On Kaufmann's Theory of the Impact of the Pianoforte Hammer.

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1. *Introduction.*

The theory of the vibrations of the pianoforte string put forward by Kaufmann in a well-known paper* has figured prominently in recent discussions on the acoustics of this instrument.† It proceeds on lines radically different from those adopted by Helmholtz in his classical treatment of the subject.‡ While recognising that the elasticity of the pianoforte hammer is not a negligible factor, Kaufmann set out to simplify the mathematical analysis by ignoring its effect altogether, and treating the hammer as a particle possessing only inertia without spring. The motion of the string following the impact of the hammer is found from the initial conditions and from the functional solutions of the equation of wave-propagation on the string. On this basis he gave a rigorous treatment of two cases: (1) a particle impinging on a stretched string of infinite length, and (2) a particle impinging on the centre of a finite string, neither of which cases is of much interest from an acoustical point of view. The case of practical importance treated by him is that in which a particle impinges on the string near one end. For this case, he gave only an *approximate* theory from which the duration of contact, the motion of the point struck, and the form of the vibration-curves for various points of the string could be found.

There can be no doubt of the importance of Kaufmann's work, and it naturally becomes necessary to extend and revise his theory in various directions. In several respects, the theory awaits fuller development, especially as regards the harmonic analysis of the modes of vibration set up by impact, and the detailed discussion of the influence of the elasticity of the hammer and of varying velocities of impact. Apart from these points, the question arises whether the approximate method used by Kaufmann is sufficiently accurate for practical purposes, and whether it may be regarded as applicable when, as in the pianoforte, the point struck is distant one-eighth

* 'Annalen der Physik,' vol. 54 (1895).

† 'Proc. Phys. Soc.,' London, February, 1913; 'Nature,' 1913 and 1914, and 'Proc. Royal Institution,' 1914 (Prof. G. H. Bryan). See also D. C. Miller, 'Science of Musical Sounds,' Macmillan & Co., p. 207 (1915).

‡ 'Sensations of Tone,' English Translation by Ellis, p. 76, and Appendix V.

or one-ninth of the length of the string from one end. Kaufmann's treatment is practically based on the assumption that the part of the string between the end and the point struck remains straight as long as the hammer and string remain in contact. *Primâ facie*, it is clear that this assumption would introduce error when the part of the string under reference is an appreciable fraction of the whole. For the effect of the impact would obviously be to excite the vibrations of this portion of the string, which continue so long as the hammer is in contact, and would also influence the mode of vibration of the string as a whole when the hammer loses contact. A mathematical theory which is not subject to this error, and which is applicable for any position of the striking point, thus seems called for.

In the present communication, it will be shown how the general case of an inelastic particle impinging at *any* specified point on the string may be dealt with rigorously, and the magnitudes of the forces exerted during impact and the duration of contact may be calculated. To illustrate the method, a number of actual cases have been worked out numerically, and an attempt is made to compare the indications of theory with the results found in experiment.

2. *Analytical Theory.*

Two possible methods for dealing with the problem under consideration suggest themselves. The first is the rigorous application of the functional solutions of the equation of wave propagation after the manner of Kaufmann, taking into account the multiple reflections that occur at the particle and at the two extremities of the string. As different expressions have to be used for the motion on the two sides of the striking point, it is obvious that such a treatment would be extremely cumbrous, and indeed of impracticable length. The other method that suggests itself is that of expressing the motion that ensues on impact as the resultant of the inharmonic vibrations of the string having a load attached to it at the striking point so long as the hammer is in contact with it, and thereafter as a free periodic vibration of the usual kind. Here also a difficulty arises. For the inharmonic trigonometrical series expressing the force exerted by the particle on the string is not uniformly convergent in the neighbourhood of the discontinuities in the function expressing its sum, and it is thus not practicable directly to carry out an accurate summation of its terms to enable the duration of contact between particle and string to be found. By a judicious combination of the two methods indicated above, however, a fairly simple and straightforward process may be evolved for finding the graph of the force exerted by the hammer on the string for any position of the striking point, and thus determining the duration of contact.

In the following analysis μ is the linear density of the string, l, a, b represent respectively the length of the whole string and of the two parts into which the striking point divides it, and m is the mass of the impinging particle.

The free periods of the loaded string are determined by the values λ_1, λ_2, λ_3, etc., satisfying the equation*

$$\mu \sin \lambda_\gamma l = m\lambda_\gamma \sin \lambda_\gamma a \sin \lambda_\gamma b. \tag{1}$$

The displacement ζ at any point of the loaded string is expressed by the infinite series,

$$\zeta = \sum_{\gamma=1}^{\gamma=\infty} \phi_\gamma \sin c\lambda_\gamma t \frac{\sin \lambda_\gamma x \sin \lambda_\gamma b}{\sin \lambda_\gamma a \sin \lambda_\gamma (l-x)}, \tag{2}$$

where c is the velocity of transverse waves on the string, and the alternative expressions refer to the two parts of the string.

The constants ϕ_1, ϕ_2, etc., in (2) have to be found from the initial conditions. The system is initially without velocities or displacements except in regard to the particle of mass m, which has a velocity v. Using the notation adopted in Art. 101 of Lord Rayleigh's 'Theory of Sound,' vol. 1, we find that the displacement ζ_0 of the point struck at any instant during the impact is given by the expression

$$\zeta_0 = \sum_{\gamma=1}^{\gamma=\infty} \sin c\lambda_\gamma t \frac{mv \sin^2 \lambda_\gamma a \sin^2 \lambda_\gamma b}{c\lambda_\gamma \int \rho u_\gamma^2 \, dx}, \tag{3}$$

where

$$\int \rho u_\gamma^2 \, dx = m \sin^2 \lambda_\gamma a \sin^2 \lambda_\gamma b + \mu \left[\int_0^a \sin^2 \lambda_\gamma x \sin^2 \lambda_\gamma b \, dx + \int_a^l \sin^2 \lambda_\gamma a \sin^2 \lambda_\gamma (l-x) \, dx \right]. \tag{4}$$

Integrating (4) and simplifying by the aid of the relation given in (1), we obtain

$$\zeta_0 = \sum_{\gamma=1}^{\gamma=\infty} \frac{2v \sin c\lambda_\gamma t / c\lambda_\gamma}{1 + \mu/m \, (a/\sin^2 \lambda_\gamma a + b/\sin^2 \lambda_\gamma b)}. \tag{5}$$

The force $-m\dfrac{d^2\zeta_0}{dt^2}$ exerted by the particle on the string is therefore given by the equation

$$-m\frac{d^2\zeta_0}{dt^2} = \sum_{\gamma=1}^{\gamma=\infty} \frac{2mvc\lambda_\gamma \sin c\lambda_\gamma t}{1 + \mu/m \, (a/\sin^2 \lambda_\gamma a + b/\sin^2 \lambda_\gamma b)}. \tag{6}$$

Equation (6) expresses the force exerted by the particle on the string as the sum of an infinite series of simple circular functions of the time, whose frequencies are the same as that of the vibrations of the loaded string, and whose relative magnitudes depend only on the ratio of the mass of the

* Lord Rayleigh's 'Theory of Sound,' Art. 136.

hammer to the mass of the string, and the ratio in which the striking point divides the string. As remarked above, a difficulty arises in attempting to carry out a numerical summation of the series for all values of t, owing to the discontinuous nature of the function which the sum represents. This difficulty may, however, be evaded, and the work of numerical computation greatly simplified, by finding the magnitudes of the discontinuities in the function and the instants of time at which they occur directly from the principles of wave-propagation. This may be done in the following way.

At time $t = 0$ the particle impinges on the wire, and two discontinuous changes of velocity, each equal to v (the initial velocity of the particle), travel out, one on each side of the string, reach the ends of the string in due course, are reflected, and return again, much in the same way as in the theory of the Helmholtzian vibrations of a bowed string. When a discontinuity reaches the particle it is reflected out again without alteration of magnitude. This follows from the fact that the particle constitutes in effect a region of infinite density on the string. Thus, at intervals of time $2\,a/c$, $4\,a/c$, etc., and again at intervals $2\,b/c$, $4\,b/c$, $6\,b/c$, etc., counting from the commencement of the impact, we have discontinuous changes of velocity reaching and being reflected from the particle. At these instants the *acceleration* of the particle suffers corresponding discontinuous changes. At the commencement of the impact the pressure exerted by the particle on the string is equal to $2\,\mu cv$, as can be shown directly from the principle of the conservation of momentum. Accordingly, at the instants $t = 2\,a/c$, $4\,a/c$, etc., and also at the instants $t = 2\,b/c$, $4\,b/c$, etc., the pressure exerted suffers discontinuous *increases* of $2\,\mu cv$. (If in any case the reflected waves from the two ends arrive simultaneously at the striking point, the pressure would increase by $4\,\mu cv$.) In the intervals between the arrival of the reflected waves the pressure exerted by the particle would decrease in a continuous manner defined by equation (6) above.

Now the question is, what would the series in (6) represent at the points of discontinuity, that is, when t is put equal to $2\,a/c$, $4\,a/c$, or $2\,b/c$, $4\,b/c$, etc.? It is clear that by taking a sufficient number of terms, we should obtain a sum which converges to the *mean* value of the function at these points, and for this purpose a much smaller number of terms (say 10 or 12) would suffice than would be necessary if we attempted to compute the form of the function itself in the neighbourhood of the discontinuities. Having ascertained the mean value of the function at a point of discontinuity, its two actual limiting values follow at once by adding and subtracting half the magnitude of the discontinuity, that is, μcv. (If two reflections arrive simultaneously at the striking point in any case, we have respectively to add and subtract $2\,\mu cv$.)

The graph of the continuous part of the function may be then filled in in free hand by joining up the alternate ends of successive discontinuities by smooth curves.

3. *Numerical Computation of the Force exerted by the Hammer.*

To illustrate the application of the foregoing theory to actual cases, the authors have carried out a numerical computation of the magnitudes and frequencies of the components of force exerted by the impinging particle on the string for a particular ratio of their masses ($m/\mu l = 1{\cdot}684$), and for 27 different positions of the striking point situated at intervals along the string between its end and the centre. In the light of the practical experience gained, it may be useful to indicate briefly how the work may be arranged, so as to secure sufficient accuracy with the minimum of labour.*

The first step is the determination of the values of λ_γ, which satisfy equation (1). If the mass m of the hammer be zero, these values are π/l, $2\pi/l$, $3\pi/l$, etc. On the other hand, if m be very large, λ_1 tends to zero, and the higher roots fall into two groups which tend to the limits π/a, $2\pi/a$, $3\pi/a$, etc., and π/b, $2\pi/b$, $3\pi/b$, etc., respectively. For any actual value of m, the lowest root λ_1 of equation (1) has to be found by trial. The work may be lightened when it has to be done for a large number of positions of the striking point, by first finding λ_1 by trial for three or four positions of the striking point at wide intervals on the string, and then finding it for the other points by graphical interpolation and testing the values thus obtained by actual substitution in the equation (1). The calculation of the higher roots of this equation is a simpler task. For, to a first approximation, λ_γ is equal to either $p\pi/a$ or $q\pi/b$, where p and q are integers. To a second approximation, λ_γ is equal to $p\pi/a$ or $q\pi/b$, plus a correction which is either $\mu/m\pi p$ or $\mu/m\pi q$. (In special cases, where $p\pi/a = q\pi/b$, the correction to the value of λ_γ is $\mu(p+q)/m\pi pq$.) The values of λ_4, λ_5, λ_6, etc., thus obtained are generally sufficiently accurate, but the valnes of λ_2, λ_3 may require further improvement by actual test against equation (1), especially when the striking point is close to the end of the string.

The second step is the determination of the magnitudes of the components of force from equation (6). The fact that $\sin\lambda_\gamma a$ and $\sin\lambda_\gamma b$ appear both in equation (1) and in (6), helps to save some labour. Except when $\gamma = 1$, one of these quantities, that is either $\sin\lambda_\gamma a$ or $\sin\lambda_\gamma b$, is generally small in comparison with the other, and the magnitude of the component of force in (6) is practically determined by the smaller of the two quantities, which alone

* The authors wish here to acknowledge the assistance they have received from Mr. Durgadas Banerji, M.Sc., in carrying out part of the numerical work.

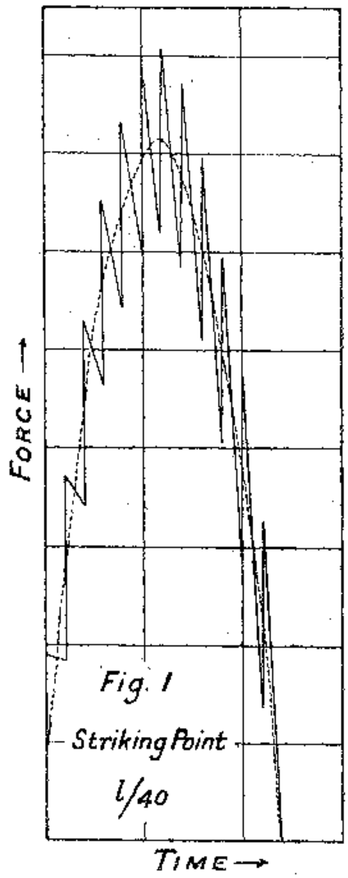
requires to be determined with accuracy. This may be found from (1), so as *exactly* to satisfy this equation, the approximate value of λ_r (ascertained as indicated in the preceding paragraph) being used for the computation of the other quantities in the equation. The results thus obtained may be substituted in the denominator of (6), and enable the force-components to be accurately determined.

It is of interest to notice the manner in which the computed magnitudes and frequencies of the components of force are found to depend on the position of the striking point. Since the frequencies of the components are those of a string carrying a load at the point of impact, they form a series of which the terms (in accordance with a well-known principle) are separated by the natural frequencies of the unloaded string. The frequency of the rth component of force is thus intermediate between that of the rth and $(r-1)$th harmonics of the string, and is a maximum when the striking point is at a node of the rth harmonic and a minimum when at a node of the $(r-1)$th harmonic.

The magnitudes of the components of force similarly fluctuate, but to a much larger extent than their frequencies, and in the opposite direction. Thus the rth component of force is zero when the striking point is at a node of the rth harmonic, and is a maximum when it is a node of the $(r-1)$th harmonic, being largest when the node is that nearest the end of the string. For example, the ninth component of force which is zero when the striking point is at $l/9$, rises rapidly to a large maximum when it is shifted to $l/8$, and falls quickly again to a small fraction of this value when it is further shifted to $l/7$ or $l/6$ or $l/5$. It shows similar fluctuations when the striking point is moved still nearer the centre of the string, becoming zero when the striking point is at $2l/9$ or $3l/9$, and reaching large values when it is at $2l/8$, $3l/8$, or $4l/8$, but these maxima are not so great as the first. The manner in which the frequencies and magnitudes of the components of force change when the striking point approaches very near one end of the string is specially worthy of notice. Here the first component (which elsewhere is generally much larger than the rest) falls off rapidly in magnitude, and as the striking point is brought nearer and nearer the end, the second, third, and higher components increase, and become in succession the largest in magnitude, and each in turn then falls off, giving precedence, as it were, to the next component in the series. The maximum value of each component is reached when its frequency falls to the minimum value and begins again to rise steeply.

4. *Graphical Determination of the Duration of Contact.*

Using the computed values of the magnitudes and frequencies of the components of force, the graph representing the discontinuous fluctuations with time of the force exerted by the impinging particle on the string may be plotted in the manner explained in a previous section. The point at which the graph cuts the axis of time gives at once the duration of contact. Figs. 1 and 2 represent the force-time graphs for the two cases in which the striking point is at $l/40$ and at $l/9$ respectively. These diagrams are given to illustrate the manner in which the form of the graphs and the duration of contact alter as the striking point is removed further and further from the end. In both of these cases, the hammer loses contact with the string long before the wave started by the impact has had time to reach the farther end of the string and to return after reflection to the striking point. The discontinuities appearing in the graphs accordingly represent the successive reflections of the waves between the striking point and the *nearer* end of the string. It will be noticed that in fig. 1 12 discontinuities appear which are close together, and in fig. 2 there are only six which are further apart. The effect of removing the striking point further from the end of the string is to *increase* the duration of contact in a continuous manner, so long as the number of discontinuities in the graph remains unaltered. At the successive stages, however, at which the number of discontinuities in the graph decreases by unity, the duration of contact *decreases* by a finite amount in a discontinuous manner, these jumps being the greater and further apart, the more distant the striking point is from the end.



It is thus clear that the duration of contact is a discontinuous function of the position of the striking point.

When the striking point is so far removed from the end of the string that reflections from both ends of the string have to be taken into account, then either a decrease or an increase in the duration of contact may take place in a discontinuous manner: the former when the number of discontinuities in the graph goes up by unity with the change in the position of the striking

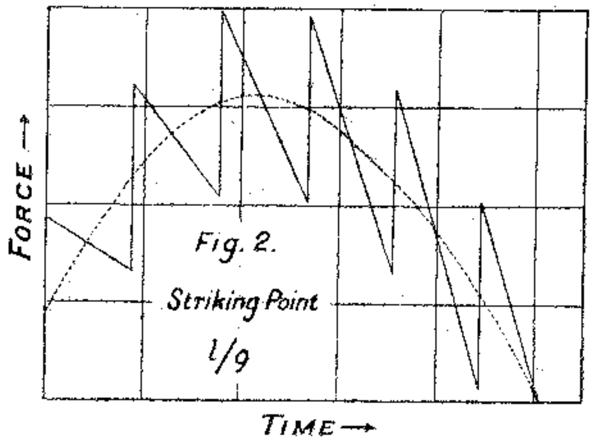


Fig. 2.

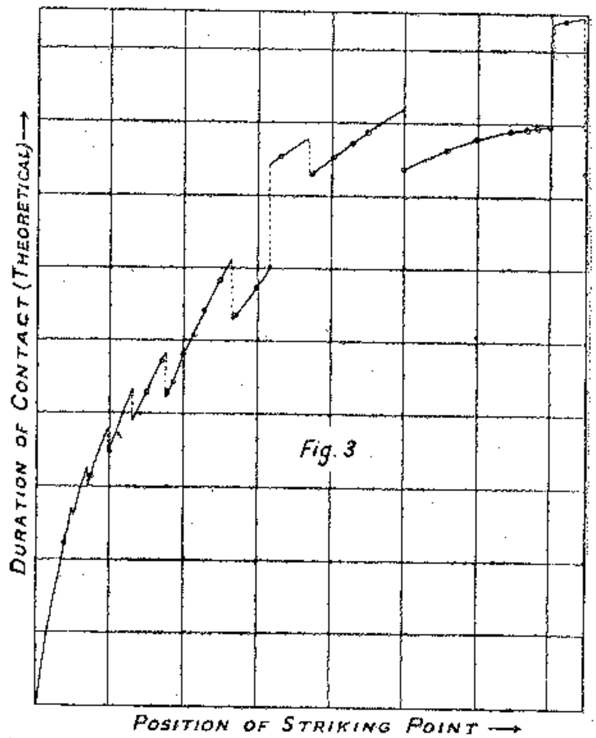


Fig. 3

point, and the latter when it goes down by the same amount. The whole course of values for the duration of contact for 27 different positions of the striking point between the end and centre of the string is shown in fig. 3.

The duration of contact found by this method agrees exactly with that found from Kaufmann's treatment for the case in which the striking point is at the centre of the string. There is also sensible agreement of the duration of contact with that found by the approximate method of Kaufmann when the striking point is sufficiently close to the end of the string; say at $l/40$ or $l/20$. Elsewhere, Kaufmann's approximate treatment fails.

Figs. 1 and 2 also illustrate another feature which is worthy of notice. The dotted lines in the two diagrams represent the graphs of the force exerted by the impinging particle on the string, if the more rapid periodic fluctuations which may be regarded as due to the vibrations of the shorter segment of the string are neglected. It is seen that the two dotted curves are distinctly unsymmetrical in shape and follow in fact the outline of a damped harmonic curve. This is in agreement with the approximate theory given by Kaufmann.

5. *Some Experimental Results.*

The result indicated by the foregoing theory that the duration of contact is a discontinuous function of the position of the striking point appeared worthy of an experimental test. This was attempted in the following way. A steel wire 150 cm. long, was nickeled and silver-plated so as to ensure its surface giving good electric contact, and stretched over the bridges of a sonometer. The linear density of the wire was 0·095 grm. per centimetre. A small solid brass cylinder was mounted at the end of a light pivoted shaft, and could be caused to impinge transversely on the wire. Immediately on impact, the cylinder and shaft of the hammer completed an electric circuit through the wire and a sensitive Leeds and Northrup ballistic galvanometer. Three or four readings of the throw of the galvanometer were taken for each of a large number of positions of the striking point, and the average struck. To secure that the variable electric resistance at the point of contact should not influence the results, a non-inductive resistance of half a megohm was included in the circuit. The results for two different masses of the impinging cylinder have been plotted in fig. 4.

It is obvious from the experimental curves that the duration of contact does not continuously increase as the striking point is removed further and further from the end of the string, but that it is subject to rapid fluctuations, the magnitude of which increases as the striking point is moved away from the end. Of the two curves shown in fig. 4, the upper relates to the case of an impinging cylinder having a mass of 24 grm., and the second of a cylinder having a mass of 16 grm., the figures in each case including a correction for the inertia of the shaft calculated after the manner of Kaufmann. The ratio of the mass of the hammer to the mass of the string for the upper of the two curves in fig. 4 is the same as that for which the theoretical

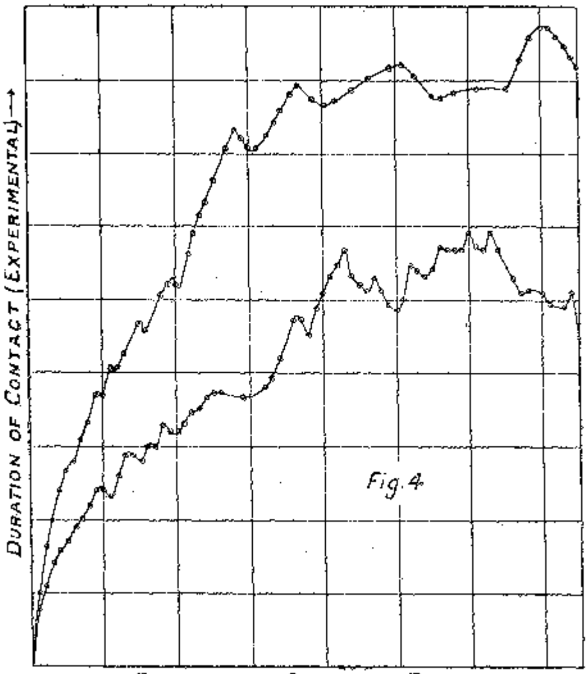


Fig. 4.

computations have been carried out and plotted in fig. 3, and the general resemblance between the computed and observed curves is obvious. Why there is not a much closer agreement is an open question.

It must be remembered that, in many respects, the experimental arrangements do not strictly reproduce the conditions assumed in the theoretical calculations. The finite size of the cylinder and elastic flexure of the shaft, the stiffness of the wire and its yielding at the ends, and the effect of gravity on the motion of the impinging cylinder, are factors which probably influence the results in an appreciable degree. There is no doubt, however, that the experimental results shown in fig. 4, broadly speaking, confirm the correctness of the theoretical results, and the suitability of the method of calculation set out in the paper.

From an acoustical point of view, a further development that would be of interest is the theoretical determination (and comparison with experiment) of the manner in which the amplitudes of the fundamental and higher harmonic components of the vibration of the string excited by impact depend on the mass of the hammer and the position of its striking point over the entire possible range.* It may be remarked that equations (2) and (5) of the paper determine the motion at every point of the string *during the continuance of the impact.* The motion, after the hammer has left the string, would have to be separately determined. This could no doubt be found from the known displacements and velocities at various points of the string at the instant the hammer leaves it. One way of doing this is by a geometrical method similar to that used by Kaufmann. A better method, however, would be to express the motion resulting from impact in terms of the forces exerted by the impinging particle and the duration of contact.† The graphs from which the duration of contact is determined would be very useful in this connection, as the necessary integrations could be carried out mechanically with the aid of these curves.

6. *Summary and Conclusion.*

The present paper is chiefly concerned with a revision and extension of the theory of the impact of the pianoforte hammer developed by Kaufmann. It is pointed out that the approximate method used by Kaufmann, in which he assumes that the part of the string between the striking point and the nearer end remains straight so long as the hammer is in contact is unsatisfactory, and cannot be regarded as applicable when this part of the string forms an appreciable fraction of the whole. It is shown how this assumption may be dispensed with, and the general case, in which an inelastic particle impinges at any point on the string, may be dealt with rigorously and subjected to numerical computation. The manner in which the force exerted by the impinging particle varies is determined, partly from the functional solutions of the equations of wave-propagation, and partly from the theory of normal vibrations. Numerical computations have been carried out of the components of force for a particular ratio of the masses of the string, and of the impinging particle, and for 27 different positions of the striking point, and the duration of contact is deduced by a graphical method. An interesting result obtained is that the duration of contact does not continually increase as the striking point is removed away from the end of the string, but is subject to discontinuous fluctuations. Experimental work, broadly speaking, confirms this indication of theory.

The present paper is preliminary to a more complete investigation, in which the authors hope to develop fully various points in the theory of the

* The choice of the position of the striking point actually adopted in the pianoforte is no doubt determined by its influence on the useful effect produced by the impact of the hammer. See G. H. Berry, 'Phil. Mag.,' April, 1910.

† Arts. 128 and 130, Rayleigh's 'Theory of Sound' vol. 1.

pianoforte, especially (1) the manner in which the amplitudes of the component partial vibrations of the string depend on the position of the striking point; (2) the effect of the mass and elasticity of the hammer; (3) the characteristics of the coupled vibrations of the string and sound-board of the pianoforte; (4) the manner in which these vibrations decay by communication of energy to the atmosphere; and (5) the theoretical determination of the quality of pianoforte tone. Further work on these points, and experimental tests on an actual pianoforte, are now in progress.

In conclusion, the authors wish to thank Dr. Gilbert Walker, F.R.S., who has taken a lively interest in their work, for his kindness in bringing it before the Royal Society.

Plane Strain: the Direct Determination of Stress.

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(1) The advantages of the direct determination of the stresses in an elastic solid have been pointed out by Prof. J. H. Michell*. The principal line of attack in the case of plane strain has been by aid of the well-known stress function method, by which the stresses are determined from a single stress function χ, of x and y only, satisfying

$$\nabla_1^4\chi = 0,$$

where
$$\nabla_1^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}. \tag{1}$$

The lines along which further advance might have been expected, and the difficulties which have been met with, are discussed by Prof. A. E. H. Love.†

It seems to the writer, however, that a point of very considerable importance has been overlooked, viz., that the stress function method gives a set of stresses which can in most cases be resolved into two distinct sets, each of which leads to strains satisfying the identical relations between the strain components.

For example, if ϕ is a plane harmonic function, $\chi = (x^2+y^2)\phi$ satisfies $\nabla_1^4\chi = 0$, and the stresses derived from this by the well-known formulæ

$$\widehat{xx} = \frac{\partial^2\chi}{\partial y^2}, \qquad \widehat{yy} = \frac{\partial^2\chi}{\partial x^2}, \qquad \text{and} \qquad \widehat{xy} = -\frac{\partial^2\chi}{\partial x\partial y}, \tag{2}$$

* 'Proc. Lond. Math. Soc.,' vol. 31, p. 100.

† 'Math. Theory of Elasticity,' 2nd Edition, p. 211.