2, ϕ_1, ϕ_2, and α_1, α_2, β_1, β_2, as an approximate solution of the equation of motion and leave us in addition a relation between these "constants" which must be satisfied for steady motion to be possible.

We now proceed to solve the equations. This may, of course, be done in the usual way in terms of certain determinants, but it is more instructive to proceed by an approximate method retaining only terms up to the order of smallness desired. Assuming that α_1, α_2, β_1, β_2 (the coefficients of variable spring) are small, it is obvious from the equations that A_2, B_2 are small compared with A_1, B_1. Further ϕ_2 is a very small quantity compared with θ_2 (which is finite and large), since the former is proportional to k, the coefficient of friction, which must itself be small for maintenance to be possible. We are therefore justified in neglecting ϕ_2 and writing the last two of equations (8) thus:—

$$\begin{aligned} \theta_2 A_2 &= A_1(\beta_2 - \beta_1) + B_1(\alpha_1 - \alpha_2), \\ \theta_2 B_2 &= A_1(\alpha_1 + \alpha_2) + B_1(\beta_1 + \beta_2), \end{aligned} \tag{9}$$

These equations give us the subsidiary components of motion A_2, B_2, in terms of the principal parts A_1, B_1. Substituting these values in the first two of equations (8), we have

$$\begin{aligned} \theta_1 A_1 - \phi_1 B_1 &= \frac{A_1}{\theta_2}\Big[(\alpha_1^2 + \beta_1^2) + (\alpha_2^2 + \beta_2^2) + 2(\alpha_1\alpha_2 - \beta_1\beta_2) \Big] \\ &\quad + \frac{B_1}{\theta_2}\Big[2(\alpha_1\beta_2 + \alpha_2\beta_1) \Big] \\ \phi_1 A_1 + \theta_1 B_1 &= \frac{A_1}{\theta_2}\Big[2(\alpha_1\beta_2 + \alpha_2\beta_1) \Big] \\ &\quad + \frac{B_1}{\theta_2}\Big[(\alpha_1^2 + \beta_1^2) + (\alpha_2^2 + \beta_2^2) - 2(\alpha_1\alpha_2 - \beta_1\beta_2) \Big]. \end{aligned} \tag{10}$$

Writing

$$\begin{aligned} \theta_1 - \frac{1}{\theta_2}\Big[(\alpha_1^2 + \beta_1^2) + (\alpha_2^2 + \beta_2^2) \Big] &= \theta, \\ \frac{2}{\theta_2}(\alpha_1\alpha_2 - \beta_1\beta_2) &= a, \\ \frac{2}{\theta_2}(\alpha_1\beta_2 + \alpha_2\beta_1) &= b \end{aligned}$$

we have

$$\begin{aligned} (\theta - a)A_1 &= (\phi_1 + b)B_1, \\ (\theta + a)B_1 &= -(\phi_1 - b)A_1. \end{aligned} \tag{11}$$

The solution of this is

$$\frac{B_1}{A_1} = \frac{\theta - a}{\phi_1 + b} = \frac{b - \phi_1}{\theta + a} \tag{12}$$

and the eliminant, *i. e.*, the relation between the constants involved which must be satisfied for maintenance to be possible, is

$$\theta^2 - a^2 = b^2 - \phi_1^2. \tag{13}$$

It is interesting to consider a few special cases. For instance put

$$\beta_1 = \beta_2 = 0 \text{ and } \alpha_1 = \alpha_2 = \alpha/2.$$

From (9) we find then that

$$A_2 = 0 \text{ and } B_2 = A_1\alpha/2.$$

Also

$$\theta_1 + \frac{\phi_1^2}{\theta_1} = \alpha^2/\theta_2. \tag{14}$$

From (14) it is evident that θ_1 and ϕ_1 are both of the order α^2/θ_2, *i. e.*, both the friction and adjustment of frequency for resonance must be correct to the second order of small quantities. The equations leave the actual amplitude of the motion indeterminate. In practice both the necessary adjustment of frequency and the determinateness of the amplitude would be secured by the fact that n is not absolutely constant, in other words by the variation of spring existing in free oscillations of sensible amplitude. It will be seen that the term B_2 is small compared with A_1, since the variable part of the spring is small compared with the permanent spring: nevertheless the term B_2 plays a very important part in the maintenance of the motion, being, as is evident from the foregoing equations, the vehicle for the supply of the requisite energy to the system.

It will be seen that the equations cannot be satisfied if $\theta_1 = 0$, as $\theta_1 + (\phi_1^2/\theta_1)$ then becomes infinitely large. According to these equations therefore, resonance is possible only when θ_1 has a small but a definite positive value: *i. e.*, when the frequency of the *free* oscillations is very slightly greater than the sum of the half-frequencies of the two imposed variations of spring. This conclusion would no doubt have to be modified in view of two factors which we have not so far taken into account. First, the neglected terms in the motion which are trigonometrical functions of $(p_1 + 3p_2)t$ and $(3p_1 + p_2)t$, etc., etc. The two terms $(p_1 + 3p_2)t$, $(3p_1 + p_2)t$ which are of the order α when compared with the fundamental motion $(p_1 + p_2)t$ cannot actually assist in maintaining it. This can be shown from very simple considerations. For one thing, they are not introduced by the action of *both* the components of variable spring.

The first is due to one component and the second is due to the other. The components in the restoring force due to their action and which are trigonometrical functions of $(p_1 + p_2)t$ are in such a *phase* that their effect is equivalent to a small increase in the frequency of free oscillations of the system which is some importance when θ_1 is very small.

The second factor which we have to take into account is the variation of spring existing in free oscillations of large amplitude. This, as in the case of oscillations maintained by a single variable spring, when expanded is found to contain both constant and periodic terms. The former are equivalent to a direct increase in the natural frequency of the system, and the latter profoundly modify the effect of the impressed forces and the phases of the respective components in the motion when the amplitude is at all sensible.

Theory of Summationals of the Second and Higher Types.

As typical of the summationals of the second type we may take the case $n_1 = 2p_1 + p_2$ (nearly). The equation of motion is

$$\ddot{U} + k\dot{U} + n^2 U = 2U \begin{bmatrix} \alpha_1 \sin 2p_1 t + \alpha_2 \sin 2p_2 t \\ + \beta_1 \cos 2p_1 t + \beta_2 \cos 2p_2 t \end{bmatrix}. \qquad (1)$$

We may to start with put

$$U = A_1 \sin (2p_1 + p_2)t + B_1 \cos (2p_1 + p_2)t + \text{etc.} + \text{etc.} \qquad (15)$$

As before, we can only solve the equation by taking certain additional terms on the right of (15). A reference to the diagram of periodicities shows that we have to take three additional pairs of terms to get an approximate solution. We may thus write

$$\begin{aligned} U = {} & A_1 \sin (2p_1 + p_2)t + B_1 \cos (2p_1 + p_2)t \\ & + A_2 \sin (2p_1 - p_2)t + B_2 \cos (2p_1 - p_2)t \\ & + A_3 \sin p_2 t + B_3 \cos p_2 t \\ & + A_4 \sin 3p_2 t + B_4 \cos 3p_2 t. \end{aligned} \qquad (16)$$

Of course, the terms in A_1 and B_1 are the largest in amplitude. In substituting the right-hand side of (16) for U in equation (1) and writing down in the results, we may use the following abbreviations:

$$\begin{aligned} n^2 - (2p_1 + p_2)^2 &= \theta_1, & k(2p_1 + p_2) &= \phi_1, \\ n^2 - (2p_1 - p_2)^2 &= \theta_2, & k(2p_1 - p_2) &= \phi_2, \\ n^2 - p_2^2 &= \theta_3, & kp_2 &= \phi_3, \\ n^2 - 9p_2^2 &= \theta_4, & 3kp_2 &= \phi_4. \end{aligned} \qquad (17)$$

By equating the various sine and cosine terms on either side of (1) after substitution, we have

$$
\begin{aligned}
\theta_1 A_1 - \phi_1 B_1 &= \beta_1 A_3 + \alpha_1 B_3 + \beta_2 A_2 + \alpha_2 B_2,\\
\phi_1 A_1 + \theta_1 B_1 &= -\alpha_1 A_3 + \beta_1 B_3 - \alpha_2 A_2 + \beta_2 B_2,\\
\theta_2 A_2 - \phi_2 B_2 &= \beta_2 A_1 - \alpha_2 B_1 - \beta_1 B_3 + \alpha_1 B_3,\\
\phi_2 A_2 + \theta_2 B_2 &= \alpha_2 A_1 + \beta_2 B_1 + \alpha_1 A_3 + \beta_1 B_3,\\
\theta_3 A_3 - \phi_3 B_3 &= \beta_1 A_1 - \alpha_1 B_1 - \beta_1 A_2 + \alpha_1 B_2 + \beta_2 A_4 - \alpha_2 B_4,\\
\phi_3 A_3 + \theta_3 B_3 &= \alpha_1 B_1 + \beta_1 B_1 + \alpha_1 A_2 + \beta_2 B_2 + \alpha_2 A_4 + \beta_2 B_4,\\
\theta_4 A_4 - \phi_4 B_4 &= \beta_2 A_3 + \alpha_2 B_3,\\
\phi_4 A_4 + \theta_4 B_4 &= -\alpha_2 A_3 + \beta_2 B_3.
\end{aligned}
\tag{18}
$$

From the last four equations in (18) it will be seen that a further simplification can be effected. For A_4 and B_4 are of the order α_2, β_2 in comparison with A_3 and B_3, and the terms $\beta_2 A_4$, $\alpha_2 B_4$, $\alpha_2 A_4$ and $\beta_2 B_4$ on the right-hand side of the fifth and sixth equations in (18) can therefore be neglected in comparison with all other terms involved. We are therefore finally left with six equations only

$$
\begin{aligned}
\theta_1 A_1 - \phi_1 B_1 &= \beta_1 A_3 + \alpha_1 B_3 + \beta_2 A_2 + \alpha_2 B_2,\\
\phi_1 A_1 + \theta_1 B_1 &= -\alpha_1 A_3 + \beta_1 B_3 - \alpha_2 A_2 + \beta_2 B_2,\\
\theta_2 A_2 - \phi_2 B_2 &= \beta_2 A_1 - \alpha_2 B_1 - \beta_1 A_3 + \alpha_1 B_3,\\
\phi_2 A_2 + \theta_2 B_2 &= \alpha_2 A_1 + \beta_2 B_1 + \alpha_1 A_3 + \beta_1 B_3,\\
\theta_3 A_3 - \phi_3 B_3 &= \beta_1 A_1 - \alpha_1 B_1 - \beta_1 A_2 + \alpha_1 B_2,\\
\phi_3 A_3 + \theta_3 B_3 &= \alpha_1 A_1 + \beta_1 B_1 + \alpha_1 A_2 + \beta_1 B_2.
\end{aligned}
\tag{19}
$$

These equations contain only terms having three periodicities, *i. e.*, the principal part of the maintained motion of frequency $(2p_1 + p_2)/2\pi$ and two others, subsidiary to it, which are given by the two nearest admissible points on the periodicity diagram, *i. e.*, of frequencies $(2p - p_2)/2\pi$ and $p_2/2\pi$ respectively. These components are derived from the principal motion by the action on it of the variations in spring, and serve to maintain it permanently in the presence of dissipative forces. They are both small compared with the main motion provided α_1, α_2, β_1, β_2, are small. Fig. 6 represents the vibration curve of this type of summational, and the component of frequency $p_2/2\pi$ (*i. e.*, half that of the graver fork) is very obvious to inspection.

Equations (19) when solved give us values for the five ratios B_1/A_1, B_2/A_1, B_3/A_1, A_2/A_1 and A_3/A_1 and leave us in addition a relation between the "constants" involved in the equations which must be satisfied for steady maintenance to be possible. The solution of the equations by the method of determinants is really a formidable business and is in fact unnecessary: an approximate method of solving them may be used which gives results quite as accurate as the equations themselves. We notice that A_2, B_2, A_3, B_3 are small quantities relatively to A_1, B_1, and further, since k, the coefficient of friction, is necessarily very small for maintenance to be possible, the quantities ϕ_2, ϕ_3 are negligible in comparison with θ_2 and θ_3. The first step in solving the equations (19) is therefore to simplify the last four of them and obtain approximate values for A_2, B_2, A_3, B_3.

We thus have

$$\theta_2 A_2 = \beta_2 A_1 - \alpha_2 B_1 + \frac{1}{\theta_3}\left(\overline{\alpha_1^2 - \beta_1^2}A_1 + 2\alpha_1\beta_1 B_1\right),$$

$$\theta_2 B_2 = \alpha_2 A_1 + \beta_2 B_1 + \frac{1}{\theta_3}\left(2\alpha_1\beta_1 A_1 - \overline{\alpha_1^2 - \beta_1^2}B_1\right),$$

$$\theta_3 A_3 = \beta_1 A_1 - \alpha_1 B_1 + \frac{1}{\theta_2}\left(\overline{\alpha_1\alpha_2 - \beta_1\beta_2}\,A_1 + \overline{\alpha_1\beta_2 + \alpha_2\beta_1}B_1\right),$$

$$\theta_3 \beta_3 = \alpha_1 A_1 + \beta_1 B_1 + \frac{1}{\theta_2}\left(\overline{\alpha_1\beta_2 + \alpha_2\beta_1}A_1 - \overline{\alpha_1\alpha_2 - \beta_1\beta_2}B_1\right). \quad (20)$$

The equations give us the values of the subsidiary components of motion in terms of the principal part and of the coefficients of the variable spring by which they are produced.

Substituting these values in the first two of equations (19) we have

$$\theta_1 A_1 + \phi_1 B_1 = \frac{A_1}{\theta_2\theta_3}\left[\theta_2(\alpha_1^2+\beta_1^2)+\theta_3(\alpha_2^2+\beta_2^2)+2\beta_2(\alpha_1^2-\beta_1^2)+4\alpha_2\alpha_1\beta_1\right]$$
$$+\frac{B_1}{\theta_2\theta_3}\left[4\beta_2\alpha_1\beta_1 - 2\alpha_2(\alpha_1^2-\beta_1^2)\right],$$

$$\phi_1 A_1 + \theta_1 B_1 = \frac{A_1}{\theta_2\theta_3}\left[4\beta_2\alpha_1\beta_1 - 2\alpha_2(\alpha_1^2-\beta_1^2)\right]$$
$$+\frac{B_1}{\theta_2\theta_3}\left[\theta_2(\alpha_1^2+\beta_1^2)+\theta_3(\alpha_2^2+\beta_2^2)-2\beta_2(\alpha_1^2-\beta_1^2)-4\alpha_2\alpha_1\beta_1\right]. \quad (21)$$

Writing

$$\theta_1 - \frac{1}{\theta_2\theta_3}\left[\theta_2(\alpha_1^2+\beta_1^2) + \theta_3(\alpha_2^2+\beta_2^2)\right] = \theta,$$

$$\frac{1}{\theta_2\theta_3}\left[2\beta_2(\alpha_1^2-\beta_1^2) + 4\alpha_2\alpha_1\beta_1\right] = a,$$

$$\frac{1}{\theta_2\theta_3}\left[4\beta_2\alpha_1\beta_1 - 2\alpha_2(\alpha_1^2-\beta_1^2)\right] = b,$$

equations (21) may be put into the form

$$(\theta - a)A_1 = (\phi_1 + b)B_1,$$
$$(\theta + a)B_1 = -(\phi_1 - b)A_1. \qquad (22)$$

It will be seen that equations (22) are identical in *form* with equations (11) obtained for the summational of the first type and the further discussion must proceed on much the same lines.

The solution is

$$\frac{B_1}{A_1} = \frac{\theta - a}{\phi_1 + b} = \frac{b - \phi}{\theta + a} \qquad (23)$$

and the eliminant is

$$\theta^2 - a^2 = b^2 - \phi_1^2. \qquad (24)$$

We may as before consider the special case in which $\beta_1 = \beta_2 = 0$ and $\alpha_1 = \alpha_2 = \alpha$

$$\theta_2 A_2 = -\alpha B_1 + \alpha^2 A_1/\theta_3,$$
$$\theta_2 B_2 = \alpha A_1 - \alpha^2 B_1/\theta_3,$$
$$\theta_3 A_3 = -\alpha B_1 + \alpha^2 A_1/\theta_2,$$
$$\theta_3 B_3 = \alpha A_1 - \alpha^2 B_1/\theta_2,$$
$$\frac{B_1}{A_1} = \frac{\theta_1\theta_2\theta_3 - \alpha^2(\theta + \theta_3)}{\phi_1\theta_2\theta_3 - 2\alpha^3} = \frac{-2\alpha^3 - \phi_1\theta_2\theta_3}{\theta_1\theta_2\theta_3 - \alpha^2(\theta_2 + \theta_3)},$$
$$[\theta_1\theta_2\theta_3 - \alpha^2(\theta_2 + \theta_3)]^2 = 4\alpha^6 - \phi_1^2\theta_2^2\theta_3^2. \qquad (25)$$

For maintenance to be theoretically possible in this case, the frictional coefficient k should be of the third order of small quantities and the adjustment of frequency must therefore be accurate up to the same order.

Cases of summationals of higher orders can be worked out in a similar manner, the approximation being carried to a higher and higher degree as we rise up the series.

Theory of Differentials.

The solution of the equation of motion in the case of the first differential is obviously to be obtained in this case by merely writing A_1, B_1, θ_1, ϕ_1 for A_2, B_2, θ_2, ϕ_2 and vice versa, in equations (8) obtained for the summational of the first type. We thus have

$$\theta_1 A_1 - \phi_1 B_1 = A_2(\beta_2 - \beta_1) + B_2(\alpha_1 - \alpha_2),$$
$$\phi_1 A_1 + \theta_1 B_1 = A_2(\alpha_1 + \alpha_2) + B_2(\beta_1 + \beta_2),$$
$$\theta_2 A_2 - \phi_2 B_2 = A_1(\beta_2 - \beta_1) + B_1(\alpha_1 + \alpha_2),$$
$$\phi_2 A_2 + \theta_2 B_2 = A_1(\alpha_1 - \alpha_2) + B_1(\beta_1 + \beta_2), \qquad (26)$$

where

$$U = A_1 \sin (p_1 - p_2)t + B_1 \cos (p_1 - p_2)t$$
$$+ A_2 \sin (p_1 + p_2)t + B_2 \cos (p_1 - p_2)t + \text{etc.} \quad (27)$$

and

$$\theta_1 = n^2 - \overline{p_1 - p_2}^2, \quad \theta_2 = n^2 - \overline{p_1 + p_2}^2,$$
$$\phi_1 = k(p_1 - p_2) \text{ and } \phi_2 = k(p_1 + p_2).$$

Proceeding as before, we put

$$\theta_2 A_2 = A_1(\beta_2 - \beta_1) + B_1(\alpha_1 + \alpha_2),$$
$$\theta_2 B_2 = A_1(\alpha_1 - \alpha_2) + B_1(\beta_1 + \beta_2), \quad (28)$$

$$\theta_1 A_1 - \phi_1 B_1 = \frac{A_1}{\theta_1}\left[(\alpha_1^2 + \beta_1^2) + (\alpha_2^2 + \beta_2^2) - 2(\alpha_1\alpha_2 + \beta_1\beta_2) \right]$$
$$+ \frac{B_1}{\theta_2}\left[2(\alpha_1\beta_2 - \alpha_2\beta_1) \right],$$

$$\phi_1 A_1 + \theta_1 B_1 = \frac{A_1}{\theta_2}\left[2(\alpha_1\beta_2 - \alpha_2\beta_1) \right]$$
$$+ \frac{B_1}{\theta_2}\left[(\alpha_1^2 + \beta_1^2) + (\alpha_2^2 + \beta_2^2) + 2(\alpha_1\alpha_2 + \beta_1\beta_2) \right]. \quad (29)$$

It will be seen that equations (29) are of the same general form as (10) with certain modifications. The terms on the right-hand sides of (10) and (29) can be derived from each other by writing α_2 for $-\alpha_2$ and vice versa. The further reduction of equations (29) may be proceeded with in the usual way.

The outstanding feature of the differential types is the relative difficulty of isolating and maintaining them successfully. No doubt this is partly due to the fact that the differentials of any given order are of much lower frequency than the summationals of the same order, and they lie therefore generally in the very region of frequencies in which simpler types of resonance are maintained far more powerfully over wide ranges. These latter are maintained by preference and extinguish the differentials. The foregoing however does not appear to be a complete explanation. Possibly the following further considerations must also be taken into account in explaining the relative poverty of differentials. In the mathematical discussion, it was shown that the subsidiary components of motion introduced under the action of the variable spring, themselves enabled the principal motion to be maintained, and the relative amplitudes and phases of these components were determined on the assumption that n^2, the free spring of the system, was a constant. It was however indicated that in practice this was not strictly the case, and in fact the success or otherwise of the experiments, *i. e.*, the steady maintenance of vibration in a certain amplitude, is dependent on the quantity n^2 not being itself absolutely a constant. For large displacements n^2 is greater than for small displacements and the equation of motion when modified to take account of this fact may be written thus,

$$\ddot{U} + k\dot{U} + [n^2 + mU^2 - \overline{2\alpha_1 \sin 2p_1t + \beta_1 \cos 2p_1t} - \overline{2\alpha_2 \sin 2p_2t + \beta_2 \cos 2p_2t}]U = 0. \quad (30)$$

The quantity mU^2 has been added to the third term within the brackets to represent the increase of spring for large displacements in a *symmetrical* system. It is obvious that when expanded for any periodic solution of U, mU^2 will give us both constant and periodic terms. The latter, *i. e.*, the periodic terms, would be of various frequencies, and of them the most important would be those which are sines or cosines of $2p_1t$ or $2p_2t$, since they would directly modify the action of the components of the variable spring. It is therefore quite evident that the amplitudes and phases of the subsidiary components of motion would not be the same as when mU^2 is omitted. Some components would tend to increase at the expense of others. Instances of such action have already been furnished in my previous publications when discussing maintenance by a simple variation of spring.

What we may expect to find is that when mU^2 is taken into account in the equation of motion, the subsidiary components which maintain differentials are less effective than they would otherwise be, whereas, in the case of summationals, they would be more effective. For, in the former case, some of them at least are of frequency higher than that of the maintained motion, in the latter they are invariably less, and the components of lower frequencies are encouraged at the expense of those of higher frequencies.

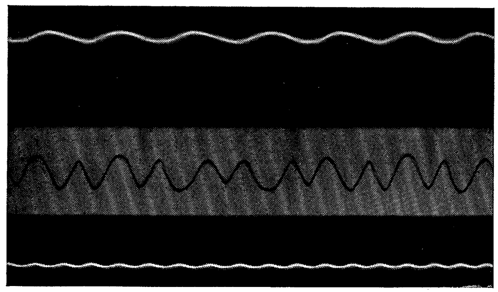
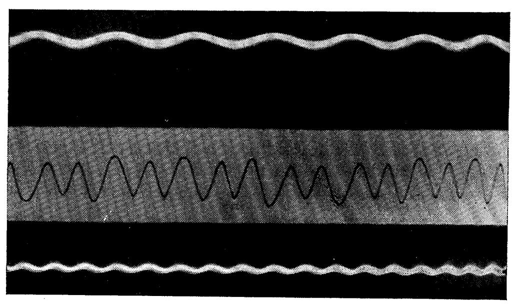


FIG. 3.



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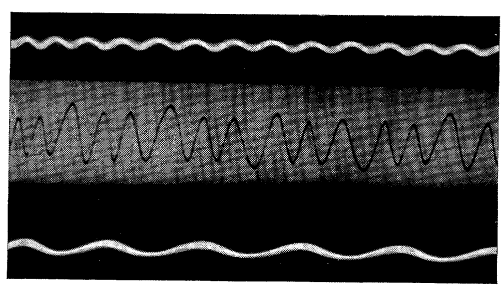

FIG. 5.

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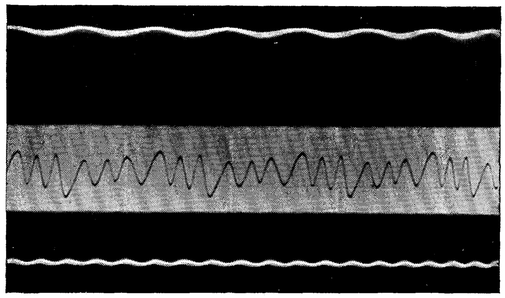

FIG. 6.

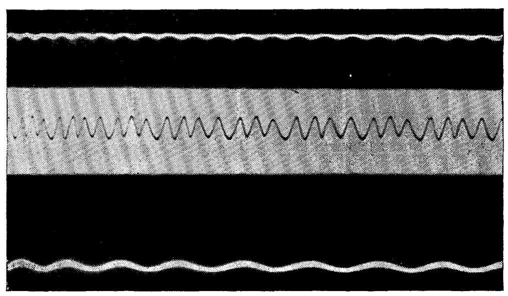

FIG. 7.

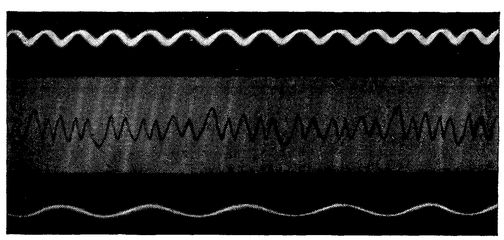

FIG. 8.

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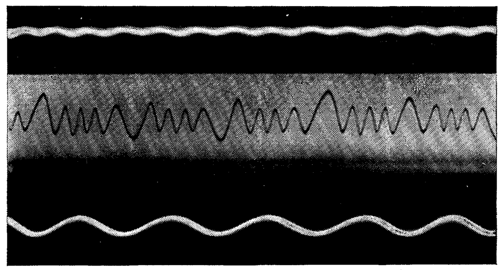

FIG. 9.

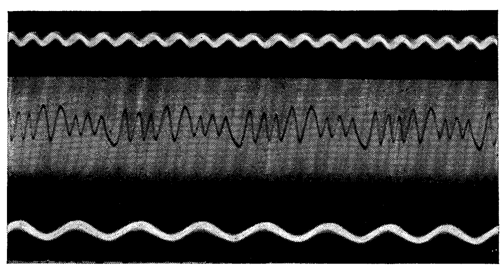

FIG. 10.

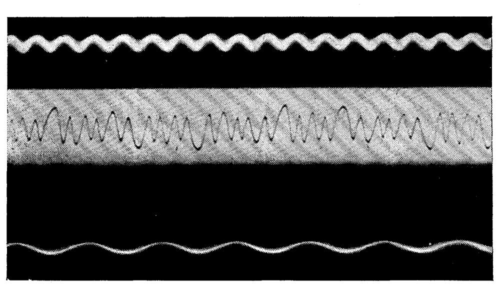

FIG. 11.

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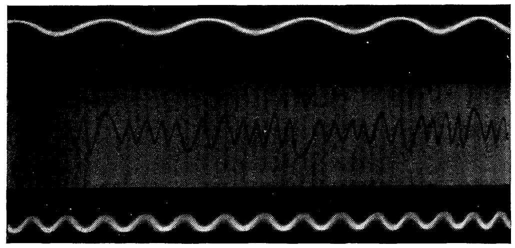

FIG. 12.

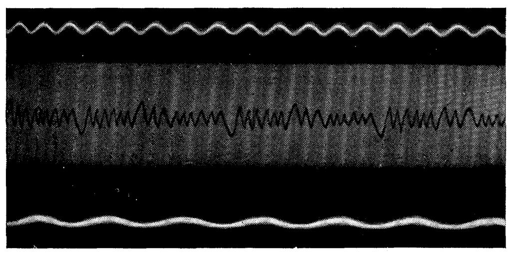

FIG. 13.

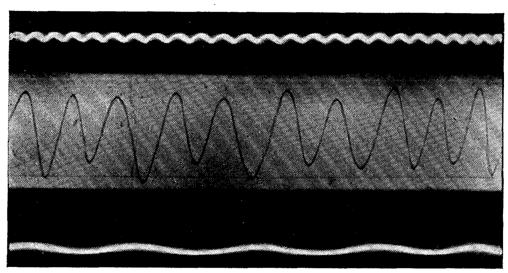

FIG. 14.

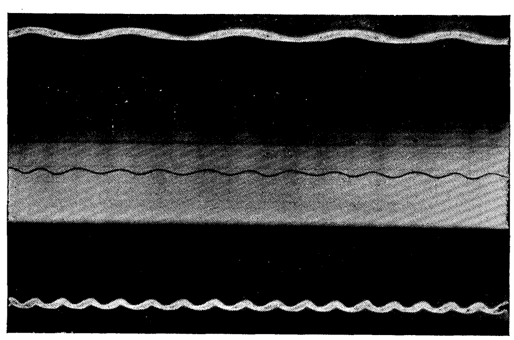

FIG. 15.

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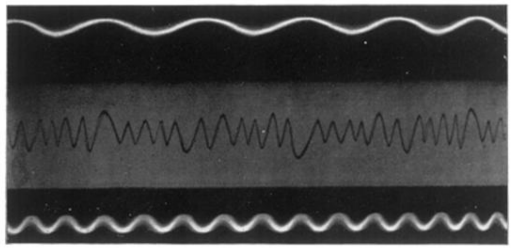


FIG. 12.

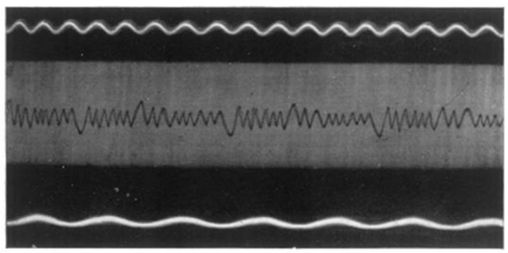


FIG. 13.

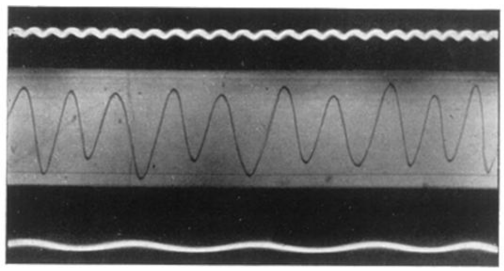


FIG. 14.

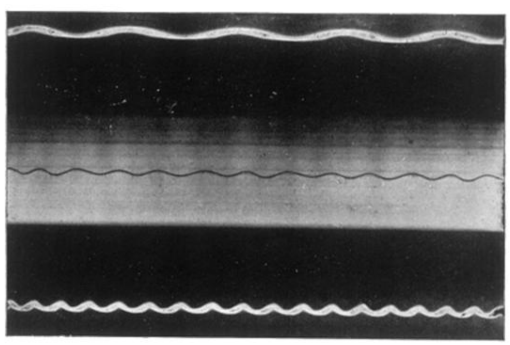
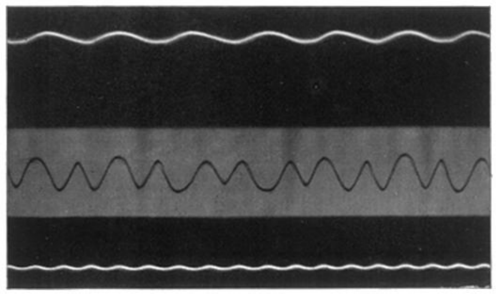


FIG. 15.



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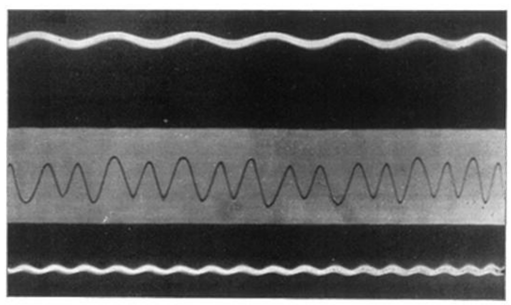

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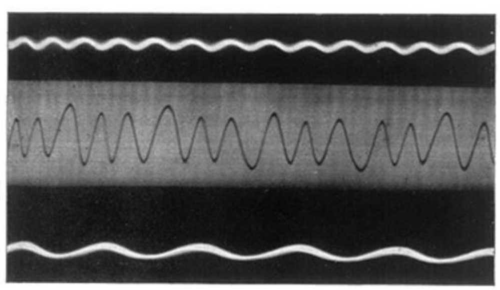

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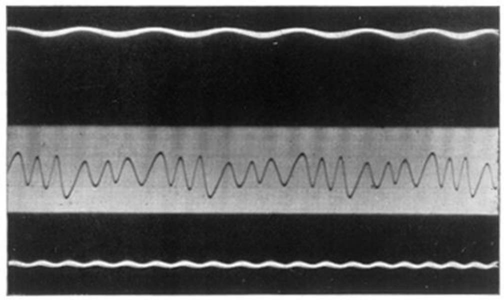

Fig. 6.

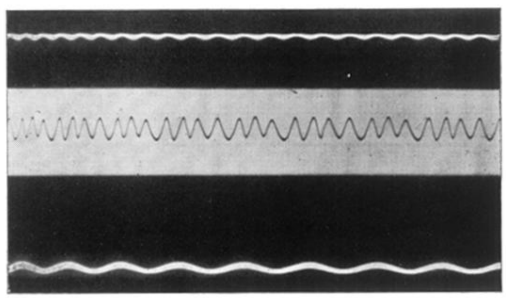

Fig. 7.

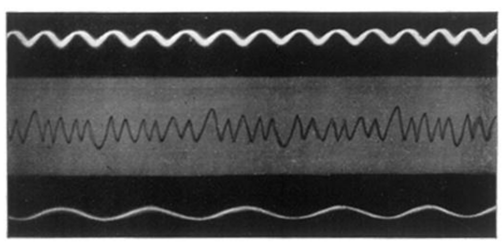

Fig. 8.

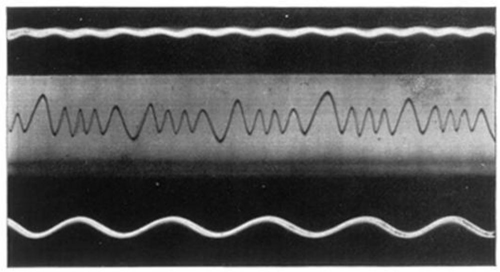

FIG. 9.

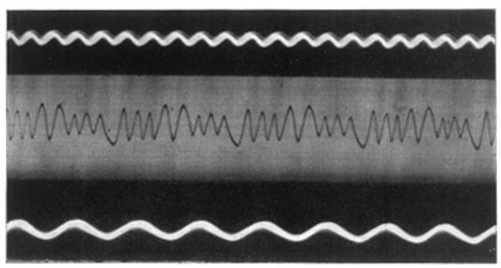

FIG. 10.

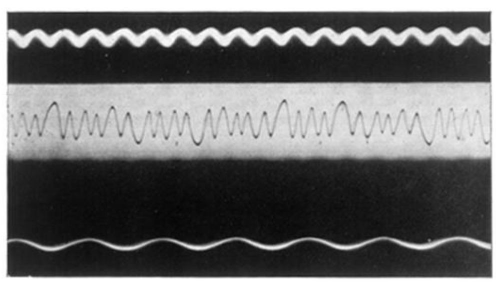

FIG. 11.