Chapter 1

Magnetism of Charged Particles

1.1 Introduction

All particles in nature can be divided into Bosons and Fermions according to their statistics. Electrons, protons and neutrons are all Fermions. An atom, which contains all three, can also be treated as a single (composite) particle. Whether the composite is Bosonic or Fermionic depends on the total number of its constituents. For example, He^4 contains two electrons, two protons and two neutrons and thus it is a Boson. But the isotope He^3 is a Fermion. Fermions obey the Pauli exclusion principle while there is no such restriction on Bosons. As a result, a collection of Bosons behaves quite differently from a collection of Fermions. A good example is the dramatic difference between a super-conductor and an ordinary metal. The electrical conductivity in ordinary metals can be understood in terms of the properties of Fermions (i.e. electrons); in contrast, super-conductivity can be understood in terms of Cooper pairs which are Boson-like.

Here, we carefully analyze the response of Bosonic particles to an external magnetic field. We discuss also the effect of the boundary on the partition function of the system. In this chapter, we restrict our discussion to N non-relativistic particles (interacting or non-interacting). As with any interacting many particle system the response is hard to compute exactly ; yet general arguments (due to B. Simon) exist which show that the response of a system of N spinless Bosons in an external magnetic field is always diamagnetic. We conclude this section by pointing out an interesting connection between the diamagnetism of spinless Boson systems and Brownian motion. This chapter is a summary of old results so as to introduce the background and motivate the work of chapter 2.

1.2 Classical Approach To Magnetism

Van Leeuwen's theorem [1] states that a classical gas of charged point particles is non-magnetic. First we will try to view this theorem pictorially [2] and later present a mathematical argument. The trajectory of a charged particle confined to a box

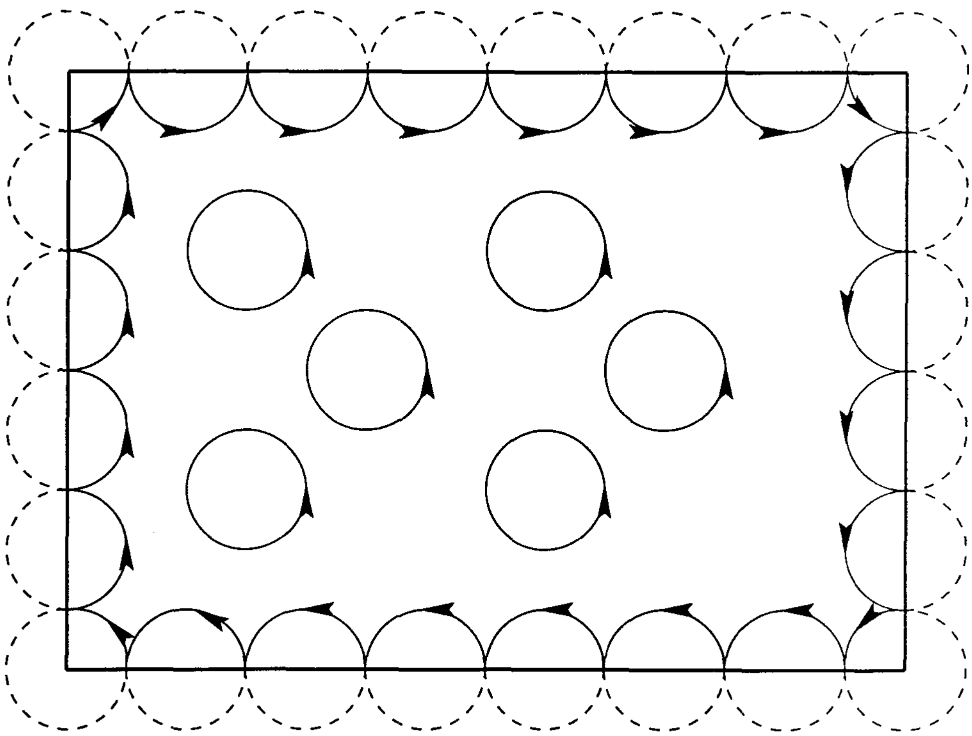

Figure 1.1: *2d motion of a charged particle in a rectangular box in a magnetic field. The internal orbits are traversed anti-clockwise sense while the boundary orbits in clockwise sense.*

and moving in an external magnetic field applied in the z direction is helical ; when projected onto a plane perpendicular to the z axis it is circular. As is evident from the figure (1.1), there are two kinds of orbits ; one set of orbits is completely inside the rectangular box while the other set "bounces around" at the perfectly reflecting boundary. The circular orbits well within the box contribute a positive magnetic moment. However, there is a negative contribution from the orbits near the boundary which exactly cancels this positive magnetic moment resulting in a zero net magnetic moment. This result is true even if the particles are interacting and is known as the Bohr-Van Leeuwen Theorem.

In statistical mechanics, the central quantity of interest is the partition function Z which is related to the free energy $\mathbf{F}$. Applying Boltzmann statistics, we get

$$Z = e^{-\beta F} = \int e^{-\beta H(r,p)} \, d^3r d^3p \tag{1.1}$$

Here, the Hamiltonian in the presence of an external magnetic field is given by

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e\vec{A}}{c}\right)^2 + V(r) \tag{1.2}$$

and $V(r)$ is a one body potential. The external magnetic field $\vec{B}$ is related to the vector potential $\vec{A}$ through $\vec{B} = \vec{\nabla}$ x $\vec{A}$. Now, changing the variable of integration $\vec{p}$ to $\vec{q} = \vec{p} - \frac{e\vec{A}}{c}$ (note that the limits of integration $-\infty$ to ∞ are not affected by this change of variables and the Jacobian of the transformation is unity.) it is easy to see that

$$Z(A) = Z(0) \tag{1.3}$$

Therefore, the magnetization vanishes because the free energy in the presence of the magnetic field is the same as without the magnetic field. It shows that an explanation of magnetic moment in thermal equilibrium must be sought in quantum mechanics.

This argument presented above for a single particle extends easily to the N particle case as follows. The N particle Hamiltonian is given by

$$H_N = \sum_{i=1}^{N} \frac{1}{2m_i}\left(\vec{p_i} - \frac{e_i\vec{A}(r_i)}{c}\right)^2 + \sum_{i=1}^{N} V(r_i) + \sum_{i<j} W(|\vec{r_i} - \vec{r_j}|) \tag{1.4}$$

Here $V(r_i)$ is the one body potential and $W(|\vec{r_i} - \vec{r_j}|)$ is the two body pairwise interaction.

The partition function of this system can be written as

$$Z_N(A) = \frac{1}{N!} \int\int \prod_{i=1}^{N} d^3p_i \ d^3r_i \exp(-\beta H_N) \qquad (1.5)$$

Again, with change of variables, this equation (1.5) becomes

$$\begin{aligned} Z_N(A) &= \frac{1}{N!} \int\int \prod_{i=1}^{N} m_i^3 \ d^3v_i \ d^3r_i \exp\left[-\beta\left(\sum_i \frac{1}{2}m_i v_i^2 + V(r_i)\right) - \beta \sum_{i<j} W(|\vec{r}_i - \vec{r}_j|)\right] \\ &= Z_N(0) \qquad (1.6) \end{aligned}$$

Therefore, the phase space distribution function $P_N(r_i, v_i)$ is given by

$$P_N = \frac{1}{Z_N \ N!} \prod_{i=1}^{N} m_i^3 \ \exp\left[-\beta\left(\sum_i \frac{1}{2}m_i v_i^2 + V(r_i)\right) - \beta \sum_{i<j} W(|\vec{r}_i - \vec{r}_j|)\right] \qquad (1.7)$$

Clearly, this distribution function is independent of the magnetic field B. The same is true for the expectation value of any gauge invariant phase space function (such as the current density at a particular point). In other words, the statistical mean of any function of the variables v_i and r, which does not involve B explicitly is unaltered by the application of a magnetic field. This completes the proof' of Van Leeuwen's theorem which states that classically the thermodynamic properties of a collection of charged particles are unaffected by a magnetic field.

1.3 Quantum Mechanical Approach To Magnetism

The quantum mechanical solution of the single charged spinless particle Hamiltonian in an external magnetic field is due to Landau [3]. The degenerate energy levels are $E_n = \hbar\omega_c(n + 1/2)$, $\omega_c = \frac{eB}{mc}$. Here, we have neglected the motion along the z direction. The degeneracy is related to the translational symmetry in the problem. The orbit may be located anywhere in the plane. The degeneracy (being a dimensionless quantity) is given by $g = \frac{L_x L_y}{2\pi l^2} = \frac{eBA}{hc} - \frac{\Phi}{\Phi_0}$, where $1 = \sqrt{\frac{\hbar c}{eB}}$ is the magnetic length. A is the area of the rectangular geometry of sides L_x and L_y and $\phi_0 = hc/e$ is the flux quantum. Notice that both the degeneracy and the cyclotron energy arc linear in B. Now, the partition function of a single particle is given by

$$\begin{aligned} Z(B) &= \frac{eBA}{hc} \sum_{n=0}^{\infty} e^{-\beta\ (n+1/2)\hbar\omega_c} \\ \frac{Z(B)}{Z(0)} &= \frac{\beta\mu B}{\sinh(\beta\mu B)} \end{aligned} \tag{1.8}$$

Here, $\mu = \frac{e\hbar}{2mc}$ is the magnetic moment. The free energy is defined as $F(B) = -\frac{1}{\beta}\log Z(B)$. Hence, the magnetization is given by

$$\begin{aligned} M &= -\frac{\partial F}{\partial B} = \frac{1}{\beta}\frac{\partial \log Z}{\partial B} \\ &= -\mu \left[\coth(\beta\mu B) - \frac{1}{\beta\mu B}\right] \end{aligned} \tag{1.9}$$

Notice that the diamagnetic contribution to magnetization vanishes in the classical limit ($\hbar \to 0$, $\beta \to 0$). This can also be verified by replacing the sum *(1.8)* in the partition function by a continuous integral

$$Z(B) = \frac{eBA}{h} \int_0^{\infty} dn\ e^{-\beta\ n\hbar\omega_c} \tag{1.10}$$

A simple scaling argument shows that the partition function is independent of the magnetic field, resulting in zero susceptibility. It is easy to check from equation *(1.9)* that the diamagnetic susceptibility is non-zero and ***negative*** definite. In the limiting case $\beta\mu B \ll 1$, we find $\chi = -\frac{\beta\mu^2}{3}$, independent of B. It is also interesting to note from equation *(1.8)* that $Z(B) \leq Z(0)$. We will see later that this is a general feature of spinless Bose systems.

1.4 Electric Susceptibility vs Magnetic Susceptibility

The response of a system to an electric field is quite different from the response of a magnetic field. In fact, Langevin theory of susceptibility predicts the electric susceptibility to be positive [4]. Here, we will describe the use of thermodynamic perturbation theory [5, 6] to calculate the change in the free energy as a result of the perturbation of its quantum energy levels by the electric field. The electric field appears in the Hamiltonian as a potential. We treat this potential as a perturbation. The standard perturbation theory gives the energy levels of the Hamiltonian H = $H_0 + V$ as

$$E_n = E_n^0 + V_{nn} + \sum_m{}' \frac{| V_{nm} |^2}{E_n^0 - E_m^0}, \tag{1.11}$$

where E_n^0 is the unperturbed energy level and V_{nm} are the matrix elements of the perturbing energy. Now, the free energy can be written as

$$e^{-\beta F} = \sum_n e^{-\beta E_n} \tag{1.12}$$

Therefore, the free energy is given by

$$F = F_0 + \sum_n V_{nn}\ w_n + \sum_n \sum_m{}' \frac{| V_{nm} |^2\ w_m}{E_n^0 - E_m^0} - \frac{\beta}{2} \sum_n V_{nn}^2 w_n + \frac{\beta}{2} \left(\sum_n V_{nn} w_n \right)^2 \tag{1.13}$$

After a little algebra, the change in the free energy can be written as

$$\Delta F = \overline{V_{nn}} - \frac{1}{2} \sum_n \sum_m{}' \frac{| V_{nm} |^2\ (w_m - w_n)}{E_n^0 - E_m^0} - \frac{\beta}{2} \overline{(V_{nn} - \overline{V_{nn}})^2} \tag{1.14}$$

Here, the bar denotes a statistical averaging over the Gibbs distribution $w_n = \exp(-\beta(E_n^0 - F_0))$. All the second order terms in this expression are negative, since $(w_m - w_n)$ has the same sign as $(E_n^0 - E_m^0)$. Thus, the correction to the free energy in the second order approximation is negative and the zero field electric susceptibility is always positive.

1.5 Diamagnetic Inequality for Spinless Bose Systems

Till now we have discussed the implication of quantum mechanics for a single particle. In fact it is easy to carry out the analysis for N free particles. But for an interacting case one cannot analytically solve the energy spectrum. However, there exist some exact results regarding the response of N spinless Bosons towards an external arbitrary (homogeneous or inhomogeneous) magnetic field. B. Simon [7] showed quite generally the universal diamagnetic behaviour of interacting spinless Bosons. We reproduce his arguments below for the ground state (T= 0 limit). The interacting Hamiltonian in this case

$$H(A) = -\sum_{i=1}^{N} \frac{1}{2m_i} \left(\nabla_i - \frac{ie\vec{A}(\vec{r_i})}{c} \right)^2 + \sum_{i<j} V(|\vec{r_i} - \vec{r_j}|) + \sum_{i=1}^{N} V(r_i) \quad (1.15)$$

Let $\Psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N)$ be the wave function. Since $|\Psi|^2 = \Psi^* \Psi$, we note that using 3N-dimensional gradients

$$\begin{aligned} (|\Psi|)\nabla(|\Psi|) &= |Re(\Psi^* \nabla \Psi)| = |Re[\Psi^*(\nabla - ieA)\Psi]| \\ &\leq |\Psi||(\nabla - ieA)\Psi| \end{aligned} \quad (1.16)$$

Therefore, with x denoting the N vectors $\{\vec{r}_1, ...\vec{r}_N\}$

$$|\nabla|\Psi||^2(x) \leq |(\nabla - ieA)\Psi|^2(x). \quad (1.17)$$

It follows that

$$\int dx \left[(\vec{\nabla}|\Psi|)^2 + V(x)|\Psi|^2 \right] \leq \int dx \left[|(\nabla - ie\vec{A})\Psi|^2 + V(x)|\Psi|^2 \right] \quad (1.18)$$

If we choose Ψ above to be the ground state wave function for $H(A)$, the R.H.S is equal to $E(A)$, whereas by the variational principle for ground state energies, the L.H.S is greater than E(0). It follows that

$$E(0) \leq E(A) \quad (1.19)$$

Therefore, the ground state energy of spinless Bose systems in the presence of an external magnetic field is always greater than that without the magnetic field. Notice that this result is really a very strong one in the sense that it is true for an arbitrarily interacting system in an arbitrary magnetic field (homogeneous or inhomogeneous). This result regarding the ground state has been extended to finite temperature case also [8]. An alternative proof of this result can be found in reference [9]. Notice that the above proof fails for Fermions. The replacement of Ψ by $|\Psi|$ used above is not allowed by Fermi statistics. Even spinless Fermions do not obey this inequality. For a counterexample choose a spherically symmetric potential V and concentrate on a particular eigenstate n = 1. Then, energies of $l \neq 0$ states decrease in lowest order perturbation theory for a suitable choice of $\vec{A}$. This shows that the above inequality fails for Fermions. In chapter two, we generalize this diamagnetic inequality for N particles to a field theory in two dimensions.

1.6 Brownian Motion and Magnetism

In this section we discuss the connection [9, 10] between Brownian motion and magnetism. This connection helps us compute the probability distribution of areas enclosed by the path of a particle diffusing on a plane and provides an alternative proof of the diamagnetic inequality for spinless Bosons. Here we summarize the work by Sinha and Samuel described in reference [9].

We consider a diffusing particle of mass m on a plane at time $\tau = 0$. If the particle returns to its starting point at time $\tau = \beta$, it encloses an area. We want to compute the conditional probability that it encloses a given area A. The closed Brownian path in the plane can be designated by $\{\vec{x}(\tau), 0 \leq \tau \leq \beta, \vec{x}(0) = \vec{x}(\beta)\}$. Now. if $\mathcal{A}[\vec{x}(\tau)]$ is the algebraic area enclosed by the path $\vec{x}(\tau)$, then the normalized probability distribution of areas $P(A)$ is given by

$$P(A) = \langle \delta(\mathcal{A}[\vec{x}(\tau)] - A) \rangle \tag{1.20}$$

The angular bracket denotes that the expectation value is with respect to the Wiener measure. For example, the expectation value of any functional f $[\vec{x}(\tau)]$ in this measure is given by

$$\langle f[\vec{x}(\tau)] \rangle = \frac{\int \mathcal{D}[\vec{x}(\tau)] f[\vec{x}(\tau)] \exp\left[-\int_0^\beta \left(\frac{m}{2}\frac{d\vec{x}}{d\tau} \cdot \frac{d\vec{x}}{d\tau} d\tau\right)\right]}{\int \mathcal{D}[\vec{x}(\tau)] \exp\left[-\int_0^\beta \left(\frac{m}{2}\frac{d\vec{x}}{d\tau} \cdot \frac{d\vec{x}}{d\tau} d\tau\right)\right]} \tag{1.21}$$

The generating function $\tilde{P}(B)$ of the distribution is given by the Fourier transform of $P(A)$

$$\tilde{P}(B) = \int P(A)\ \exp\left(\frac{ieBA}{\hbar c}\right)\ dA = \left\langle e^{\frac{ieBA}{\hbar c}}\right\rangle \tag{1.22}$$

Notice that $B\mathcal{A}$ can be expressed as

$$B\mathcal{A} = \int_0^\beta \vec{A}(\vec{x}) \cdot \frac{d\vec{x}}{d\tau} d\tau \tag{1.23}$$

where $\vec{A}(\vec{x})$ is any vector potential whose curl is a homogeneous magnetic field. With all these ingredients, equation *(1.22)*can be written as

$$\tilde{P}(B) = \frac{\int \mathcal{D}[\vec{x}(\tau)] \exp\left[-\int_0^\beta \left(\frac{m}{2}\frac{d\vec{x}}{d\tau}\cdot\frac{d\vec{x}}{d\tau}d\tau\right) + \frac{ie}{\hbar c}\int_0^\beta \left(\vec{A}\cdot\frac{d\vec{x}}{d\tau}d\tau\right)\right]}{\int \mathcal{D}[\vec{x}(\tau)] \exp\left[-\int_0^\beta \left(\frac{m}{2}\frac{d\vec{x}}{d\tau}\cdot\frac{d\vec{x}}{d\tau}d\tau\right)\right]} \tag{1.24}$$

Inspection reveals that

$$\tilde{P}(B) = \frac{Z(B)}{Z(0)} \tag{1.25}$$

where $Z(B)$ is the partition function for a quantum particle in an external homogeneous magnetic field. Therefore, the equation *(1.25)*relates Brownian motion and magnetism. This relation has been used to compute the distribution of solid angles for diffusion of particle on a sphere [9]. If the partition function of the magnetic system is known, then we can compute $\tilde{P}(B)$ and hence the distribution function $P(A)$. Also, notice that by definition $\tilde{P}(B) \leq 1$ which immediately implies that

$$Z(B) \leq Z(0) \tag{1.26}$$

and therefore

$$F(B) \geq F(0) \tag{1.27}$$

Thus, the free energy in presence of the magnetic field is always higher than the free energy without the magnetic field. Note that this inequality was obtained without explicitly computing the partition function by using the analogy with Brownian motion. The zero-field susceptibility $\chi = -\mathrm{d}^2 F(B)/\partial B^2|_{B=0}$ of the magnetic system is related to the variance of the distribution of areas in the diffusion problem as follows.

$$\chi = \frac{1}{\beta}[\ln(\tilde{P}(B))]''|_{B=0} = -\frac{e^2}{\beta}\overline{(A-\overline{A})^2} = -\frac{e^2}{\beta} Var A \leq 0 \tag{1.28}$$

Since, the variance is always positive, it follows that the zero-field susceptibility is negative. This alternative formulation of diamagnetic inequality through this Brownian motion has been used [9] to prove the diamagnetism of N spinless Bosons.

1.7 Conclusion and Discussion

Diamagnetism is essentially a quantum effect. A single electron by itself will have a permanent magnetic moment arising from its spin and induced magnetic: moment from its orbital motion. Even for the simplest atom namely hydrogen, the orbital motion of the electron around the proton will give rise to an induced diamagnetic susceptibility. The permanent moment arising from the same electron will give rise to paramagnetic susceptibility. The diamagnetic susceptibility cannot, of course, be measured separately, but can be estimated theoretically.

From the pictorial proof of Van Leeuwen's theorem, it might appear that the role of a boundary is crucial. One might wonder then about the validity of this theorem in the case of a particle moving on a sphere (having no boundary) subject to a magnetic field created by a monopole of quantized strength at the center of the sphere. However, an explicit computation on a sphere [9] confirms that the boundaries are unnecessary for the validity of Van Leeuwen's theorem. Following reference [9], the partition function for this case for a monopole of strength g/e can be written as $Z_g = \sum_{j=|g|}^{\infty}(2j+1) \quad \exp(-\beta/2[j(j+1) - g^2])$. In the high temperature ($\beta \to 0$) classical limit the sum can be replaced by an integral and one finds that $Z_g = Z_0$ and hence one recovers Van Leeuwen's theorem.

To summarize, in this chapter we have reviewed the magnetic properties of particles with emphasis on spinless Bosons. In particular, we have noted that the free energy of a spinless Bose system in an external magnetic field is higher than the free energy without the magnetic field.

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Chapter 2

Field Theoretic Treatment of Charged Bosons in a Magnetic Field

2.1 Introduction

In the previous chapter, we have noted the diamagnetic inequality for N spinless Bosons. This diamagnetism of spinless Bosons is a universal property in the sense that it is independent of whether the applied external magnetic field is homogeneous or inhomogeneous. Also, this diamagnetism is independent of the interaction between particles at any finite temperature. In this chapter, we generalize this inequality to field theory.

In field theory (which describes systems with an infinite number of degrees of freedom) charged spinless Bosons are described by complex scalar fields. One might therefore expect that charged scalar fields would also show diamagnetic behaviour. However, while this conjecture is plausible, the result is by no means obvious. In quantum field theory one has to deal with the problem of divergences and their regularization. In fact as we will see later (see Appendix A), the QED vacuum is diamagnetic, contrary to what a naive argument would suggest. With this motivation, we study the magnetic behaviour of scalar field theories in two spatial dimensions.

This chapter is organized as follows. The first part deals with finite temperature free scalar field theory in the presence of an external homogeneous magnetic field. Here, we explicitly calculate the partition function and the free energy as a function of the applied magnetic field in two dimensions. This free energy expression is formally divergent. Using a suitable regularization scheme, we compute the *difference* in the free energy (with and without the magnetic field) and obtain a finite answer. This difference is shown to be positive, thus establishing the diamagnetic behaviour of free charged scalar fields.

In section **3,** we discuss the diamagnetism of interacting charged scalar fields. In this theory, we cannot evaluate the partition function explicitly. Nevertheless we prove the universal diamagnetism of scalar fields by assuming a finite momentum cutoff in the theory. This *non-pertwrbative* treatment can be adapted to any number of spatial dimensions. In section 4, we give our conclusions. All the results in this chapter have earlier been presented in [1].

2.2 Computation of Free energy in Free Field Theory

In this section we calculate the free energy of free scalar fields in the presence of an external uniform magnetic field. For ease of presentation we work here first in two spatial dimensions. The interesting physics takes place in the plane normal to the applied field.

Let Φ be a complex scalar field which describes charged spinless Bosons. The Lagrangian density of a free charged scalar field in the presence of a constant homogeneous external magnetic field is given by

$$\mathcal{L} = (D_\mu \Phi)^* (D^\mu \Phi) - m^2 (\Phi^* \Phi) \tag{2.1}$$

where $\mu = 0, 1, 2,$

$$D_\mu = \partial_\mu - \mathrm{i}eA_\mu \tag{2.2}$$

and m and e are the mass and charge respectively. (We set $\hbar = 1$ and $c = 1$). Now, we write the complex field in term of two real fields Φ_1 and Φ_2.

$$\Phi = \frac{\Phi_1 + i\Phi_2}{\sqrt{2}}, \Phi^* = \frac{\Phi_1 - i\Phi_2}{\sqrt{2}} \tag{2.3}$$

This theory has a global U(1) symmetry and therefore a conserved Noether charge Q, given by

$$Q = \int d^2x \ (\Pi_1\Phi_2 - \Phi_1\Pi_2) \tag{2.4}$$

where

$$\Pi_i = \partial_0\Phi_i \tag{2.5}$$

The Hamiltonian density of the system is given by

$$\mathcal{H} = \frac{1}{2}(\Pi_1^2 + \Pi_2^2) + \frac{1}{2}(\nabla\Phi_1)^2 + \frac{1}{2}(\nabla\Phi_2)^2 + \frac{1}{2}(m^2 + e^2A^2)(\Phi_1^2 + \Phi_2^2) - \vec{j}\cdot\vec{A}, \tag{2.6}$$

where the current density j is given by

$$\vec{j} = -\mathrm{e} \ (\Phi_1\vec{\nabla}\Phi_2 - \Phi_2\vec{\nabla}\Phi_1). \tag{2.7}$$

We now suppose that the external magnetic field is uniform in the $x - y$ plane.

We choose the temporal gauge ($A_0 = 0$). The constant magnetic field B is

$$B = \partial_x \ A_y - \partial_y \ A_x \tag{2.8}$$

where A, and A, are independent of t.

The action of this theory is

$$S = \int_0^\beta \int d^2x \ d\tau \ \mathcal{L}(\Phi, \Phi^*, \mathrm{A}), \tag{2.9}$$

where τ is the imaginary time variable which runs from 0 to β ($=1/(k_BT)$), the inverse temperature. The action defined above is quadratic and so the partition function can be evaluated exactly. As is usual in finite temperature field theory [2], we impose periodic boundary conditions for Bosonic fields

$$\Phi(\mathbf{x}, 0) = \Phi(\mathbf{x}, \beta). \tag{2.10}$$

Now, the partition function of this theory can be written as

$$\begin{aligned} Z(B) &= \int \mathcal{D}[\Pi_1]\mathcal{D}[\Pi_2] \int \mathcal{D}[\Phi_1]\mathcal{D}[\Phi_2] \exp \Big[\int d\tau\, d^2x \\ &\quad \Big(i\Pi_1 \frac{\partial \Phi_1}{\partial \tau} + i\Pi_2 \frac{\partial \Phi_2}{\partial \tau} - \mathcal{H}(\Phi_1, \Phi_2, \Pi_1, \Pi_2) \\ &\quad + \mu(\Phi_2 \Pi_1 - \Phi_1 \Pi_2)\Big)\Big] \end{aligned} \tag{2.11}$$

Here μ is the chemical potential associated with the conserved charge[1] Q. We pick the gauge in which the vector potential **A** is (-By, 0), and expand the complex scalar field in terms of modes adapted to the present situation. These modes solve the Klein-Gordon equation in an external magnetic field. The eigenfunctions are labeled by one discrete (1) and one continuous (p,) quantum number and the spectrum depends on 1 only. In the gauge we choose, the modes are plane waves in the x direction and harmonic oscillator (i.e. Gaussian) wave functions in the y direction.

The spectrum is given by

$$\omega_l^2 = m^2 + (2l+1)eB, \quad l = 0, 1, 2.......\infty \tag{2.12}$$

The degeneracy of these states is $eAB/2\pi$ (independent of l), where A is the area of the system. So, these modes can be thought of as quantized harmonic oscillators. Expanding the fields Φ_1 and Φ_2 in these modes the system reduces to a collection of harmonic oscillators with frequency ω_l.

By standard manipulations [2], we get the free energy as

[1] The charge density (Q/A, where A is the area of the system) has to be contrasted with the normal number density. The charge density refers here to the difference between the particle density and the anti-particle density and hence can take any sign while the number density, by definition, is always positive. In that sense μ is not the usual chemical potential used in the Grand Canonical Ensemble.

$$
\begin{aligned}
F(B) &= 2\pi \mathrm{eA}B\sum_{l=0}^{\infty}\left[\omega_l + \frac{1}{\beta}\ln(1-\exp(-\beta(\omega_l-\mu))\right. \\
&\left. + \frac{1}{\beta}\ln(1-\exp(-\beta(\omega_l+\mu))\right]
\end{aligned} \tag{2.13}
$$

The first term in the square brackets corresponds to the zero point fluctuation of the vacuum and the other two terms are finite temperature contributions of the particles and anti-particles respectively.

Notice that this zero point energy is divergent due to the summation of infinite number of modes (Landau levels). In conventional field theory, this infinite zero point energy is always discarded; since it can be re-absorbed in a suitable redefinition of the zero of energy. This is justified in the sense that the infinite zero point energy is unobservable. However, the change in zero point energy caused by changes in external parameters is finite and observable. So, according to Casimir's [3] idea, the physical vacuum energy can be defined as the difference between the zero point energy corresponding to vacuum configurations with constraints and the one corresponding to free vacuum configurations. This definition must be supplemented in general with a regularization prescription in order to obtain a convergent expression.

2.2.1 Zero Temperature Field Theory

The zero temperature free energy of the system in the presence of a constant magnetic field is given by

$$F_0(B) = 2\pi A\ eB\sum_{l=0}^{\infty}\ \omega_l, \tag{2.14}$$

where $\omega_l^2 = \mathrm{m}^2 + (21+1)eB$. Obviously, this sum diverges. In order to obtain a finite answer, we need to impose a cutoff L in the sum (2.14). Then the free energy becomes

$$F_0(B,L) = 2\pi A\ eB\sum_{l=0}^{L}\ \omega_l. \tag{2.15}$$

The free energy in the absence of the magnetic field at zero temperaturo is given by the divergent expression

$$F_0(0) = 2\pi A \int_0^{\infty} pdp \sqrt{(p_x^2 + p_y^2) + m^2} \tag{2.16}$$

We regularize this expression by imposing a cutoff **A.** Then the free energy (2.16) becomes

$$F_0(0,\Lambda) = 2\pi A \int_0^{\Lambda} pdp \sqrt{(p_x^2 + p_y^2) + m^2} \tag{2.17}$$

In order to compare the free energies in equations (2.15) and (2.17), we choose the cutoffs L and **A** in such a way that both systems have the same number of modes. Such a procedure can be justified on physical grounds if one imagines that the magnetic field is turned on [2].

Counting the modes up to the L-th Landau level we find

$$2\pi A \; eB \sum_{l=0}^{L} 1 = 2\pi A \; eB(L+1) \tag{2.18}$$

Similarly, for the momentum cutoff up to **A** we get the modes without the magnetic field as

$$2\pi A \int_0^{\Lambda} pdp = \pi \; A\Lambda^2 \tag{2.19}$$

Equating these gives us

$$\Lambda^2 = 2 \; eB \; (L+1) \tag{2.20}$$

Now, the free energy in absence of the magnetic field (which depends on magnetic field through the momentum cutoff) is given by

$$F_0(0,L) = 2\pi A \int_0^{\Lambda(B)} pdp \sqrt{p^2 + m^2} \tag{2.21}$$

The difference between the two free energies is given by

$$\Delta F(B,L) = F_0(B,L) - F_0(0,L) \tag{2.22}$$

---

[2]This idea of matching of modes is used in solid state physics to compute the specific heat of solids at low temperatures in the Debye model (see for example, *Solid State Physics* by N. W. Ashcroft and N. D. Mermin (Hott, Rinehart and Wiston, 1976)).

We define $f(B) = F_0(B, L)/2\pi A$, $\tilde{f}(B) = F_0(0, L)/2\pi A$ and $\Delta f(B) = f(B) - \tilde{f}(B)$. Numerically evaluating these sum and integral (see figure 2.1 and figure 2.2) one can show that for finite L, $\Delta f(B)$ the difference between two large quantities is positive. As the cutoff L goes to infinity, $\Delta f(B)$ becomes the difference between two infinities. In this limit we find that $\Delta f(B)$ tends to a finite value. Thus, the susceptibility at zero temperature in the relativistic case is non-zero. This vacuum susceptibility can be physically interpreted as due to virtual currents.

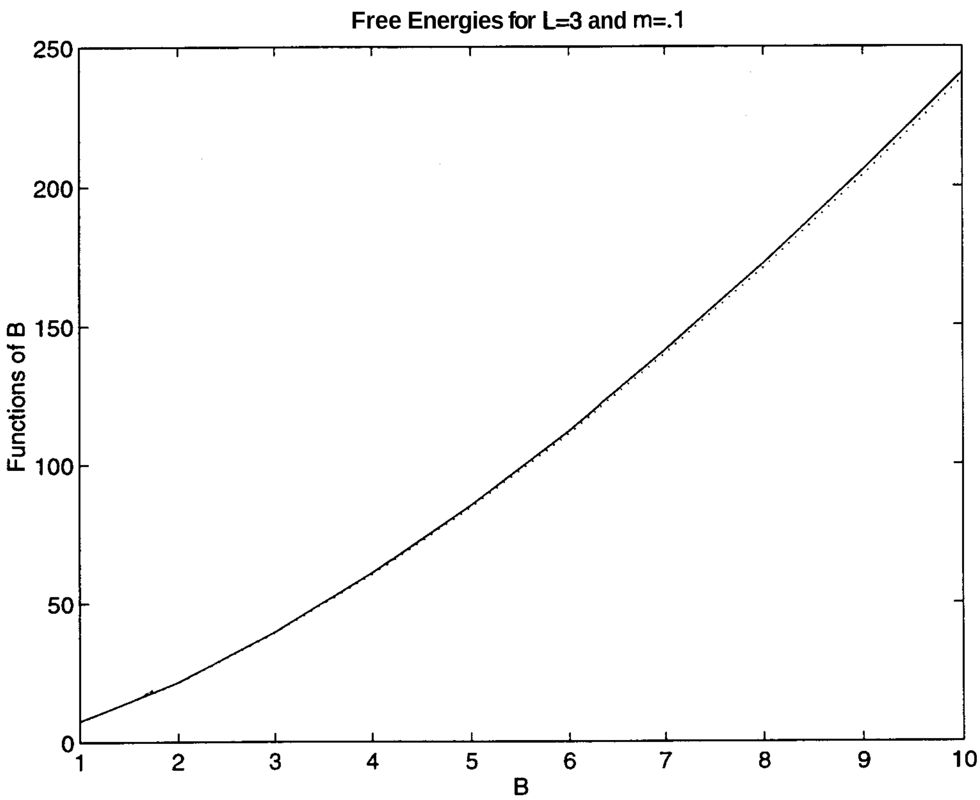

Figure 2.1: *The functions* $f(B)$ *and* $\tilde{f}(B)$ *have been plotted against* B *for* $L = 3$ *and* $m = .1$. *The solid line is* $f(B)$ *while the dotted line refers to* $\tilde{f}(B)$.

We now show analytically that $\Delta F(B)$ is positive i.e. the vacuum is diamagnetic. Note that

$$\Delta f(B) \;=\; f(B) - \tilde{f}(B) = \sum_{l=0}^{\infty} a_l(B, m) \tag{2.23}$$

where $a_l(B, m)$ is given by

$$a_l(B,m) = eB\left[\sqrt{m^2+(2l+1)eB} - \int_0^1 d\alpha\, \sqrt{m^2+2(l+\alpha)eB}\right] \tag{2.24}$$

Introducing a dimensionless quantity $\rho = \frac{eB}{m^2}$ the above equation becomes

$$a_l(\rho) = m^3\rho\left[\sqrt{1+(2l+1)\rho} - \int_0^1 d\alpha\, \sqrt{1+2(l+\alpha)\rho}\right] \tag{2.25}$$

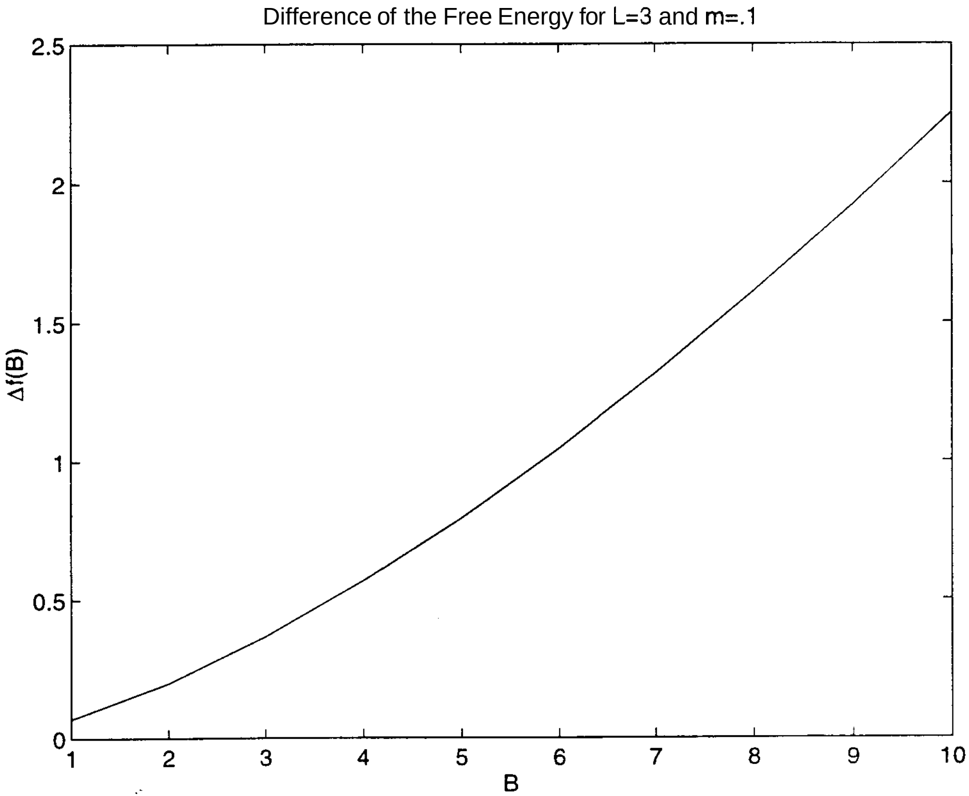

Figure 2.2: *The functions* $\Delta f(B) = f(B) - \tilde{f}(B)$ *has been plotted against* B. *The values of* L *and* m *are the same as figure 2.1. Note that the scale in this figure in y direction is much expanded compared to figure 2.1.*

The positivity of $a_l(\rho)$ for each l can be proved geometrically. Defining $z_l = (1+2l\rho)/2\rho$ and $f(\alpha) = \sqrt{z_l+\alpha}$, the coefficient $a_l(\rho)$ can be rewritten in terms of $c_l(\rho)$ as

$$c_l(\rho) = \frac{a_l(\rho)}{\sqrt{2}\,\rho^{3/2}} = f(1/2) - \int_0^1 d\alpha\, f(\alpha). \tag{2.26}$$

Since, the function $f(\alpha)$ is convex, the area under the tangent drawn at $\alpha = 1/2$ is greater than the area under the curve (see the figure 2.3). But the area under the tangent is equal to the area under the dotted curve which is equal to $f(1/2)$. This

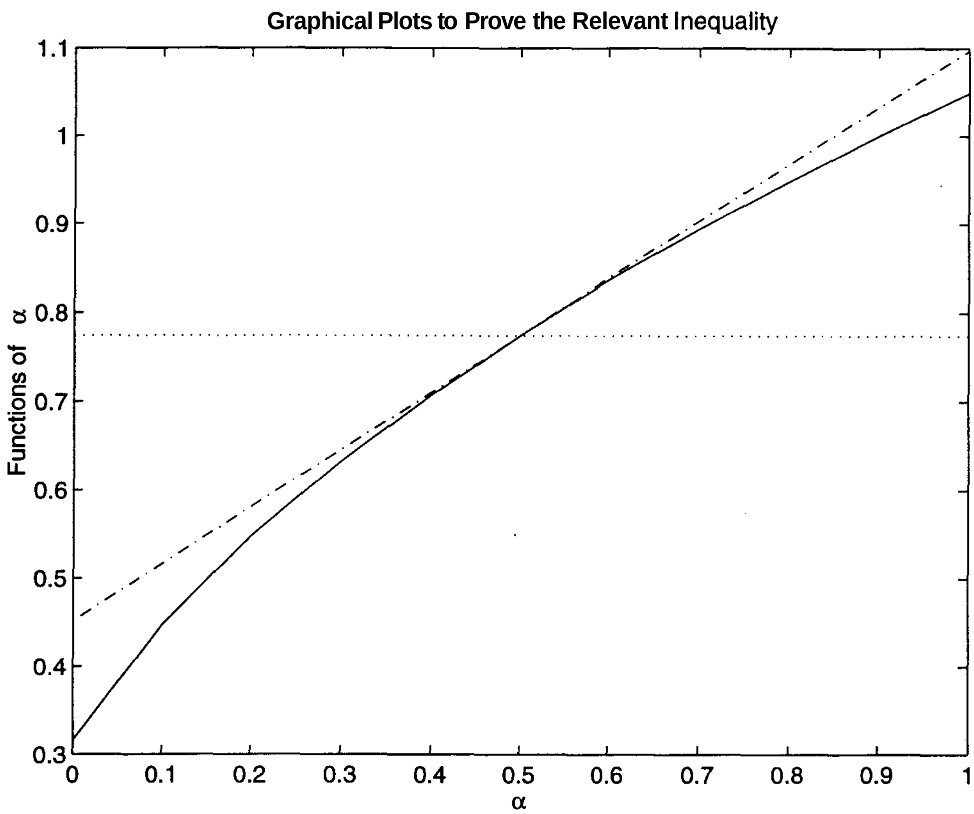

Figure 2.3: *The full line is the curve $f(\alpha) = (.1 + \alpha)^{1/2}$. The dashed line is the tangent to the above curve at $\alpha = .5$. It is straightforward to see that the area under the curve is less than the area under the tangent. Also the area under the tangent is the same as the area of the rectangle. The area of this rectangle is given by $f(1/2) \times 1 = f(1/2)$. Hence, the positivity of $a(.1)$ is proved. This can be generalized to any positive value of z_l.*

shows that $c_l(\rho)$ is positive.

To show the convergence of the sum (2.23), we note that

$$\int_0^1 d\alpha\, f(\alpha) \;\leq\; \left[f(1/2) \;-\; \frac{f(0)+f(1)}{2} \right] \tag{2.27}$$

Now, applying mean value theorem twice one can easily show that

$$\int_0^1 d\alpha\, f(\alpha) \leq \frac{1}{16(z_l+\alpha)}3/2, \tag{2.28}$$

for some α. Thus, the coefficient $c_l(\rho)$ is positive for each l and the sum converges, hence the diamagnetic inequality is established.

Low mass limit:

In this section we want to discuss the behaviour of the leading term of the free energy in the low mass limit and its consequences. This is defined by the condition $\rho \gg 1$. It is easy to see from the zero temperature free energy that the magnetization in the low mass limit is given by

$$M(B) \sim -\sqrt{B} \tag{2.29}$$

So, the susceptibility in this limit is given by

$$\chi(B) \sim -\frac{1}{\sqrt{B}} \tag{2.30}$$

which diverges as B goes to zero. This divergence of the susceptibility is reflects the fact that the free energy $F(B) \sim B^{3/2}$. This variation of the free energy in the massless limit can also be understood from dimensional arguments as follows. Since we are working in natural units $\hbar = 1$ and c = 1, then [m] $\sim [L]^{-1}$. Then the free energy density (i.e. per unit area) varies as $[L]^{-3}$. However. the dimension of B is $[L]^{-2}$. So, the massless limit restricts the free energy density variation with magnetic field B to $B^{3/2}$ only. This feature of the susceptibility has already been noticed in the magnetized pair Bose gas [4].

2.2.2 Finite Temperature Field Theory

Now, for the finite temperature case one can regularize the free energy through the same mode matching method and write down the free energy difference in dimensionless form as before

$$\Delta F(B) = F(B) - F(0) = \sum_{l=0}^{\infty} b_l(\rho, \delta, \zeta) \tag{2.31}$$

where,

$$b_l(\rho,\delta,\zeta) = \frac{\rho}{\delta}\left[g(\rho,l,1/2) - \int_0^1 d\alpha\; g(\rho,l,\alpha)\right]. \tag{2.32}$$

The dimensionless variables are defined as $\delta = \beta m$ and $\zeta = \beta\mu$.

The coefficient $g(\rho, l, \alpha)$ is given by

$$\begin{aligned} g(\rho,l,\alpha) &= \log\left(1-\exp(-\delta(\sqrt{1+2(l+\alpha)\rho}-\zeta))\right)+ \\ & \log\left(1-\exp(-\delta(\sqrt{1+2(l+\alpha)\rho}+\zeta))\right). \end{aligned} \tag{2.33}$$

Now, defining $z_l = \frac{\delta^2(1+2l\rho)}{2\rho}$ we can rewrite the equation (2.33) as

$$\begin{aligned} g(\rho,l,\alpha) &= \log\left(1-\exp(-(\sqrt{z_l+\alpha}-\zeta))\right)+ \\ & \log\left(1-\exp(-(\sqrt{z_l+\alpha}+\zeta))\right). \end{aligned} \tag{2.34}$$

The function $g(\rho, l, \alpha)$ is convex, so the zero temperature argument applies unchanged. It follows that the free energy satisfies the following inequality

$$F(B) \geq F(0) \tag{2.35}$$

Thus the response of the system to the magnetic field will be diamagnetic.

2.3 Interacting Field Theory

In this section we want to extend the diamagnetic inequality to the self-interacting field theory case including the dynamical interaction between scalar fields. The partition function of this charged self-interacting field theory in the presence of the magnetic field can be written as

$$Z(B) = \int\int \mathcal{D}[\Phi]\mathcal{D}[\Phi^*]\, \exp(-S(\Phi,\Phi^*,A)), \tag{2.36}$$

where the action S is defined as

$$S = \int\int d^2x\, d\tau \left[(D_\mu\Phi)(D^\mu\Phi)^* + m^2(\Phi^*\Phi) + V(\Phi^*\Phi)\right]. \tag{2.37}$$

The action is not quadratic and $Z(B)$ cannot be evaluated in closed form. Nevertheless, we can show that the response of the system to an external magnetic field is diamagnetic. Since the formal expression for the partition function may not exist (the integrals may not exist) we impose a cut off in momentum space. The functional integral in (2.36) signifies that one only integrates over those field configurations whose Fourier transforms have support within a sphere of radius Λ_1 in momentum space. The partition function then explicitly depends on Λ_1. We do not explicitly indicate the Λ_1 and μ dependence of $Z(B,\Lambda_1,\mu)$ below.

We divide the action into two parts S_0 and S_{int}, where S_0 is the action in the absence of the external field.

$$S = S_0 + S_{int}, \tag{2.38}$$

where

$$S_0 = \int\int d^2x \; d\tau \left[(\partial_\mu\Phi)(\partial^\mu\Phi)^*) + m^2(\Phi^*\Phi) + V(\Phi^*\Phi)\right], \tag{2.39}$$

$$S_{int} = \int\int d^2x \; d\tau \left[-ie(\partial_\mu\Phi)(A^\mu\Phi^*) + ie(A_\mu\Phi)(\partial^\mu\Phi^*) + e^2(\mathbf{A}\cdot\mathbf{A})(\Phi\Phi^*)\right]. \tag{2.40}$$

Notice that $\exp(-S_0)$ is a positive measure on the space of field configurations. The ratio $Z(B)/Z(0)$ can therefore be regarded as the expectation value of $\exp(-S_{int})$. Since $\exp(-S_{int})$ is an oscillatory function whose modulus is less than or equal to 1, we conclude that

$$\frac{Z(B)}{Z(0)} = \ll \exp - S_{int} \gg \leq 1 \tag{2.41}$$

This implies that

$$F(B) \geq F(0) \tag{2.42}$$

This result is an *exact* and *non-perturbative* one. Hence, it is more general and stronger than perturbative results (see reference [5]).

In this derivation, we have not assumed any form for the vector potential. So, the result derived above is true for *both homogeneous or inhomogeneous* magnetic fields of any strength. Since β is arbitrary, the result holds at *all temperatures*. The argument presented here works for any arbitrary interaction $V(\Phi^*\Phi)$ (Generally, it is assumed that $V(\Phi^*\Phi)$ is a smooth function, for instance, a polynomial). Also, in field theory one would also require that the interaction $V(\Phi^*\Phi)$ be renormalizable. In two dimensions this would restrict the interaction to $(\Phi^*\Phi)^p$ only. From simple power counting, one can notice easily that the value of $p \leq 3$. In condensed matter physics, where a natural cutoff exists, higher order powers of $(\Phi^*\Phi)$ may also be present.

Up to now we have considered the cases of charged scalar fields interacting through a potential. It is also possible to consider interaction mediated by a dynamical electromagnetic field A,. The fields in the system are now Φ (charged scalar fields) and A,. If one applies an external magnetic field A_{ext} then the full Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F^2 + (D_\mu\Phi)^*(D^\mu\Phi) - m^2(\Phi^*\Phi) - V(\Phi^*\Phi) \tag{2.43}$$

where $D_\mu = \partial_\mu - ieA_\mu^{ext} - ieA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu$.

The argument given above can be modified as follows. The definition of S_0 and S_{int} changes slightly with some additional terms.

$$\begin{aligned} S_0 &= \int\int d^2x\, d\tau \Big[-\frac{1}{4}F^2 + (\partial_\mu - ieA_\mu)\Phi^*(\partial^\mu + ieA^\mu)\Phi \\ &\qquad + m^2(\Phi^*\Phi) + V(\Phi^*\Phi)\Big], \end{aligned} \tag{2.44}$$

and

$$\begin{aligned} S_{int} &= \int\int d^2x\, d\tau \Big[-ie(\partial_\mu\Phi)(A^\mu_{ext}\Phi^*) + ie(A_\mu^{ext}\Phi)(\partial^\mu\Phi^*) \\ &\quad + e^2\left((\mathbf{A}^{ext}\cdot\mathbf{A}^{ext})(\Phi\Phi^*) + (A_\mu\Phi)(A^\mu_{ext}\Phi^*) + (A_\mu^{ext}\Phi^*)(A^\mu\Phi^*)\right)\Big] \end{aligned} \tag{2.45}$$

Again one can repeat the same argument to establish the diamagnetic inequality by noting that $\exp(-S_0)$ is a positive measure and the ratio $Z(B)/Z(0)$ as an expectation value of $\exp(-S_{int})$. This universal inequality follows from basic principles and does not depend on the details of the interaction.

2.4 Conclusion and Perspective

The response of a system to an electric field is completely different from its response to a magnetic field. The basic difference between the responses of a system on application of an electric field or a magnetic field lies in the structure of the Hamiltonian of the system.

The Lagrangian of a system in the presence of an electric field can be written as

$$\mathcal{L} = (D_0\Phi)^*(D_0\Phi) - (\nabla\Phi)^*(\nabla\Phi) - m^2(\Phi^*\Phi) - V(\Phi^*\Phi) \tag{2.46}$$

where

$$D_0 = \partial_0 - ieA_0 \tag{2.47}$$

For statistical mechanics to make sense, the Hamiltonian H must be independent of time. We choose the gauge so that the vector potential is time independent and the Hamiltonian is

$$\begin{aligned} \mathcal{H} \ &= \ (\Pi^*)(\Pi) + (\nabla\Phi)^*(\nabla\Phi) + m^2(\Phi^*\Phi) + V(\Phi^*\Phi) \\ &\quad -ie\left[(\Pi^*)(A_0\Phi) - (\Pi)(A_0\Phi^*)\right] \end{aligned} \tag{2.48}$$

The electric field appears in the Hamiltonian through A_0 terms. Now, from finite temperature second order perturbation [6] theory, one can show easily that the free energy of the system always decreases with the electric field. Hence, the dielectric susceptibility is always positive in thermal equilibrium.

But in the case of a magnetic field the Hamiltonian contains both linear and quadratic terms in A. The net effect of an applied magnetic field is not a *priori* clear. However, as our analysis makes clear, for charged scalar field theories the net effect is always diamagnetic.

In the case of spinless Bosons, there is no spin magnetic moment and hence the system always has a higher energy in a magnetic field than in the absence of magnetic field. It has been already pointed out [7, 8, 9, 10] that there is no corresponding theorem for fermions.

Let us consider some illustrative examples of Spinless Bose systems. One obvious example in the laboratory is Cooper pair formation in super-conductors which shows perfect diamagnetism (known as the Meissner [11] effect) below the critical temperature. Cooper pairs also exist in Neutron Stars [12] where the magnetic field is very high compared to any laboratory field. Of course, the operators which create and destroy Cooper pairs are not strictly Bose operators, so this is only an analogy[3]. Pions would be suitable candidates for application of our theory with π^+ and π^- regarded as the particles and anti-particles. Pions are massive ($mc^2 = 139.5673$ Mev), obey Bose-Einstein statistics and they possess no spin.

Before we end we would like to comment on a recent work in the literature [5] on ultra-relativistic hot scalar plasma. The authors study the scalar electrodynamics by re-summation methods in perturbation theory. This treatment does not allow for self-interaction of the charged scalar field. By re-summing a thermal loop expansion, they found that the magnetic permeability of this hot scalar plasma is diamagnetic at distances greater than a cutoff length scale determined by the charge and the temperature. This perturbative result is consistent with our results. However, our result of diamagnetism of a charged scalar field is more general in the sense that it takes self-interaction of the field into account and is non-perturbative.

[3]The analogy mentioned here is at the following level. The (pairing) operators $b_k^\dagger$ and $b_{k'}$ do satisfy the usual Bosonic commutation relations for $k \neq k'$ but ${b_k}^{\dagger 2} = b_k^2 = 0$ for $k = k'$. This is obviously the Pauli exclusion principle restriction.

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