APPENDIX A

Deflection of Null Ray in Higher Dimensional Black Hole Spacetime

This is a simple calculation based on standard procedure that we (Bhawal B. and Mani H.S., 1988, unpublished) did to arrive at an expression for the measure of the deflection of null ray propagating in higher dimensional black hole spacetime. We found that, under certain assumptions, the general expressions obtained for even and odd number of dimensions differ from each other.

In any static, spherically symmetric higher dimensional spacetime of the form

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega_n^2, \qquad (A.1)$$

where $d\Omega_n^2$ is given by Eq.(II.3), one may consider the orbit of the null ray to be in the equatorial plane (i.e. where all polar angles $\theta_i = \pi/2$ for i = $2, 3, \ldots, n$), since the field is isotropic.

Then proceeding in the standard way [Weinberg, 1972], one arrives at the following expression for the deflection of null ray by the gravitational field.

$$\Delta\theta_1 = 2|T| - \pi \qquad (A.2)$$

$$T = \int_r^\infty A^{1/2}(r)\left[\left(\frac{r}{b}\right)^2 \frac{B(b)}{B(r)} - 1\right]^{-1/2} \frac{dr}{r} \qquad (A.3)$$

where b is the distance of the closest approach to the central point.

Let us now choose the background metric to be the higher dimensional Schwarzschild-de Sitter spacetime [Dianyan, 1988] given by Eq.(A.1) and

$$B(r) = A^{-1}(r) = 1 - \frac{r_0^m}{r^m} - \frac{2\Lambda r^2}{(m+1)(m+2)}, \qquad r_0^m = 2GM \qquad (A.4)$$

Λ is the cosmological constant. m = D − 3.

Integrating T for the above expressions of A and B is very difficult. We assume that throughout the orbit of the null ray, r (or, b) is much greater than r_0. Then, since A is also a very small quantity, one may write

$$A(r) = B^{-1}(r) = 1 + \frac{r_0^m}{r^m} + \frac{2\Lambda r^2}{(m+1)(m+2)} \tag{A.5}$$

Then T may be calculated to be

$$T \simeq \int_b^\infty \frac{dr}{r}\left[\frac{r^2}{b^2}-1\right]^{-1/2}\left[1+\frac{r_0^m}{2r^m}+\frac{r_0^m}{2r^{m-2}b^m}\frac{b^{m-1}+rb^{m-2}+\cdots+b^{m-1}}{r+b}\right.$$
$$\left. - \frac{2\Lambda r^2}{(m+1)(m+2)}\right] \tag{A.6}$$

Making change of variable to $x = b/r$, we obtain the deflection of the null ray to be

$$\begin{aligned}\Delta\theta_1 =& \frac{r_0^m}{b^m}\int_0^1 \frac{dx}{(1-x^2)^{1/2}}\left[x^m + \frac{1+x+\cdots+x^{m-1}}{1+x}\right]\\ &- \int_0^1 \frac{dx}{x^2(1-x^2)^{1/2}}\frac{4\Lambda b^2}{(m+1)(m+2)}\end{aligned} \tag{A.7}$$

The last term containing Λ blows up. Therefore, from now onwards we set $\Lambda = 0$ or equivalently, we confine our discussion to ordinary higher dimensional Schwarzschild spacetime.

For $\Lambda = 0$,

$$\Delta\theta_1 = \frac{r_0^m}{b^m}\int_0^1 \frac{dx}{\sqrt{1-x^2}}\left[\frac{1+x+\cdots+x^{m+1}}{1+x}\right] \tag{A.8}$$

When D is odd (or, m + 1 is odd)

$$1+x+\cdots+x^{m+1} = (1+x)(1+x^2+x^4+\cdots x^m). \tag{A.9}$$

When D is even (or, m + 1 is even)

$$1+x+\cdots+x^{m+1} = 1+(1+x)(x+x^3+x^5+\cdots x^m). \tag{A.10}$$

So, when D is odd, deflection of null ray is given by

$$\Delta\theta_{1O} = \frac{r_0^m}{b^m}\int_0^1 \frac{dx}{\sqrt{1-x^2}}\left[1+x^2+x^4+\cdots+x^m\right] \tag{A.11}$$

The corresponding expression for D even is given by

$$\Delta\theta_{1E} = \frac{r_0^m}{b^m}\int_0^1 \frac{dx}{\sqrt{1-x^2}}\left[\frac{1}{1+x} + x + x^3 + x^5 + \cdots + x^m\right]. \tag{A.12}$$

All the integrals appearing in the above expressions can be written in terms of Beta or Gamma functions as shown below. For any value of m = p (say)

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}x^p = \frac{1}{2}B\left(\frac{p+1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{p}{2}+1\right)}. \tag{A.13}$$

The following expressions can, therefore, be obtained

$$\Delta\theta_{1O} = \frac{\pi GM}{b^m}\left[1 + \sum_{k=1}^{m/2}\frac{(2k-1)(2k-3)\cdots 3.1}{2^k k!}\right] \tag{A.14}$$

$$\Delta\theta_{1E} = \frac{4GM}{b^m} + \frac{GM}{b^m}\sum_{k=1}^{(m-1)/2}\frac{2^{k+1}k!}{(2k+1)(2k-1)\cdots 5.3.1}. \tag{A.15}$$

The expression of $\Delta\theta_{1E}$ for $D = 4$ tallies with the standard expression obtained in four dimensional analysis ($= 4GM/b$).

If one wants to compare the values of deflections in two Schwarzschild spacetimes of different dimensions, one may do so by choosing units c = G = 1 and dimensionless variable for the radial coordinate. It is not easy to arrive at any general conclusion since the result crucially depends on the value of b as well as the corresponding: number of dimensions.

APPENDIX B

Alternative Scalar Wave Solutions in The Witten Bubble Background

Here, we present alternative forms of solutions of both the radial and τ-equations, obtained by different procedures, for the Klein-Gordon equation in the Witten Bubble spacetime. Since these solutions are not convenient for formulating scattering and other problems discussed in chapter IV, we have not used them in our work. However, for the sake of completeness and for possible use elsewhere, we describe these here.

(i) τ-Equation

The r-part of the separated Mein-Gordon equation, which we will denote here as T, instead of $T^{\ell}_{i\omega}$, satisfies Eq.(IV.7).

Let us do the coordinate transformation

$$\tau \rightarrow i\left(\frac{\pi}{2} - \tau'\right) \tag{B.1}$$

This is equivalent to the Euclidean continuation of Eq.(IV.7). Then introducing the variable

$$p = \cos\tau', \tag{B.2}$$

we can write Eq.(IV.7) as

$$(1-p^2)\frac{d^2T^E}{dp^2} - 3p\frac{dT^E}{dp} + \left(\alpha - \frac{\ell(\ell+1)}{1-p^2}\right)T^E = 0 \tag{B.3}$$

where by T^E, we represent the Euclidean continuation of function T. Also,

$$\alpha = -\omega^2 - 1. \tag{B.4}$$

Defining $$Z = (1-p^2)^{-\ell/2}T^E \tag{B.5}$$

we get

$$(1-p^2)\frac{d^2Z}{dp^2} - p(2\ell+3)\frac{dZ}{dp} + [k(k+2) - \ell(\ell+2)]Z = 0 \tag{B.6}$$

where we have chosen

$$\alpha = k(k+2). \qquad (B.7)$$

Equation(B.6) can now be written as

$$(1-p^2)\frac{d^2Z}{dp^2} - p(2\mu+1)\frac{dZ}{dp} + \lambda(\lambda+2\mu)Z = 0 \qquad (B.8)$$

by defining

$$\begin{aligned} \mu =& \ell + 1 \\ \lambda =& k - \ell. \end{aligned} \qquad (B.9)$$

This is the standard Gegenbauer equation, which has two solutions expressed in terms of hypergeometric series.

$$C^{\mu}_{\lambda}(p) = \frac{\Gamma(2\mu+\lambda)}{\Gamma(\lambda+1)\Gamma(2\mu)} F(-\lambda, \lambda+2\mu; \mu+\frac{1}{2}; \frac{1-p}{2}) \qquad (B.10)$$

$$D^{\mu}_{\lambda}(p) = 2^{-1-\lambda}\frac{\Gamma(\mu)\Gamma(2\mu+\lambda)}{\Gamma(\mu+\lambda+1)} F(\mu+\frac{1}{2}\lambda, \mu+\frac{\lambda}{2}+\frac{1}{2}; \mu+\lambda+1; p^2) \qquad (B.11)$$

Therefore, using Eq.(B.5), we get two solutions for Eq.(B.3)

$$T^E_1 = (\sin\tau')^{\ell} C^{\mu}_{\lambda}(\cos\tau') \qquad (B.12)$$

$$T^E_2 = (\sin\tau')^{\ell} D^{\mu}_{\lambda}(\cos\tau') \qquad (B.13)$$

Performing the reverse transformation of Eq.(B.1), or equivalently, continuing back to the Minkowski solutions, we obtain

$$T_1 = (\cosh\tau)^{\ell} C^{\mu}_{\lambda}(-i\sinh\tau) \qquad (B.14)$$

$$T_2 = (\cosh\tau)^{\ell} D^{\mu}_{\lambda}(-i\sinh\tau) \qquad (B.15)$$

(ii) Radial Equation

We can get a Frobenius series solution of Eq.(IV.14), if we assume it first to be of the form

$$\mathcal{R} \sim (r-R_0)^q \sum_{n=0}^{\infty} a_n (r-R_0)^n$$

Then substituting this in Eq.(IV.14), we obtain the following equation

$$
\begin{aligned}
0 =& q(q-1)z^{q-2}\sum_{0}^{\infty} a_n z^n + 2qz^{q-1}\sum_{1}^{\infty} n a_n z^{n-1} + z^q \sum_{2}^{\infty} n(n-1)a_n z^{n-2} \\
&+ \left(1 + \frac{3z}{2R_0} + \frac{5z^2}{4R_o^2} + \frac{27z^3}{8R_0^3} + \cdots\right)\left[qz^{q-2}\sum_{0}^{\infty} a_n z^n + z^{q-1}\sum_{1}^{\infty} n a_n z^{n-1}\right] \qquad (B.16) \\
&+ \frac{\omega^2+1}{2R_0}\sum_{0}^{\infty}\left(-\frac{z}{2R_0}\right)^n z^{q-1}\sum_{0}^{\infty} a_n z^n,
\end{aligned}
$$

where $z = \mathrm{r} - R_0$.

Equating the coefficients of different powers of z, we obtain q $= 0$ and can determine different a,, so that the solution turns out to be

$$
\mathcal{R}_1 = a_0\left[1 - \frac{\omega^2+1}{2R_0}(r-R_0) + \frac{\omega^4+6\omega^2+5}{16R_0^2}(r-R_0)^2 - \cdots\right]. \qquad (B.17)
$$

A second solution can be found to be of the form

$$
\mathcal{R}_2 = \ln(r-R_0)\sum_{0}^{\infty} a_n (r-R_0)^2 + \sum_{0}^{\infty} b_n (r-R_0)^n \qquad (B.18)
$$

where a,, b_n are constants to be determined from Eq.(B.15).

The first solution behaves properly throughout the range of the variable r, whereas, the second solution blows up at r $= R_0$.

APPENDIX C

A Special Kind of Coordinate Transformation

Coordinate transformations like Eq.(IV.15) are widely used in many situations both in flat and curved spacetimes to bring the radial equation to the Schrodinger form, e.g. the 'tortoise' coordinates in the Schwarzschild spacetime. However, these were considered to be just some mathematical operation. Their actual significance does not seem to have been discussed in the literature. Here, we will attempt to give a general basis for this.

For a metric in which the Klein-Gordon equation is separable and g_{00}, $g_{\mu\nu}$ are independent of time, one can always obtain the radial equation in Schrodinger form just by choosing a null coordinate system.

In a static spacetime, if one solves the Klein-Gordon equation for a massive scalar field, one obtains the following eigenvalue equation after separating out the temporal part which will be of the form $e^{+i\omega t}$,

$$\frac{1}{\sqrt{-g}} g_{00} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) + g_{00} m^2 \Phi = \omega^2 \Phi \qquad (C.1)$$

Now, if it is a two dimensional metric

$$ds^2 = -A(r) dt^2 + B(r) dr^2, \qquad (C.2)$$

let us try to get a null-vector η_i by introducing a new coordinate $r*$,

$$\eta^i = (1, \frac{dr}{dr^*})$$

so that,

$$\eta^i \eta_i = -A + (\frac{dr}{dr^*})^2 B = 0$$

$$\text{or,} \qquad \frac{dr^*}{dr} = \sqrt{\frac{B}{A}}. \qquad (C.3)$$

$$\text{Then} \quad ds^2 = A(-dt^2 + dr^{*2})$$

and Eq.(C.1) becomes

$$-\frac{d^2\Phi}{dr^{*2}} + m^2 A\Phi = \omega^2\Phi \tag{C.4}$$

Only the mass term contributes to the effective potential. For m = 0, this is just a free wave solution.

In a general dimensional spacetime, if $g_{\alpha\alpha}$, where $\alpha \neq t$, r be r-dependent, then there will be an extra first derivative term in Eq.(C.4). This first derivative term can be easily eliminated by suitably defining a new radial function and the Schrodinger equation can be obtained. The r-dependence of $g_{\alpha\alpha}$ will actually contribute to the effective potential of this equation.

If $g_{\alpha\alpha}$ is also time-dependent, the eigenvalue equation of Φ will not be of the form (C.1). But one can easily see that this will not create any problem in getting a Schrodinger form by choosing a null coordinate.

REFERENCES

Appelquist T., Chodos A. and Freund P.G.O., 1987, eds. Modern Kaluza Klein Theories, (Addison Wesley Publishing Co.).

Bailin D. and Love A., 1987, Rep. Prog. Phys., 50, 1087.

Balbinot *R.* and *Barlith* A., 1989, Class. Qu. Grav., 6, 195.

Barrow J.D. and *Cotsakis* S., 1991, Phys. Lett., **258B**, 299.

Bhawal *B.,1990*, Phys. Rev., D42, 449.

Bhawal B. and Kar S., 1992, Phys. Rev., **D46**,2464

Bhawal B. and Mani H.S., 1992, submitted for publication.

Bhawal B. and Vishveshwara C.V., 1990, Phys. Rev., **D42**, 1996.

Birrel N.D. and Davies P.C.*W.*, 1982, Quantum Fields in Curved Space, (Cambridge University Press, Cambridge).

Boulware D. and Deser S., 1985, Phys. Rev. Lett., 55, 2656.

Boulware D., Strominger A., Tomboulis E., 1984, in Quantum Theory of Gravity, ed: Christensen S.. (Hilger, London).

Brill D.R. and *Matlin* M.D., 1989, Phys. Rev., D39, 3151.

Callan C.G., *Myers R.C.* and Perry *M.J.*, 1989, Nucl. Phys., **B311**, 673.

Candelas P. and *Weinberg* S., 1984, Nucl. Phys., **B237**, 397.

Cho Y.M., *1975*, J. Math. Phys., 16, 2029.

Coleman S., 1977, Phys. Rev., D15, 2929.

Coleman S., 1988, Nucl. Phys., **B298**, 178.

Cremmer E. and Julia B., 1979, Nucl. Phys., **B159**, 141.

Cremmer E. and Scherk J., 1976, Nucl. Phys., **B103**, 399; **B108**, 409.

Deruelle N. and *Madore* J., 1987, Phys. Lett., **186B**, 25.

Deser S., 1986. in *Supersymmetry* and its applications, eds. Gibbons G.W., Hawking S.W., Townsend P.K., (Cambridge Univ. Press).

Deser S. and *Soldate* M., 1988, Nucl.Phys., **B311**, 739.

Deser S., Kay J., Stelle K.S., 1977, Phys. Rev. Lett., 38, 527.

Dianyan X., 1988, Class. Quantum Grav., 5, 871.

Dirac P.A.M., 1938, Proc. Roy. Soc., **A167**, 148.

Duff M.J., *Nilsson* B.E.W., Pope C.N., 1986, Phys. Rep., **130**,1.

Edelstein L., Vishveshwara C.V., 1970, Phys. Rev., **D1**, 3514.

Einstein A., 1916, Ann. Der Phys., 49.

Frolov V.P., Mazzitelli F.D., Paz J.P., 1989, Phys. Rev., D40, 948.

Gerlach U.H., 1983, Phys. Rev., **D28**, 761.

Gibbons G.W. and Perry, M.J., 1976, Phys. Rev. Lett., 36,985.

Gibbons G.W. and Perry, M.J., 1978, Proc. R. Soc. Lond., **A358**, 467.

Gibbons G.W. and Ruback P.J., 1986, Phys. Lett., **171B**, 390.

Gonsalez-Diaz P., 1990, Phys. Lett., **247B**, 251.

Goroff M.H. and *Sagnotti* A., 1986, Nucl. Phys., **B266**, 709.

Green M.B., Schwarz J.H., Witten E., 1987, Superstring Theory, (Cambridge University Press, Cambridge).

Grisaru M.T., *Nieuwenhuizen* P.V., *Vermasaren* J.A.M., 1976, Phys. Rev. Lett., 37. 1662; ibid, 37, 1669.

Gross D.J., Perry *M*.J. and Yaffe L.G., 1980, Phys. Rev., D25,330.

Hartle B., Hawking S.W., 1976, Phys. Rev., **D13**, 2188.

Hartle B., *Horowitz* G.T., 1981, Phys. Rev., **D24**, 257.

Hawking S.W., 1974, Nature (London), 248, 30.

Hawking S.W., *1975*, Comm. Math. Phys., 43, 199.

Hawking S.W., 1981, Comm. Math. Phys., 80, 421.

Hawking S.*W*. and Ellis G.F.R., 1973, The Large Scale Structure of Spacetime, (Cambridge Univ. Press).

Henriques A.B., 1986, Nucl. Phys., **B277**, 621.

Howard K.W., 1984, Phys. Rev., **D30**, 2532.

Ishihara H., *1986*, Phys. Lett., **179B**, 217.

Israel *W*., 1986, Phys. Rev. Lett., 57, 397.

Iyer B.R., Iyer S. and Vishveshwara C.V., 1989, Class. Quantum Grav., 6, 1627.

Jianjun *X*. and Sicong J., 1991, Mod. Phys. Lett., 6, 2197.

Lanczos C., 1938, Ann. Math., 39, 842.

Lovelock D., 1971, J. Math. Phys., 12, 498.

Lovelock D., 1972, J. Math. Phys., 13, 874.

Madore J., 1985, Phys. Lett., **110A**, 289.

Madore J., 1986, Class. Qu. Grav., 3, 361.

Maeda *K.*, 1986, Phys. Lett., **166B**, 59.

Matlin M.D., 1991, Ph.D thesis, Maryland University, College Park.

Merzbacher E., 1970, Quantum *Mechanics*,2nd ed., p.123, (John Wiley and sons, NY).

Mignemi S., 1986, Mod. Phys. Lett., **A1**, 337.

Mukherjee S. *and* Paul B., 1990, Phys. Rev., **D42**, 2595.

Müller-Hoissen F., 1985, Phys. Lett., **163B**, 106.

Müller-Hoissen F., 1986, Class. Qu. Grav., **3,** 665.

Myers R.C., 1987, Nucl.Phys., **B289**, 701.

Myers R.C. and Perry M.J., 1986, Ann. Phys., 172, 304.

Myers R.C. and Simon J.Z., 1988, Phys. Rev., **D38**, 2434.

Narlikar J.V. and *Padmanabhan* T., 1986, Gravity, Gauge Theories and Quantum Cosmology, (D. Reidal Publishing Co., Dordrecht, Holland).

Penrose R., *1979*, in *General Relativity* : An Einstein Centenary Survey, eds. Hawking S.W. and Israel W., (Cambridge University Press).

Poisson E., 1991, Phys. Rev., **D43**, 3923.

Regge T., Wheeler J.A., *1957*, Phys. Rev., 108, 1063.

Sahdev D., 1984, Phys. Rev., **D30**, 2495.

Salam A. and Strathdee J., 1982, Ann. Phys.(N.Y.),**141**, 316.

Scherk J. and *Schwarz* J.H., 1974, Nucl. Phys., **B81**, 118.

Scherk J. and *Schwarz* J.H., 1975, Phys. Lett., **57B**, 463.

Shafi Q. and Wetterich C., 1985, Phys. Lett., **152B**,51.

Simon J.Z., 1990, Phys. Rev., **D41**, 3720.

Simon J.Z., 1991, Phys. Rev., D43, 3308.

Simon J.Z., 1992, Phys. Rev., **D45**, 1953.

Sokolowski L.M., Golda Z.A., Litterio M., Amendola L., 1991, Int. J. Mod. Phys., A6, 4517.

Starobinsky A.A., 1980, Phys. Lett., **91B**, 99.

Starobinsky A.A., 1983, Sov. Astron. Lett., 9, 302.

Stelle K.S., 1977, Phys. Rev., **D16**, 953; GRG, 9,353.

Strominger A., 1984, Phys. Rev., D30, 2257.

Strominger A., 1989, in Baby Universes [TASI proceedings, Brown University, 1988], Particles, Strings and Supernovae, eds. Jevicki A. and Tan C.I., (World Scientific).

Tangherlini F.R., 1963, Nuovo Cimento, 77, 636.

Teitelboim C. and Zanelli J., 1987, in Constraint's Theory and Relativistic Dynamics [Proc. of Workshop in Florence, Italy, 1986], eds. Longhi G., Lusanna L., (World Scientific).

Tomimatsu A. and Ishihara H., 1987, J. Math. Phys., 28, 2720.

Vishveshwara C.V., 1970, Phys. Rev., **D1**, 2870.

Voloshin M.B., Kobzarev I.Y., Okun L.B., 1975, Sov. J. Nucl. Phys., **20**, 644.

Weinberg S., 1972, Gravitation and Cosmology, (John Wiley and Sons. NY).

Weinberg S., 1983, Phys. Lett., **B125**, 265.

Weyl H., 1919, Ann. Phys.(Leipzig), 59, 101.

Wheeler J.T., 1986, Nucl. Phys., **B268**, 737; ibid, **B273**, 732.

Wiltshire D.L., 1986, Phys. Lett., **169B**, 36.

Wiltshire D.L., 1988, Phys. Rev., **D38**, 2445.

Witten E., 1981a, Comm. Math. Phys., 80, 381.

Witten E., 1981b, Nucl. Phys., **B186**, 412.

Witten E., 1982, Nucl. Phys., **B195**, 481.

Yoshimura M., 1986, Prog. Theor. Phys. Suppl., **86**, 208.

Zumino B., 1986, Phys. Rep., 137, 109.

Zwiebach B., 1985, Phys. Lett., **B156**, 315.