Chapter 1

Introduction.

In this thesis we study some aspects of gravitational instability in an expanding universe. The universe is modeled as a system of nonrelativistic particles interacting only through Newtonian gravity. In the unperturbed universe these particles are uniformly distributed and they move according to Nubble's law. We want to study the evolution of disturbances in such a universe.

We first briefly discuss the relevance of this problem in the context of cosmology.

1.1 Why Study Gravitational Instability ?

It is believed from many lines of observational evidence that on very large scales the universe is homogeneous and isotropic and most models of the universe are based on this assumption. Combined with general relativity this assumption provides a model which explains the observed expanding universe. It also provides a background for understanding the observed light element abundances and the isotropic 3deg K microwave background. References [24],[27], [31] and [33] discuss the different cosmological models in detail. It is observed that the large scale distribution of matter in the universe exhibits structures. (see for example section 3 of Peebles (1993) and references given there). One then has to explain the observed large scale structure of the universe in a manner consistent with the homogeneous and isotropic models of the universe.

It is believed that the observed large scale structure in the universe is the result of the growth of some initially small disturbance in the uniform background universe. This growth is supposed to be due to the process of gravitational instability. Thus the reason for studying the growth of gravitational instability in an expanding universe is to understand the formation of the observed large scale structure in the universe. Newtonian dynamics gives a good description of cosmology on scales much smaller than the horizon in the matter dominated era (see for example Peebles 1980). It is during this epoch that the analysis discussed in this thesis is valid. Also, all non-gravitational forces have been ignored. This is primarily because it is expected that at large scales the gravitational force will dominate. Observations on various scales show that the mass inferred from the observed light is much smaller than the mass inferred from dynamics. This indicates that dynamics on various scales may be dominated by matter that cannot be seen. This matter component is often modeled as weakly interacting particles which in the matter dominated era may be treated as particles interacting only gravitationally.

1.2 Linear Evolution.

Any particle in the system described earlier will move under the gravitational force of all the other particles. Thus to solve the evolution of the system we have to simultaneously solve for the motion of all the particles. This is not possible in general.

We may simplify matters a little by treating the distribution of particles as a continuum. In this description the force on any particle is due to a smooth distribution of matter and the residual direct particle-particle scattering is not taken into account. If we consider the phase space of one particle, the state of this system can now be described by a function (the distribution function) on this space. The distribution function is the number density of particles in phase space and its evolution is governed by the coupled Liouville-Poisson equations.

When the particles are uniformly distributed and they move according to Hubble's law, the equations are much simpler and they reduce to a scalar equation for the Hubble constant (or equivalently the scale factor). This can be solved.

One next studies disturbances where the deviations from the background universe are small. This is characterised by a small number ϵ. It is assumed that the fractional change in the density (relative to the background) is of the order of ϵ and it is assumed that the deviations from Hubble flow are of this order too.

In addition we only consider situations where initially the velocity dispersion at any point is zero (i.e. all the particles at any point have the same velocity, a single streamed flow). With these assumptions the state of the system can be described by the density and velocity at every point, and the evolution is given by the hydrodynamic equations without pressure (HD equation) i.e. 1.continuity equation 2. Euler equation and 3. Poisson equation. This approach may be described as the single stream approximation.

The equations for the evolution are both non-linear and nonlocal. If we linearize the equations keeping only terms which are linear in ϵ, we obtain a set of linear equations that can be easily solved. As a result of this analysis it is found that the fractional density contrast at any point evolves as

$$\delta(x,t) = D_1(t)\delta_1(x) + D_2(t)\delta_2(x)$$

i.e. the fractional density contrast at any point can be broken up into two parts each of which evolves independently. There is a growing part with subscript 1 and an decreasing part with subscript 2. The linear analysis also tells us that any initial vorticity decays because of the expansion and if the initial vorticity is zero it remains so. The vorticity remains zero as long as the flow is single streamed, even in the non-linear regime.

Based on this one reaches the conclusion that the decaying mode of the density perturbation and the initial vorticity may be neglected in the study of large scale structure formation. The resulting picture is that one only deals with the part of the initial disturbance that grows. At any point this density perturbation just gets.scaled as a function of time and the overdense regions get more overdense and the underdense regions get more underdense. If one tries to extrapolate this picture for arbitrarily long periods of evolution one finds that the results cease to have any physical significance. This is because at places where the density is less than the background density the density keeps on decreasing and becomes negative and it is not possible to physically interpret this negative density. One expects the linear theory of qerturbations to give results which correctly describe the process of gravitational instability in the regime when the higher order perturbations are much smaller than the linear terms.

References [24],[27], [27], [31] and [33] discuss the linear evolution of cosmological perturbation in detail.

1.3 Zel'dovich approximation

Zel'dovich (1970) suggested a different way of looking at linear perturbation. He suggested that, instead of considering the evolution of the density, we should consider the trajectories of the particles. If one does a linear perturbation in the trajectories of the particles one gets a map from the initial position of the particles to their position after a time t. This map turns out to be such that in suitable coordinates it corresponds to the free motion of the particles where they keep on moving with their initial velocity. In this map time is replaced by the growing mode $D_1(t)$ and in the initial stages of the evolution it reproduces the results of the linear theory for the evolution of the density. This has the advantage that the density never becomes negative and one can physically interpret the results even at arbitrarily late times. Following the evolution of the particles t'o later times one finds that the trajectories of the 'particles intersect. When this happens the flow becomes multi-streamed (i.e., many particles with different velocities at the same point) and the density at the points where this occurs becomes infinite. These particles then pass through one another and the multi-streamed region increases. As a result vorticity and pressure develop and the single strem equations are no longer valid.

Based on this picture Zel'dovich suggested that the regions where this singularity first develops will be the places where the first structures form. He also predicted that the first structures to form will result from a one dimensional collapse and the resulting structures will be like pancakes i.e. two dimensional.

If one relaxes the assumptions made and allows for the presence of,multi streamed initial conditions one finds that it does not change the outcome very much. The initial velocity dispersion will dampen the growth of the linear perturbations on small scales but it will have no effect on the large scales. Also the picture of the evolution that one gets from the Zel'dovich approximation will still hold and the effect of the initial velocity dispersion will be to thicken the pancakes that are formed.

For a comprehensive discussion on the Zel'dovich approximation the reader is referred to a review by Shandarin and Zel'dovich (1989).

1.4 Perturbative non-linear evolution,

The linear theory gives a good description of the early stages of the evolution of the initially small disturbances. As the evolution proceeds the disturbances grow and they become non-linear and the whole perturbative approach is no longer applicable. We expect that there will be an intermediate epoch when the disturbances are weakly non-linear and they may be adequately described by taking into account the higher order terms of the perturbative expansion. We next discuss some issues that may be studied in the weakly non-linear regime.

As discussed earlier the results of linear evolution are local, but as Newtonian gravity is non-local we expect the quasilinear evolution of the disturbance to be non-local. The streaming of the particles will also result in non-local effects. These non-local effects can be studied by going to higher orders in the perturbative expansion.

The large scale structure in the universe may be described by the statistical properties of the matter distribution and the peculiar velocities. In the linear theory the evolution of these statistical properties is rather simple and the evolution can be described by scaling the initial statistical properties with appropriate powers of the growing mode. Initial conditions where the density fluctuation is a random Gaussian field are of particular interest. In such cases the initial statistical properties are fully described by the two point correlation function and all the higher correlations are zero. These are the simplest possible initial conditions and most models of inflation predict such initial conditions (see chapter 17, Peebles 1993 and references therein). In the linear evolution all the higher correlations (e.g. three point correlation function) remain zero. But as the evolution proceeds the density field is no longer Gaussian and one expects the higher correlations to develop. It is possible to study this perturbatively in the weakly non-linear regime.

If the initial conditions are such that the flow is single streamed, it remains single streamed in the linear evolution. But it is known that as the evolution proceeds the flow becomes multi-streamed. This is another effect that motivates the study of the higher order terms in the perturbative expansion.

It is also required to investigate the higher order terms in the perturbative expansion to determine the regime when the linear results are valid. It also tells us about the validity of the whole perturbative approach.

Many of these issues have been extensively studied and there exists a considerable amount of literature on this subject. Peebles (1980) discusses the lowest order non-linear term for the evolution of the density. This is based on the HD equations and as expected the result is non-local. Peebles used this to calculate the skewness induced in the density distribution for the situation when the initial density fluctuation is Gaussian. The induced three point correlation function has been calculated at the lowest order of non-linearity by Fry (1984). His calculations were based on the HD equations and he concluded that the three point correlation function has the hierarchical form i.e. it can be expressed in terms of products of the two point correlation function. Fry (1984) has also studied the trispectrum which is the Fourier transform of the four point correlation function. Inagaki (1991) has used the BBGKY hierarchy equations to study the bispectrum which is the Fourier transform of the three point correlation function. He has calculated the bispectrum at the lowest order of non-linearity for Gaussian initial conditions. There has been a lot of interest in studying the evolution of the moments of the density field using the HD equation. For Gaussian initial conditions all moments except for the second moment are initially zero. These moments become non-zero due to the non-linear evolution. Goroff et. al. (1986) have investigated the evolution of the first five moments of the density field averaged over a Gaussian ball. Bernardeau (1992) has found a general method for calculating any moment of the density field. Bernardeau (1994) has also discussed the evolution of the skewness of the density field averaged with a top hat filter. The evolution of the moments of the smoothed density field has also been studied by Bouchet et. al. (1992).

The non-linear corrections to the two point correlation function has also been widely studied. The lowest order non-linear correction to the two point correlation function has been calculated by Juszkewicz (1981), Vishniac (1983), and Makino, Sasaki and Suto (1992).

Fry (1994) has calculated the higher order non-linear corrections to the two point correlation function. All these calculations have been done using the single stream approximation.

Unlike the linear evolution, the non-linear evolution exhibits coupling of the various scales. This has been examined analytically by Suto and Sasaki (1991) and Makino et al. (1992). This issue has also been studied by Juszkewicz, Sonoda and Barrow (1984) and Hansel et. al. (1985), and for the particular case of the CDM initial condition it has been studied by Coles (1990), Jain and Bertschinger (1994), and Baugh and Efstathiou (1994).

The non-linear evolution can also be studied by considering a perturbative expansion of the trajectories of the particles, This leads to the calculation of non-linear modifications of the Zel'dovich approximation. This alternative approach (Lagrangian perturbation theory) has been discussed by Moutarde et. al. (1991), Bouchet et. al. (1992) and Buchert (1994).

1.5 Relation with other methods.

Foremost amongst the methods used for studying gravitational instability in an expanding universe is N-body simulations. In most of these simulations the matter content of the part of the universe that is being modeled is in the form of N representative particles. These particles are supposed to represent the fluid elements at the various points. In the N body simulation the motion of these particles is studied. There are various methods for calculating the force on each particle. For example in the particle-mesh (PM) code the matter distribution is smoothened by associating all the mass with a grid and then using this to calculate the force. This removes the effect of direct particle-particle interaction and makes the calculation faster. Although N-body simulations allow one to follow the evolution of the disturbances to the strongly non-linear regime they have their restrictions.

In any particular simulation there will be an upper length scale associated with the size of the part of the universe being simulated. At the other end of the spatial scales the limitation will come from the resolution of the grid being used. There will also be a limitation on the resolution in mass because the total mass of the universe will be associated with N representative particles.

The gravitational evolution is non-local and hence it is necessary to study the effects of these various spatial cut-offs discussed above. It is also necessary to study the effect of the limited mass resolution.

Also, when dealing with statistical quantities, one has to do an average over a sufficiently large number of realisations to obtain a good estimate of the actual quantity. N-body simulations require intensive computing and one may not have that many realisations available resulting in uncertainties in the values of statistical quantities. This is particularly true at scales comparable to the largest scale being considered in the simulation. In many cases it may be possible to complement the information obtained from N-body simulations with perturbative results. In addition the perturbative results can also be used to analyze and understand the results of N-body simulations. (For a discussion on N-body simulations the reader is referred to section 8.10 of Padmanabhan (1993) and references given there.)

There are many other models for the process of gravitational instability. These are mostly based on the results of linear theory, and unlike linear theory, they give results that can be physically interpreted even in the late stages of evolution. These models include the adhesion model (Gurbatov, Saichev and Shandarin 1989), the frozen flow approximation (Matarrese et. al. 1992) and the frozen potential approximation (Bagla & Padmanabhan 1994; Bernard, Sherrer and Villumsen 1993).

1.6 The scope of this thesis

In this thesis we address some of the issues involved in gravitational clustering in the weakly non-linear regime. This study is based on the BBGKY hierarchy. These equations govern the evolution of the ensemble averaged distribution functions. These are functions in phase space which have the statistical information about the disturbances. This hierarchy has an infinite sequence of equations, the first one being coupled to the second, the second to the third, etc. If one wants to deal with this hierarchy one usually has to use some assumptions so that one can deal with only a finite number of equations. Here we treat the BBGKY hierarchy perturbatively. This allows us to keep only a finite number of the equations in a self-consistent manner.

We use the velocity moments of the equations of the BBGKY hierarchy to perturbatively study the evolution of the two and three point correlation functions in a universe with $\Omega = 1$. We consider initial conditions where the density field is a random Gaussian variable and we also assume that the flow is initially single streamed. We first calculate, at the lowest order of non-linearity, the induced three point correlation function.

The BBGKY hierarchy is also valid in the highly non-linear regime. In this regime one has to assume some truncation scheme. Davis and Peebles (1977) have considered a particular truncation scheme to close the hierarchy. Their scheme is based on the assumption that the three point correlation function has the 'hierarchical' form where it can be expressed as a product of the two point correlation functions. One of the issues we investigate in this thesis is the validity of such an assumption where the three point correlation function is supposed to have a local dependence on the two point correlation function.

The evolution of the two point correlation function is influenced by the three point correlation function through the tidal force. We use the lowest order induced three point correlation function to calculate the lowest order non-linear correction to the two point correlation function. Although we initially have a single streamed flow it will become multi-streamed as the evolution proceeds. Unlike the other calculations, the equations we use are valid even in the multistreamed regime and our results would include effects of multi-streaming, if any, at the lowest order of non-linearity. We also investigate how the various scales affect each other because of the non-local nature of the terms. We investigate the evolution of the pair velocity in the weakly non-linear regime.

Hamilton et. al. (1991) have proposed that the process of gravitational instability in a $\Omega = 1$ universe has certain scaling properties. This is based on the results of N-body simulations. Nityananda and Padmanabhan (1994) have examined the possible origin of this scaling relation. These arguments are based on an interpolation between the evolution in the linear regime and in the stable clustering regime. Similar scaling relations have also been studied by Peacock and Dodds (1994) and Mo, Jain and White (1995). In this thesis we use the perturbative calculations to investigate whether the proposed scaling relations are valid in the weakly non-linear regime.

The Zel'dovich approximation (ZA) is a good model for gravitational dynamics in the weakly non-linear regime and it has the advantage that the equations for the evolution are much simpler compared to the full gravitational dynamics. This allows us to calculate the correlation functions at any order of perturbation without any great difficulty. In chapter VI of this thesis we use ZA to study the evolution of the two and three point correlation functions. We compare the results obtained using ZA to those obtained using perturbative gravitational dynamics and we use ZA to study the limitations of the perturbative approach itself.

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Chapter 2

Formalism.

2.1 The system and the evolution parameter.

Consider a system of a large number (N) of collisionless particles interacting through Newtonian gravity. The Lagrangian for such a system is

$$L = \frac{m}{2}\sum_a \left(\frac{dr^a}{dt}\right)^2 + \frac{m^2 G}{2}\sum_{a\neq b}\frac{1}{\mid r^a - r^b \mid}, \tag{2.1}$$

where r^a_μ refers to the 'μ' Cartesian component of the 'a' particle. When there is no subscript it refers to the vector r^a . We transform to a time dependent co-ordinate system with new co-ordinate 'x', with

$$r^a_\mu(t) = S(t)x^a_\mu(t)\ , \tag{2.2}$$

where $S(t)$ is a function of time. The Lagrangian becomes

$$L = \frac{mS^2}{2}\sum_a \left(\frac{dx^a}{dt}\right)^2 + \frac{m^2 G}{2S}\sum_{a\neq b}\frac{1}{\mid x^a - x^b \mid} - \frac{mS}{2}\frac{d^2 S}{dt^2}\sum_a (x^a)^2\ . \tag{2.3}$$

The extra potential can be interpreted as arising from the change to an accelerating co-ordinate system. The change to the expanding co-ordinate system also introduces a term which is a total time derivative of some quantity and can be dropped from the Lagrangian. The function S(t) (scale factor) is dimensionless and is chosen such that it satisfies the well known equation of Newtonian cosmology

$$\frac{d^2 S}{dt^2} = -\frac{4\pi G\rho}{3S^2} \tag{2.4}$$

where ρd^3x is the mass that would be in the volume d^3x if all the particles were uniformly distributed.

The Lagrangian then becomes

$$L = \frac{mS^2}{2}\sum_a \left(\frac{dx^a}{dt}\right)^2 + \frac{m^2 G}{2S}\sum_{a\neq b}\frac{1}{\mid x^a - x^b \mid} + \frac{2\pi G\rho m}{3S}\sum_a (x^a)^2\ . \tag{2.5}$$

If the particles are all uniformly distributed, the attractive force of gravity is exactly canceled by the repulsive harmonic oscillator force described by the last term in equation (2.5). In this case, if all the particles start with $\frac{dx^a_\mu}{dt} = 0$ then the solution is $x^a_\mu(t) = x^a_\mu(t_0)$ i.e. the co-ordinate system moves with the particles. Hence we see that 'x' is a comoving co-ordinate system and 'ρ' is the comoving density which remains a constant. In this case the system corresponds to a part of a homogeneous and isotropic universe where all the dynamical information is in $S(t)$. We now consider how a system with an initial configuration slightly different from the above mentioned one evolves.

For convenience time is replaced by a parameter λ ,where

$$d\lambda = \frac{dt}{S(t)^2}. \tag{2.6}$$

The use of this parameter instead of the cosmic time was first introduced by Doroshkevich et. al. (1980). The condition that the action should be invariant

$$A = \int L dt = \int L' d\lambda \tag{2.7}$$

defines the new Lagrangian

$$L' = \frac{m}{2}\sum_a\left(\frac{dx^a}{d\lambda}\right)^2 + \frac{Sm^2G}{2}\sum_{a\neq b}\frac{1}{|x^a - x^b|} + \frac{2\pi SG\rho m}{3}\sum_a (x^a)^2 . \tag{2.8}$$

For evolution in λ the Hamiltonian is .

$$H = \frac{1}{2m}\sum_a (p^a)^2 - \frac{Sm^2G}{2}\sum_{a\neq b}\frac{1}{|x^a - x^b|} - \frac{2\pi SG\rho m}{3}\sum_a (x^a)^2 , \tag{2.9}$$

where

$$p^a_\mu = m\frac{dx^a_\mu}{d\lambda} \tag{2.10}$$

is the momentum conjugate to x^a_μ . The main advantage of using λ instead of the cosmic time is that no $S(t)$ appears explicitly in (2.10). As a result the equation of motion for a particle, which is

$$\frac{d^2x^a_\mu}{d\lambda^2} = SGm\sum_b\frac{x^b_\mu - x^a_\mu}{|x^a - x^b|^3} + \frac{4}{3}\pi SG\rho x^a_\mu , \tag{2.11}$$

resembles the equation of motion of a particle in an inertial reference frame with a time dependent force. If instead we use cosmic time as the evolution parameter, a term with the product of the momentum and the derivative of the scale factor appears in the equation of motion. This term which looks like the frictional force has been avoided by using the parameter λ.

The relation between this momentum and the peculiar velocity is

$$p_\mu^a = mSv_\mu^a. \tag{2.12}$$

In terms of λ equation (2.4) becomes

$$\frac{d}{d\lambda}\left(\frac{1}{S^2}\frac{dS}{d\lambda}\right) = -\frac{4\pi G\rho}{3}. \tag{2.13}$$

Any solution of this equation is given by a parabola

$$\frac{1}{S} = \frac{2\pi G\rho}{3}\left[(\lambda+\lambda_1)^2 + K\right]. \tag{2.14}$$

Since the range of λ can be chosen arbitrarily, we set $\lambda_1 = 0$. There are three possibilities $K < 0$, $K = 0$ and $K > 0$. Figure 2.1 shows a plot of $\frac{1}{S(\lambda)}$ as a function of λ for the different cases. For the case with $K = 0$ we have

$$S(\lambda) = \frac{3}{2\pi G\rho\lambda^2} \tag{2.15}$$

with λ going from $-\infty$ to 0. In the case with $K < 0$ the allowed range for λ is $-\infty$ to $-\sqrt{K}$. In both these case we find that the universe keeps on expanding and $\frac{1}{S} \to 0$, or $S \to \infty$ as $\lambda \to -\sqrt{K}$. The case $K = 0$ corresponds to a universe with $\Omega = 1$ and $K < 1$ corresponds to a universe with $\Omega < 1$. For the case where $K > 0$ the allowed range of λ is $-\infty$ to ∞. We find that the universe expands to a maximum value of the scale factor.

$$S_{max} = \frac{3}{271\text{-}G\rho K} \tag{2.16}$$

when λ has the value zero and then starts to contract when λ goes over to the positive side. This corresponds to the case where $\Omega > 1$. In this thesis for all calculations we assume $\Omega = 1$ and equation (2.15) is used for $S(\lambda)$.

2.2 The BBGKY hierarchy and evolution of reduced distribution functions.

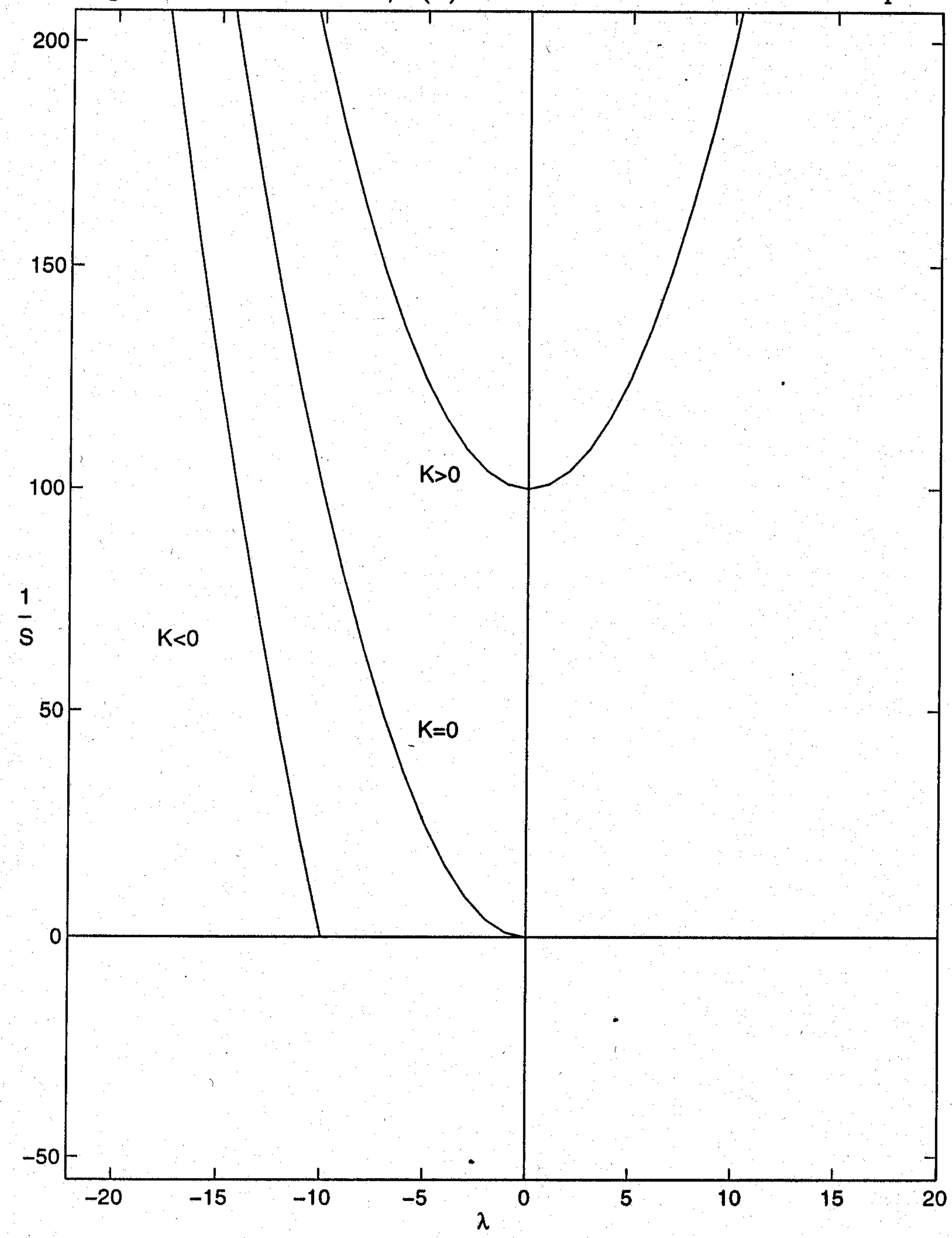
It is assumed that :

(1)There is a large spatial scale on which the universe is homogeneous and isotropic.

(2)Volumes of this size located at different parts of the universe are independent realisations of the same physical processes with different initial conditions. Such volumes can be 'assembled' to form an ensemble.

The system defined in the previous section is a model for one member of such an ensemble. We treat the system in the continuum (or fluid) limit where we ignore the direct residual two

Figure 2.1: This shows $1/S(\lambda)$ as a function of λ for the various possible cases.



body forces between the particles. This system can be described by a distribution function on phase space. This function $f(x,p,A)$ gives the humber of particles in the volume of the phase space $d^3x d^3p$ at the point (x,p) at the instant A. The evolution of this distribution function is governed by the Vlasov equation

$$\frac{\partial}{\partial\lambda}f(x,p,\lambda) + \frac{p_\mu}{m}\frac{\partial}{\partial x_\mu}f(x,p,\lambda) + \frac{4}{3}\pi SG\rho x_\mu \frac{\partial}{\partial p_\mu}f(x,p,\lambda)$$
$$+ SGm^2 \int f(x',p',\lambda)\frac{x'_\mu - x_\mu}{|x' - x|^3}\frac{\partial}{\partial p_\mu} f(x,p,\lambda) d^3x' d^3p' = 0 \qquad (2.17)$$

The ensemble described above can be used to define an ensemble averaged M point distribution function $\rho_M(x^1,p^1,x^2,p^2,..,x^M,p^M,A)$ defined as

$$\rho_M(x^1,p^1,x^2,p^2,..,x^M,p^M,\lambda) = < f\left(x^1,p^1,\lambda\right) ... f\left(x^M,p^M,\lambda\right) >, \qquad (2.18)$$

where the angular brackets indicate an average over all the systems in the ensemble.

This function gives the joint probability density of finding a particle in the volume $d^3x^1 d^3p^1$ at the point (x^1,p^1) and in the volume $d^3x^2 d^3p^2$ at the point (x^2,p^2) and in the volume $d^3x^2 d^3p^2$ at the point (x^3,p^3), etc. at the instant A. The evolution of this distribution functions is governed by the BBGKY hierarchy For a detailed discussion of this subject the reader is referred to Peebles (1980) and references given there.

In the fluid limit the equations of the BBGKY hierarchy can be easily derived using equation (2.17) and the definition of the distribution functions. Here we explicitly derive the first equation of the BBGKY hierarchy which governs the evolution of the ensemble averaged one point distribution function which is defined as

$$\rho_1(x,p,\lambda) = < f(x,p,\lambda) > . \qquad (2.19)$$

Differentiating this with respect to A and using equation (2.17) we have

$$\frac{\partial}{\partial A}\rho_1(x,p,A) = < \frac{\partial}{\partial\lambda}f(x,p,A) > = -\frac{p_\mu}{m}\frac{\partial}{\partial x_\mu} < f(x,p,\lambda) > \qquad (2.20)$$
$$- \frac{4}{3}\pi SG\rho x_\mu \frac{\partial}{\partial p_\mu} < f(x,p,\lambda) > - SGm^2 \int \frac{\partial}{\partial p_\mu} < f(x,p,\lambda) f(x',p',\lambda) > \frac{x'_\mu - x_\mu}{|x' - x|^3} d^3x' d^3p'$$

Using the numbers 1,2,3 etc. to denote points in phase space i.e. $(1,2,3)$ instead of $(x^1,p^1,x^2,p^2,x^2,p^2)$, and using the notation

$$X^{ab}_\mu = \frac{x^a_\mu - x^b_\mu}{|x^a - x^b|^3} \qquad (2.21)$$

we can write this equation as

$$\frac{\partial}{\partial\lambda}\rho_1(1,\lambda)+\frac{p^1_\mu}{m}\frac{\partial}{\partial x^1_\mu}\rho_1(1,\lambda)+Sm^2G\frac{\partial}{\partial p^1_\mu}\int\rho_2(1,2,\lambda)\,X^{21}_\mu d^3x^2d^3p^2$$

$$+\quad\frac{4}{3}\pi SG\rho m x^1_\mu\frac{\partial}{\partial p^1_\mu}\rho_1(1,\lambda)=0. \tag{2.22}$$

This is the first equation of the BBGKY hierarchy. At this stage we should point out that later on in the text we shall use the numbers $1,2,3$ etc to denote points in real space. The usage will be clear from the context.

We should note that the equation for the one point distribution function involves the two point distribution function. This is because the gravitational force is determined by the distribution of the rest of the matter. Similarly, the equation for the two point distribution function involves the three point distribution function and the equation for the three point distribution function involves the four point distribution function and so on. We next present the equation for the two point distribution function. The derivation of this equation is similar to that presented for equation (2.22). In this equation the index a is to be summed over the values 1 and 2.

$$\frac{\partial}{\partial\lambda}\rho_2(1,2,\lambda)+\frac{p^a_\mu}{m}\frac{\partial}{\partial x^a_\mu}\rho_2(1,2,\lambda)+\frac{4}{3}\pi SG\rho m x^a_\mu\frac{\partial}{\partial p^a_\mu}\rho_2(1,2,\lambda)$$

$$+\quad Sm^2G\frac{\partial}{\partial p^a_\mu}\int\rho_3(1,2,3,\lambda)\,X^{3a}_\mu d^3x^3d^3p^3=0. \tag{2.23}$$

The following equation is the third equation of the BBGKY hierarchy. In this equation the index a is to be summed over the values 1, 2 and 3.

$$\frac{\partial}{\partial\lambda}\rho_3(1,2,3,\lambda)+\frac{p^a_\mu}{m}\frac{\partial}{\partial x^a_\mu}\rho_3(1,2,3,\lambda)+\frac{4}{3}\pi SG\rho m x^a_\mu\frac{\partial}{\partial p^a_\mu}\rho_3(1,2,3,\lambda)$$

$$+\quad Sm^2G\frac{\partial}{\partial p^a_\mu}\int\rho_4(1,2,3,4.\lambda)\,X^{4a}_\mu d^3x^4d^3p^4=0,. \tag{2.24}$$

This hierarchy continues and in the fluid limit we have an infinite hierarchy of equations..

The condition that the disturbances are statistically homogeneous and isotropic allows us to simplify the functional form of the distribution functions. For example, homogeneiety implies that the ensemble averaged one point distribution function cannot have any spatial dependence and isotropy implies that it cannot depend on the direction of the momentum vector i.e. $\rho_1(x,p,\lambda)=\rho(|p|,\lambda)$. Henceforth we shall use $f(p,\lambda)=\rho_1(|p|,\lambda)$ for the ensemble averaged one point distribution function.

For special initial conditions it is possible to truncate the hierarchy at some level, the error from the terms dropped being of a higher order in some small parameter compared to the terms retained. In order to do this explicitly it is convenient to work in terms of the reduced distribution functions defined below.

The probability density for finding a particle at x^1 with momentum p^1 and another with position x^2 and momentum p^2 has a contribution from the one point distribution function. This is $f(1)f(2)$. The reduced two point distribution function is defined by the equation

$$\rho_2(1,2) = f(1)f(2) + c(1,2) . \tag{2.25}$$

The reduced three and four point distribution functions are similarly defined by

$$\rho_3(1,2,3) = f(1)f(2)f(3) + \sum_P f(1)c(2,3) + d(1,2,3) , \tag{2.26}$$

$$\rho_4(1,2,3,4) = f(1)f(2)f(3)f(4) + \sum_P f(1)f(2)c(3,4)$$

$$+ \sum_P c(1,2)c(3,4) + \sum_P f(1)d(2,3,4) + e(1,2,3,4) , \tag{2.27}$$

where $\sum_P$ means a sum over all cyclic permutations of the position indices. Equations (2.22),(2.23) and (2.24) and the definition of the reduced distribution'functions can be combined to obtain equations for the evolution of the reduced distribution functions. Written in terms of the reduced distribution functions, the first equation of the BBGKY hierarchy is

$$\frac{\partial}{\partial\lambda} f(1,\lambda) + Sm^2G\frac{\partial}{\partial p^1_\mu} \int c(1,2,\lambda) X^{21}_\mu d^3x^2 d^3p^2 = 0 . \tag{2.28}$$

It should be noted that the part of the force which arises because of the transformation to a time dependent co-ordinate system and has the form of an inverted simple harmonic oscillator is canceled by the force due to the reducible part of the two point distribution function $f(p^1)f(p^2)$. This cancellation is expected, and in the case when the universe is absolutely uniform and all the particles move with the Hubble flow, this cancellation leads to the result that the ensemble averaged one point distribution function remains unchanged as the evolution proceeds. A similar cancellation occurs in the equations for the higher distribution functions too.

We next consider the equation for the reduced two point distribution function. This depends on a pair of points (e.g.,. 1 and 2) and in the equation below the index a refers to any one of these points. For a fixed vale of a the index a' refers to the other member of the pair (e.g. for $a = 1, a' = 2$). Also the index a is to be summed over the allowed values whenever it appears twice. Using this notation, we have for the reduced two point distribution function

$$\frac{\partial}{\partial\lambda}c(1,2,\lambda)+\frac{p^a_\mu}{m}\frac{\partial}{\partial x^a_\mu}c(1,2,\lambda)$$

$$+\ Sm^2G\frac{\partial}{\partial p^a_\mu}f(a)\int c\left(a',3,\lambda\right)X^{3a}_\mu d^3x^3d^3p^3$$

$$+\ Sm^2G\frac{\partial}{\partial p^a_\mu}\int d(1,2,3,\lambda)\,X^{3a}_\mu d^3x^3d^3p^3=0. \qquad (2.29)$$

Similarly, the three point distribution function depends on three points **i.e.** the vertices of a triangle which may be labeled *1,2* and *3.* We use the index a to denote any one of the vertices of the triangle and we use the index a'' to refer to both the other vertices simultaneously (e.g.,. if $a=1, a''=(2,3)$). We also use the indices a_1' and a_2' to individually refer to the other two vertices for a fixed value of a (e.g.,. for $a=1$ we have $a_1'=2, a_2'=3$ and $a_1'=3, a_2'=2$). The possible set of values for the indices a' and a'' are shown in the table below.

a	1	2	3
a''	2,3	3,1	1,2

The various values over which the symbols a, a_1' and a_2' are to be summed over whenever they appear together are shown below.

a	1	1	2	2	3	3	
a_1'	2	3	3	1	1	2	
a	6	3	2	1	3	2	1

Using this notation we have for the reduced three point distribution function

$$\frac{\partial}{\partial\lambda}d(1,2,3,\lambda)+\frac{p^a_\mu}{m}\frac{\partial}{\partial x^a_\mu}d(1,2,3,\lambda)$$

$$+\ Sm^2G\frac{\partial}{\partial p^a_\mu}f(a)\int d\left(a'',4,\lambda\right)X^{4a}_\mu d^3x^4d^3p^4$$

$$+\ Sm^2G\frac{\partial}{\partial p^a_\mu}c\left(a,a_1'\right)\int c\left(a;,4,\lambda\right)X^{4a}_\mu d^3x^4d^3p^4$$

$$+ \quad Sm^2 G \frac{\partial}{\partial p^a_\mu} \int e(1,2,3,4,\lambda)\, X^{4a}_\mu d^3x^4 d^3p^4 = 0 \,. \tag{2.30}$$

2.3 Initial Conditions.

We next specify the initial conditions which we are going to evolve using these equations. These initial conditions have to be specified for the ensemble averaged statistical quantities and they have to be such that we can have some meaningful evolution using only a few equations of the whole hierarchy.

We choose initial conditions such that in any member of the ensemble the deviation of the particles from the uniform distribution is small. The fractional density perturbation at any point is of order e (a small number). The deviations from the Hubble flow are also of this order. It is also assumed that initially in any member of the ensemble all the particles at any one point have the same velocity. In other words, initially all the members of the ensemble have a single streamed flow. When we average over the whole ensemble we have different velocities from the different members of the ensemble contributing at the same point. Thus initially, the velocity dispersion at a fixed point or a fixed separation will arise due to the spread in velocities across the various members of the ensemble and there will be no contribution due to the spread in velocities at a point in a particular member of the ensemble. As the evolution proceeds multi-streaming will occur in the different members of the ensemble. In the multistreamed regime the velocity dispersion will have a contribution from both the spread in velocities amongst the various members of the ensemble as well as the spread in velocities at a point in any member of the ensemble itself.

We should also point out that because the disturbances being considered are statistically homogeneous (i.e. there is no preferred origin) we can replace the ensemble average by an average over the whole of space.

Using these assumptions we can estimate orders of magnitude for the initial values of various moments of the distribution functions as powers of' e. We give this for some of the moments we encounter later. These initial conditions correspond to a situation where the linear theory of density perturbations (Peebles 1980) can be applied. In the first equation n is the mean number density of particles per unit comoving volume.

$$\int f(1)\, d^3p^1 = n \; (nm = \rho) \,, \tag{2.31}$$

From isotropy we have

$$\int p^1_\mu f(1)\, d^3p^1 = 0 \,, \tag{2.32}$$

This is true for all odd velocity moments of the one point distribution function. The second velocity moment of the one point distribution function corresponds to the velocity dispersion (across the ensemble) at a point. We define various moments, displayed using angular brackets, as follows

$$\int \left(p_\mu^1\right)^2 f(1)\, d^3p^1 = n < \left(p_\mu^1\right)^2 >_1 \sim \epsilon^a . \tag{2.33}$$

For the moments of the two point distribution function we have the definitions

$$\int c(1,2)\, d^3p^1 d^3p^2 \;=\; n^2 \xi\left(x^1, x^2\right) \sim \epsilon^2 , \tag{2.34}$$

$$\int p_\mu^i c(1,2)\, d^3p^1 d^3p^2 \;=\; n^2 < p_\mu^i >_2 \left(x^1, x^2\right) \sim \epsilon^2 , \tag{2.35}$$

$$\int p_\mu^i p_\nu^j c(1,2)\, d^3p^1 d^3p^2 \;=\; n^2 < p_\mu^i p_\nu^j >_2 \left(x^1, x^2\right) \sim \epsilon^2 , \tag{2.36}$$

where i and j take values 1 and, 2 and i$\neq$j and ξ is the two point correlation function. All other moments of 'c' are of higher order in ϵ. It should be noted that we use the subscript below the angular bracket to denote the distribution function whose moment is being referred to (e.g.,. $< p_\mu^a >_2$ is the first moment of the two point distribution function).

We also present similar expressions defining our notations for some of the moments of the three point distribution function

$$\int d(1,2,3)\, d^3p^1 d^3p^2 d^3p^3 \;=\; n^3 \zeta\left(x^1, x^2, x^3\right) \sim \epsilon^3 \tag{2.37}$$

$$\int p_\mu^i d(1,2,3)\, d^3p^1 d^3p^2 d^3p^3 \;=\; n^3 < p_\mu^i >_3 \left(x^1, x^2, x^3\right) \sim \epsilon^3 , \tag{2.38}$$

$$\int p_\mu^i p_\nu^j d(1,2,3)\, d^3p^1 d^3p^2 d^3p^3 \;=\; n^3 < p_\mu^i p_\nu^j >_3 \left(x^1, x^2, x^3\right) \sim \epsilon^3 , \tag{2.39}$$

$$\int p_\mu^i p_\nu^j p_\sigma^k d(1,2,3)\, d^3p^1 d^3p^2 d^3p^3 \;=\; n^3 < p_\mu^i p_\nu^j p_\sigma^k >_3 \left(x^1, x^2, x^3\right) \sim \epsilon^3 , \tag{2.40}$$

where i, j and k run over the values 1,2,3 and i$\neq$j$\neq$k. All other moments of 'd', the reduced three point distribution function, are of higher order in a. All moments of 'e', the reduced four point distribution function, are also of higher order in a. We also use ζ to denote the three point correlation function, and χ to denote the four point correlation function.

The initial conditions are all specified at some instant λ_0.

2.4 Perturbative evolution and linear theory.

We now want to see how the various moments of the reduced distribution functions evolve from the given initial conditions. We treat the problem perturbatively by initially keeping terms only up to the lowest order in ϵ and solving the equations and then putting in the contribution from the higher order terms as corrections.

We first deal with the reduced two point distribution function c. We proceed by taking velocity moments of the evolution equation for c. The zeroth moment of equation *(2.29)* is

$$\frac{\partial}{\partial\lambda}\xi(1,2,\lambda)+\frac{1}{m}\frac{\partial}{\partial x_\mu^a}<p_\mu^a>_2(1,2,\lambda)=0\,. \qquad (2.41)$$

This equation relates the evolution of the two point correlation function to the divergence of the first moment of 'c'. This equation does not contain Newton's constant and is purely kinematical and it is the continuity equation for pairs. Using $x, = x_\mu^2 - x_\mu^1$ and

$$j_\mu(x,\lambda)=\frac{1}{m}(<p_\mu^2>_2-<p_\mu^1>_1) \qquad (2.42)$$

this equation may be written as

$$\frac{\partial}{\partial\lambda}\xi(x,\lambda)+\partial_\mu j_\mu=0\,. \qquad (2.43)$$

The quantity $j_\mu(x, A)$ is the pair current at the separation x and epoch A, and its divergence gives the rate of change of the two point correlation function at that separation. Another way of interpreting this equation is to average it over a sphere of comoving radius r denoted by $V(r)$. This then gives us

$$\int_{V(r)}\frac{\partial}{\partial\lambda}\xi(x,\lambda)d^3x=-\int_{V(r)}\partial_\mu j_\mu d^3x\,. \qquad (2.44)$$

The second integral can be converted to a surface integral. Also from isotropy we know that both ξ and $j,$ are spherically symmetric functions. We then have

$$j_\mu(r,\lambda)4\pi r^2=-\frac{\partial}{aX}4\pi\int_0^r\xi(x,\lambda)x^2dx\,. \qquad (2.45)$$

which may also be written as

$$j_\mu(x,\lambda)=-\frac{x_\mu}{3}\frac{\partial}{\partial\lambda}\bar{\xi}(x,\lambda) \qquad (2.46)$$

where $\bar{\xi}(x, A)$ is the two point correlation function averaged over a sphere of radius x

$$\bar{\xi}(x,\lambda)=\frac{3}{x^3}\int_0^x\xi(y,\lambda)y^2dy\,. \qquad (2.47)$$

These are various ways of writing the same pair continuity equation. This equation has two unknown functions both of order ϵ^2 and therefore we cannot ignore any of them. We cannot solve this equation either. We look at the evolution of the first moment which is given by the first moment of equation *(2.29)*

$$\frac{\partial}{\partial\lambda} < p_\mu^a >_2 + \frac{1}{m}\frac{\partial}{\partial x_\nu^b} < p_\mu^a p_\nu^b >_2 (1,2,\lambda) - SmG\rho \int \xi \left(a',3\right) X_\mu^{3a} d^3x^3$$

$$- \quad SmG\rho \int \zeta(1,2,3,\lambda) X_\mu^{3a} d^3x^3 = 0 . \tag{2.48}$$

The last two terms have been obtained by doing the 'p' integral by parts and dropping the surface term. This will be done in the equations for all the other moments qlso. The second term in the above equation is due to the streaming of particles. The effect of the gravitational force is in the last two terms. If we are calculating the force at position 1, the matter distribution causing the force may be correlated with the position of the second point in the two point correlation function i.e. 2. The force term containing the two point correlation function arises because of this effect. The matter distribution causing the force may be correlated to the to particles at both the positions 1 and 2 and this effect is in the term containing the three point correlation function. The equations for the higher moments of the two point distribution function have a similar structure.

We next take the divergence of the above equation , and differentiate the continuity equation with respect to λ, and combine the two to get an equation involving the two point correlation function and the second moment of 'c'

$$\frac{\partial^2}{\partial\lambda^2}\xi - \frac{1}{m^2}\frac{\partial^2}{\partial x_\nu^b \partial x_\mu^a} < p_\mu^a p_\nu^b >_2 - 8\pi SG\rho\xi = -f_1 , \tag{2.49}$$

where,

$$f_1(1,2,\lambda) = SG\rho \frac{\partial}{\partial x_\mu^a} \int \zeta(1,2,3,\lambda) X_\mu^{3a} d^3x^3 . \tag{2.50}$$

This equation still has two unknown functions of order ϵ^2. We take the evolution equation for the second moment of 'c'. This equation is

$$\begin{aligned}
\frac{\partial}{\partial\lambda} &< p_\mu^a p_\nu^b >_2 (1,2,\lambda) + \frac{1}{m}\frac{\partial}{\partial x_\sigma^c} < p_\mu^a p_\nu^b p_\sigma^c >_2 (1,2,\lambda) \\
&- \frac{m^2 GS}{n^2} \int \left(\delta_{\sigma\mu}^{ca} p_\nu^b + \delta_{\sigma\nu}^{cb} p_\mu^a\right) f(c)\, c\left(c',3\right) X_\sigma^{3c} d^3x^3 d^9p \\
&- \frac{m^2 GS}{n^2} \int \left(\delta_{\sigma\mu}^{ca} p_\nu^b + \delta_{\sigma\nu}^{cb} p_\mu^a\right) d(1,2,3) X_\sigma^{3c} d^3x^3 d^9p = 0 .
\end{aligned} \tag{2.51}$$

Taking divergence with respect to both the free indices we have

$$\frac{\partial^3}{\partial\lambda \partial x_\nu^b \partial x_\mu^a} < p_\mu^a p_\nu^b >_2 \quad + \quad \frac{1}{m}\frac{\partial^3}{\partial x_\sigma^c \partial x_\nu^b \partial x_\mu^a} < p_\mu^a p_\nu^b p_\sigma^c >_2$$

$$+ \quad 8\pi SmG\rho \frac{\partial}{\partial x_\mu^a} < p_\mu^a >_2 = m^2 f_2 \,, \qquad (2.52)$$

where,

$$f_2(1,2,\lambda) = 2SGn \frac{\partial^2}{\partial x_\nu^b \partial x_\mu^a} \int < p_\mu^a >_3 (1,2,3,\lambda) X_\nu^{3b} d^3x^3 \,. \qquad (2.53)$$

Differentiating equation (2.49) with respect to A and using equations (2.52) and (2.41) we have the equation for the two point correlation function

$$\frac{\partial^3}{\partial\lambda^3}\xi - 8\pi G\rho \left[S\frac{\partial}{\partial\lambda}\xi + \frac{\partial}{\partial\lambda}(S\xi) \right] = f_2 - f_3 - \frac{\partial}{\partial\lambda} f_1 \,, \qquad (2.54)$$

where,

$$f_3(1,2,\lambda) = \frac{1}{m^3} \frac{\partial^3}{\partial x_\sigma^c \partial x_\nu^b \partial x_\mu^a} < p_\mu^a p_\nu^b p_\sigma^c >_2 (1,2,3,\lambda) \,. \qquad (2.55)$$

In this equation the only unknown function of order ϵ^2 is the two point correlation function ξ. The functions **i.e.** f_1, f_2 and f_3 are of higher order in ϵ. Initially we neglect terms of higher order in ϵ and deal with an equation correct to order ϵ^2 only.

As the system evolves the higher order terms become important and they have to be considered. They can be thought of as giving rise to corrections to the lowest order solution.

Keeping terms of order ϵ^2 only equation (2.54) becomes

$$\frac{\partial^3}{\partial\lambda^3}\xi - 8\pi G\rho \left[S\frac{\partial}{\partial\lambda}\xi + \frac{\partial}{\partial\lambda}(S\xi) \right] = 0 \,. \qquad (2.56)$$

This is a third order differential equation for the two point correlation function.

For an $\Omega = 1$ universe this is

$$\frac{\partial^3}{\partial\lambda^3}\xi - \frac{24}{\lambda^2}\frac{\partial}{\partial\lambda}\xi + \frac{24}{\lambda^3}\xi = 0 \,, \qquad (2.57)$$

which has solutions of the form

$$\xi(1,2,\lambda) = \left(\frac{\lambda}{\lambda_0}\right)^{-4} F_1 + \left(\frac{\lambda}{\lambda_0}\right) F_2 + \left(\frac{\lambda}{\lambda_0}\right)^6 F_3 \,, \qquad (2.58)$$

where F_1, F_2 and F_3 are functions of x^1 and x^2. The two point correlation function at λ_0 is expressed in terms of these functions which have to be given as initial conditions. We have three initial conditions because we have a third order differential equation. Instead of these three functions one could have given the two point **correlation** function and the first two moments of 'c' at λ_0 as initial conditions.

One can derive the same result by evolving one realisation of the ensemble using the linear perturbation **theory** and then calculating the correlation function **.** The growing mode for density perturbations, usually denoted by $D_1(t)$, grows proportional to the scale factor and the decaying mode ,usually denoted by $D_2(t)$ is proportional to t^{-1}. In terms of λ this is λ^3. The three modes of growth for the two point correlation function correspond to D_1^2, D_1D_2 and D_2^2 (Peebles 1980), which is also what we get above.

If the two point correlation function starts as a mixture of the three modes, after some time it will be dominated by the growing mode $D_1^2(\lambda)$. For most purposes it suffices to just keep this mode. If we consider a situation where only the growing mode is present we can introduce a potential $\phi(x^1, x^2)$. All the quantities of interest can, to order ϵ^2, be expressed in term of this potential.

$$\phi\left(x^1, x^2\right) = \phi\left(x^1 - x^2\right), \tag{2.59}$$

$$\xi\left(x^1, x^2, \lambda\right) = \frac{1}{2}\frac{\lambda_0^5}{\lambda^4}\nabla^4\phi\left(x^1 - x^2\right), \tag{2.60}$$

$$< p_\mu^a >_2 \left(x^1, x^2, \lambda\right) = m\frac{\lambda_0^5}{\lambda^5}\frac{\partial}{\partial x_\mu^a}\nabla^2\phi\left(x^1 - x^2\right), \tag{2.61}$$

$$< p_\mu^a p_\nu^b >_2 \left(x^1, x^2, \lambda\right) = 2m^2\frac{\lambda_0^5}{\lambda^6}\frac{\partial^2}{\partial x_\mu^a x_\nu^b}\phi\left(x^1 - x^2\right). \tag{2.62}$$

In the above equations the ∇^2 is with respect to either x^1 or x^2, and, a, b = 1, 2 with $a \neq b$. It can be checked that the above relations are consistent with all of the moment equations (i.e. 2.41,2.48 and 2.52) of the two point distribution function at order ϵ^2.

The potential ϕ is proportional to the correlation of the gravitational potential at the two points x^1 and x^2 and has dimensions L^4T^{-1}. If the other modes are present one can introduce potentials for them too. This is not considered here.

A similar treatment can also be done for (2.30), the third equation of the hierarchy. This equation governs the three point distribution function. We follow a sequence of operations very similar to that described above for the two point distribution function. We do this below to obtain an equation for perturbatively evolving the three point correlation function.

The zeroth moment of equation (2.30) is

$$\frac{\partial}{\partial\lambda}\zeta(1,2,3,\lambda) + \frac{1}{m}\frac{\partial}{\partial x_\mu^a} < p_\mu^a >_3 (1,2,3,\lambda) = 0. \tag{2.63}$$

This is the triplet continuity equation and it is similar to the pair continuity equation. The first moment of equation (2.30) is

$$\frac{\partial}{\partial\lambda} < p_\mu^a >_3 + \frac{1}{m}\frac{\partial}{\partial x_\nu^b} < p_\mu^a p_\nu^b >_3 (1,2,3,\lambda)$$

$$- \; SmG\rho \int \zeta\,(a'',4)\, X_\mu^{4a} d^3x^4 - SmG\rho \int \chi(1,2,3,4)\, X_\mu^{4a} d^3x^4$$

$$- \; SmG\rho\,\xi\,(a,a_1') \int \xi\,(a_1',4)\, X_\mu^{4a} d^3x^4 = 0\,. \qquad (2.64)$$

Taking divergence of equation *(2.64)* **we have**

$$\frac{\partial^2}{\partial\lambda\partial x_\mu^a} < p_\mu^a >_3 + \frac{1}{m}\frac{\partial^2}{\partial x_\nu^b \partial x_\mu^a} < p_\mu^a p_\nu^b >_3 + 12\pi SmG\rho\zeta = m\,(f_4 + f_5)\,, \qquad (2.65)$$

where

$$f_4(1,2,3,\lambda) = SG\rho\frac{\partial}{\partial x_\mu^a}\left(\xi\,(a,a_1',\lambda)\int \xi\left(a_2',4,\lambda\right) X_\mu^{4a} d^3x^4\right) \qquad (2.66)$$

and

$$f_5(1,2,3,\lambda) = SG\rho\frac{\partial}{\partial x_\mu^a}\int \chi(1,2,3,,4,\lambda)\, X_\mu^{4a} d^3x^4\,. \qquad (2.67)$$

Differentiating equation *(2.63)* **with** λ **and using equation** *(2.65)* **we have**

$$\frac{\partial^2}{\partial\lambda^2}\zeta - \frac{1}{m^2}\frac{\partial^2}{\partial x_\nu^b \partial x_\mu^a} < p_\mu^a p_\nu^b >_3 - 12S\pi G\rho\zeta = -\,(f_4 + f_5)\,. \qquad (2.68)$$

The second moment of equation *(2.30)* **is**

$$\frac{\partial}{\partial\lambda} < p_\mu^a p_\nu^b >_3 + \frac{1}{m}\frac{\partial}{\partial x_\sigma^c} < p_\mu^a p_\nu^b p_\sigma^c >_3 (1,2,3,\lambda)$$

$$- \; \frac{Sm^2G}{n^3}\int \left(\delta_{\sigma\mu}^{ca} p_\nu^b + \delta_{\sigma\nu}^{cb} p_\mu^a\right) f(c)\, d\left(c'',3\right) X x_\sigma^{4c} d^3x^4 d^{12}p$$

$$- \; \frac{Sm^2G}{n^3}\int \left(\delta_{\sigma\mu}^{ca} p_\nu^b + \delta_{\sigma\nu}^{cb} p_\mu^a\right) c\left(c,c_1'\right) c\left(c_2',4\right) X_\sigma^{4c} d^3x^4 d^{12}p$$

$$- \; \frac{Sm^2G}{n^3}\int \left(\delta_{\sigma\mu}^{ca} p_\nu^b + \delta_{\sigma\nu}^{cb} p_\mu^a\right) e(1,2,3,4)\, X_\sigma^{4c} d^3x^4 d^{12}p = 0\,. \qquad (2.69)$$

Taking divergence with respect to both the free indices we have

$$\frac{\partial^3}{\partial\lambda\partial x_\nu^b \partial x_\mu^a} < p_\mu^a p_\nu^b >_3 \; + \; \frac{1}{m}\frac{\partial^3}{\partial x_\sigma^c \partial x_\nu^b \partial x_\mu^a} < p_\mu^a p_\nu^b p_\sigma^c >_3$$

$$+ \; 16\pi SmG\rho\frac{\partial}{\partial x_\mu^a} < p_\mu^a >_3 = m^2\,(f_6 + f_7)\,, \qquad (2.70)$$

where,

$$f_6(1,2,3,\lambda) = \frac{2SG}{n^3}\frac{\partial^2}{\partial x_\nu^b \partial x_\mu^a}\int p_\nu^b c\left(a,a_1'\right) c\left(a_2',4\right) X_\mu^{4a} d^3x^4 d^{12}p \tag{2.71}$$

and

$$f_7(1,2,3,A) = 2SGn\frac{\partial^2}{\partial x_\nu^b \partial x_\mu^a}\int < p_\nu^b >_4 (1,2,3,4,\lambda)\, X_\mu^{4a} d^3x^4 . \tag{2.72}$$

Using equation *(2.63)* this becomes

$$\frac{\partial^3}{\partial\lambda\partial x_\nu^b \partial x_\mu^a} < p_\mu^a p_\nu^b >_3 + \frac{1}{m}\frac{\partial^3}{\partial x_\sigma^c \partial x_\nu^b \partial x_\mu^a} < \bar{p}_\mu^a \bar{p}_\nu^b \bar{p}_\sigma^c >_3$$

$$- 16\pi Sm^2 G\rho\frac{\partial}{\partial\lambda}\dot{\zeta} = m^2 (f_6 + f_7) , \tag{2.73}$$

which when combined with equation *(2.68)* gives

$$\frac{\partial^3}{\partial\lambda^3}\zeta + \frac{1}{m^3}\frac{\partial^3}{\partial x_\sigma^c \partial x_\nu^b \partial x_\mu^a} < \bar{p}_\mu^a \bar{p}_\nu^b \bar{p}_\sigma^c >_3 - \pi G\rho\left[16S\frac{\partial}{\partial\lambda}\zeta + 12\frac{\partial}{\partial\lambda}(S\zeta)\right]$$

$$= (f_6 + f_7) - \frac{\partial}{\partial\lambda}(f_4 + f_5) \cdot , \tag{2.74}$$

The third moment of equation *(2.30)* is

$$\frac{\partial}{\partial\lambda} < p_\mu^a p_\nu^b p_\sigma^c >_3 + \frac{1}{m}\frac{\partial}{\partial x_\gamma^d} < p_\mu^a p_\nu^b p_\sigma^c x_\gamma^d >_3 (1,2,3,\lambda) \tag{2.75}$$

$$- \frac{Sm^2G}{n^3}\int\left(\delta_{\gamma\mu}^{ea} p_\nu^b p_\sigma^c + \delta_{\gamma\nu}^{eb} p_\mu^a p_\sigma^c + \delta_{\gamma\sigma}^{ec} p_\mu^a p_\nu^b\right) f(e)\, d\left(e'',4\right) X x_\gamma^{4e} d^3x^4 d^{12}p$$

$$- \frac{Sm^2G}{n^3}\int\left(\delta_{\gamma\mu}^{ea} p_\nu^b p_\sigma^c + \delta_{\gamma\nu}^{eb} p_\mu^a p_\sigma^c + \delta_{\gamma\sigma}^{ec} p_\mu^a p_\nu^b\right) c\left(e,e_1'\right) c\left(e_2',4\right) X_\gamma^{4e} d^3x^4 d^{12}p$$

$$- \frac{Sm^2G}{n^3}\int\left(\delta_{\gamma\mu}^{ea} p_\nu^b p_\sigma^c + \delta_{\gamma\nu}^{eb} p_\mu^a p_\sigma^c + \delta_{\gamma\sigma}^{ec} p_\mu^a p_\nu^b\right) e(1,2,3,4)\, X_\gamma^{4e} d^3x^4 d^{12}p = 0 .$$

Taking divergence with all the three position co-ordinates we have

$$\frac{\partial^4}{\partial\lambda\partial x_\sigma^c \partial x_\nu^b \partial x_\mu^a} < \bar{p}_\mu^a \bar{p}_\nu^b \bar{p}_\sigma^c >_3 + \frac{1}{m}\frac{\partial^4}{\partial x_\gamma^d \partial x_\sigma^c \partial x_\nu^b \partial x_\mu^a} < \bar{p}_\mu^a \bar{p}_\nu^b \bar{p}_\sigma^c \bar{x}_\gamma^d >_3$$

$$+ 12\pi SmG\rho\frac{\partial^2}{\partial x_\mu^a x_\nu^b} < p_\mu^a p_\nu^b >_3 = m^3 (f_8 + f_9 - f_{10} - f_{11}) , \tag{2.76}$$

where,

$$f_8(1,2,3,\lambda) = \frac{3SG}{n^3 m}\frac{\partial^3}{\partial x^c_\sigma \partial x^b_\nu \partial x^a_\mu}\int p^b_\nu p^a_\mu c\left(c, c_1'\right) c\left(c_2', 4\right) X^{4c}_\sigma d^3x^4 d^{12}p\,, \quad (2.77)$$

$$f_9(1,2,3,\lambda) = 3SG\frac{n}{m}\frac{\partial^3}{\partial x^c_\sigma \partial x^b_\nu \partial x^a_\mu}\int < p^a_\mu p^b_\nu >_4 X^a_{4c} d_3 x_4\,, \quad (2.78)$$

$$fio(1,2,3,\lambda) = \frac{12\pi SG\rho}{m^2}\frac{\partial^2}{\partial x^a_\mu \partial x^a_\nu} < p^a_\mu p^a_\nu >_3 (1,2,3,\lambda)\,, \quad (2.79)$$

and

$$f_{11}(1,2,3,\lambda) = \frac{12\pi SG\rho}{m^2} < p^a_\mu p^a_\nu >_1 \frac{\partial^2}{\partial x^a_\mu \partial x^a_\nu}\zeta(1,2,3,\lambda)\,. \quad (2.80)$$

Using equation *(2.68)* this becomes

$$\frac{\partial^4}{\partial\lambda \partial x^c_\sigma \partial x^b_\nu \partial x^a_\mu} < p^a_\mu p^b_\nu p^c_\sigma >_3 + \frac{1}{m}\frac{\partial}{\partial x^d_\gamma \partial x^c_\sigma \partial x^b_\nu \partial x^a_\mu} < p^a_\mu p^b_\nu p^e_\sigma x^d_\gamma >_3 \quad (2.81)$$

$$+ \quad 12\pi S m^3 G\rho\left[\frac{\partial^2}{\partial\lambda^2}\zeta - 12S\pi G\rho\zeta + (f_4 + f_5)\right] = m^3(f_8 + f_9 - f_{10} - f_{11})\,.$$

Differentiating equation *(2.74)* with λ and using equation *(2.81)* we obtain the equation for the three point correlation function

$$\frac{\partial^4}{\partial\lambda^4}\zeta - 40\pi G\rho S\frac{\partial^2}{\partial\lambda^2}\zeta - 40\pi G\rho\frac{dS}{d\lambda}\frac{\partial}{\partial\lambda}\zeta - 12\pi G\rho\left[\frac{d^2S}{d\lambda^2} - 12\pi G\rho S^2\right]\zeta$$

$$= \quad f_{12} + f_{11} + f_{10} - f_9 - f_8 + \frac{\partial}{\partial\lambda}(f_7 + f_6) + \left[12\pi G\rho S - \frac{\partial^2}{\partial\lambda^2}\right](f_5 + f_4)\,, \quad (2.82)$$

where,

$$f_{12}(1,2,3,\lambda) = \frac{1}{m^4}\frac{a}{\partial x^d_\gamma \partial x^c_q \partial x^b_\nu \partial x^a_\mu} < p^a_\mu p^b_\nu p^c_\sigma x^d_\gamma >_3\,. \quad (2.83)$$

The functions f_4 to f_{12} are of order ϵ^4 or higher. To order ϵ^3 we have a **fourth** order differential equation for the three point correlation function

$$\frac{\partial^4}{\partial\lambda^4}\zeta \quad - \quad 40\pi G\rho S\frac{\partial^2}{\partial\lambda^2}\zeta - 40\pi G\rho\left(\frac{dS}{d\lambda}\right)\frac{\partial}{\partial\lambda}\zeta$$

$$- \quad 12\pi G\rho\left[\frac{d^2S}{d\lambda^2} - 12\pi G\rho S^2\right]\zeta = 0\,, \quad (2.84)$$

which for an $\Omega = 1$ universe becomes

$$\frac{\partial^4}{\partial\lambda^4}\zeta - \frac{60}{\lambda^2}\frac{\partial^2}{\partial\lambda^2}\zeta + \frac{120}{\lambda^3}\frac{\partial}{\partial\lambda}\zeta + \frac{216}{\lambda^4}\zeta = 0\,. \quad (2.85)$$

The solution of this equation can be written as

$$\zeta(1,2,3,\lambda) = \lambda^{-6}F_4 + \lambda^{-1}F_5 + \lambda^4 F_6 + \lambda^9 F_7 \tag{2.86}$$

where F_4, F_5, F_6, F_7 are functions of x^1, x^2 and x^3.

Thus we have obtained four modes $D_1^3, D_1^2 D_2, D_1 D_2^2$ and D_2^3 for the evolution of the three point correlation function. This is as expected and it corresponds to what we would have got if we had used the linear theory of density perturbations to evolve some initial density perturbations and then calculated the three point correlation function and compared it to the initial three point correlation function of the density field. One could do a similar treatment for the higher correlation functions too.

The solution we obtained for the two point correlation function will be valid as long as the ϵ^3 terms may be neglected. As the evolution proceeds the contribution from the higher order terms will increase and they will modify the evolution of the two point correlation function. The evolution of the higher order functions f_1, f_2 and f_3 is calculated by solving to lowest order the equation for these quantities. For example, to lowest order the function f_1 will be of order ϵ^3, and its evolution is governed by equation (2.82). These functions are then to be incorporated as known functions into the equations for the two point correlation function. These equations then have to be solved to obtain the two point correlation function to a higher order. This method can in principle be used to caltulate higher order terms for the other correlation functions also.

The perturbative approach is expected to break down when $\epsilon \frac{D_1(\lambda)}{D_1(\lambda_0)} \sim 1$.

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