CHAPTER 3

General Results for Poisson Noise on Image Intensity Correlations

3.1 Introduction

Image intensity correlation techniques have proved useful in imaging through turbulent media. For example, in speckle interferometry one is concerned with reconstructing high resolution stellar images which have been affected by the earths turbulent atmosphere. For bright sources one is concerned with the noise due to the random medium itself. At low light levels, in addition to this noise, noise due to the photonic nature of light needs to be considered. The Poisson fluctuations in the number of photons detected introduce bias terms which dominate at low light levels. These bias terms are present even for time independent images and must be compensated for. The basic reason for the existence of these bias terms is that the average of a product of random variables is not (except when they are statistically independent) the product of their averages. Consider a detector which detects n photons with average n. The probability distribution $P(n)$ is the Poisson distribution:

$$P(n) = e^{-\bar{n}} \frac{\bar{n}^{n}}{n!} \tag{3.1}$$

For this distribution (we denote the Poisson average by $\overline{}$) the estimator of $\bar{n}$ is n. However, the estimator of $\bar{n}^2$ is not n^2 since $\overline{n^2} = \bar{n}^2 + \bar{n}$. The $\bar{n}$ term represents the bias term which dominates at low light levels ($\bar{n} < 1$). In speckle interferometry low light levels mean stars fainter than about 13'th magnitude. The unbiased estimator for $\bar{n}^2$ is obtained by subtracting the bias term. Hence the unbiased estimator of $\bar{n}^2$ is n^2-n. For one detector it is well-known[17] that the unbiased estimator for $\bar{n}^m$ is $n(n-1)\cdots(n-m+1)$. We denote unbiased estimator by UE{ }. Thus

$$UE\{\bar{n}^m\} = n(n-1)\cdots(n-m+1) \tag{3.2}$$

The variance on this estimator involves the knowledge of the expectation

$$\overline{[UE\{\bar{n}^m\}]^2} \tag{3.3}$$

In the case of a detector array with average number of detected photons $\bar{n}_i$ in the i'th pixel one needs unbiased estimators for products like $\bar{n}_i \bar{n}_j \bar{n}_k$. Here, bias terms come from coincident pixels: i=j etc. as the Poisson fluctuations in different pixels are statistically independent. In the bispectrum method of phase recovery proposed by Weigelt[18] the bispectrum

$$\bar{n}_u \bar{n}_v \bar{n}_{-u-v} \quad ; \; n_u = \sum_i n_i e^{iux_i} \tag{3.4}$$

is used where x_i is the poisition of the i'th pixel. Wirnitzer[19] has shown that the unbiased estimator for the bispectrum is

$$UE\{\bar{n}_u \bar{n}_v \bar{n}_{-u-v}\} = n_u n_v n_{-u-v} - n_u n_{-u} - n_v n_{-v} - n_{u+v} n_{-u-v} + 2n_0$$

One way to obtain unbiased estimators is as follows. Replace $\bar{n}_u \bar{n}_v \cdots$ by $n_u n_v \cdots$ and take the Poisson average $\overline{n_u n_v \cdots}$. This contains lower order bias terms like $\bar{n}_u \bar{n}_v$ which cannot be replaced straightaway by $n_u n_v$ as the latter contains the bias $\bar{n}_{u+v}$. In this way one can systematically go on compensating bias terms until on the right hand side all terms are of the form $\overline{n_u n_v \cdots}$ with a single Poisson average. The average of the squared modulus of the bispectrum involves averages of products of the n_u's upto sixth order. It turns out that a straight forward calculation generates hundreds of terms (434 in this case) , which conspire by miraculous cancellation yielding only a handful of terms (33 in the case of bispectrum)[1,14] in the end. In the course of our study of this problem certain general patterns and rules became apparent and they are presented in this chapter. In section 3.2 we state our results for a general N'th order correlation and give a diagramatic representation for book-keeping. The Appendix A contains a derivation of these results.

3.2 General Results for Poisson Noise on image correlations

In this section we state our results on Poisson statistics for frequency domain correlations of the n_u's although the results are easily extended to a general N'th order correlation (real or frequency domain).

Rule a) Average of N'th order products: Consider the following special cases:

$$N=1 \qquad \bar{n}_u = \bar{n}_u$$

$$N=2 \qquad \overline{n_u n_v} = \bar{n}_u \bar{n}_v + \bar{n}_{u+v}$$

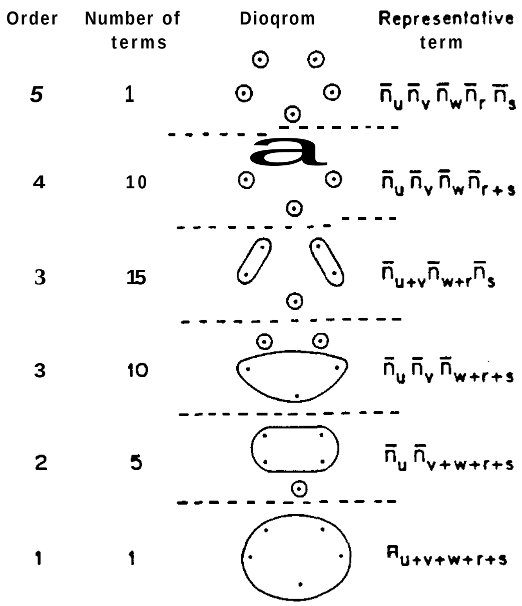
$$N=3 \qquad \overline{n_u n_v n_w} = \bar{n}_u \bar{n}_v \bar{n}_w + \bar{n}_u \bar{n}_{v+w} + \bar{n}_v \bar{n}_{u+w} + \bar{n}_w \bar{n}_{u+v} + \bar{n}_{u+v+w}$$

The general rule underlying these special cases is the following. Consider all N subscripts (Fourier components in above example). Then form m clusters out of these N subscripts in all possible ways. For a specific way of partitioning sum all the subscripts within a cluster. Use these m sums as subscripts for $\bar{n}$'s to obtain one m'th order term in the $\bar{n}_u$'s. Here a cluster must contain at least one element. For N=2 case there is only one way of forming two clusters which gives the first term and only one way of forming a single cluster which gives the second term. Thus there are (according to this rule) seven second order terms in the average of $\overline{n_u n_v n_w n_r}$: $\bar{n}_u \bar{n}_{v+w+r}$, $\bar{n}_v \bar{n}_{u+w+r}$, $\bar{n}_w \bar{n}_{u+v+r}$, $\bar{n}_r \bar{n}_{u+v+w}$

$\bar{n}_{u+v} \bar{n}_{w+r}$, $\bar{n}_{u+w} \bar{n}_{v+r}$, $\bar{n}_{u+r} \bar{n}_{v+w}$

Note that two kinds of clusters can be formed out of four symbols. One is partitioning four as one and three and the second is as two and two. These give four and three terms respectively. We depict a cluster by a loop around the symbols involved. In Fig 3.1 we show diagramatically the 52 terms involved in the average of $\overline{n_u n_v n_w n_r n_s}$.

Fig 3.1



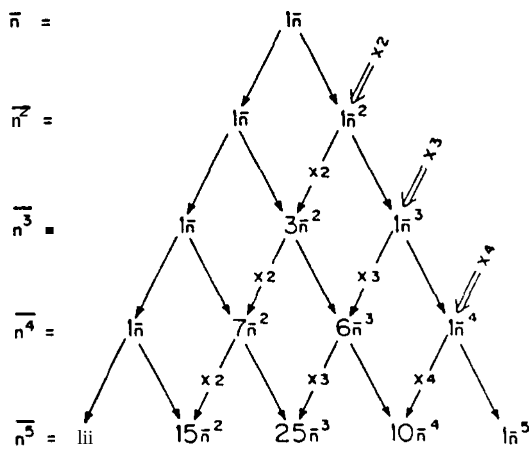


Fig 3.2

Consider just one pixel i.e. $u=v=0$ special case of this rule. We get

$$\overline{n^N} = \sum_{m=1}^{N} {}^N B_m \bar{n}^m \qquad (3.5)$$

where ${}^N B_m$ is the number of ways of forming m distinct clusters out of N symbols. The ${}^{N}B_{\ell}$'s can also be generated on the lines of Pascals triangle as shown in Fig 3.2 . The rule for generating the triangle is also shown in the figure. The numerical coefficients in the triangle give the number of terms in the more general multipixel case. One can verify this for N=5 case shown in the triangle and in Fig 3.1.

Rule b) Unbiased estimators: In the Fourier domain unbiased estimators are given by a generalization of the well-known one pixel rule (Eq 3.2):

$$UE\{\bar{n}_u\} = n_u$$

$$UE\{\bar{n}_u \bar{n}_v\} = n_u(n_v - a_{v\to u}) = n_u n_v - n_{u+v}$$

$$UE\{\bar{n}_u \bar{n}_v \bar{n}_w\} = n_u(n_v - a_{v\to u})(n_w - a_{w\to u} - a_{w\to v})$$

where the symbol $a_{v\to u}$ indicates addition of v to u. The N'th order unbiased estimator is obtained by induction. The N'th order estimator is related to the (N-1)'th through a multiplicative factor

$$UE\{\bar{n}_{u_N} \bar{n}_{u_{N-1}} \cdots \bar{n}_{u_1}\} = \Big(n_{u_N} - \sum_{m<N} a_{u_N\to u_m}\Big) UE\{\bar{n}_{u_{N-1}} \cdots \bar{n}_{u_1}\} \qquad (3.6)$$

Note that the ordering is just to avoid overcounting; the result is independent of the order in which the u's show up.

Rule c) Noise on the unbiased estimators: The Poisson average of the square modulus of the unbiased estimator in Eq 3.6 has an even simpler diagramatic interpretation. First consider special cases:

$$\overline{|UE\{\bar{n}_u\}|^2} = \bar{n}_u \bar{n}_{-u} + \bar{n}_0$$

$$\overline{|UE\{\bar{n}_u \bar{n}_v\}|^2} = \bar{n}_u \bar{n}_{-u} \bar{n}_v \bar{n}_{-v} + \bar{n}_{u-v} \bar{n}_{-u} \bar{n}_v + \bar{n}_0 \bar{n}_v \bar{n}_{-v} + \bar{n}_0 \bar{n}_u \bar{n}_{-u}$$
$$+ \bar{n}_u \bar{n}_{-v} \bar{n}_{v-u} + \bar{n}_{u-v} \bar{n}_{v-u} + \bar{n}_0^2 \qquad (3.7)$$

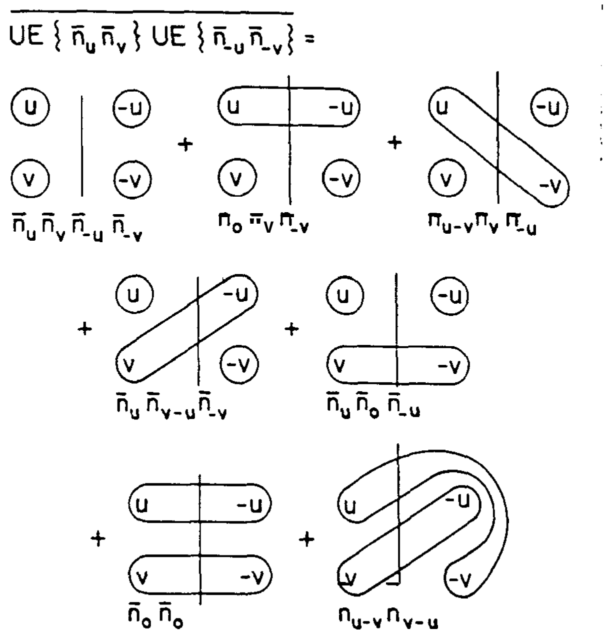


***Fig* 3.3**

We have shown in Fig 3.3 how the terms in Eq 3.7 arise. First we write in a column all the subscripts of the N'th order product for which the averaged square modulus is sought. Then we draw a vertical line and write all the conjugate (negative sign in case of frequencies) subscripts to the right of this line. Note that there are 2N symbols in all. The 2N'th order term is a product of 2N $\bar{n}_u$'s with these subscripts. For the next lower order term we draw a loop around two of the 2N subscripts. However, there are constraints to be observed. The two subscripts must be on different sides of the vertical line. The N^2 terms in the (2N-1)'th order are obtained by drawing one two loop in all possible (N^2) ways and using these 2N–1 subscripts (remember that when we draw a loop we add the subscripts involved). For example, with N=2 we have four third order terms in Eq 3.7.

Successive lower order terms are obtained by drawing more and more two- loops. We call these as two-lobps because as a part of our diagramatic rule for the variance only two symbols enter a loop as opposed to the rule for the average where any number of symbols are allowed to enter a loop. Furthermore the two symbols entering a loop must be on different sides of the verical line. Also two two-loops should not have a symbol in common for this would mean a three loop. One can verify that this rule yields all 33 terms for the bispectrum averaged modulus already known in the literature.[1,14] The # of ways of drawing m two loops with the above mentioned constraints is $\frac{N!^2}{m!(N-m)!^2}$. This is the number of terms in the (2N-m)'th order in the $\bar{n}_u$'s. As a special case the averaged square of the N'th order one pixel estimator Eq 3.2 is

$$\overline{[UE\{\bar{n}^N\}]^2} = \bar{n}^{2N} \sum_{m=0}^{N} \frac{(N!)^2}{m!\,[(N-m)!]^2}\,\bar{n}^{-m} \qquad (3.8)$$

The constraints on the way one draws two loops are reflections of the fact that the set of subscripts on either side of the line represent unbiased estimators and so do not permit drawing loops within themselves. This is a very special property of the Poisson statistics considered here. Take a product of any number of the $\bar{n}_u$s and any number of unbiased estimator of the kind $UE\{\bar{n}_u\cdots\}$. Then the average of such an expression is given by rule a) with the constraint that in any unbiased sector no loop can be drawn around more than one symbol. For example, the average

$$\overline{UE\{\bar{n}_u\bar{n}_v\cdots\bar{n}_w\}\,UE\{\bar{n}_p\bar{n}_q\cdots\bar{n}_r\}}$$

is obtained just like $\overline{UE\{\bar{n}_u\bar{n}_v\cdots\bar{n}_w\}\,UE\{\bar{n}_{-u}\bar{n}_{-v}\cdots\bar{n}_{-w}\}}$ with $p, q, \cdots, r$ playing the role of the right hand subscripts.

Rule d) Extension to a general correlation of the N'th order: So far we stated results for Fourier components **i.e.** functions of the pixel counts with weights e^{iux_i}. The results are applicable not only to weight fuctions of the form $\sum W_i \bar{n}_i$ but also to even more (most) general weight function

$$\sum_{i_1,\cdots,i_N} W_{i_1\cdots i_N} \bar{n}_{i_1} \cdots \bar{n}_{i_N} \qquad (3.9)$$

The results for this most general weight function are obtained by treating the indices on the Ws like the subscripts in the case of Fourier weights. Instead of reading the rule as "add the subscripts " we read "set the two indices equal". When we set two or more indices equal to say i only one $\bar{n}_i$ appears. For the special case of Fourier weights this ammounts to adding the frequencies. More specifically, the unbiased estimator of the quantity is obtained by replacing the product $\bar{n}_{i_1} \cdots \bar{n}_{i_N}$ by $n_{i_1} \cdots (n_{i_N} - \sum_{m \neq N} \delta_{i_N i_m})$ where the symbol $a_{v \to u}$ is now replaced by a Kronecker δ symbol. To obtain the averaged square modulus of the unbiased estimator of the quantity Eq 3.9 repeat proceedure c) above with the indices of W playing the role of u, v ... and the indices of W playing the role of the conjugate subscripts -u, -v

For example, with $N=2$

$$\overline{UE\{\sum_{ij} W_{ij}\bar{n}_i\bar{n}_j\}\, UE\{\sum_{kl} W^*_{kl}\bar{n}_k\bar{n}_l\}} = \sum_{ijkl} W_{ij}W^*_{kl}\bar{n}_i\bar{n}_j\bar{n}_k\bar{n}_l$$

$$+\sum_{ijk} W_{ij}(W^*_{ik}+W^*_{ki}+W^*_{jk}+W_{kj})\bar{n}_i\bar{n}_j\bar{n}_k + \sum_{ij} W_{ij}(W^*_{ij}+W^*_{ji})\bar{n}_i\bar{n}_j$$

3.3 CONCLUSIONS

We have derived results of reasonable generality on the Poisson noise, due to the photonic nature of light, on image intensity correlations. The results may be very useful if correlations of high orders need to be considered. Even for lower order correlations the generality of the weight functions considered allows one to discuss issues related to optimum weight function one should use to extract a particular piece of information from the correlations.

APPENDIX A3

We have used the well known technique of moment generating functions to the derive results presented in this chapter. First we give a derivation of one pixel results for simplicity and as an illustration of the proceedure followed. The relevance of single pixel results is that the numerical coefficients in this case give the number of terms in the general case.

Let n be a Poisson variable with probability distribution

$$P(n) = e^{-\bar{n}} \frac{\bar{n}^n}{n!} \tag{3.10}$$

We define the moment generating function as

$$\psi(\lambda) = \overline{e^{\lambda n}} = \exp\{\bar{n}(e^{\lambda}-1)\} \tag{3.11}$$

Average of n^N is obtained by taking the N'th derivative of the moment generating function with respect to λ, and then setting $\lambda=0$. It is easy to verify by successive differentiation the rule for numerical coefficients is that given by the triangle rule (Fig 3.2).

Let

$$\hat{d} = \frac{d}{d\lambda} \quad \text{then} \quad \hat{d}^N \psi(\lambda)\Big|_{\lambda=0} = \overline{n^N} \tag{3.12}$$

The operator

$$\hat{D} = e^{-\lambda}\hat{d} \tag{3.13}$$

satisfies

$$\hat{D}^N \psi(\lambda) = \bar{n}^N \psi(\lambda) \tag{3.14}$$

Note that in **Eq** 3.14 λ can take any value while in **Eq 3.12** it is zero. The interplay between the operators $\hat{d}$ and $\hat{\mathcal{D}}$ respecting the constraint $\lambda=0$ is the key element of our derivation. When we have expression in $\bar{n}$ and want to get it in terms of n we start replacing $\bar{n}$ by $\hat{\mathcal{D}}$ and then write this in terms of $\hat{d}$ and vice versa. This interconversion of $\hat{\mathcal{D}}$ and $\hat{d}$ allows one to obtain unbiased estimators and the noise on them. Consider the operator

$$\hat{\mathcal{D}}^N=(e^{-\lambda}\hat{d})^N \tag{3.15}$$

which acting on ψ gives $\bar{n}^N$ as an eigenvalue. The unbiased estimator for $\bar{n}^N$ is obtained by expanding $(e^{-\lambda}\hat{d})^N$ in such a way that in any term the $\hat{d}$'s appear to the right of $e^{-\lambda}$. Such expression can be readily obtained

$$(e^{-\lambda}\hat{d})^N=e^{-N\lambda}(\hat{d}-N+1)\cdots(\hat{d}-1)\hat{d} \tag{3.16}$$

Now the operator $\hat{d}$ or any polynomial in $\hat{d}$ is meaningful only with the constraint $\lambda=0$. Therefore one can remove $e^{-N\lambda}$ (which is unity when $\lambda=0$) and get the unbiased estimator of $\bar{n}^N$ by replacing $\hat{d}$ by n:

$$UE\{\bar{n}^N\}=n(n-1)\cdots(n-N+1) \tag{3.17}$$

which is a well-known result. To get the average of the square of this we start with the operator

$$[\hat{d}(\hat{d}-1)\cdots(\hat{d}-N+1)]^2 \tag{3.18}$$

which is equivalent to $[n(n-1)\cdots(n-N+1)]^2$. The idea now is to express this in terms of the operator $\hat{\mathcal{D}}$ in such a way that all factor in powers of e^{λ} appear to the left (this a way of making things valid for all λ and thus circumventing the constraint on in Eq **3.12**. From Eq **3.16 we** get

$$[\hat{d}(\hat{d}-1)\cdots(\hat{d}-N+1)]^2=e^{N\lambda}(\hat{d}+N)(\hat{d}+N-1)\cdots(\hat{d}+1)\hat{\mathcal{D}}^N$$

One has to expand the operator $(\hat{d}+1)\cdots(\hat{d}+N)$ in powers of $\hat{D}$.
But as $\hat{D}^m = e^{-m\lambda}\hat{d}(\hat{d}-1)\cdots(\hat{d}-m+1) = \frac{\cancel{e^{-m\lambda}\hat{d}!}}{(\hat{d}-m)!}$ (the last relation is purely notational). So let

$$(\hat{d}+1)\cdots(\hat{d}+N) = \sum_{m=0}^{N} {}^{N}A_m \frac{\hat{d}!}{(\hat{d}-m)!} = \sum_{m=0}^{N} {}^{N}A_m \, e^{m\lambda} \hat{D}^m \qquad (3.19)$$

One can obtain the following recursion relations for the coefficients ${}^{N}A_m$ by relating $(\hat{d}+N)\cdots(\hat{d}+1)$ to $(\hat{d}+N-1)\cdots(\hat{d}+1)$:

$${}^{N}A_0 = N\left({}^{N-1}A_0\right), \quad {}^{N}A_N = {}^{N-1}A_{N-1}$$

$$\textit{otherwise} \quad {}^{N}A_m = (N+m)\left({}^{N-1}A_m\right) + {}^{N-1}A_{m-1} \qquad (3.20)$$

One can show that the diagramatic rule for the variance satisfies this recursion i.e.

$${}^{N}A_m = \frac{(N!)^2}{(N-m)!\,(m!)^2} \qquad (3.21)$$

as given by the two loop rule.

In the case of a pixel array one needs to introduce one generator, say λ_i, for every pixel. The moment generating fuction is product of the moment generating functions for individual pixels as the Poisson fluctuations n_i are independent:

$$\Psi(\{\lambda_i\}) = \overline{\exp\{\sum_i \lambda_i n_i\}}$$

$$= \exp\{\sum_i \bar{n}_i (e^{\lambda_i}-1)\} \qquad (3.22)$$

Any N'th order statistics is of the form

$$\sum_{i_1,\cdots,i_N} W_{i_1\cdots i_N} \, \overline{n_{i_1}\cdots n_{i_N}} \qquad (3.23)$$

where the N dummy indices i's take all possible pixel lables. The operator equivalent of this general correlation is

$$\sum_{i_1,\cdots,i_N} W_{i_1\cdots i_N} \, \hat{D}_{i_1}\cdots\hat{D}_{i_N} \qquad (3.24)$$

where analogous to the one pixel case we define the operators $\hat{d}_i$.

and $\hat{\mathcal{D}}_i$ for every pixel:

$$\hat{d}_i = \frac{\partial}{\partial \lambda_i} \quad ; \hat{\mathcal{D}}_i = e^{-\lambda_i} \hat{d}_i \tag{3.25}$$

The unbiased estimator for the correlation Eq 3.23

$$\sum_{i_1,\dots,i_N} W_{i_1\dots i_N}\, n_{i_1}(n_{i_2}-\delta_{i_2 i_1})\cdots(n_{i_N}-\sum_{m<N}\delta_{i_N i_m}) \tag{3.26}$$

follows from the identity

$$\hat{\mathcal{D}}_{i_1}\hat{\mathcal{D}}_{i_2}\cdots\hat{\mathcal{D}}_{i_N} = e^{-\sum_j \lambda_{i_j}}\, \hat{\partial}_{i_1}(\hat{\partial}_{i_2}-\delta_{i_2 i_1})\cdots(\hat{\partial}_{i_N}-\sum_{m<N}\delta_{i_N i_m}) \tag{3.27}$$

The squared modulus of the unbiased estimator Eq 3.26 is equivalent to the operator (and all $\lambda_i=0$ afterwards)

$$\sum_{\substack{i_1,\dots,i_N\\ l_1,\dots,l_N}} W_{i_1\dots i_N} W^*_{l_1\dots l_N}\, \hat{\partial}_{i_1}\cdots(\hat{\partial}_{i_N}-\sum_{m<N}\delta_{i_N i_m})\hat{\partial}_{l_1}\cdots(\hat{\partial}_{l_N}-\sum_{m<N}\delta_{l_N l_m}) \tag{3.28}$$

To get the average we express this in terms of the operator $\hat{\mathcal{D}}$'s in such a way that all e^{λ_i} factors are to the left and then ignore such factors. Such expression is

$$\sum_{\substack{i_1,\dots,i_N\\ l_1,\dots,l_N}} W_{i_1\dots i_N} W^*_{l_1\dots l_N}\, e^{\sum_k \lambda_{i_k}}\, \hat{\mathcal{D}}_{i_1}\cdots\hat{\mathcal{D}}_{i_N}\, e^{\sum_j \lambda_{l_j}}\, \hat{\mathcal{D}}_{l_1}\cdots\hat{\mathcal{D}}_{l_N} \tag{3.29}$$

By repeated application of the identity

$$\hat{\mathcal{D}}_{i_N} e^{\sum_j \lambda_{l_j}} = e^{\sum_j \lambda_{l_j}} \left(\hat{\mathcal{D}}_{i_N} + e^{-\lambda_{i_N}} \sum_j \delta_{i_N l_j}\right) \tag{3.30}$$

the above expression becomes (ignore all e^{λ_i} occuring on the left)

$$\sum_{\substack{i_1,\dots,i_N\\ l_1,\dots,l_N}} W_{i_1\dots i_N} W^*_{l_1\dots l_N} (\hat{\mathcal{D}}_{i_1}+\sum_j \delta_{l_j i_1})(\hat{\mathcal{D}}_{i_2}-\delta_{i_2 i_1}+\sum_j \delta_{l_j i_2})\cdots \times$$
$$\times(\hat{\mathcal{D}}_{i_N}-\sum_{m<N}\delta_{i_N i_m}+\sum_j \delta_{l_j i_N})\hat{\mathcal{D}}_{l_1}\cdots\hat{\mathcal{D}}_{l_N} \tag{3.31}$$

We note that the diagramatic rules stated in this chapter follow from interpreting the way the δ's appear. The $\hat{\mathcal{D}}$'s can now be replaced by $\bar{n}$'s.