

Bose–Einstein condensation: Where many become one and so there is plenty of room at the bottom

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Classically identical particles become quantum mechanically indistinguishable. Satyendra Nath Bose taught us, in 1924, how to correctly count the distinct microstates for the indistinguishables, and for a gas of light quanta (later photons), whose number is *not conserved*, e.g., can vary with temperature, he gave a proper derivation of Planck's law of black body radiation. Einstein, in 1925, generalized the Bose statistics to a quantum gas of material particles whose *number is now fixed*, or conserved, e.g., ^4He , and thus opened a new direction in condensed matter physics: He showed that for low enough temperatures (~ 1 Kelvin and below), a macroscopic number of the particles must accumulate in the lowest one-particle state. This degenerate gas with an extensively occupied single one-particle state is the Bose–Einstein condensate, now called BEC. (Fragmented BEC involving a multiplicity of internal states of non-scalar Bose atoms is, however, also realizable now). Initially thought to be a pathology of an ideal non-interacting Bose system, the BEC turned out to be robust against interactions. Thus, the Bose–Einstein condensation is a quantum phase transition, but one with a difference – it is a purely quantum statistical

effect, and requires no inter-particle interaction for its occurrence. Indeed, it happens in spite of it. The condensate fraction, however, diminishes with increasing interaction strength – to less than ten per cent for ^4He . The BEC turned out to underlie superfluidity, namely that the superfluid may flow through finest atomic capillaries without any viscosity. Interaction, however, seems essential to superfluidity. But, the precise connection between BEC and the superfluidity remains elusive. Thus, for example, we may have superfluidity in two-dimensions where there is no condensate! Seventy years later now, the BEC has come alive with the breakthrough in 1995 when near-ideal BEC was created in dilute alkali gases of ^{87}Rb and ^{23}Na atoms *cooled in the gaseous state* down to nanokelvins and localized in a trap. There are reasons why we ought to be mindful of the BEC – if only because here even the interaction between the particles is tunable at will – the sign as well as the strength of it. BEC has now become an ideal laboratory for basic and condensed matter experiments, and for high resolution applications. Properly viewed, it is indeed a new state of matter. This article is about the saga of BEC that really began with Einstein in 1925.

Keywords: Bose–Einstein condensation, bosonic stimulation, quantum gas statistics.

LET me begin with what may seem like an apology, which it is certainly not, and a touch of history of Bose–Einstein condensation, that it is. The point is that if we go strictly by the calendar, then the Bose–Einstein condensation does not belong in the miracle year of 1905, which is being observed as the World Year of Physics by many learned bodies around the world. In fact, Bose–Einstein condensation came in twenty years too late, in the year 1925 when Einstein, already famous at 45, derived the condensate for a degenerate quantum gas of permanent particles, i.e., of fixed number, such as helium (^4He), as a necessary consequence following from the novel quantum statistics¹ proposed a year earlier in 1924 by the young Indian lecturer, Satyendra Nath Bose at 25, then at the Dacca University, for the gas of light quanta (later photons). Bose had shown how to correctly count the distinct

distributions (microstates or complexions) of indistinguishable objects (particles) among the distinguishable boxes (phase-space cells) with no restrictions on the occupation numbers. Actually, Bose had proposed this new quantum statistics in an attempt to give a logical derivation of the celebrated Planck law of black body radiation (Figure 1), without the *ad-hoc* assumptions that Planck had to make. His paper was, however, turned down by the editors of the *Philosophical Magazine*. Convinced that he was right, Bose sent his manuscript to Einstein. It was Einstein who first saw the important conceptual advance – the indistinguishability of identical particles – in Bose's work. At his (Bose's) request, contained in the letter accompanying the manuscript, Einstein personally translated it in German and got it published speedily in *Zeitschrift fuer Physik*, the prestigious physics journal of the time. (It is of certain historical note that in his letter young Bose had addressed Einstein as 'Respected master' in the typically Indian tradition of Ekalavya. They met only later in 1925 in Berlin. Einstein also promoted Bose's work in the Prussian Academy of Sciences). Ein-

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stein at once saw the deep connection between Bose's view of the Planck thermal radiation as a gas of massless light quanta and an ideal quantum gas of identical material particles of non-zero mass, such as helium or hydrogen. The crucial point was that of indistinguishability of the identical quantum particles and the quantum-statistically correct way of counting them. As an act of pure transference, Einstein applied²⁻⁵ the quantum statistics of Bose to a gas of identical particles, but with the proviso that, unlike the case of photons, the number of particles must be conserved now. Hence, the introduction of a chemical potential that, at low enough temperatures, led to the condensation⁴ (that now bears his name jointly with the name of Bose, see Figure 1). This condensation was much to the disbelief of many leading physicists of the time; among them were Planck, Schrödinger, and Einstein himself perhaps. Einstein never returned to Bose-Einstein condensate (BEC) after 1925.

But, all this was about twenty years after the miracle year of 1905. The question then is why include BEC in this centennial issue. One obvious answer is that BEC was a landmark in the developments that followed the quantum revolution unfolding at the turn of the 20th century. C. N. Yang has remarked to the effect that just BEC alone would have assured a place for Einstein, and certainly for Bose, in the annals of physics. There is, how-

ever, to my mind, yet another, more efficient cause for its inclusion here. While it is true that BEC was not born in the miracle year of 1905, it did have a miraculous resurrection seventy years later in 1995, nearer our own times, when the near-ideal BEC was realized experimentally in a dilute gas of alkali atoms at a few billionths of a degree Kelvin (nanokelvins) above the absolute zero of temperature! For a review, see refs 6-8. This was an ideal condensation (one hundred per cent!) – a dream for a quantum condensed matter physicist! Indeed, the term BEC came into being just then – for this new state of matter. The saga of BEC that really began in 1925 with Einstein, lived through 1995 to our own times, and will certainly live on far beyond. See the timeline at the end. Thusly justified, let us turn now to the physics of BEC.

Bose-Einstein condensation: The phenomenon

Normally, a gas such as the air in a room has its molecules distributed over a broad range of energy-momentum. This is familiar from the Maxwell distribution of velocities known from the 19th century. The velocity distribution is smooth and sparse – the occupancy number of an elementary molecular phase space cell ($d\mathbf{p}d\mathbf{q} \sim h^3$) being typically <1 . As the temperature is lowered, however, the situation begins to change. The gas condenses into liquid and then into the solid state, through successive phase transitions at which some of the thermodynamic quantities become singular. All these transitions involve interactions among the particles, e.g., the long-range van der Waals attraction. Something much more subtle happens for certain gases, such as helium that remains fluid down to the absolute zero of temperature under its own vapour pressure. (Helium (^4He) can be solidified only under pressure of about 25 atmospheres. This is because of its low atomic mass and weak inter-atomic attraction, giving it a large zero point energy – we call it a quantum liquid). The helium (^4He), however, does undergo a phase transition at a critical temperature $T_c = 2.18 \text{ K}$ – a second order phase transition at which its specific heat at constant volume has a logarithmic singularity (I -shaped and hence the name I -point for the transition). A non-interacting (ideal) Bose gas has a gentler singularity, namely a cusp – a third order transition in the sense of Ehrenfest. The interaction drives it to the second order. Very spectacularly, the lower-temperature phase (called He II) turns out to be superfluid (with zero viscosity) while the higher-temperature phase (He I) remains normal. At a deeper level, however, for $T < T_c$, the velocity distribution ceases to be sparse, and a finite fraction of the ^4He atoms accumulates in the zero-momentum state. *This macroscopic, extensive occupation of the single one-particle state is the Bose-Einstein condensation, or BEC for short. It is driven not by any inter-particle attraction, but is a purely quantum statistical effect. Much of this was, of course, not known in 1925. Einstein was led*

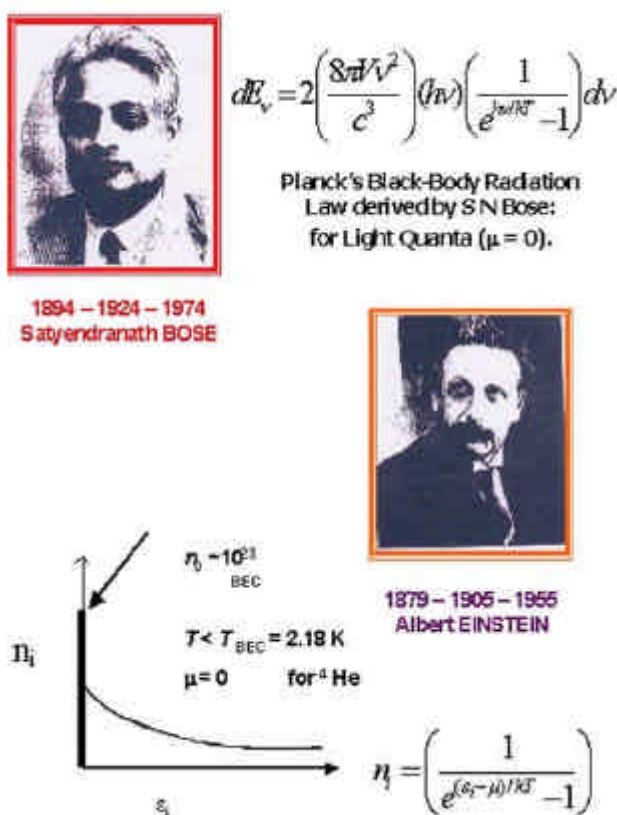


Figure 1. Bose-Einstein Condensation and Quantum Statistical Phase Transition: From the photon gas to a gas of indistinguishable molecules.

to the condensation from his close examination of the ideas that followed from the new quantum statistics that Bose had proposed in the previous year, 1924, for the gas of light quanta in the black body radiation. Einstein clearly saw the deep concept of indistinguishability of identical particles that was implicit in Bose's derivation of the Planck radiation law. All he really had to do then was to replace the gas of massless photons (with the relativistic dispersion relation $p = h\mathbf{n}/c$) by the gas of material particles (with non-zero mass and the non-relativistic dispersion relation $E = p^2/2m$), and to introduce a chemical potential μ to ensure a fixed particle number N . A brief account of essentially his derivation of the condensate, and the underlying quantum gas statistics, is given below.

BEC and the quantum gas statistics

All statistics is about counting. And the quantum gas statistics is about counting the indistinguishables. It is concerned with finding the most probable distribution of the gas molecules over the molecular phase space subject to certain given constraints or subsidiary conditions, e.g., given total energy and the total number of particles. The term probability is used here in the sense of Planck, namely, that the probability of a macrostate (coarse grained macroscopic description) is determined by (proportional to) the number of distinct microstates (fine-grained microscopic descriptions or complexions) that are consistent with it (assuming that these microstates are degenerate in energy and equally probable). This is a problem in combinatorics – in how many ways (W) can we distribute N objects (particles) among Z boxes (phase-space cells) with the i^{th} box containing n_i objects. For the classical case (classical gas statistics) the boxes are, of course, distinct, but the N objects, though identical, are also distinguishable, and we have the classical Boltzmann result, $W_c = N!/\prod_i n_i!$. Here, permuting (exchanging) the n_i objects within the i^{th} box obviously creates no new microstates; but a permutation involving exchange of the particle between different boxes does create new microstates, and must be counted as such. Now, let the classically identical objects become quantum mechanically indistinguishable, as it must be in the case of quantum gas statistics. Then the permutations of the objects even between different boxes must be discounted – indeed, a permutation of the indistinguishables generates no new microstates!. Then, the Boltzmannian number of microstates (complexions) for the classical case, W_c ($\gg 1$), must be replaced in the quantum case by the correctly counted $W_Q = 1$. This is the essence of indistinguishability and of the quantum statistics that Einstein had made use of.

BEC derived

Proceeding with Einstein, consider an ideal (non-interacting) gas of indistinguishable molecules for which the

molecular phase-space lying in the energy shell $E_v \pm 1/2\Delta E_v$ has the number of elemental phase-space cells Z_v given by

$$Z_v = (2pV/h^3)(2m)^{3/2} E_v^{1/2} dE_v, \quad (1)$$

(h being the volume of the elementary phase space cell after Planck).

We can now distribute N_v of the indistinguishable molecules among the Z_v distinct cells in W_v ways, where

$$W_v = (N_v + Z_v - 1)! / N_v! (Z_v - 1)! \quad (2)$$

(To see that this is so, just imagine placing Z_v partitions separating the N_v objects arranged on a line, and count the number of ways of doing this. This is essentially same as Bose's way of defining a microstate in terms of the set of occupation numbers of the cells; or equivalently, distributing the distinguishable (Z_v) cells among the indistinguishable (N_v) molecules). It is assumed here that $Z_v \gg 1$, which is true for a gas extended over a large volume with a phase-space that is a continuum. The total number of microstates for the N indistinguishable molecules distributed over the total phase-space is then

$$W = \prod W_v. \quad (3)$$

The rest follows the standard exercise in maximizing the associated entropy (S) function subject to the subsidiary conditions, or constraints, of the given number (N) and energy (E):

$$\begin{aligned} S &= k_B \ln W, \\ N &= \sum N_v, \\ E &= \sum E_v N_v. \end{aligned} \quad (4)$$

The constraints are to be imposed through the introduction of the corresponding Lagrange multipliers. Maximization of the entropy function is facilitated in the thermodynamic limit ($N \rightarrow \infty$, $V \rightarrow \infty$, with $N/V =$ number density; $n =$ constant) through the Stirling approximation for the otherwise tyrannical factorials, $\ln N! \approx N \ln N - N$, for $N \gg 1$. This straightforwardly leads to the distribution:

$$N_v = Z_v \frac{1}{e^{b(E_v - \mu)} - 1}, \quad (5)$$

with

$$N = \sum N_v = \sum \frac{Z_v}{e^{b(E_v - \mu)} - 1}, \quad (6)$$

where $b = 1/k_B T$.

Note that the expression for Z_v hides in it the single-particle density-of-states factor which depends on the dimensionality.

Several observations can now be made on eqs (5) and (6) that eventually gave the BEC⁴:

- (a) The expression on the right-hand-side of eq. (5) is singular for $E_v = \mathbf{m}$. The singularity is, however, integrable for a 3D gas, but logarithmically divergent in 2D.
- (b) The chemical potential must be negative, including zero, as N_v has to be a non-negative number.
- (c) The chemical potential must increase towards zero as the temperature is lowered for a given N , while the right-hand-side of eq. (6) decreases continuously.
- (d) At a certain critical value T_c , the chemical potential vanishes and remains stuck at zero then on for lower temperatures.
- (e) Below the critical temperature, the right-hand-side of eq. (6) becomes less than N for a given T : the thermal distribution now cannot hold all of N bosons in the thermally excited non-zero energy states – there is an over-population!
- (f) The excess population must necessarily accumulate in the singularity at the lowest single-particle state, i.e., the zero momentum state. This is how Einstein had argued, and was thus led to condensation, and to the condensate fraction. The above-the-condensate fraction remains as a saturated ideal gas (vapour) in equilibrium with the condensate.

The critical condition ($\mathbf{m}=0$) for the condensation is best expressed in terms of the phase-space density:

$$n\lambda_{dB}^3 \geq 2.612, \quad (7)$$

$$\lambda_{dB} = h/(2Mk_B T)^{1/2}.$$

This is also called the condition for quantum degeneracy. The equality sign in eq. (7) holds at the critical temperature. Here λ_{dB} is the thermal de Broglie wavelength, and $n = N/V$ is the number density. (This condition for BEC can be re-stated as that the mean inter-particle spacing be less than the de Broglie wavelength ensuring appreciable overlap of the thermal wavepackets, that makes the indistinguishability effective).

It is clear from eq. (6) that there is no BEC in a 2D gas where the density-of-states has a non-zero value at zero energy. This is unlike the case of a 3D gas where the density-of-states vanishes at the bottom, i.e., at the zero of the single-particle energy. It is the wholeness of the BEC (a single macroscopic object) that accommodates the excess plurality (population) in the single zero-momentum state. It is in this sense that there is plenty of room at the bottom.

Einstein clearly realized that BEC is a purely-quantum statistical effect. He did not refer to the condensation as a phase transition. Einstein, however, had the mental picture of the condensate fraction in equilibrium with the above-

the-condensate fraction much as the saturated vapour is in equilibrium with the liquid phase under isothermal conditions. Though, in a BEC the phase separation is in the momentum space.

Several other things also followed naturally from his derivation: The Nernst Theorem was satisfied (entropy vanished at the zero of temperature as there was a single state – the BEC); the Gibbs paradox was obviated, without recourse to any *fixing* or *correction*, such as dropping the factorial $N!$ from the Boltzmann way of counting. This made the entropy correctly additive.

Note on indistinguishability

Two objects may be said to be indistinguishable if they are merely two different states of the same underlying entity. This, of course, happens naturally in a quantum-field description where the particles are the excitations of an underlying field – just its internal movements. At a somewhat heuristic level, one can understand the quantum indistinguishability of the classically identical particles. Classically, it is possible in principle to keep track of the identity of the particles as they are being permuted – here permutation is viewed as a process. The continuous tracking makes it always possible to know which is which. quantum mechanically, however, there are no trajectories and thus keeping track of the identical particles is forbidden in principle. Hence their indistinguishability. This is as operational as one can get. There remains, however, a question: is there a degree of indistinguishability, e.g., two particles differing arbitrarily slightly in their masses. Is approximate indistinguishability meaningful, or must it be an absolute condition? Some of these questions had occurred to Einstein³ – in the form of a paradox involving a mixture of two gases with slightly differing molecular masses. Einstein also examined thermal fluctuations of the number in the Bose system (fluctuations, and, of course, invariances being his abiding interests). He found the wave noise (an interference effect) in addition to the shot noise-rediscovered many a time since.

Finally, a note on BEC in a spatially localized quantum gas. This is relevant to the BEC now realized in the optical and magnetic traps⁶⁻⁹. Here the usual condition $Z_v \gg 1$ for the phase-space elements on an energy shell is clearly not satisfied. One must do the fully quantum treatment using the occupation number representation for the Bose system with its second quantized creation/annihilation operators⁸.

Generalization of BEC

The BEC derived by Einstein was only for an ideal gas of non-interacting scalar bosons, extended uniformly in the 3-dimensional (3D) space. Generalization has since been considered and in some cases realized: (a) fragmented

BEC¹⁰⁻¹² – extrinsically into non-overlapping regions of coordinate or momentum space; and intrinsically in terms of their internal (hyperfine) spin structures, or even their macroscopic quantum mechanical phases. Then, there is the question of their thermodynamic stability – repulsive interaction as in ⁴He has been shown to disfavour fragmentation for reasons of the energetics of exchange interactions¹¹; (b) Dimension – there is no BEC in two dimensions, as can be readily seen from the fact that the sum of the thermal occupation numbers (see eq. (6)) over the molecular states for the system then diverges at all temperatures for chemical potential $\mu = 0$. The absence of BEC in two dimensions, of course, follows from a rather general theorem in condensed matter physics; (c) Localized condensates⁶⁻⁹ – BEC has now been realized in dilute alkali atomic gases in harmonic traps, magnetic and optical; (d) Interactions¹³ – interacting bosons, for example ⁴He, has been treated extensively by the many-body theorists. Interactions (repulsive) deplete the condensate; (e) Condensate fraction – neutron scattering¹⁴ has been an experimental technique of choice for determining the condensate fraction in ⁴He. In neutron scattering with high energy-momentum transfer rates, the struck atoms of the condensed system are excited to energies much greater than the binding energy, and thus the scattered cross-section gives directly the momentum distribution (the so-called Compton profile) of the system. The macroscopic occupation of the zero momentum single-particle state (the BEC) should now show up as sharp delta-function singularity (peak) in the measured momentum distribution, which is, however, yet to be clearly seen. It is essentially a *deep inelastic scattering* in the context of condensed matter. A novel method to demonstrate that a BEC really exists is the technique of quantum evaporation¹⁵. Here, a collimated beam of phonons injected into the sample causes evaporation of the atoms from the sample in a single-excitation to single-atom process. The angular distribution of the evaporated atoms was then inverted to show that there indeed is an accumulation of the atoms in the zero-momentum state – a BEC.

Bose–Einstein correlation

Closely related to BEC is the phenomenon of Bose–Einstein correlation (bec), where bosons of the same kind emitted from nearby sources get correlated in energy (E) and momentum (P), or time (t) and space (x). This is best seen with reference to the scattering of the two Bose particles at an ideal 50:50 beam splitter as depicted in Figure 2.

For the one-boson incoming state, we have

$$a_1^+ |vac\rangle \rightarrow (ra_3^+ + ta_4^+) |vac\rangle,$$

$$a_2^+ |vac\rangle \rightarrow (ra_3^+ - ta_4^+) |vac\rangle,$$

in obvious notation, where a_i^+ is the Bose creation operator for the i th channel, and r and $-t$ the (real) elements of the scattering matrix for the beam splitter.

Now, for the two-boson incoming state, we have

$$a_1^+ a_2^+ |vac\rangle \rightarrow (r^2(a_3^+)^2 - t^2(a_4^+)^2 - rta_3^+ a_4^+ + rta_4^+ a_3^+) |vac\rangle.$$

Thus, for our beam splitter with $r = t$, we have only the doubly occupied outgoing states inasmuch as a_3^+ and a_4^+ commute for bosons. (For fermions, of course, the reverse is true, and we will have only the singly occupied outgoing states). This flocking of bosons of the same kind is, of course, crucial to BEC and to bosonic stimulation¹⁶.

BEC and bosonic stimulation

This is closely connected with the $(1 + N)$ factor that multiplies the probability of a scattering event in which a Bose particle is scattered into a single-particle state that already has N bosons of the same kind. (This is, indeed, the ‘crazy idea’ that had intrigued Bose and Saha in the papers of Einstein and Ehrenfest¹⁷ and of Pauli¹⁸ written in the context of the probability of Compton scattering that

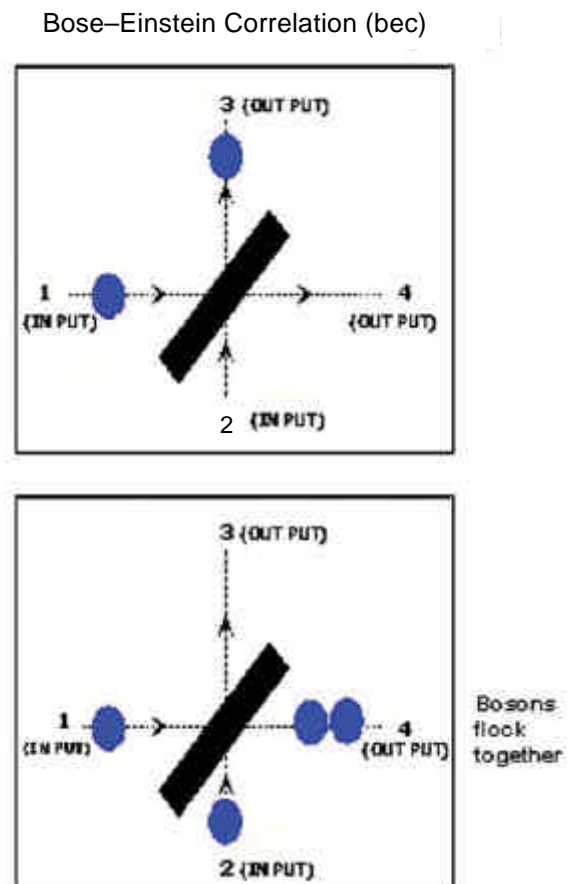


Figure 2. Bosons emitted from nearby sources tend to be correlated in P and E or x and t .

depended on the radiation density at the scattering frequency that would arise in the process). Bosonic stimulation is clearly involved in the kinetics of BEC growth, and may be used to amplify certain extremely weak effects in condensed matter.

Macroscopic wavefunction for BEC

Einstein had considered only the quantum statistics and the resulting thermodynamics of BEC. Fritz London¹⁹ introduced the idea of BEC as a single macroscopic quantum object. The proximity of $T_{\text{BEC}} = 3.13$ K, calculated for ⁴He regarded as an ideal Bose gas, and the observed $T_\lambda \approx 2.18$ K, led him to identify the lambda transition with the BEC. This eventually led to a non-linear Schrödinger-like equation, the Gross–Pitaevskii equation^{7,8}, for the macroscopic matter–wave function $\Psi(r)$, of a single coordinate r , describing the BEC in a realistic Bose system with interactions (e.g., ⁴He):

$$i\hbar \frac{\partial}{\partial t} \mathbf{y}_0(r,t) = -\frac{\hbar^2}{2m} \nabla^2 \mathbf{y}_0(r,t) + g |\mathbf{y}_0(r,t)|^2 \mathbf{y}_0(r,t), \quad g = \frac{4\mathbf{p} \hbar^2 a}{m}, \quad (8)$$

with $\Psi_0(r, t) = \text{Complex order parameter} = \sqrt{n_0(r, t)} e^{i\theta_0(r, t)}$, $n_0(r, t) = \text{BEC number density}$; $\mathbf{q}_0(r, t) = \text{phase}$, $v = \text{velocity} \equiv (\hbar/m) \nabla \mathbf{q}_0(r, t)$, where a is the s -wave scattering length that parametrizes the self-interaction ($a > 0$ for repulsive interaction).

Ψ_0 describes BEC

Space-time coherent phenomena – interference, diffraction
 Superfluid flow through capillaries, past obstacles
 Quantized vortices ($h/m = \text{quantum of circulation}$)
 Tunneling through barrier
 Non-linear Matter Waves (4 wave mixing)
 Collective excitations (the sounds).
 At $T = 0$, the condensate fraction < 1 , but the superfluid fraction = 1.

BEC localized in traps

When Einstein derived the condensation for an ideal (non-interacting) gas of particles obeying the quantum (Bose) statistics, he had in mind helium, hydrogen (which are actually strongly interacting), and also the gas of electrons (wrongly, as electrons actually obey the Fermi–Dirac statistics, of course, not known then). A BEC with high condensate fraction requires a high phase–space density without having to encounter the adverse effects of strong interactions that not only deplete the condensate, but also

actually pre-empt it by causing solidification. This suggests low densities and correspondingly low enough temperatures, and of course, low-mass atoms. This is precisely what has been achieved in the dilute gaseous alkali atom BECs, with typically ~ 10 orders of magnitude lower than the normal condensed matter density and a temperature ~ 100 nanokelvins. Thus, the domain of BEC has been extended far beyond helium, or shall we say, outside the heliocentric boundary in the laboratory. Extreme BECs are suspected in the cores of compact astrophysical objects, and in the cosmological vacua. Very recently,²⁰ superfluidity (and by implication BEC) has been demonstrated in solid ⁴He. Given below is the Zoo of BECs:

- ⁴He (the inert noble, but Nobel-active gas)
- H (spin -polarized hydrogen), an example of effectively spin-half Bose gas
- Excitonic condensates
- Composite bosons, e.g. ($e^- - e^-$); (³He – ³He). . .
- Alkali atomic (bosonic) isotopes ⁸⁷Rb, ⁸⁵Rb, ⁷Li, ²³Na, . . .
- Alkali molecules (fermionic-isotope pairs) ⁴⁰K₂, ⁶Li₂
- Protonic/neutronic and pion condensates – neutron star interior.
- Cosmological condensates – field vacua.

Also, listed below are some parameter values typical of BECs in the laboratory for neutral bosonic alkali atoms:

Temperature	: 500 nk – 2 μ k
Number density	: $10^{14} - 10^{15} \text{ cm}^{-3}$
Total number	: $10^3 - 10^7 - 10^9$
Size and shape	: 10–50 μ m spherical 15 μ m \times 300 μ m cigar shaped
Cooling cycle time	: few \times seconds – few \times minutes.

Open problems

Some of these are: (a) Connection between BEC and superfluidity; (b) Interaction and dimensionality; (c) Fragmented BEC for composite bosons with internal structure; (d) Kinetics of BEC growth; (e) BEC and decoherence; (f) Amplification of weak effects, e.g., the extremely small rotational magnetic moments expected of hydrogen molecules may add up coherently to give a large macroscopic magnetic polarization in H₂ BEC!; (g) Bosonic stimulation – one may even speculate about the decay rate of a radioactive nucleus being enhanced many-fold if embedded in the BEC of one of its bosonic decay products; (h) BEC being a superfluid solid – a supersolid.

Timeline (fuzzy and annotated) of BEC

- 1900 • Planck’s Quantum Hypothesis; Planck’s Law of Black Body Radiation (‘... happy guesswork ...’).

- 1924
- Planck's Law and Light Quantum Hypothesis; Satyendra Nath Bose, *Zeit. f. Phys.*, 1924, **26**, 178. The pre-factor $2 \times (4\delta\bar{r}^2/c^3)$ also derived.
 - Beginning of Quantum Statistics: Loss of identity of light quanta. A new way of counting the indistinguishables.
 - Photons indistinguishable but the phase space cells distinct.
 - Photon number not conserved: Chemical potential $m=0$
- Quantum theory of the monatomic ideal gas, A. Einstein, *Preuss. Akad. Wiss.*, 1925, p. 3.**
- 1925
- Extension of the Bose Statistics: Particle Number Conserved – Chemical potential $m \neq 0$, e.g., Helium (^4He).
 - Gibbs paradox (the tyranny of $N!$) resolved, and the Nernst Theorem obeyed.
 - Startling consequences: Macroscopic occupation of the lowest single-particle state – Bose–Einstein Condensation (BEC).
 - Purely quantum statistical phase transition *sans* interaction.
 - Initial reaction to BEC: Einstein ... '... that is only by the way ...', Planck ... frankly disbelieved it, Schrödinger ... suspected an error in it.
- 1926
- P. A. M. Dirac²¹ gave antisymmetric wavefunction for fermions (^3He) obeying Pauli's exclusion principle with the occupation numbers restricted to 0 and 1 (the exclusive Fermi–Dirac statistics); and symmetric wavefunction for the bosons (^4He) obeying inclusive statistics with occupation numbers not restricted (the inclusive Bose–Einstein statistics). Matters of statistics were clarified by 1927.
- 1928
- W. Hendrik Keesom: He I (normal helium) – He II (superfluid helium) phase transition – the *I*-transition at a critical temperature $T_c = 2.18$ K.
- 1938
- Pyotr Kapitza: Discovers superfluidity of helium (^4He). (Earlier in 1911 Heike Kamerlingh Onnes had discovered superconductivity in Sn).
 - Fritz London Hypothesis: Superfluidity of ^4He a manifestation of BEC. A macroscopic wavefunction proposed for this phase. Now called the order parameter.
- 1940
- W. Pauli²² derived spin-statistics connection from special relativity and quantum mechanics: bosons for integer spin and fermions for half-integer spins.
- 1948
- N. N. Bogoliubov¹³: First microscopic theory of interacting Bose-gas: ^4He – Superfluidity and BEC connected. Depletion of BEC due to strong interactions in liquid ^4He .
- 1956
- O. Penrose and Lars Onsager²³: First estimation of BEC fraction $\sim 10\%$ for ^4He .
- 1957
- Bardeen–Cooper–Schrieffer (BCS) theory of superconductivity: Condensation of Bose-like Cooper pairs in the zero momentum state.
- 1966
- Seminal suggestion of P. Hohenberg and P. Platzman initiates probing of the condensate fraction by high-energy (epithermal) neutron scattering – momentum distribution (Compton Profile). But conflicting results for BEC fraction¹⁴.
- 1972
- Condensation of bosonic pairs of fermionic ^3He .
- 1980s
- Advances in laser cooling and trapping of neutral alkali atoms down to microkelvins; Steven Chu and William D. Phillips; and Claude Cohen-Tannoudji.
- BEC SAGA: 70 years after 1925 and end of helio-centricity⁶⁻⁸**
- 1995
- BEC RESURRECTED MIRACULOUSLY Eric A Cornell (NIST) Wolfgang Ketterle (MIT) and Carl E. Wieman (JILA +Univ. Colorado) obtain BEC in dilute gases of ^{87}Rb alkali atoms at ~ 20 nK (0.00000002 K) and ^{23}Na . BEC fraction $\sim 100\%$, the ideal value.
- ~nanokelvin*
- ~1999*
- New State of Matter: TUNABLE CONDENSATE
 - Coherent matter waves – atom laser
 - Bosonic stimulation
 - Non-linear matter – wave interaction: 4 WM
 - Quantum phase transition: BEC in optical lattice
 - Interaction tunable through Feshbach resonance
- ~2003*
- Fermionic atom pairs (Composite bosons): $^{40}\text{K}_2$, ^6Li molecular condensates.
- ~2004*
- Close encounters: Cold collisions for scattering length \ll de Broglie wavelength.
 - BEC (real-space pairs) – to – BCS (momentum-space pairs) crossover in fermionic systems
 - Molecular BEC: Chemistry with cold coherent matter; Photo-association of atoms into molecules.
 - Highest spatial and spectral resolutions; sensitive detectors (possibly for gravitational waves?).
 - BEC on a microchip.
 - BEC: A 'laboratory' for testing condensed matter models of strongly interacting systems, e.g., Mott insulator to superfluid transition.

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