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Oscillations of stretched strings

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A point in the theory of the forced oscillations of stretched strings,–Experimental study of this point,–Localised change of phase at nodes.–Stroboscopic study of the small motion at nodes.

Donkin falls into a curious error in his treatment of the problem of the forced oscillations of stretched strings (*Acoustics*, 2nd Edn., pp. 121–124). The question is 'to find the motion of a string when a given point on it is subjected to a given obligatory vibration, dissipation of energy being taken into account'—this last being necessary to make the theoretical treatment represent the facts truly. The problem is worked out by Lord Rayleigh (*Theory of Sound*, I, pp. 197–199) and by Donkin on the assumption that each element of the string is retarded by a force proportional to its velocity. Donkin, having obtained a rigorous and rather complicated expression giving the motion at any point, proceeds to simplify and discuss the result. His approximation, while giving correctly the amplitude of the motion at points not near a node, makes out the phase quite wrong.

Starting with the assumption that the obligatory motion at the point $x = b$, is $p \sin nt + q \cos nt$ his final approximate result for points not near a node is $\sin \theta (p \sin nt + q \cos nt)/(\sin^2 \phi + \delta_0^2 \cos^2 \phi)^{1/2}$ [for the meaning of the symbols, I refer the reader to the original]. Since in the above given expression $\sin \theta$ and the denominator do not involve the time, it follows that the phase of the oscillation for points for which $\sin \theta$ is positive, is the same as that of the obligatory motion. This is very different from what a study of the general theory of resonance and a consideration of the question of supply of energy to the vibrating string lead us to expect. The exact step in his approximation which introduces the error is putting $\tan \Phi = q/p$ where

$$\tan \Phi = (q\sigma_0 \sin \phi + p\delta_0 \cos \phi)/(p\sigma_0 \sin \phi - q\delta_0 \cos \phi).$$

To make the error clear, we may, without loss of generality put $q = 0$. Then $\tan \Phi = \delta_0 \cot \phi/\sigma_0 = \beta\phi \cot \phi = c\phi \cot \phi/2n$. Though the damping factor c is very small, we cannot, as Donkin does, put it equal to zero. For, the coefficient of the term $\cot \phi$ is very large and when $\Phi = i\pi$ at the exact stage of resonance, becomes infinite. At this stage $\tan \Phi$ becomes ∞ and $\Phi = \pi/2$, whereas Donkin would have $\tan \Phi = 0$ and therefore $\Phi = 0$.

A general idea of the correct facts of the case can be had from very simple considerations. Since the energy requisite for the maintenance of the vibration is supplied at one point of the string and is communicated to the different sections of the string through the nodes, these are not points of absolute rest but have a small motion. Since on opposite sides of a node, at some distances from it, the velocities and displacements are in opposite directions, i.e. in opposite phases, the small motions at and near the node must necessarily be in intermediate phases.* But the point at which the obligatory motion is imposed is when the tension is adjusted so that resonance occurs, itself situated at or near a node. It follows therefore that the imposed obligatory motion and the resultant vibration of the greater part of the string are in general in very different phases.

Experimental study of this point

I have devised some experimental methods of investigating the phase relations between the obligatory motion imposed at one point of the string and the resultant general vibration of the string. In this note only the two best will be mentioned.

Direct method: The obligatory motion at one point is secured by the ordinary arrangement of attaching the extremity of a stretched string to the prong of a transversely-vibrating tuning fork. A short length of the string is brightly illuminated and by the aid of a lens an image of it is formed in the field of view of a stroboscopic disc. A steel mirror attached to a prong of the tuning fork serves to reflect the light issuing from a horizontally-held illuminated slit, and a second lens focusses the reflected light into a linear image in the field of view. The two lines of light are then brought into juxtaposition; on starting the tuning fork which is electrically maintained, they spread themselves into bands of light. These again resolve themselves into slowly-moving lines of light when the stroboscopic disc is set rotating at a suitable speed. It can then, generally speaking, be seen at once that they are not in the same phase. As the tension of the string is gradually raised from some value below that necessary for resonance to one above it, a continuous and complete reversal of phase in the motion of the string can be observed. At or about the point of maximum resonance one of the lines is a quarter of an oscillation in advance of the other.

Method of Lissajous figures: As the vibration of the tuning fork is parallel to that of the string, a special device has been employed in order to adapt this method to the case in hand.

*It cannot be predicated from *a priori* considerations what exactly the phase of the vibration is at a point half way between the loops, i.e. a node. This must be left for dynamical investigation to decide. But if it can be assumed that the phase change is symmetrical, it follows that the vibration at the nodes differs in phase from the vibration at a loop by a quarter of an oscillation. This point will be investigated in the fifth section of this paper.

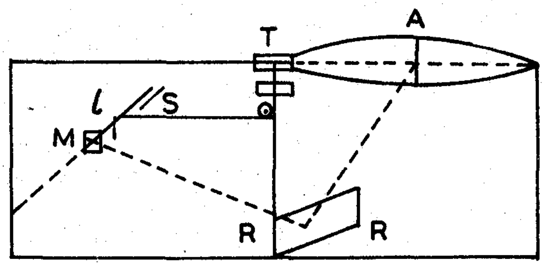

Figure 1.

A small mirror, M is fixed vertically (figure 1) to the end of a light brass lever L. This lever is pivoted upon a vertical axis and is actuated by a stout thread which with one end attached to a prong of the tuning fork, T passes under a pulley and then being fixed at the other extremity to the lever, keeps the latter tightly pressed against a spring S. When the tuning fork (which is a massive one) is maintained in oscillation, the mirror M executes oscillations in a plane perpendicular to the excursions of the prongs of the tuning fork, the phases of the former and the latter being identical—this is secured by the inextensibility of the connecting thread. One point A on the string maintained in vibration is brightly illuminated throughout its excursion by means of a cylindrical lens, and the luminous line produced thereby is viewed by reflection, first at the fixed mirror RR and then at the oscillating mirror M. From the Lissajous figure (circle, ellipse or straight line) seen under these circumstances, the phase relation between the vibration of the tuning fork and that of the string (over the major part of it) can at once be inferred. In particular, the phase difference is found to be a quarter of an oscillation when resonance was about a maximum.

Localised change of phase at the nodes

We now proceed to the mathematical discussion of the problem in hand. The expression for the displacement at every point of the string maintained in vibration, as obtained by Lord Rayleigh in his investigation (*Theory of Sound*, **I**, pp. 197 to 199) is $\gamma(R_x/R_b)\cos(pt + E_x - E_b)$ where $\tan E_x = \exp(\beta x) - \exp(-\beta x)/\exp(\beta x) + \exp(-\beta x)\cot \alpha x$ corresponding to an obligatory motion $\gamma \cos pt$ at the point $x = b$. For the meaning of the other symbols, I refer the reader to the original. Since β is small $E_x - E_b = \tan^{-1}(\beta x \cot \alpha x) - \tan^{-1}(\beta b \cot \alpha b)$ and this may be put equal to zero except when $\cot \alpha x$ or $\cot \alpha b$ is very large.

Two cases arise: (*a*) If αb is nearly equal to $i\pi$, where i is any integer, the string between $x = 0$ and $x = b$ is thrown into strong vibration, and the value of $E_x - E_b$ is in general not negligible. In the special case $\alpha\beta = i\pi$, $E_x - E_b = \pi/2$, except at the points on the string where αx is nearly equal to any multiple of π, (these points being the nodes of the forced oscillation). We have already in the previous section considered experimental methods of verifying this result.

(*b*) If αx is nearly equal to $m\pi, m$ being any integer, this relation giving the positions of the nodes, $(E_x - E_b)$ would again be not negligible. Its value in this case would of course depend on whether αb does or does not nearly satisfy the relation $\alpha b = i\pi$. In order however to trace the changes of $E_x - E_b$ with the change in the value of x we may simply assume b to be constant. It then appears that for points for which $\alpha x = i\pi$, its value differs by $\pi/2$ from that at points at considerable distances from these. In other words, the small residual vibration at the nodes differs in phase by $\pi/2$ from the vibration of practically all the rest of the string; if the point observed is not quite a node but slightly to one side of it, the difference of phase between its vibration and the general vibration of the string is smaller but still not negligible.

Stroboscopic study of the small motion at nodes

We now proceed to consider experimental methods of verifying the result (*b*) of the last section. It will be seen that in the second section of the present paper we have already discussed such methods for the particular case in which the node under observation is the one at or near which the obligatory motion is imposed. It therefore remains to deal with the other cases.

Taking Lord Rayleigh's result for the displacement at every point given in the last section, $\gamma(R_x/R_b)\cos(pt + E_x - E_b)$, we can, since $R_x^2 = \sin^2 \alpha x + (k^2x^2/4a^2)\cos^2 \alpha x$, and $\tan E_x = -(kx/2a)\cot \alpha x$, write it as $(\gamma/R_b)[\sin \alpha x \cos(pt - E_b) + (kx/2a)\cos \alpha x \sin(pt - E_b)]^2$; or, changing the origin of time, as $(\gamma/R_b)(\sin \alpha x \cos pT + (kx/2a)\cos \alpha x \sin pT)$. Since k is small, the second term is negligible except at or near the nodes, these being the points where the first term is zero. The two terms differ in phase by a quarter of an oscillation. Taking this expression for the displacement at every point on the string, I have plotted (figure 2) the positions of the string at and near a node (other than the one at $a = 0$ i.e. the fixed extremity) at successive intervals of one eighth of an oscillation. The string is supposed to be parallel to the foot of the page.

The string is maintained in a vibration in two loops, and the central node is observed, preferably through a magnifying glass, under intermittent illumination, the source of which is the spark of an induction coil worked with a tuning fork as an interrupter. The frequency of this tuning fork is approximately twice that of the tuning fork maintaining the oscillation of the string and it is maintained independently of the latter by passing through its electromagnet a part or the whole of the current running through the primary of the induction coil. It is observed that the frequency of this tuning fork, and therefore also the frequency of its beats with the other, can be varied somewhat by varying the pressure of the platinum wire and the contact breaker. Under the periodic illumination secured by this arrangement, the string is seen simultaneously in two slowly moving

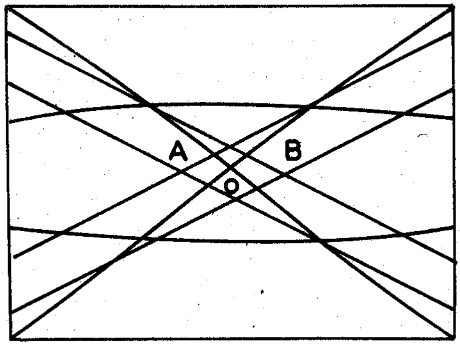

Figure 2.

positions. It is then observed that *the point of intersection of these two lines*, in other words, the fictitious node, instead of being stationary, *executes a periodic movement of large amplitude parallel to the string*. The characteristics of this movement can be best understood by reference to figure 2. The two positions of the string seen under the periodic illumination represent two stages of the oscillation in opposite phases. At $T = 0$ the positions intersect at O, this being the point half way between the loops, in other words, the real node of the oscillation. An eighth of an oscillation later, the point of intersection has moved towards the left to the point A. Then at $T = (\pi/2p)$ the point of intersection is completely off the field; theoretically it should have moved off to the position of the loop, in other words, by a quarter of the total length of the string. It now reappears on the right and in another eighth of an oscillation moves up to B. At $T = (2\pi/p)$ the lines are again in their first positions.

That the phase of the small motion at the node differs by $\pi/2$ from the motion at the loops of the string can be proved in the following way. If this be not true, the expression for the displacement, instead of being equal to $(\gamma/R_b)(\sin \alpha x \cos pT + (kx/2a) \cos \alpha x \sin pT$ will be of the type $(\gamma/R_b)(\sin \alpha x \cos pT + (kx/2a) \cos \alpha x \cos pT - E)$ where E is not equal to $\pi/2$. On plotting this expression in the manner of figure 2, it is found that in this case, as in the former, the point of intersection of the two positions of the string in opposite phases executes periodic movements of large range parallel to the string. But the important difference is that in this case the motion of the point is unsymmetrical, its velocity when at a given distance from O on one side and approaching it, being very much greater than its velocity when at the same distance from O on the other side and receding from it. Now, so far as can be seen, there is no effect of this kind in the actual experiment, the motion observed being symmetrical and of the type plotted in figure 2.

It can therefore be regarded as definitely proved by experiment that the expression for the displacement at any point on the string is of the type $(\gamma/R_b)(\sin \alpha x \cos pT + (kx/2a) \cos \alpha x \sin pT)$ the two terms differing in phase by a quarter of an oscillation.