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Some remarkable cases of resonance*

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The general principal of resonance is that a periodic force acting on an oscillatory system may set up and maintain a large amplitude of vibration when the periods of the force and of the system are approximately equal, even though in other cases the amplitude might be so small as to be negligible. In the present paper I propose to discuss some remarkable cases which form apparent exceptions to this law of equality of periods, that is, in which we have marked resonance when the periods of the impressed force and of the system do *not* stand to each other in a relation of approximate equality.

The first of these is the well known case of double frequency, the theory of which was first discussed by Lord Rayleigh.† In a note published in the *Phys. Rev.* for March, 1911, I promised a fuller discussion of the modification in this theory necessary to fit the results with those actually observed in experiment. I now proceed to fulfil this promise and the delay that has occurred in doing so is I feel a matter for regret. Lord Rayleigh starts with the following as his equation of motion:

$$\ddot{U} + k\dot{U} + (n^2 - 2\alpha \sin 2pt)U = 0, \tag{1}$$

and assuming that U, the displacement at any instant during steady motion, can be represented by an expression of the form

$$A_1 \sin pt + B_1 \cos pt + A_3 \sin 3pt + B_3 \cos 3pt + A_5 \sin 5pt, \tag{2}$$

proceeds to find the conditions that must be satisfied for the assumed steady motion to be possible. This he does by substituting (2) for U in the left-hand side of equation (1) and equating to zero the coefficients of $\sin pt$, $\cos pt$, etc. The conditions for the possibility of steady motion thus obtained are

$$\frac{B_1}{A_1} = \frac{\sqrt{(\alpha - kp)}}{\sqrt{(\alpha + kp)}} = \tan e, \tag{3}$$

$$(n^2 - p^2)^2 = \alpha^2 - k^2p^2. \tag{4}$$

*Preliminary notes on this subject appeared in *Nature (London)*, December 9, 1909 and February 10, 1910, and in the *Phys. Rev.*, March 1911.

†*Philos. Mag.*, April, 1883 and August, 1887, and *Theory of Sound*, Art. 68(*b*).

By a trigonometrical transformation equation (3) can be written in the form

$$kp = \alpha \cos 2e. \tag{5}$$

It appears from these equations that the phase of the oscillation maintained, i.e. e is independent of the amplitude, and that the latter quantity is indeterminate.

I attempted to verify the phase-relation given by equations (4) and (5) experimentally in the following way: The oscillating system used was a stretched string and this was maintained in motion by periodically varying its tension in the manner of Melde's experiment. Since the periodic change of double frequency in the tension of the string is imposed by the tuning-fork, the motion of the prong corresponds to the term $-2\alpha \sin 2pt$ in equation (1) and the transverse vibration of the string to the expression $(A_1^2 + B_1^2)^{1/2} \sin(pt + e)$ if the small terms in A_3, B_3, etc. are neglected. The experimental problem therefore reduces itself to the determination of the phase-relation between the motion of the string and the vibration of the prong of the exciting tuning-fork. This can be attacked by two distinct methods.

(i) *Mechanical composition of the two motions*: This is automatically effected and needs no special experimental device. For, the motion of the prong is longitudinal to the string and any point on the string near the end attached to the prong or near any other intermediate node of the oscillation has *two* rectangular motions: the first longitudinal to the string and having the same frequency as the vibration of the fork; and the second which is the general transverse oscillation of the string. The resulting motion is in a Lissajous figure and this is readily observed by attaching a fragment of a silvered bead to a point on the string near the fork.

(ii) *Optical composition of the two motions*: Furnishes a second method and this is undoubtedly the more elegant of the two. It can be effected in the following way: a small mirror is attached normally to the extremity of the prong of the fork. The plane of the oscillation of this mirror is perpendicular to that of the vibration of the string, and a point on the latter is brightly illuminated by a transverse sheet of light from a lantern or from a cylindrical lens. When the string is in oscillation the illuminated point appears drawn out into a straight line, and this is viewed by reflection first at a fixed mirror and then at the mirror attached to the vibrating prong. The illuminated point is then seen to describe a Lissajous figure which is compounded of the motions of the string and the tuning-fork.

Observing by either of the methods described above, the relation between the phases of the transverse oscillation of the string and of the motion of the prong of the tuning-fork can be closely studied, and some remarkable phenomena are noticed in this way. The principal point observed is that the phase relation is not independent of the amplitude maintained. This is best shown by using a bowed fork and starting with a large amplitude of motion and then allowing the motion to die away. The initial curve of motion and the changes in it as the motion dies away both depend on the tension of the string. When this is in excess of that required for the most vigorous maintenance, the curve is a parabolic arc convex to the fork and remains as such when the motion dies away. With a smaller tension adjusted so that the free period of the string is exactly double that of the fork, the initial curve with a large amplitude of motion is still approximately a parabolic arc convex to the fork. The damping of the motion is now more rapid and the curve reduces to an 8-shaped figure when the amplitude is very small. For the most vigorous maintenance a still smaller tension is necessary and the initial curve with a large motion is still convex to the fork, but it will now be noticed that when the amplitude falls to a very small quantity the curve passes through the 8-figure stage and tends to become *concave* to the fork. The most remarkable changes are observed when the tension is smaller still. The damping here is very large and a steady motion is only possible when the amplitude exceeds a certain minimum value. At this stage the string very rapidly comes to rest and in the final stage the curve of motion becomes a parabolic arc *concave* to the fork with a very small amplitude. For a satisfactory explanation of these phenomena it is necessary to start with a modified equation of motion which takes into account the variations of tension which exist in *free* oscillations of sensible amplitude and are proportional to the square of the motion. The equation of motion thus completed is

$$\ddot{U} + k\dot{U} + (n^2 - 2\alpha \sin 2pt + \beta U^2)\cdot U = 0. \qquad (6)$$

Substituting expression (2) for U in the left-hand side of the above given equation and putting the coefficients of $\sin pt$, $\cos pt$, etc. equal to zero, the conditions for the possibility of steady motion reduce to the form

$$kp = \alpha \cos 2e \qquad (7)$$

and

$$(n^2 - p^2 + F)^2 = \alpha^2 - k^2 p^2, \qquad (8)$$

where

$$F = \frac{3\beta}{4}(A_1^2 + B_1^2).$$

Equation (8) determines the amplitude of the motion and (7) its phase. It is evident at once that with given values for α and kp the amplitude of the motion is greatest when n is smallest and that there can be no maintenance if $\alpha < kp$. We therefore get the apparently paradoxical conclusion (which is amply verified by experiment) that the maintenance is *not* the most vigorous when the free period (for small amplitudes) of the string is double that of the fork. Another interesting inference which is confirmed by experiment is that when $n < p$ and $(n^2 - p^2)^2 > (\alpha^2 - k^2 p^2)$ maintenance is impossible unless F, i.e. also the amplitude, has a definite minimum value.

Equation (7) shows that as α is increased e, the phase of the oscillation, alters continuously. The influence of F, i.e. of the amplitude of the motion on e its phase can readily be traced from equation (8). When $n > p$, for a larger value of F we must have a large value of α and $\cos 2e$ tends more and more to assume a zero value. Writing equation (7) in the form

$$\tan e = \frac{\alpha - kp}{n^2 - p^2 + F} \tag{9}$$

it is evident that when $n > p$ and F is large, e is positive and approaches to the value $\pi/4$. This agrees with the experimental result. When $n < p$, e may be positive or negative according as F is greater or less than $(p^2 - n^2)$ and the alteration of the phase of the motion with the amplitude is most conspicuous, and this is in agreement with observation. In the extreme lower limit e tends to the value $-\pi/4$, and the curve of motion is a parabolic arc *concave* to the fork. In other words when the prong is at its extreme outward string, the string is also at its extreme outward swing, a seemingly paradoxical result not in accordance with the ordinarily received ideas of the experiment. Equation (1) may be written in the form

$$\ddot{U} + k\dot{U} + n^2 U = 2\alpha U \sin 2pt. \tag{10}$$

The right-hand side of this equation may be regarded as the *impressed* part of the restoring force acting on the system, and this is a very useful way of regarding the matter. Putting $U = P\sin(pt + e)$ to a first approximation, we can find the conditions that must exist for steady motion *directly* by equating the work done by the force represented by the right-hand side term of equation (10) to the energy dissipated in an equal time by the friction term on the left. The relation thus obtained is identical with equation (7) obtained from the complete analysis. It is observed that the right-hand side of equation (10) is

$$2\alpha P \sin 2pt \sin(pt + e)$$

and this may be written as

$$\alpha P(\cos \overline{pt - e} - \cos \overline{3pt + e}).$$

The second term within the brackets is ineffective so far as the maintenance of the motion $P\sin(pt + e)$ is concerned. We may therefore leave it out and write equation (10) in the form

$$\ddot{U} + k\dot{U} + n^2 U = \alpha P \cos(pt - e). \tag{11}$$

Written in this way it is evident that a large motion must ensue if $p = n$ and that we have here merely an example of the general principle of resonance.

Part II

I now proceed to consider some other exceedingly interesting cases of resonance under the action of forces similar in character to that in the case of double frequency considered above, but having other frequency relations to the system influenced. My experiments show that resonance may occur in the following cases of the kind:

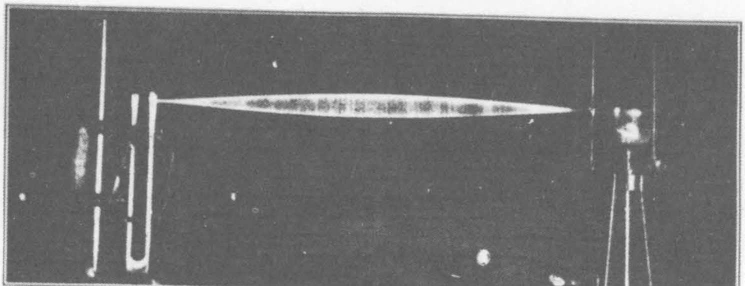
(1) When the period of the force is $\frac{1}{2}$ that of the system.
(2) When the period of the force is $\frac{2}{2}$ times that of the system.
(3) When the period of the force is $\frac{3}{2}$ times that of the system.
(4) When the period of the force is $\frac{4}{2}$ times that of the system.
(5) When the period of the force is $\frac{5}{2}$ times that of the system.
(6) When the period of the force is $\frac{6}{2}$ times that of the system.
And so on.

The most remarkable instances of these cases of resonance are furnished by a stretched string under the action of a periodically varying tension. To observe them, all that is required is that the tension of the string should be gradually increased till its free period in the fundamental mode stands in the desired relation to the period of the tuning-fork which imposes the variable tension. It will then be found that a vigorous oscillation is maintained. Figures 1, 2, 3, 4 and 5 are photographs of stretched strings maintained in the first five types of motion respectively under the action of an electrically maintained tuning-fork varying the tension periodically.

The actual frequency and the phase of the maintained motion in each of these cases can be determined by observation of the corresponding Lissajous figures, using the mechanical or optical method of composition described above for the first of these cases. The detailed results I must reserve for a future paper. One good way of studying these types of motion is to illuminate one point on the vibrating string by a transverse sheet of light from a lantern or from a cylindrical lens and to observe the line of light so produced in a revolving mirror. But the best method of all for recording the motion photographically is that by which I obtained figures 6, 7, 8, 8(a), 9 and 10 published herewith and which I now proceed to describe.

Figures 6, 7, 8, 9 and 10 refer respectively to the first five types of motion as shown in figures 1, 2, 3, 4 and 5. It will be observed that each of them shows two curves. The white curve in the black ground is a record of the motion of the tuning-fork, and the other curve which is black on a white ground is a record of the motion of a point on the string maintained in vibration. These records were obtained on a moving photographic plate in the following manner. One source of light was a horizontal slit, and the other was a vertical slit placed behind the oscillating string. Both were illuminated by sunlight and had collimating lenses in front of them. The light from the former fell upon a small mirror attached to the prong of the vibrating fork and after reflection fell upon the lens (having an aperture of 4 cm diameter) of a roughly constructed camera. The light from the vertical slit behind the vibrating string was also reflected into the camera by a fixed mirror. In the focal plane of the camera was placed a brass plate with a

(1)



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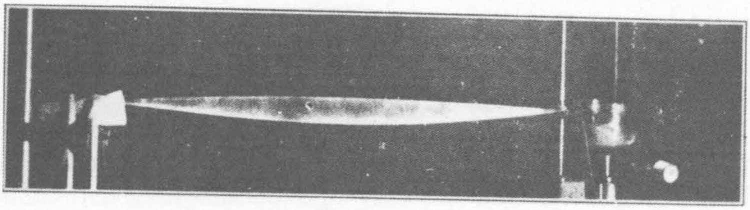

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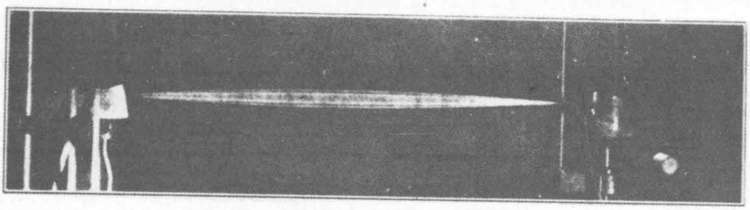

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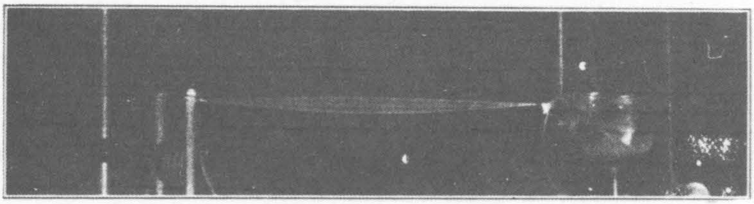

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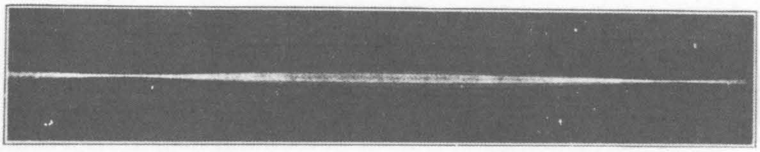

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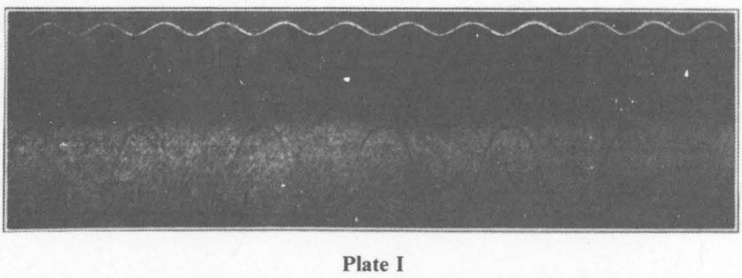
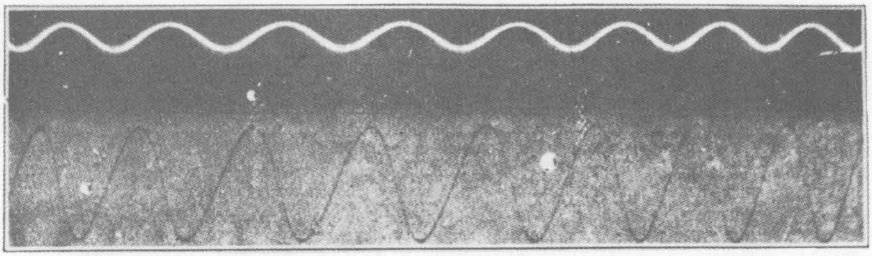


Plate I

(7)



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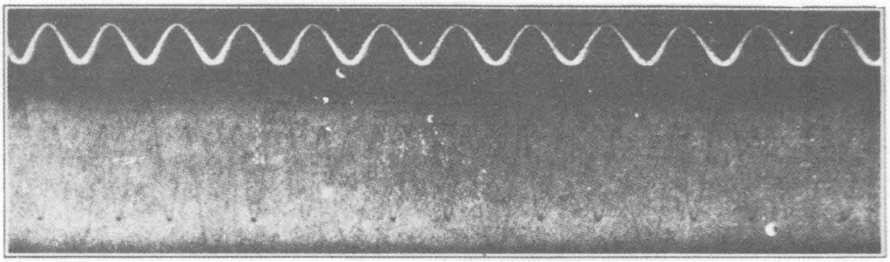
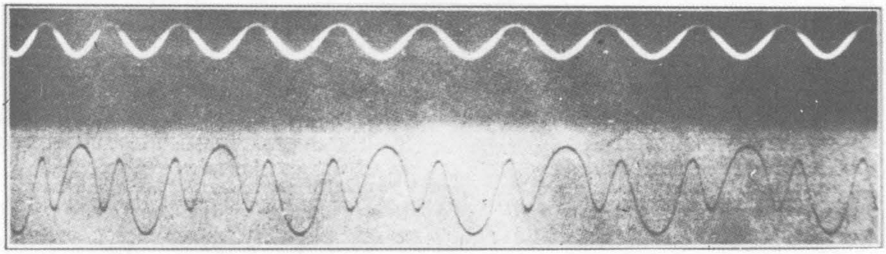
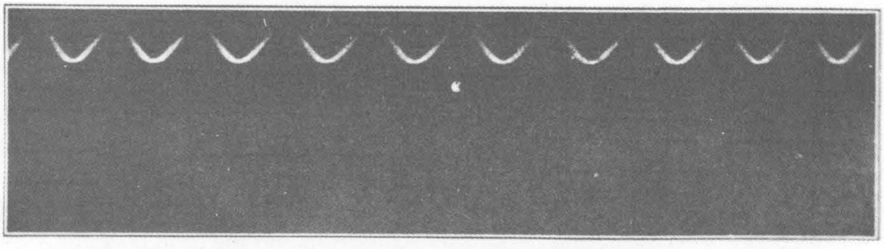


(8a)



(9)



(10)

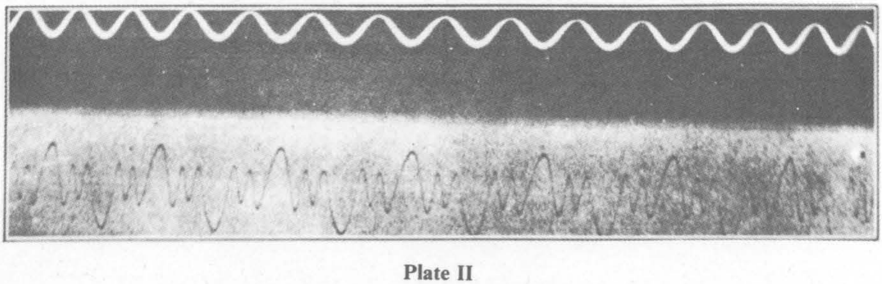


Plate II

vertical slit cut in it. The images of the horizontal and vertical slits fell, one immediately above the other, on the slit in the plate. Only a very small length of the former, i.e. practically only a point of light was allowed to fall upon the photographic plate. The dark slide which held this was moved uniformly by hand in horizontal grooves behind the slit in the focal plane of the camera, while the fork and the string were in oscillation.

Figures 1 and 6 represent the well-known case in which the string makes one oscillation for every two oscillations of the fork. This is evident in the photograph.

Figures 2 and 7 represent the next type in which the variable tension maintains an oscillation of the same frequency as its own. It will be noticed that the curvature of one of the extreme positions of the string is somewhat greater than that of the other and that the mid-point of the oscillation is somewhat displaced to one side of the middle-point of the vertical slit at which the string was set when at rest. The inference from this fact of observation is that the transverse motion of each point on the string is represented by an expression of the form

$$P \sin(2pt + e) + Q, \tag{12}$$

where Q is a constant. A motion of this type is only possible under a variable spring. For, the restoring forces acting on an element of the string at the two unequally curved extremes of its swing cannot themselves be equal and opposite (the condition necessary for a simple harmonic oscillation) unless the tensions of the string at the two extreme positions are unequal. In fact the second constant term Q in the motion is introduced under the action of the variable spring, and its importance will become evident as we proceed.

Figures 3, 8 and 8(a) represent the third type of motion in which the string makes *three* swings for every *two* swings of the fork. But it is evident from figures 8 and 8(a) that the successive swings on opposite sides are not all equal in amplitude and the influence of this is also perceptible in figure 3, having given rise to the appearance of two extra strings, which represent really the turning points of the motion. The vibration curve shown in figures 8 and 8(a) can be represented by an expression of the form

$$P \sin(3pt + e) + Q \sin(pt + e'). \tag{13}$$

The alternate increase and decrease of the amplitude of the motion of the string is evidently due to the action of the varying tension and the term $Q \sin(pt + e')$ in the motion which superposed on the first reproduces this waning and waxing effect, plays a very important part in the maintenance of the motion, as we shall see later on. Figures 4 and 9 represent the fourth type of motion in which the string makes *four* swings for every *two* swings of the fork. As before the waning and waxing of the motion under the action of the variable spring is evident in the photographs and the observed motion of the points on the string is of the type

$$P \sin(4pt + e) + Q \sin(2pt + e'). \tag{14}$$

Here as before we shall see that the second term which is introduced under the action of the variable spring plays a very important part in the maintenance of the motion.

Figures 5 and 10 represent the fifth type of motion in which the string makes five swings for every two swings of the fork. The periodic increase and decrease in the amplitude of the motion is also evident. The vibration-curve may be represented by

$$P \sin(5pt + e) + Q \sin(3pt + e'), \tag{15}$$

the second term being due to the action of the variable tension.

In the general case therefore we may assume the maintained motion to be of the form

$$P \sin(\gamma pt + e) + Q \sin(\overline{\gamma - 2}pt + e'). \tag{16}$$

The equation of motion under a variable spring may be written as

$$\ddot{U} + k\dot{U} + n^2 U = 2\alpha U \sin 2pt.$$

(See equation (10) above.)

If now we substitute (16) for U, the right-hand side of equation (10) gives us what we may regard as the impressed part of the restoring force at any instant. It may be written as

$$\alpha P[\cos(\overline{\gamma - 2}pt + e) - \cos(\overline{\gamma + 2}pt + e)] + \alpha Q[\cos(\overline{\gamma - 4}pt + e') - \cos(\gamma pt + e')]. \tag{17}$$

The work done by this force in a period of time t embracing any number of complete cycles of the variable spring is found on integration to be equal to $PQ\alpha pt \cos(\pi + e - e')$ if $\gamma > 2$ or to $2PQ\alpha pt \sin e \sin(e' - \pi)$ if $\gamma = 2$. The surplus of energy thus made available may be sufficient to counteract the loss by dissipation in the same time, i.e. to maintain the motion.

It is not difficult to make out from equations (10) and (17) that these apparently anomalous cases in reality form illustrations of the general principle of equality of periods required for resonance. For, we get a large motion when $n = \gamma p$ in the general case, and the reason for this is evident at once if we neglect the first three terms in (17) as ineffective and write equation (10) as under

$$\ddot{U} + k\dot{U} + n^2 U = -\alpha Q \cos(\gamma pt + e'). \tag{18}$$

We started on the assumption that

$$U = P \sin(\gamma pt + e) + Q \sin(\overline{\gamma - 2}pt + e').$$

Equation (18) shows that if we had neglected the second term (coefficient Q) we should have been unable to account for the resonance effect observed. Probably for a more complete discussion it would be necessary to take three terms thus:

$$U = P\sin(\gamma pt + e) + Q\sin(\overline{\gamma - 2}pt + e')$$
$$+ R\sin(\overline{\gamma + 2}pt + e_1).$$

Experiment shows however that the third term (with coefficient R) is relatively unimportant and the treatment given above may therefore be taken as a fairly close first approximation. It will be noticed from figures 7, 8, 8(a), 9 and 10 that the epochs of maximum amplitude in each case pretty closely correspond to those of minimum tension and vice versa. This is exactly what is to be expected on general considerations.

Some very curious phenomena are observed when the vibration of the string in each of the cases described above is observed through a stroboscopic disk. These and other matters I hope to detail in a future paper.

Part III

In part II, the equations of motion discussed are throughout those of a body having one degree of freedom. This was sufficient for the purpose of elucidating the leading features of each type of motion considered. But it must not be overlooked that the systems dealt with, i.e. stretched strings, have more than one normal mode of motion and this fact leads to certain exceedingly interesting complications. The phenomena observed under this heading fall into two distinct classes which I shall discuss separately.

The first class of phenomena I have designated "transitional types of oscillation." Their existence may be explained somewhat as follows: Take the case of a system maintained in one of its natural modes of vibration by periodic forces of double frequency. It is evident that the actual period of vibration would be *exactly* double the period of the acting force but the *free* period of vibration in the particular mode may differ slightly from the *forced* period of vibration. The range and extent of the permissible difference between the two is a function of the magnitude of the periodic force acting on the system. Assume now that the system has another natural mode of vibration whose frequency for free oscillations is not very far removed from that of the first and that the magnitude and frequency of the periodic force acting on the system is such that the ranges of the two natural modes of vibration for maintenance by forces of double frequency partly overlap and the force actually at work falls within the overlapping part. It is evident that in such a case the system would vibrate with a frequency equal to exactly half that of the acting force, but the *mode* of vibration would not be either of the natural modes but something intermediate between the two. These "transitional" types or modes of motion possess special experimental interest in the case of stretched strings as they can be readily observed and studied. It is not at all difficult for instance to maintain a "transition" mode of oscillation intermediate between the ordinary modes with three and four ventral segments respectively, by suitably adjusting the tension and varying it periodically by the aid of a tuning-fork. The frequency of the motion would everywhere be exactly half that of the fork and the motion at each point strictly "simple harmonic," but there would be no "nodes" or points of rest visible. Such a type of motion presents a very remarkable appearance when examined under intermittent illumination of periodicity slightly different from that of the tuning-fork. The two intersecting positions of the string seen undergo a periodic cycle of changes enclosing alternately three and four ventral segments.

The phenomena observed in the experiment described above can be explained on the supposition that the displacement at each point can be represented by the equation

$$y = a \sin \frac{3\pi x}{l} \sin (pt + e) + b \sin \frac{4\pi x}{l} \sin (pt + e'). \tag{19}$$

Equation (19) suggests that the phase of the motion is not the same at all points of the string. In fact working by the optical and mechanical methods described in the first part of this paper I observed very remarkable variations of phase over the length of the string. It appeared that in some cases e and e' differed by as much as $\pi/2$.

Of course we should get "transitional types" of oscillation with the vibrations of higher frequencies maintained by periodic forces which were discussed in part II of this paper, but they are not so marked as in the case of double frequency since the frequency-ranges become smaller as we go up the series.

The second class of phenomena observed cannot be fully discussed within the limits of the present paper and I shall have to content myself with briefly indicating their nature. In part II of this paper I showed that a variable tension or "spring" may under suitable circumstances maintain an oscillation of a frequency standing in any one of a series of ratios to its own frequency. If the system which is subject to the variable "spring" or tension has itself a series of natural modes or frequencies, it would evidently be possible for two or more modes of vibrations to be set up simultaneously with the respective frequencies and we would find a "simple harmonic" variation of tension maintaining a *compound* vibration. The special interest of this in the case of stretched strings consists of the fact that the natural frequencies of the system themselves form a harmonic series, and we may also have oscillations set up independently by one and the same force in rectangular planes and the compound character of the motion would be rendered visible by the curved paths of points on the string. These curves would in fact be identical with or analogous to the respective Lissajous figures and I hope with a future paper to publish several photographs which I have taken of compound vibrations maintained in this manner by a single tuning-fork. Two of these will be found published with my note in *Nature* (*London*), February 10, 1910.