CHAPTER 2

Detection of parity of a binary star in triple correlation speckle interferometry: limiting faintness

2.0 Introduction

In this chapter we present focal plane calculations of the SNR for detecting the parity of a binary star. Parity is defined as the side of the brighter component assuming that the line joining the stars has already been determined by two-point correlation analysis. As mentioned before the power spectrum is not enough to reconstruct the object uniquely. In the case of a binary, the autocorrelation tells us the fluxes of the two sources and their separation but not whether the brighter source is on the left or otherwise. In other words, the position angle of the vector joining the brighter to the fainter component is ambiguous by 180^o. Our reason to consider a binary is as follows. A binary is perhaps the simplest object needing phase recovery schemes for unambiguous reconstruction. For such a simple object the natural correlations are in the focal plane. The object being simple in the focal plane, only few focal plane correlations are of interest. Fourier domain calculations of the SNR for determing individual phases of a general object involve two factors: 1) the SNR for the bispectrum, 2) a factor representing improvement due to the redundancy of phase information stored in the bispectrum. By restricting oneselves to binaries, as the following calculations show, it is possible to eliminate the intermediate step of calculating the bispectrum. Also in this case the phases themselves are of little importance as they do not individually refer to any specific feature in the source. Thus it is possible to combine information from all the N_S phases to get information about the parity of the binary. Not only do we expect the focal plane calculations to be simpler but we also expect the focal plane statistics to have better SNR. We expect

$$\text{Binary} \qquad SNR_{PARITY} \sim N_S^{1/2} \; SNR_{PHASE} \qquad (2.1)$$

In this case of a binary, the situation is quite similar to the second order statistics where focal plane correlation[3] has better SNR than that for an individual power spectrum[4] component:

$$\text{Binary} \qquad SNR_{PARITY} \sim N_S^{1/2} \; SNR_{POWER\ SPECTRUM} \qquad (2.2)$$

For a general object the symmetry in the Fourier transform and its inversion implies that SNR will be similar for individual elements in the focal and the Fourier domain. For simpler objects like a binary only a few focal plane correlations are important. These correlations can be thought of as a result of combining N_S frequency domain correlations. As a consequence these special focal plane correlations have SNR $N_S^{1/2}$ times better than the SNR for a typical frequency domain correlation.

Before we start on the SNR calculations for the parity we briefly review the physical motivation for the triple correlation method. The presentation is along the lines given in Weigelt's pioneering paper but deals with the simpler case of no atmospheric noise. In the absence of atmospheric noise the image of a binary contains just two spikes at the positions of the components Fig 2.1. In this case it is, of course, possible to measure the fluxes of the two stars and no ambiguity exists. However, we choose to discuss correlations because the results then carry over, as shown later, even in the presence of atmospheric noise. If we measure the first order statistics N_1+N_2 and the symmetric second order statistics $N_1^2+N_2^2$ and N_1N_2 then we are left with the ambiguity of the parity of the system. In Fig 2.2 we show the autocorrelation for the binary in Fig 2.1.

Fig 2.1

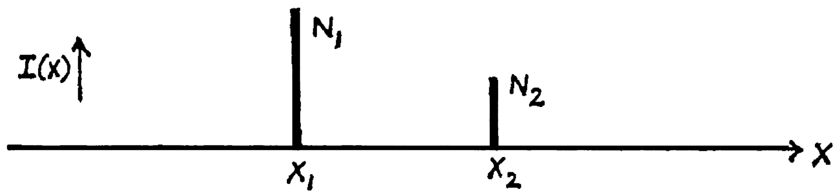
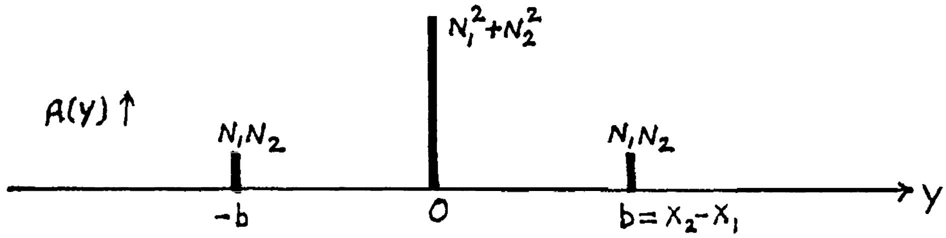


Fig 2.2



By definition the autocorrelation

$$A(Y) = \int d^2X\, I(X)I(X+Y)$$

is focal plane mean of the product $\langle I(X)I(X+Y)\rangle$. This pair correlation depends on the relative displacement Y of the two copies $I(X)$ and $I(X+Y)$ of the focal plane intensity. In the case of a binary such pair correlation, in the absence of the atmosphere, exists only for three values of the relative displacement Y. These cases are shown in Fig 2.3. To get the autocorrelation from these product functions $\langle I(X)I(X+Y)\rangle$ one has to integrate over the focal plane coordinate. The autocorrelation is a fuction of the fluxes N_1 and N_2 of the components. In the (N_1, N_2) plane the measured autocorrelation elements $N_1^2+N_2^2=A$ and $N_1N_2=B$ say, represent a circle and a pair of hyperbolas respectively. From Fig 2.4 we see that these two curves meet in

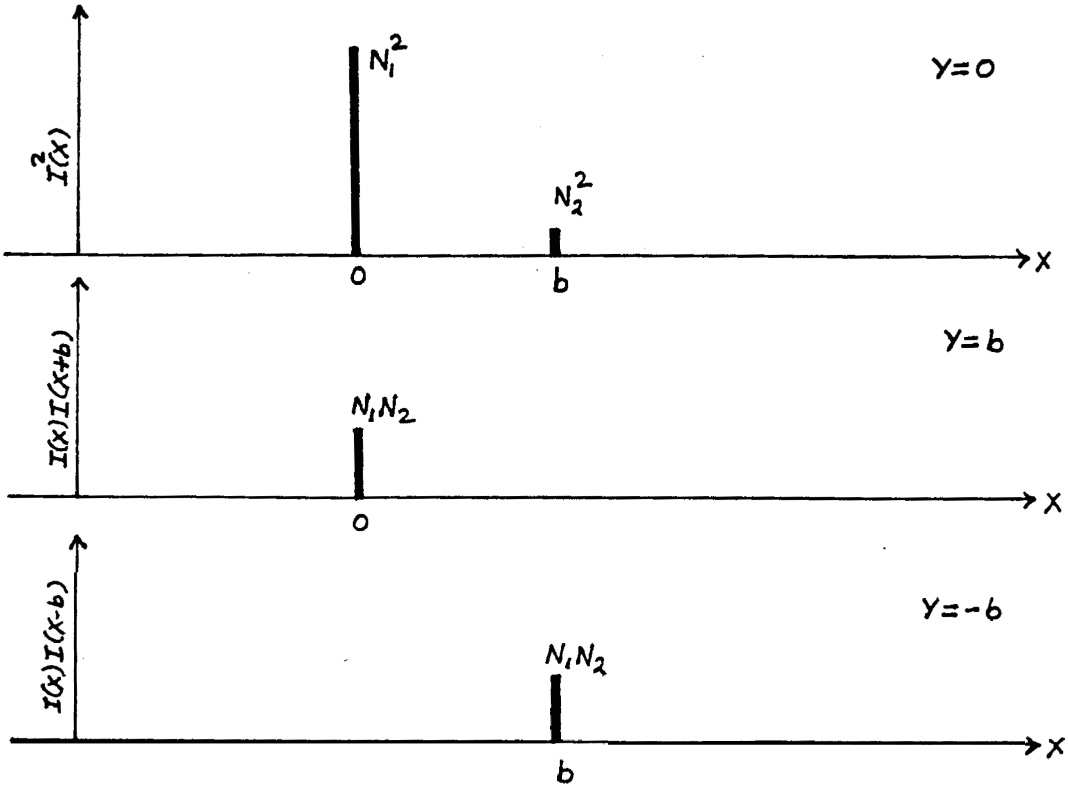


Fig 2.3 The product fuction $I(X)I(X+Y)$ is nonzero only for three values of y shoon above.

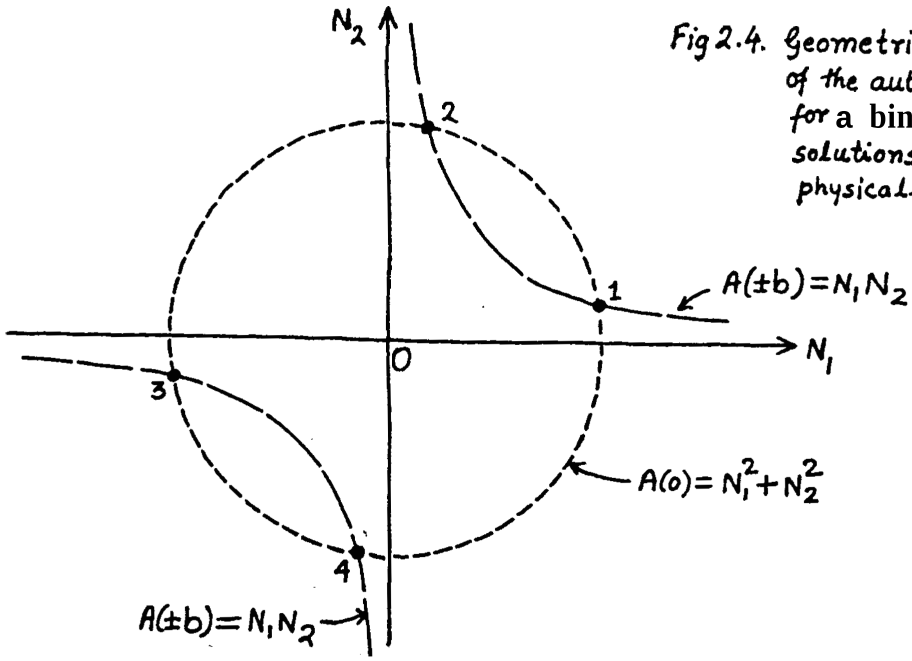


Fig 2.4. Geometrical presentation of the autocorrelation eq^{ns} for a binary: of ffic four solutions, 1 and 2 represent physically allowed solutions.

four points. Two points lie in the unphysical negative flux quadrant. The remaining two are physically allowed solutions and represent the ambiguity of the parity (these solutions are symmetric about the $N_1 = N_2$ line). The total flux which is a straight line $N_1 + N_2 = C$ say, does not remove this ambiguity. All these correlations are symmetric in N_1 and N_2 so can not tell the parity.

One may say that why make a big deal about parity, after all moon does not look strange when seen upside down through an astronomical telescope. Consider then a linear triple star shown in Fig 2.5 with the same inter star separation. In this case it

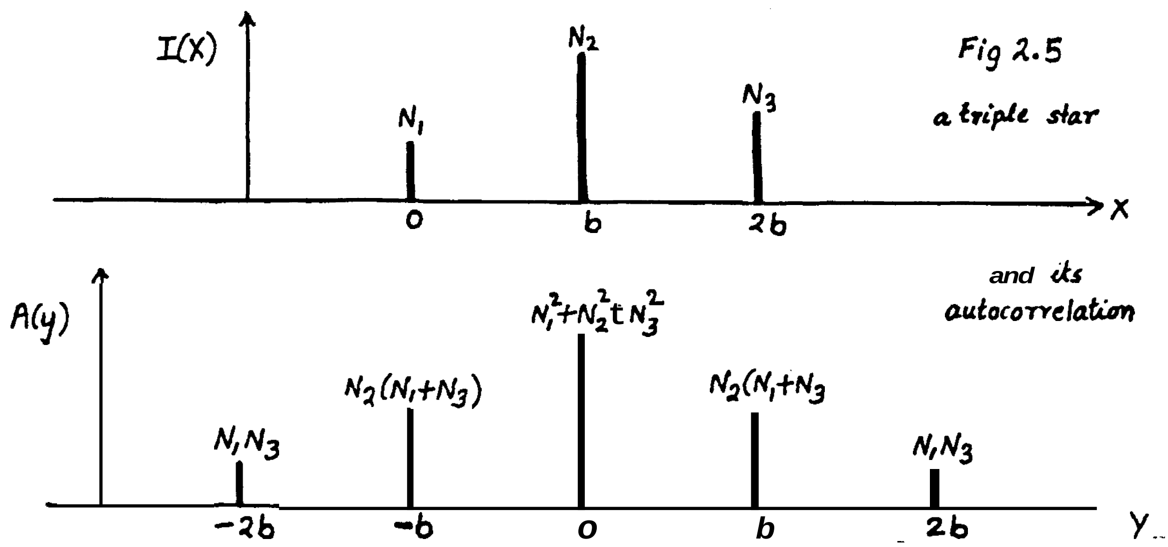


Fig 2.5
a triple star
and its
autocorrelation

can be shown that there are eight solutions to the autocorrelation equations (Fig 2.5) of which four are in the unphysical octant of negative fluxes in the space of (N_1, N_2, N_3). The remaining four can be grouped into two sets. The sets differ in the intensities of the three stars and within a set there is the ambiguity of parity. To be specific let

$$N_1^2 + N_2^2 + N_3^2 = A \quad ; \quad N_2(N_1+N_3) = B \quad \text{and} \quad N_1 N_3 = C$$

then

$$N_2 = \pm\left[\frac{1}{2}(A+2C) \pm \frac{1}{2}\sqrt{(A+2C)^2 - 4B}\,\right]^{1/2}$$

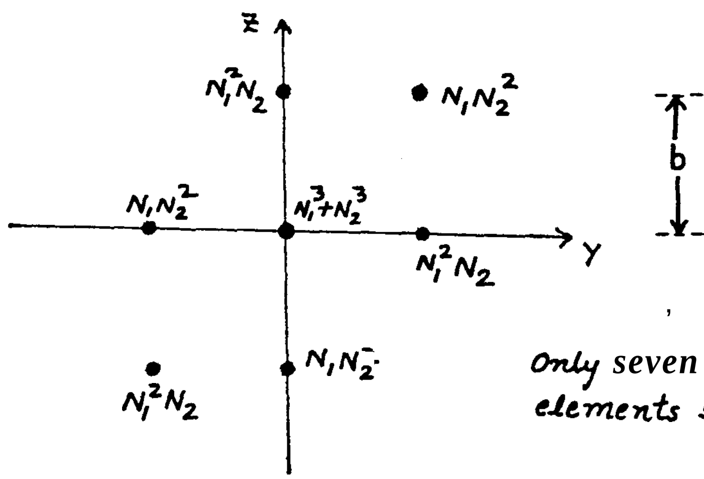
$$N_1 = \frac{B}{2N_2} \pm \left[\frac{B^2}{N_2^2} - 4C\right]^{1/2} \qquad N_3 = \frac{B}{2N_2} \mp \left[\frac{B^2}{N_2^2} - 4C\right]^{1/2}$$

In this example the additional ambiguity has its origin in redundant separations. If a source contains isolated point sources with nonredundant separations then parity is the only ambiguity left over after measuring its autocorrelation (assuming that the later can be determined with good SNR). The interesting question of multiple solutions of the autocorrelation is not addressed here. We consider the parity as much a representative of the ambiguities in the reconstruction from the autocorrelation as astronomically important in itself. In astronomical situations where the environment is known at other wavelengths (radio for example) the knowledge about the parity of the binary (the central enigine and a one sided jet, for instance) may be crucial in interpretation. In gravitational lens pair delay, again, the identification is important.

The triple correlation removes this ambiguity. The reason why triple correlation works becomes clear if we note that the intermediate two-product fuctions in Fig 2.3 are just delta functions for some values of the relative displacement of the two copies of the image. If we now make one more copy of the image and convolve it with a delta function due to previous two copies then we recover the source structure without any ambiguity. This was the intuitive step that lead Weigelt to propose the triple correlation method to solve the phase problem in speckle interferometry. The original focal plane triple correlation method has the name "speckle masking". The basic motivating principle was that for simple enough sources one could get an intermediate delta function by a suitable choice of relative displacement. In the presence of atmospheric noise the result of such intermediate step will be the point spread function with some noise due to random overlap of the two copies. The triple correlation method is, however, not limited to simple sources but should also work for complex objects. In Fig 2.6 we show the triple correlation for the binary shown in Fig 2.1. Note that the central element is symmetric in N_1 and N_2 while other six terms are asymmetric in N_1 and N_2. If we concentrate on the top horizontal segment we see that the binary is unambiguously restored. The overall $N_1 N_2$ is the sign of the delta function in the intermediate pair correlation. One can combine the six elements to get a single antisymmetric statistic $N_1^2 N_2 - N_1 N_2^2$ which removes the ambiguity of parity.

Fig2.6
Triple correlation for a binary

no atmospheric noise



Only seven nonzero elements shown ●

2.1 Model of the point spread function

The triple correlation is a third order statistic in the intensities. The noise on the triple correlation contains terms of third, fourth, fifth and sixth order in the intensities. The square of the triple correlation is of the sixth order in intensities. The other lower order terms in the noise appear because of photonic nature of light (see Eq 2.19). The intensities themselves are second order quantities considering the fields as the basic quantities. So a rigorous calculation should involve sixth, eighth, tenth and twelfth order integrals in the fields. A simpler model is therefore welcome. Low flux results based on field-correlations are presented in chapter 5. In the following calculations we make reasonable approximations about the point spread function (PSF). First of all, we take the seeing disk to be uniform and neglect all "edge" effects which we discuss in chapter 5. Secondly, we divide the focal plane into pixels with the diffraction limited size. Since the telescope aperture acts as a filter for spatial frequencies we expect the intensities over a pixel to be correlated. We, therefore, approximate the intensity correlations in the focal plane as follows. Intensity over any pixel is regarded as uniform and intensities over different pixels are uncorrelated. Thus in our model the point source response is completely specified by intensities μ_i's at the i'th pixel. The μ_i's are statistically independent and have the same distribution for all i within the seeing disk.

We further assume that the intensity at

any pixel varies in time with a Rayleigh (exponential) distribution. We do consider other distributions in Section 2.5 but as the following argument shows the Rayleigh distribution for the μ_i's is perhaps the natural choice. Any point in the focal plane receives complex fields from roughly N_S different correlation patches in the pupil plane. The resultant of such addition of large number of complex fields is a complex number whose real and imaginary parts have a Gaussian distribution because of the central limit theorem. The intensity which is the modulus of the resultant field is therefore distributed according to the Rayleigh statistics (Rayleigh, I, 491):

$$P(\mu)\, d\mu = (d\mu/\langle\mu\rangle) \exp(-\mu/\langle\mu\rangle)$$

$$\text{or} \quad \langle\mu^m\rangle = m!\langle\mu\rangle^m \quad \text{m: nonnegative integer} \tag{2.3}$$

As one moves away from a point in the focal plane the intensity will start getting decorrelated with that at the first point. So strictly speaking the intensity within a speckle-sized 'pixel will show variations and the statistics may deviate from the Rayleigh statistics which holds for intensity at one point. Since the focal plane intensity correlation length is of the order of the pixel size we expect the statistics to be close to Rayleigh. Deviations from this statistics can be checked by more detailed calculations dealing with field correlations (chapter 5). The present statistical model allows us to deal with the intensities themselves, thus reducing the order of correlations one has to deal with. We also assume that the fluctuation in the number of photons detected in a pixel are described by a Poisson distribution with the instantaneous intensity as the mean. Strictly speaking there are correlations of the Brown-Twiss type but as mentioned in the introduction (section 0.1 page 4) these are negligible for speckle interferometric observations. So our averaging proceedure involves two steps. In the first step we do the Poisson average for a given focal plane intensity distribution. In the second step we do the classical averaging. We denote the Poisson average by $\overline{}$ and the classical average by $\langle\ \rangle$. Later, in Section 2.7 we show that above model for the system response correctly reproduces the known results for the autocorrelation of a binary.

2.2 Parity statistics for a binary: expression

In our model for the focal plane image of a point source intensities over different pixels are uncorrelated. However, another point source within the isoplanatic patch will give an exactly similar but shifted intensity pattern. In the high flus limit the speckle patterns due to the two stars have the sane relative intensity as the true binary. In this chapter we use the word 'speckle' for contribution to pixel intensity due to a single star. So all pixels, except near the edge of the seeing disk, receive two speckles: one each due to the two stars. We are mainly concerned with cases where the binary separation is smaller than the seeing disk and so neglect any edge effects. More specifically, we denote the speckle intensity due to star 1 (on the left) at the pixel i by μ_i. This pixel also receives a speckle due to star 2. The intensity of this speckle is same (appropriately scaled) as the intensity of the speckle due to star 1 at the (i-b)'th pixel where b is the binary separation. The speckle at the i'th pixel due to star 2 has the intensity $\nu_{i-b} = \frac{\mathcal{N}_2}{\mathcal{N}_1} p_{i-b}$ where $\frac{\mathcal{N}_2}{\mathcal{N}_1}$ is the true ratio of the intensity of star2 to that of star 1. If we denote the total intensity at the i'th pixel by $\overline{n}_i$ then

$$\overline{n}_i = \mu_i + \nu_{i-b} = \mu_i + \frac{\mathcal{N}_2}{\mathcal{N}_1}\mu_{i-b} \quad ; \langle \mu \rangle = \mathcal{N}_1 \qquad (2.4)$$

Since the μ_i's are statistically independent two $\overline{n}_i$'s say $\overline{n}_i$ and $\overline{n}_j$ are also statistically independent unless either i=j or j=i±b. The correlation between $\overline{n}_i$ and $\overline{n}_{i\pm b}$ is due to a pair of speckles common to these pixels. This pair of speckles has the same relative intensity as the binary. It can be easily checked that in our model (which neglects edge or gradient effects) the general focal plane triple correlation

$$\langle T_{kj} \rangle = \sum_i \langle \overline{n}_i \, \overline{n}_{i+k} \, \overline{n}_{i+j} \rangle \qquad (2.5)$$

is symmetric in the average per speckle source strengths $\mathcal{N}_1$ and $\mathcal{N}_2$ except for following six cases for which T_{kj} is asymmetric in $\mathcal{N}_1$ and $\mathcal{N}_2$:

1) $k=b, j=0$ 2) $k=0, j=b$ 3) $k=-b, j=-b$ (2.6)

4) $k=b, j=0$ 5) $k=-b, j=0$ 6) $k=0, j=-b$ (2.7)

It can be seen from the definition that the three triple correlations in Eq 2.6 are identical, and the same is true of the three triple correlations in Eq 2.7. Note that this is true without the average denoted by $\langle \quad \rangle$ and therefore these three values are identical in all realizations. So there is no gain in SNR by a factor of $\sqrt{3}$ on combining these three values. Out of the two sets of triplets we choose only two statistically independent terms

$$T_{ob} = \sum_i \bar{n}_i^2 \bar{n}_{i+b} \quad ; \; T_{bb} = \sum_i \bar{n}_i \bar{n}_{i+b}^2$$

It is possible to combine these two terms so as to get a single parity statistic that is antisymmetric in x_1 and x_2. Thus the only third order correlation that contains the parity information is:

$$\langle \bar{P} \rangle = \sum_i \langle (\bar{n}_i^2 \bar{n}_{i+b} - \bar{n}_i \bar{n}_{i+b}^2) \rangle \tag{2.8}$$

where n_i is number of photons in the i'th pixel, b is binary separation. Note that the expresion in Eq 2.8 is unbiased under Poisson fluctuations i.e. the unbiased estimator of the parity statistics is

$$UE\{\bar{P}\} = \quad P = \sum_i (n_i^2 n_{i+b} - n_i n_{i+b}^2) \tag{2.9}$$

We can write this as

$$P = \sum_i p_i \quad ; \quad p_i = n_i^2 n_{i+b} - n_i n_{i+b} \tag{2.10}$$

where p_i can be looked at as contribution due to i'th pair of pixels consisting of the i'th and (i+b)'th pixel. The rest of this chapter deals with SNR for the parity statistics, Eq 2.9, derived above.

2.3 SNR for the Parity detection at low light levels:

Results on the SNR for this parity statistic for general light levels are presented in Section 2.6. The algebraic complexity makes the general case physically less transparent. Here we, therefore, treat the case of low light levels for which the algebra is simpler and the physical origin of various contributions to the SNR clearer. At low flux the number of pairs of pixels giving nonzero values of parity is small as compared to the total number of pairs possible. It will be shown later that their overlap contributes to the fourth and higher order terms in the variance and not to the lowest third order. Since we approximate the seeing disk by a uniform disk and since the overlap in the different pairs of pixels does not contribute in the third order it is enough to consider a representative pair of pixels with the binary separation. The second simplification is that such a representative pair of pixels must not receive all the three photons in one pixel. As the table 2.2 shows one of these pixels must receive one photon and the other two. It is therefore enough to use the truncated Poisson distribution shown in table 2.1 which gives. the probability P(n) of detecting n photons when the mean is $\bar{n}$.

Table 2.1 Truncated Poisson distribution

n	0	1	2
P(n)	$1-\bar{n}+\frac{1}{2}\bar{n}^2$	$\bar{n}-\bar{n}^2$	$\frac{1}{2}\bar{n}^2$

The SNR due to one such pair must then be multiplied by $N_s^{1/2}$ to get the SNR for parity statistics.

Consider, then, a pair of pixels with the binary separation, with intensity $\bar{n}$ and $\bar{n}_1$ for one atmospheric realization. We suppress the subscript i and denote $\bar{n}_i$ by $\bar{n}$ $\bar{n}_{i+mb}$ by $\bar{n}_{mb}$. In all meaningful correlations b is the basic displacement. Later on we shall denote $\bar{n}_{mb}$ by $\bar{n}_m$ for simplicity.

The parity statistics for this representative pair of pixels becomes $p = n^2 n_1 - n n_1^2$ which takes values with the probabilities shown in the tables 2.2 and 2.3 respectively.

Table 2.2 Parity as function of and

$n \backslash n_1$	0	1	2
0	0	0	0
1	0	0	-2
2	0	2	0

Table 2.3 Probability distribution for Parity

Parity	Probability
2	$\frac{1}{2}\bar{n}^2\bar{n}_1$
-2	$\frac{1}{2}\bar{n}\bar{n}_1^2$
0	remainder

It is clear from table 2.3 that the Poisson average and variance (upto third order) of the parity statistics for one pair is

$$\bar{p} = \bar{n}^2\bar{n}_1 - \bar{n}\bar{n}_1^2$$
$$\overline{p^2} - \bar{p}^2 \sim \overline{p^2} = 2(\bar{n}^2\bar{n}_1 + \bar{n}\bar{n}_1^2) \qquad (2.11)$$

One should have subtracted $\bar{p}^2$ but this of sixth order in the flux per speckle and neglected here. However, the atmospheric average remains to be done. Now the pixel on the left contains a speckle due to star 1 with intensity μ and the pixel on the right contains corresponding speckle due to star 2 with intensity $\nu = \frac{N_2}{N_1}\mu$. In addition to this correlated pair the pixel on the left will contain a speckle due to star 2 with intensity ν_- and the one on the right will contain a speckle due to the star 1 with intensity μ_1. The quantities with different subscripts are uncorrelated to each other. On performing the atmospheric average we get for the per pair parity average

$$\langle \bar{p} \rangle = \{\langle \mu^3 \rangle - 3\langle \mu^2 \rangle \langle \mu \rangle + 2\langle \mu \rangle^3\} \frac{N_2(N_1 - N_2)}{N_1^2} \qquad (2.12)$$

Note that the Eq 2.12 does not make any assumptions about the distribution of μ and allows us to discuss other statistics as well. We now assume (see Section 2.1) the intensities of the speckles to be distributed according to the Rayleigh statistics. We discuss other interesting cases in Section 2.5). The average and the variance of the per pair parity is:

$$\langle \bar{p} \rangle = 2\mathcal{N}_1 \mathcal{N}_2 (\mathcal{N}_1 - \mathcal{N}_2) \qquad (2.13)$$

$$\langle \bar{p}^2 \rangle = 4(2\mathcal{N}_1^3 + 7\mathcal{N}_1^2 \mathcal{N}_2 + 7\mathcal{N}_1 \mathcal{N}_2^2 + 2\mathcal{N}_2^3) \qquad (2.14)$$

This gives our estimate for the low flux SNR for the parity

$$SNR_{PARITY} = \frac{q^{3/2} M^{1/2} N_S^{1/2} \mathcal{N}_1 \mathcal{N}_2 (\mathcal{N}_1 - \mathcal{N}_2)}{[2\mathcal{N}_1^3 + 7\mathcal{N}_1^2\mathcal{N}_2 + 7\mathcal{N}_1\mathcal{N}_2^2 + 2\mathcal{N}_2^3]^{1/2}} \qquad \mathcal{N}_1, \mathcal{N}_2 < 1 \qquad (2.15)$$

where M frames of data are used, q is detector efficiency (optics+quantum). Note that for one realization this is consistent with our calculation in the frequency domain Eq 1.17. and expectation Eq 2.1 relating SNR for parity to the SNR for phase (take $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_1 - \mathcal{N}_2 \sim \mathcal{N}$. Priliminary results on the low flux SNR were presented at the NOAO-ESO conference[11] on 'High resolution imaging by interferometry' (Karbelkar 1988).

2.4 Parity detection in the presence of sky background

In this section we consider the effect of a uniform sky background of K photons on the average, per pixel per exposure. In the previous section it was seen that only events registering one photon in one pixel and two in the other contribute to the parity information. In the absence of sky background noise these photons come from the binary and so contain information about the parity of the binary. Sky background can mimic parity events: as an extreme example all the three photons may come from the background. Assuming uniform background, such spurious events will have their negatives so on an average there is no signal due to the background. But, of course, there will be additional noise due to such spurious events. To calculate the fluctuations in parity due to photon noise one should convolve the Poisson fluctuations in the photons from the binary (as before) with the fluctuations in the photons from the background. Since sum of Poisson fluctuations is again a Poisson fluctuation with the sum of the means the calculation of probabilities in table 2.3 continues to hold with

$$\bar{n}_i = \mu_i + \nu_{i-b} + K \tag{2.16}$$

instead of Eq 2.4 which is true for no background. Assuming that the speckles have Rayleigh distribution as before we get for one pair of pixels:

$$\langle \bar{p} \rangle = 2\mathcal{N}_1 \mathcal{N}_2 (\mathcal{N}_1 - \mathcal{N}_2) \tag{2.17}$$

$$\begin{aligned} \langle \bar{p}^2 \rangle = {} & 8\mathcal{N}_1^3 + 14\mathcal{N}_1^2\mathcal{N}_2 + 14\mathcal{N}_1\mathcal{N}_2^2 + 8\mathcal{N}_2^3 + 8K(\mathcal{N}_1^2 + 3\mathcal{N}_1\mathcal{N}_2 + \mathcal{N}_2^2) \\ & + 12K^2(\mathcal{N}_1 + \mathcal{N}_2) + 4K^3 \end{aligned} \tag{2.18}$$

Stationarity for the statistics of the μ_i's was used in getting Eq 2.18. Strictly speaking one must subtract the square of the averaged parity but this is of the sixth order and will be taken care of in section 2.6 where all orders are considered. Combining Eq 2.17 and Eq 2.18 we get the SNR for the parity in the presence of sky background.

2.5 Effect of various intensity distributions for pixels

We now check that the general formula Eq 2.12 gives correct results in other limiting cases of the statistics for the pixel intensities. Consider a long exposure image. The intensities on the pixels are positive. However as result of a large number of speckles one may get, to the zeroth order, a constant intensity as in the case of long exposure images. To higher order one could approximate the intensities to be Gaussianly distributed around the mean because of the central limit theorem. The variance of this Gaussian component has to be much smaller than the mean so that the unphysical negative intensities predicted by the approximation have negligible probability. Such a distribution is represented by $\mu = 1 + x$ where x is a zero mean Gaussian with $\langle x^2 \rangle \ll 1$. For this distribution

$$\langle \mu \rangle = 1 \quad ; \quad \langle \mu^2 \rangle = 1 + \langle x^2 \rangle \quad ; \quad \langle \mu^3 \rangle = 1 + 3\langle x^2 \rangle$$

and there is no parity signal as can be verified by direct substitution in Eq 2.12. The Gaussian distribution may arise in another way. Consider a more complex source than a binary. Now every pixel gets a speckle due to every component. The resultant of such large number of independent Rayleigh distributions is a Gaussian distribution. As is well known for Gaussian distribution the bispectrum is identically zero. Note that the expression in the curly brackets Eq 2.12 is just the third moment about the mean. Whatever be the statistics of the pixel intensities, this third moment must be nonvanishing for parity detection.

2.6 SNR for parity detection at general light levels

The variance for the parity statistics contains terms in the fourth, fifth and the sixth order in addition to the third order considered already. We summarize the results here while the details are left to the appendix **A**. Starting with the unbiased parity statistic

$$P = \sum_i \left(n_i^2 n_{i+b} - n_i n_{i+b}^2 \right)$$

where n_i is the number of photons recorded in the i'th pixel in a realization of the atmospheric noise. We first perform the Poisson averge for the square of the parity to get

$$\begin{aligned}
\overline{P^2} = & \left[\sum_i \left(\bar{n}_i^2 \bar{n}_{i+b} - \bar{n}_i \bar{n}_{i+b}^2 \right) \right]^2 \\
& + \sum_i \Big\{ 2 \bar{n}_i^2 \bar{n}_{i+b} + 2 \bar{n}_i \bar{n}_{i+b}^2 \\
& \quad + 4 \bar{n}_i^3 \bar{n}_{i+b} + 4 \bar{n}_i \bar{n}_{i+b}^3 - 4 \bar{n}_i^2 \bar{n}_{i+b}^2 - 4 \bar{n}_i \bar{n}_{i+b}^2 \bar{n}_{i+2b} \\
& \quad + \bar{n}_i^4 \bar{n}_{i+b} + \bar{n}_i \bar{n}_{i+b}^4 + 4 \bar{n}_i^2 \bar{n}_{i+b}^2 \bar{n}_{i+2b} + 4 \bar{n}_i \bar{n}_{i+b}^2 \bar{n}_{i+2b} \\
& \quad - 2 \bar{n}_i^2 \bar{n}_{i+b} \bar{n}_{i+2b}^2 - 8 \bar{n}_i \bar{n}_{i+b}^3 \bar{n}_{i+2b} \Big\}
\end{aligned} \qquad (2.19)$$

Note that the sixth order term is just the classical variance and the Poisson contribution exists only in the lower orders. Also note that the lower order terms are third, fourth and fifth order in intensities and are due to photonic nature of light.

Here we give an interpretation of the Poisson contribution to the variance. We note that the square of parity statistics (in the form **Eq** 2.10) is

$$\sum_{i,j} p_i p_j \qquad (2.20)$$

Now since n_i's are independent Poisson variables for a realization of intensity distribution $\overline{n}_i$, in the first stage of (Poisson) averaging and p_j are independent unless j=i or j=i±b. The Poisson average of the above expresssion is

$$\sum_{i,j} \overline{p_i p_j} = (\sum_i \overline{p}_i)^2 + \sum_i (\overline{p_i^2} - \overline{p}_i^2) + 2 \sum_i (\overline{p_i p_{i+b}} - \overline{p}_i \overline{p}_{i+b}) \qquad (2.21)$$

The first term is the square of the classical parity statistics. The second term is a variance term of the kind discussed before while considering the low flux noise. The third term includes the effects of overlapping pairs. For example, consider events where three pixels with interpixel separation equal to the binary register one, two and one photons respectively. The first pair of pixels contributes -2 while the pair of middle and the extreme right pixels contributes +2 to the parity. Such an event has a net parity zero. However, in the previous calculations the two pairs were taken to contribute independently to the parity so that the variance due to them summed up. The event under consideration has zero parity and should not have contributed to the variance. Such an excess counting must be corrected for. The probability of such an event is in the fourth order and contributes $- 4A\overline{n}_1^2 \overline{n}_2$ (table 2.4d). Just as the third order Poisson events were tabulated in tables 2.2 and 2.3 the higher order Poisson contributions can be worked out. These higher order contributions and the events generating them are given in table 2.4. For general light levels it is necessary to retain fifth order terms in the truncated Poisson distribution (table 2.4a). Events that contribute to the parity statistics are shown in table 2.4b. One has to consider two kinds of events.

TABLE 2.4a PARTIAL POISSON PROBABILITIES

n	Partial Probability P_n	Exact
0	$1-\bar{n}+\frac{1}{2}\bar{n}^2-\frac{1}{6}\bar{n}^3+\frac{1}{24}\bar{n}^4-\frac{1}{120}\bar{n}^5$	$e^{-\bar{n}}$
1	$\bar{n}-\bar{n}^2+\frac{1}{2}\bar{n}^3-\frac{1}{6}\bar{n}4+\frac{1}{24}\bar{n}^5$	$\bar{n}e^{-\bar{n}}$
2	$\frac{1}{2}\bar{n}^2-\frac{1}{2}\bar{n}^3+\frac{1}{4}\bar{n}^4-\frac{1}{12}\bar{n}^5$	$\frac{1}{2}\bar{n}^2e^{-\bar{n}}$
3	$\frac{1}{6}\bar{n}^3-\frac{1}{6}\bar{n}^4+\frac{1}{12}\bar{n}^5$	$\frac{1}{3!}\bar{n}^3e^{-\bar{n}}$
4	$\frac{1}{24}\bar{n}^4-\frac{1}{24}\bar{n}^5$	$\frac{1}{4!}\bar{n}^4e^{-\bar{n}}$
5	$\frac{1}{120}\bar{n}^5$	$\frac{1}{5!}\bar{n}^5e^{-\bar{n}}$

TABLE 2.4b PARITY $n^2n_1-nn_1^2$ AS A FUNCTION OF n AND n_1

$n \backslash n_1$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	-2	-6	-12	-20
2	0	2	0	-6	-16	
3	0	6	6	0		
4	0	12	16			
5	0	20				

HIGER THAN THE FIFTH ORDER $\Downarrow$

TABLE 2.4c POISSON VARIANCE: THIRD ORDER

n	n_1	p	P_3		
2	1	2	$\frac{1}{2}\bar{n}^2\bar{n}_1$	Average	$\bar{n}^2\bar{n}_1-\bar{n}\bar{n}_1^2$
I	2	-2	$\frac{1}{2}\bar{n}\bar{n}_1^2$	Variance	$2(\bar{n}^2\bar{n}_1+\bar{n}\bar{n}_1^2)$

TABLE 2.4d POISSON VARIANCE: FOURTH ORDER

n	n_1	n_2	p	p_1	P_4	Variance
2	1	0	2	0	$-\frac{1}{2}\bar{n}^2\bar{n}_1^2-\frac{1}{2}\bar{n}^3\bar{n}_1$	$-2\bar{n}^2\bar{n}_1^2-2\bar{n}^3\bar{n}_1$
1	2	0	-2	0	$-\frac{1}{2}\bar{n}^2\bar{n}_1^2-\frac{1}{2}\bar{n}\bar{n}_1^3$	$-2\bar{n}^2\bar{n}_1^2-2\bar{n}\bar{n}_1^3$
3	1	0	6	0	$\frac{1}{6}\bar{n}^3\bar{n}_1$	$6\bar{n}^3\bar{n}_1$
1	3	0	-6	0	$\frac{1}{6}\bar{n}\bar{n}_1^3$	$6\bar{n}\bar{n}_1^3$
1	2	1	-2	2	$\frac{1}{2}n\,n_1^2n_2$	$-4\bar{n}\bar{n}_1^2\bar{n}_2$ (COVARIANCE)

$$p_1 = n_1^2n_2 - n_1n_2^2$$

Fourth order contribution to

Average $\quad 0$

Variance $\quad 4\bar{n}^3\bar{n}_1+4\bar{n}\bar{n}_1^3-4\bar{n}\bar{n}_1^2\bar{n}_2-4\bar{n}^2\bar{n}_1^2$,

TABLE 2.4e: FIFTH ORDER CONTRIBUTION TO THE VARIANCE

(n, n_1, n_2)	(p, p_1)	P_5	VARIANCE
$(4,1,0)$	$(12,0)$	$\frac{1}{24}\bar{n}^4\bar{n}_1$	$6\,\bar{n}^4\bar{n}_1$
$(1,4,0)$	$(-12,0)$	$\frac{1}{24}\bar{n}\,\bar{n}_1^4$	$6\,\bar{n}\,\bar{n}_1^4$
$(3,2,0)$	$(6,0)$	$\frac{1}{12}\bar{n}^3\bar{n}_1^2$	$3\,\bar{n}^3\bar{n}_1^2$
$(2,3,0)$	$(-6,0)$	$\frac{1}{12}\bar{n}^2\bar{n}_1^3$	$3\,\bar{n}^2\bar{n}_1^3$
$(1,3,1)$	$(-6,6)$	$\frac{1}{6}\bar{n}\,\bar{n}_1^3\bar{n}_2$	$-12\,\bar{n}\,\bar{n}_1^3\bar{n}_2$ (COVARIANCE)
$(2,1,2)$	$(2,-2)$	$\frac{1}{4}\bar{n}^2\bar{n}_1\bar{n}_2^2$	$-2\,\bar{n}^2\bar{n}_1\bar{n}_2^2$
$(3,1,0)$	$(6,0)$	$-\frac{1}{6}\bar{n}^3\bar{n}_1^2-\frac{1}{6}\bar{n}^4\bar{n}_1$	$-6\,\bar{n}^3\bar{n}_1^2-6\,\bar{n}^4\bar{n}_1$
$(1,3,0)$	$(-6,0)$	$-\frac{1}{6}\bar{n}^2\bar{n}_1^3-\frac{1}{6}\bar{n}\,\bar{n}_1^4$	$-6\,\bar{n}^2\bar{n}_1^3-6\,\bar{n}\,\bar{n}_1^4$
$(1,2,1)$	$(-2,2)$	$-\frac{1}{2}\bar{n}^2\bar{n}_1^2\bar{n}_2-\frac{1}{2}\bar{n}\,\bar{n}_1^3\bar{n}_2-\boxed{\frac{1}{2}\bar{n}\,\bar{n}_1^2\bar{n}_2^2}$	$4\bar{n}^2\bar{n}_1^2\bar{n}_2+4\bar{n}\,\bar{n}_1^3\bar{n}_2+4\bar{n}\,\bar{n}_1^2\bar{n}_2^2$
$(2,1,0)$	$(2,0)$	$\frac{1}{4}\bar{n}^4\bar{n}_1+\frac{1}{2}\bar{n}^3\bar{n}_1^2+\frac{1}{4}\bar{n}^2\bar{n}_1^3$	$\bar{n}^4\bar{n}_1+2\,\bar{n}^3\bar{n}_1^2+\bar{n}^2\bar{n}_1^3$
$(1,2,0)$	$(-2,0)$	$\frac{1}{4}\bar{n}\,\bar{n}_1^4+\frac{1}{2}\bar{n}^2\bar{n}_1^3+\frac{1}{4}\bar{n}^3\bar{n}_1^2$	$\bar{n}\,\bar{n}_1^4+2\,\bar{n}^2\bar{n}_1^3+\bar{n}^3\bar{n}_1^2$

FIFTH ORDER CONTRIBUTION TO THE VARIANCE (POISSON)

$$\bar{n}^4\bar{n}_1+\bar{n}\,\bar{n}_1^4-8\,\bar{n}\,\bar{n}_1^3\bar{n}_2+4\,\bar{n}^2\bar{n}_1^2\bar{n}_2-2\,\bar{n}^2\bar{n}_1\bar{n}_2^2$$

First, events due to a single pair of pixels (with binary separation). These events are like the ones considered before, however, now one is interested in higher order contributions to the variance. Consider, for example, an event when the pixel on the left records 2 photons and the one on the right records one photon. As shown in the table 2.3 this event contributes $2\bar{n}^2 n$, in the third order to the variance. This three photon event also contributes in higher orders. To see this let us cosider only the pixel on the left. The exact probability of this pixel recording two photons is $\frac{1}{2}\bar{n}^2 e^{-\bar{n}}$. When we consider this probability order by order, the third order "partial probability" is $-\frac{1}{2}\bar{n}^3$. Note that though this is negative there is nothing to worry about because it is just a part of the probability i.e. "partial probability". The three photon event contributes to the fouth order in two ways: 1) Third order "partial" contribution from the pixel on the left and the first order "partial" contribution from the pixel on the right, 2) second order "partial" contributions from both the pixels. These give us the two terms listed in the table 2.4d. Note that the order of partial probability cannot be less than the number of photons involved. The second kind of events include overlapping pairs. Their contribution to the variance is given by the last term in Eq 2.21. Such contributions are labled "covariance" in the table. This interpretation is helpful in understanding and checking the detailed derivation described in Appendix A2

The atmospheric noise needs to be averaged and when this is done (Appendix A) we get for the variance the explicitly positive definite form:

$$
\begin{aligned}
\langle \overline{P^2} \rangle - \langle \bar{P} \rangle^2 = {} & 4N_S [2\mathcal{N}_1^3 + 7\mathcal{N}_1^2\mathcal{N}_2 + 7\mathcal{N}_1\mathcal{N}_2^2 + 2\mathcal{N}_2^3] \\
& + 4N_S [6(\mathcal{N}_1^2 - \mathcal{N}_2^2)^2 + 20\mathcal{N}_1\mathcal{N}_2(\mathcal{N}_1^2 + \mathcal{N}_2^2) + 2\mathcal{N}_1^2\mathcal{N}_2^2] \\
& + 8N_S [3\mathcal{N}_1^5 + 5\mathcal{N}_1^4\mathcal{N}_2 + 2\mathcal{N}_1^3\mathcal{N}_2^2 + 2\mathcal{N}_1^2\mathcal{N}_2^3 + 5\mathcal{N}_1\mathcal{N}_2^4 + 3\mathcal{N}_2^5] \\
& + N_S [8(\mathcal{N}_1 - \mathcal{N}_2)^6 + 124\mathcal{N}_1^2\mathcal{N}_2^2(\mathcal{N}_1 - \mathcal{N}_2)^2 + 32\mathcal{N}_1\mathcal{N}_2(\mathcal{N}_1^2 - \mathcal{N}_2^2)^2 + 8\mathcal{N}_1^3\mathcal{N}_2^3]
\end{aligned}
\tag{2.22}
$$

We note that in this expression for the variance the contribution in all orders (in $\mathcal{N}$) has a weight N_S . This is a signature of the fact that in our model the information is contained in N_S triplets of speckles. It follows that when $\mathcal{N}$ is greater than unity the sixth order terms dominate while for $\mathcal{N} < 1$ the third order terms dominate. For reasonable observational parameters $\mathcal{N} = 1$ means 13^m. So the transition from high flux to low flux estimates occurs at 13^m . In chapter 5 we consider edge effects where the information about the parity may come from uncorrelated triplets of speckles. It will be shown there that different orders have different weights in powers of N_S . The transition to low then goes through an intermediate step.

2.7 SNR for autocorrelation

The SNR for autocorrelation of a binary is well known in the literature. Here we show that the SNR estimate for the autocorrelation based on our assumptions and simplifications does agree with the known result. As before, assuming a uniform seeing disk, we treat the N_S terms in the general autocorrelation

$$\sum_i n_i\, n_{i+X} \qquad (2.23)$$

equivalent at low flux levels and consider only one representative pair

$$a_X = n_i\, n_{i+X} \qquad (2.24)$$

The pair of pixels register n_i and n_{i+X} photons respectively with average intensity $\bar{n}$ and $\bar{n}_1$ in one realization of atmospheric noise. Note that for $X \neq 0$ the statistics is unbiased under Poisson statistics. The autocorrelation gets its leading contribution (low flux) in the second order in the flux per pixel so it is enough to truncate the Poisson distribution after first order: the values of the autocorrelation and their probabilities are given in tables 2.5 and 2.6 respectively.

Table 2.5 Autocorrelation as function of and

$n \backslash n_1$	0	1
0	0	0
1	0	1

Table 2.6 Probability distribution for autocorrelation

Auto.	Probability
1	$\bar{n}\,\bar{n}_1$
0	$1-\bar{n}\,\bar{n}_1$

It is then clear that the Poisson average and variance are:

$$\bar{a}_X = \bar{n}\,\bar{n}_1 \quad ; \quad \overline{a_X^2} = \bar{n}\,\bar{n}_1 \qquad (2.25)$$

Consider the case where X equals the binary separation. Then from Eq 2.16 which gives intensities in the presence of sky background

$$\langle \bar{a}_b \rangle = \mathcal{N}_1^2 + 3\mathcal{N}_1\mathcal{N}_2 + \mathcal{N}_2^2 + 2K(\mathcal{N}_1+\mathcal{N}_2) + K^2 \qquad (2.26)$$

However, one should actually be able to measure the slight bump in $\langle \bar{a}_X \rangle$ at $X = b$ relative to its neighbours where $\langle \bar{a}_X \rangle$ takes the value say $\langle \bar{a}_{dc} \rangle$. For $X \neq 0, X \neq b$ we have

$$\langle \bar{a}_{dc} \rangle = \mathcal{N}_1^2 + 2\mathcal{N}_1\mathcal{N}_2 + \mathcal{N}_2^2 + 2K(\mathcal{N}_1+\mathcal{N}_2) + K2 \qquad (2.27)$$

One should strictly speaking calculate the variance in $a_b - a_{dc}$, however, one can determine a_{dc} from many separarions x which give independent a_{dc} and thus consider it almost noise free when compared to a_b. Thus the **SNR** for focal plane two point correlation is

$$SNR_{AUTO.} = \frac{q\, m^{1/2}\, \mathcal{N}_1 \mathcal{N}_2\, N_s^{1/2}}{[\mathcal{N}_1^2 + 3\mathcal{N}_1\mathcal{N}_2 + \mathcal{N}_2^2 + 2k(\mathcal{N}_1+\mathcal{N}_2) + K^2]^{1/2}} \qquad (2.28)$$

The scaling with $M, q, \mathcal{N}$ and N_S agrees with previous results due to Dainty (1974) who considers a binary with $\mathcal{N}_1, \mathcal{N}_2 \sim \mathcal{N}$.

2.8 Conclusion

For concreteness we consider a specific case of a 4 m telescope, optical bandwidth **100** A, exposure time of 10 ms. In this case $N_S = 1600$ and per speckle photon count of unity means $12.^M25$ star. 'In the high flux limit the sixth order terms dominate and the **SNR** is function of the relative strength r of the two components:

$$r = \mathcal{N}_2/\mathcal{N}_1 = 10^{-0.4(m_2 - m_1)} \qquad \mathcal{N}_1 > \mathcal{N}_2 > 1 \qquad (2.29)$$

$$SNR_{PARITY} = \frac{q^{3/2}\, m^{1/2}\, N_s^{1/2}\, r(1-r)}{[2 - 4r + 61r^2 + 116r^3 + 61r^4 - 4r^5 + 2r^6]^{1/2}} \qquad (2.30)$$

where m_1 and m_2 are the magnitudes of the two stars and we have taken detector efficiency (optics+quantum) $q=0.2$. This high flux **SNR** is given in table 2.7 as a function of the magnitude difference $\Delta m = m_2 - m_1$.

Table 2.7 High flux SNR for parity

Am	SNR	m	SNR
0.0	0	3.5	31
0.2	112	4.0	20
0.5	130	4.5	13
1.0	130	5.0	8
1.5	118	5.5	5
2.0	96	6.0	3
2.5	70	6.5	2
3.0	48	7.2	1

Calculations based on all orders (exact in our model) show that this gives SNR accurate to few percent if the brighter. component is brighter than 7^m and the fainter component is brighter than about 13^m. We note from the table that for bright binaries parity cannot be detected with SNR>3 if the magnitude difference is greater than 6 though it may be possible to see the binary nature in the autocorrelation. Table 2.8 gives the limiting magnitude of the fainter component, for a given magnitude of the brighter component, for which parity and autocorrelation can be detected with SNR>3.

Table 2.8 Limiting faintness

Brighter component		Limiting fainter component Parity		Autocorrelation
13.0	....	17.2	....	20.5
14.0	....	17.0	....	20.5
15.0	....	16.0	....	20.5
upto 20.0	....	None	....	20.5

For the chosen observation parameters sky background noise is unimportant and makes no difference in the limiting magnitudes.

The limiting magnitude given in table 2.8 for the autocorrelation are based on our calculations outlined in Section 2.7. In this case, keeping the 21^m per pixel sky background in mind, the SNR calculations were terminated at 20.5. We conclude from the table 2.8 that parity detection has a significantly poorer SNR than the autocorrelation. One may raise the following legitimate question. From autocorrelation, which has muchbetter SNR, one knows $\mathcal{N}_1$ and $\mathcal{N}_2$ and therefore the magnitude $|\mathcal{N}_1^2\mathcal{N}_2 - \mathcal{N}_1\mathcal{N}_2^2|$ of the parity quite well. It is the sign of parity which is unknown. So the relevant statistical question is to assign probability distribution to the signs when the observed parity value is given. This question can be answered only if one knows how the parity is distributed around its mean (which has to be consistent with the modulus of parity obtained from autocorrelation). Knowing a distribution means knowing all the moments of the variable (the parity). The complexity involved in evaluating the second moment of the parity statistics (despite a simple model) indicates the near impossibility of carring out such a task by analytical methods. Two extreme cases are, however, trivial. For large values of SNR the sign of parity is well defined. On the other hand for poor SNR the observed value might, equally well, have come from any one of the two signs. We take SNR=3 as the case where one sign has significantly greater probability than the other. We also note that we have considered an ideal situation, in reality there could be other sources of noise and present day speckle work does not attain these theoretical limits. The limits themselves are still of interest.

APPENDIX A2 SNR for parity detection at general light levels

Here we give the details of the SNR calculations for the parity statistics introduced before.

$$P = \sum_i \left(n_i^2 n_{i+b} - n_i n_{i+b}^2 \right) \quad , b=0 \tag{2.31}$$

The average of this is given by Eq 2.13. The square of the parity statistics is

$$P^2 = \sum_{i,j} n_i^2 n_{i+b} n_j^2 n_{j+b} + \sum_{i,j} n_i n_{i+b}^2 n_j n_{j+b}^2 - 2\sum_{i,j} n_i^2 n_{i+b} n_j n_{j+b}^2 \tag{2.32}$$

a) Poisson flucuations

As the Poisson fluctuations in different pixels are uncorrelated for a given intensity distribution, correlations come only when any two of the subscripts are equal. Since we restrict to $b \neq 0$ no three subscripts in any of the three terms in Eq 2.32 can be equal. So while taking the Poisson average of terms in Eq 2.32 one can partition the sunmation $\sum_{i,j}$ into four sums: 1) j distinct from i, i+b and i-b; 2) j=i; 3) j=i+b; 4) j=i-b. These partitions are mutually exclusive ($b \neq 0$) and cover all the terms implied by the original summation without any restriction over i and j. When i is distinct from j, $j \pm b$ the Poisson average of a terns takes simpler form: for example

$$\overline{n_i^2 n_{i+b} n_j^2 n_{j+b}} = \overline{n_i^2}\ \overline{n}_{i+b}\ \overline{n_j^2}\ \overline{n}_{j+b} \qquad i \text{ distinct from } j \text{ and } j \pm b.$$

One can relax the restriction on j and pretend that the Poisson average can always be split like this though this is not true for partitions other than 1). For other partitions one must first write the correct result implied by the partition and then subtract from it the above (pretend) uncorrelated average. For example

$$\sum_{i,j} \overline{n_i^2 n_{i+b} n_j^2 n_{j+b}} = \sum_{i,j} \overline{n_i^2}\, \bar{n}_{i+b}\, \overline{n_j^2}\, \bar{n}_{j+b} + \sum_i \left(\overline{n_i^4}\, \overline{n_{i+b}^2} - \overline{n_i^2}^2\, \overline{n_{i+b}^2}^2\right)$$

$$+ 2\sum_i \overline{n_i^2}\, \bar{n}_{i+2b} \left(\overline{n_{i+b}^3} - \overline{n_{i+b}^2}\, \bar{n}_{i+b}\right) \qquad (2.33)$$

Using the well-known results

$$\overline{n_i} = \bar{n}_i$$

$$\overline{n_i^2} = \bar{n}_i^2 + \bar{n}_i$$

$$\overline{n_i^3} = \bar{n}_i^3 + 3\bar{n}_i^2 + \bar{n}_i$$

$$\overline{n_i^4} = \bar{n}_i^4 + 6\bar{n}_i^3 + 7\bar{n}_i^2 + \bar{n}_i \qquad (2.34)$$

for the Poisson distribution for n_i with given mean $\bar{n}_i$ we can write the averages in terms of the given intensities. A compact notation is useful. Since except for the first way of partitioning other partitions involve only one summation over i we drop the explicit $\sum_i$ in such terms. Also we suppress the subscript i and denote n_i by n, n_{i+b} by n_1, n_{i+mb} by n_m etc. Also since i is a dummy index we have

$$\bar{n}\,\bar{n}_1^2\,\bar{n}_2 = \bar{n}_-\,\bar{n}\,\bar{n}_1 \qquad \textit{and so an.}$$

With this compact notation we have

$$\sum_{i,j} \overline{n_i^2 n_{i+b} n_j^2 n_{j+b}} = \sum_{i,j} (\bar{n}_i^2 + \bar{n}_i)\bar{n}_{i+b} (\bar{n}_j^2 + \bar{n}_j)\bar{n}_{j+b} + \bar{n}^4 \bar{n}_1 + 4\bar{n}^3\bar{n}_1^2 + 4\bar{n}^2\bar{n}_1^2\bar{n}_2$$

$$+ 2\bar{n}^2\bar{n}_1\bar{n}_2 + 6\bar{n}^3\bar{n}_1 + 6\bar{n}^2\bar{n}_1^2 + 4\bar{n}\,\bar{n}_1^2\bar{n}_2 + 7\bar{n}^2\bar{n}_1 + \bar{n}\,\bar{n}_1^2$$

$$+ 2\bar{n}\,\bar{n}_1\bar{n}_2 + \bar{n}\,\bar{n}_1 \qquad (2.35)$$

$$\sum_{i,j}\overline{n_i n_{i+b}^2 n_j n_{j+b}^2} = \sum_{i,j}\bar{n}_i(\bar{n}_{i+b}^2+\bar{n}_{i+b})\bar{n}_j(\bar{n}_{j+b}^2+\bar{n}_{j+b}) + \bar{n}\bar{n}_1^4 + 4\bar{n}^2\bar{n}_1^3 + 4\bar{n}_2^2\bar{n}_1^2\bar{n}$$
$$+2\bar{n}_2^2\bar{n}_1\bar{n} + 6\bar{n}_1^3\bar{n} + 6\bar{n}_1^2\bar{n}^2 + 4\bar{n}_2\bar{n}_1^2\bar{n} + 7\bar{n}_1^2\bar{n} + \bar{n}_1\bar{n}^2$$
$$+2\bar{n}\bar{n}_1\bar{n}_2 + \bar{n}\bar{n}_1 \qquad (2.36)$$

$$-2\sum_{i,j}\overline{n_i^2 n_{i+b} n_j n_{j+b}^2} = -2\sum_{i,j}(\bar{n}_i^2+\bar{n}_i)\bar{n}_{i+b}\bar{n}_j(\bar{n}_{j+b}^2+\bar{n}_{j+b}) - 4\bar{n}^3\bar{n}_1^2 - 4\bar{n}^2\bar{n}_1^3 - 2\bar{n}^2\bar{n}_1\bar{n}_2$$
$$-8\bar{n}\bar{n}_1^3\bar{n}_2 - 16\bar{n}^2\bar{n}_1^2 - 2\bar{n}\bar{n}_1^3 - 2\bar{n}^3\bar{n}_1 - 2\bar{n}^2\bar{n}_1\bar{n}_2 - 2\bar{n}\bar{n}_1\bar{n}_2^2$$
$$-12\bar{n}\bar{n}_1^2\bar{n}_2 - 6\bar{n}^2\bar{n}_1 - 6\bar{n}\bar{n}_1^2 - 4\bar{n}\bar{n}_1\bar{n}_2 \qquad (2.37)$$

This gives us the Poisson average of the square of the parity statistics as used in Eq 2.19 of the text

$$\overline{p^2} = [\sum_i(\bar{n}_i^2\bar{n}_{i+b} - \bar{n}_i\bar{n}_{i+b}^2)]$$
$$+\bar{n}^4\bar{n}_1 + \bar{n}\bar{n}_1^4 + 4\bar{n}^2\bar{n}_1^2\bar{n}_2 + 4\bar{n}\bar{n}_1^2\bar{n}_2^2 - 2\bar{n}^2\bar{n}_1\bar{n}_2^2 - 8\bar{n}\bar{n}_1^3\bar{n}_2$$
$$+4\bar{n}^3\bar{n}_1 + 4\bar{n}\bar{n}_1^3 - 4\bar{n}^2\bar{n}^2 - 4\bar{n}\bar{n}_1^2\bar{n}_2$$
$$+2\bar{n}^2\bar{n}_1 + 2\bar{n}\bar{n}_1^2 \qquad (2.38)$$

b) Rayleigh average upto the fifth order: Note that the third, the fourth and the fifth order terms in Eq 2.38 involve only one summation and are easier to average over the assumed Rayleigh distribution for individual speckles. We assume, as before, the seeing disk to have a uniform profile. The average of these sums can then be replaced by N_s times the average for one term. The sixth ordered terms involve double summations and need different handling, similar to the Poisson double summation. So consider three pixels within the seeing disk separated by the binary separation with contributing speckles $\bar{n}$, $\bar{n}_1$ and $\bar{n}_2$ (compact notation) given by Eq 2.4. The quantities with different subscripts are uncorrelated. The results for Rayleigh averages of correlated variables can be summarized as follow:

$$\langle \mu_i^{m_1} \nu_i^{m_2} \rangle = (m_1+m_2)!\, \mathcal{N}_1^{m_1} \mathcal{N}_2^{m_2} \qquad (2.39)$$

where $\langle\ \rangle$ denotes the Rayleigh average. The following Rayleigh averages are needed (2.40)

$$\langle \bar{n}^2 \bar{n}_1 \rangle = 2(\mathcal{N}_1^3 + 4\mathcal{N}_1^2\mathcal{N}_2 + 3\mathcal{N}_1\mathcal{N}_2^2 + \mathcal{N}_2^3)$$
$$\langle \bar{n}^3 \bar{n}_1 \rangle = 6\mathcal{N}_1^4 + 30\mathcal{N}_1^3\mathcal{N}_2 + 24\mathcal{N}_1^2\mathcal{N}_2^2 + 18\mathcal{N}_1\mathcal{N}_2^3 + 6\mathcal{N}_2^4$$
$$\langle \bar{n}^2 \bar{n}_1^2 \rangle = 4\mathcal{N}_1^4 + 16\mathcal{N}_1^3\mathcal{N}_2 + 36\mathcal{N}_1^2\mathcal{N}_2^2 + 16\mathcal{N}_1\mathcal{N}_2^3 + 4\mathcal{N}_2^4$$
$$\langle \bar{n} \bar{n}_1^2 \bar{n}_2 \rangle = 2\mathcal{N}_1^4 + 12\mathcal{N}_1^3\mathcal{N}_2 + 22\mathcal{N}_1^2\mathcal{N}_2^2 + 12\mathcal{N}_1\mathcal{N}_2^3 + 2\mathcal{N}_2^4$$
$$\langle \bar{n}^4 \bar{n}_1 \rangle = 24\mathcal{N}_1^5 + 144\mathcal{N}_1^4\mathcal{N}_2 + 120\mathcal{N}_1^3\mathcal{N}_2^2 + 96\mathcal{N}_1^2\mathcal{N}_2^3 + 72\mathcal{N}_1\mathcal{N}_2^4 + 24\mathcal{N}_2^5$$
$$\langle \bar{n}^2 \bar{n}_1^2 \bar{n}_2 \rangle = 4\mathcal{N}_1^5 + 28\mathcal{N}_1^4\mathcal{N}_2 + 72\mathcal{N}_1^3\mathcal{N}_2^2 + 68\mathcal{N}_1^2\mathcal{N}_2^3 + 24\mathcal{N}_1\mathcal{N}_2^4 + 4\mathcal{N}_2^5$$
$$\langle \bar{n}^2 \bar{n}_1 \bar{n}_2^2 \rangle = 4\mathcal{N}_1^5 + 24\mathcal{N}_1^4\mathcal{N}_2 + 44\mathcal{N}_1^3\mathcal{N}_2^2 + 44\mathcal{N}_1^2\mathcal{N}_2^3 + 24\mathcal{N}_1\mathcal{N}_2^4 + 4\mathcal{N}_2^5$$
$$\langle \bar{n} \bar{n}_1^3 \bar{n}_2 \rangle = 6\mathcal{N}_1^5 + 42\mathcal{N}_1^4\mathcal{N}_2 + 84\mathcal{N}_1^3\mathcal{N}_2^2 + 84\mathcal{N}_1^2\mathcal{N}_2^3 + 42\mathcal{N}_1\mathcal{N}_2^4 + 6\mathcal{N}_2^5$$

$\langle \bar{n}\bar{n}_1^2 \rangle$, $\langle \bar{n}\bar{n}_1^3 \rangle$, $\langle \bar{n}\bar{n}_1^4 \rangle$ and $\langle \bar{n}\bar{n}_1^2\bar{n}_2^2 \rangle$ are obtained from $\langle \bar{n}^2\bar{n}_1 \rangle$, $\langle \bar{n}^3\bar{n}_1 \rangle$, $\langle \bar{n}^4\bar{n}_1 \rangle$ and $\langle \bar{n}^2\bar{n}_1^2\bar{n}_2 \rangle$ respectively by interchanging $\mathcal{N}_1$ and $\mathcal{N}_2$.

c) Rayleigh average of the classical sixth order terms

The sixth order terms are the classical terms in the sense that even in the absence of photon noise these terms would survive. This is the reason why there are no terms in the 'sixth order with single summation: the photon noise merges into the wave noise. The double summation in the sixth order terms needs similar handling to the double summation $\sum_{i,j}$ in the Poisson case. In the Poisson case the Poisson fluctuations in different pixels were independent so the subscripts on the n's had tobe equal for correlations to arise. This meant that j=i or j=i ± b. In the Rayleigh case under consideration the correlations in the intensities also arise if two pixels are separated by the binary separation. This is because such pixels have one pair of speckles with the true intensity ratio for the binary. Thus the subscripts on the $\bar{n}$'s have to be either equal or differ by b the binary separation. This means $j=i; j=i\pm b$ or $j=i\pm 2b$ give correlation. Barring these five possibilities the Rayleigh average can be split. This of course include the above five cases for which such spliting is not possible. So when we write these five cases as they should be we subtract from them the terms with split averages: much as we did before for the Poisson statistics. The sixth order terms can be expanded as

$$
\begin{aligned}
\langle [\sum_i (\bar{n}_i^2 \bar{n}_{i+b} - \bar{n}_i \bar{n}_{i+b}^2)]^2 \rangle = {} & [\langle \sum_i (\bar{n}_i^2 \bar{n}_{i+b} - \bar{n}_i \bar{n}_{i+b}^2) \rangle]^2 \\
& + \langle \bar{n}^4 \bar{n}_1^2 \rangle + \langle \bar{n}^2 \bar{n}_1^4 \rangle + 2\langle \bar{n}^2 \bar{n}_1^3 \bar{n}_2 \rangle + 2\langle \bar{n} \bar{n}_1^3 \bar{n}_2^2 \rangle + 2\langle \bar{n}^2 \bar{n}_1 \bar{n}_2^2 \bar{n}_3 \rangle \\
& + 2\langle \bar{n} \bar{n}_1^2 \bar{n}_2 \bar{n}_3^2 \rangle - 5\langle \bar{n}^2 \bar{n}_1 \rangle^2 - 5\langle \bar{n} \bar{n}_1^2 \rangle^2 + 10\langle \bar{n}^2 \bar{n}_1 \rangle \langle \bar{n} \bar{n}_1^2 \rangle \\
& - 2\langle \bar{n}^3 \bar{n}_1^3 \rangle - 2\langle \bar{n}^2 \bar{n}_1^2 \bar{n}_2^2 \rangle - 2\langle \bar{n}^2 \bar{n}_1 \bar{n}_2 \bar{n}_3^2 \rangle - 2\langle \bar{n} \bar{n}_1^4 \bar{n}_2 \rangle \\
& - 2\langle \bar{n} \bar{n}_1^2 \bar{n}_2^2 \bar{n}_3 \rangle
\end{aligned}
\qquad (2.41)
$$

As before we consider only a representative term whenever a single summation occurs. However, since pixel-intensities are correlated if the pixel separation equals the binary separation, we need to consider four consecutive pixels (instead of the three in the Poisson case) with the binary separations.

The following averages are needed:

$$\langle \bar{n}^4 \bar{n}_1^2 \rangle = 48\mathcal{N}_1^6 + 288\mathcal{N}_1^5\mathcal{N}_2 + 960\mathcal{N}_1^4\mathcal{N}_2^2 + 672\mathcal{N}_1^3\mathcal{N}_2^3 + 432\mathcal{N}_1^2\mathcal{N}_2^4 + 192\mathcal{N}_1\mathcal{N}_2^5 + 48\mathcal{N}_2^6$$

$$\langle \bar{n}^2 \bar{n}_1^4 \rangle = 48\mathcal{N}_1^6 + 192\mathcal{N}_1^5\mathcal{N}_2 + 432\mathcal{N}_1^4\mathcal{N}_2^2 + 672\mathcal{N}_1^3\mathcal{N}_2^3 + 960\mathcal{N}_1^2\mathcal{N}_2^4 + 288\mathcal{N}_1\mathcal{N}_2^5 + 48\mathcal{N}_2^6$$

$$\langle \bar{n}^2 \bar{n}_1^3 \bar{n}_2 \rangle = 12\mathcal{N}_1^6 + 96\mathcal{N}_1^5\mathcal{N}_2 + 264\mathcal{N}_1^4\mathcal{N}_2^2 + 432\mathcal{N}_1^3\mathcal{N}_2^3 + 288\mathcal{N}_1^2\mathcal{N}_2^4 + 84\mathcal{N}_1\mathcal{N}_2^5 + 12\mathcal{N}_2^6$$

$$\langle \bar{n}\, \bar{n}_1^3 \bar{n}_2^2 \rangle = 12\mathcal{N}_1^6 + 84\mathcal{N}_1^5\mathcal{N}_2 + 288\mathcal{N}_1^4\mathcal{N}_2^2 + 432\mathcal{N}_1^3\mathcal{N}_2^3 + 264\mathcal{N}_1^2\mathcal{N}_2^4 + 96\mathcal{N}_1\mathcal{N}_2^5 + 12\mathcal{N}_2^6$$

$$\langle \bar{n}^2 \bar{n}_1 \bar{n}_2^2 \bar{n}_3 \rangle = 4\mathcal{N}_1^6 + 36\mathcal{N}_1^5\mathcal{N}_2 + 108\mathcal{N}_1^4\mathcal{N}_2^2 + 132\mathcal{N}_1^3\mathcal{N}_2^3 + 92\mathcal{N}_1^2\mathcal{N}_2^4 + 32\mathcal{N}_1\mathcal{N}_2^5 + 4\mathcal{N}_2^6$$

$$\langle \bar{n}\, \bar{n}_1^2 \bar{n}_2 \bar{n}_3^2 \rangle = 4\mathcal{N}_1^6 + 32\mathcal{N}_1^5\mathcal{N}_2 + 92\mathcal{N}_1^4\mathcal{N}_2^2 + 132\mathcal{N}_1^3\mathcal{N}_2^3 + 108\mathcal{N}_1^2\mathcal{N}_2^4 + 36\mathcal{N}_1\mathcal{N}_2^5 + 4\mathcal{N}_2^6$$

$$\langle \bar{n}^2 \bar{n}_1 \bar{n}_2 \bar{n}_3^2 \rangle = 4\mathcal{N}_1^6 + 32\mathcal{N}_1^5\mathcal{N}_2 + 84\mathcal{N}_1^4\mathcal{N}_2^2 + 120\mathcal{N}_1^3\mathcal{N}_2^3 + 84\mathcal{N}_1^2\mathcal{N}_2^4 + 32\mathcal{N}_1\mathcal{N}_2^5 + 4\mathcal{N}_2^6$$

$$\langle \bar{n}\, \bar{n}_1^2 \bar{n}_2^2 \bar{n}_3 \rangle = 4\mathcal{N}_1^6 + 36\mathcal{N}_1^5\mathcal{N}_2 + 128\mathcal{N}_1^4\mathcal{N}_2^2 + 200\mathcal{N}_1^3\mathcal{N}_2^3 + 128\mathcal{N}_1^2\mathcal{N}_2^4 + 36\mathcal{N}_1\mathcal{N}_2^5 + 4\mathcal{N}_2^6$$

$$\langle \bar{n}\, \bar{n}_1^4 \bar{n}_2 \rangle = 24\mathcal{N}_1^6 + 192\mathcal{N}_1^5\mathcal{N}_2 + 408\mathcal{N}_1^4\mathcal{N}_2^2 + 432\mathcal{N}_1^3\mathcal{N}_2^3 + 408\mathcal{N}_1^2\mathcal{N}_2^4 + 192\mathcal{N}_1\mathcal{N}_2^5 + 24\mathcal{N}_2^5$$

$$\langle \bar{n}^3 \bar{n}_1^3 \rangle = 36\mathcal{N}_1^6 + 180\mathcal{N}_1^5\mathcal{N}_2 + 504\mathcal{N}_1^4\mathcal{N}_2^2 + 1044\mathcal{N}_1^3\mathcal{N}_2^3 + 504\mathcal{N}_1^2\mathcal{N}_2^4 + 180\mathcal{N}_1\mathcal{N}_2^5 + 36\mathcal{N}_2^6$$

$$\langle \bar{n}^2 \bar{n}_1^2 \bar{n}_2^2 \rangle = 8\mathcal{N}_1^6 + 56\mathcal{N}_1^5\mathcal{N}_2 + 192\mathcal{N}_1^4\mathcal{N}_2^2 + 256\mathcal{N}_1^3\mathcal{N}_2^3 + 192\mathcal{N}_1^2\mathcal{N}_2^4 + 56\mathcal{N}_1\mathcal{N}_2^5 + 8\mathcal{N}_2^6$$

Using these averages and noting the remaining double sum is just the square of the averaged (both Poisson and Rayleigh) parity we get the variance for the parity (which can be put in the explicitely positive form, the Eq 2.22

$$\begin{aligned}\langle \bar{p}^2 \rangle - \langle \bar{p} \rangle^2 = {} & 4N_S\,[\,2\mathcal{N}_1^3 + 7\mathcal{N}_1^2\mathcal{N}_2 + 7\mathcal{N}_1\mathcal{N}_2^2 + 2\mathcal{N}_2^3\,] \\ & + 4N_S\,[\,6(\mathcal{N}_1^2-\mathcal{N}_2^2)^2 + 20\mathcal{N}_1\mathcal{N}_2(\mathcal{N}_1^2+\mathcal{N}_2^2) + 2\mathcal{N}_1^2\mathcal{N}_2^2\,] \\ & + 8N_S\,[\,3\mathcal{N}_1^5 + 5\mathcal{N}_1^4\mathcal{N}_2 + 2\mathcal{N}_1^3\mathcal{N}_2^2 + 2\mathcal{N}_1^2\mathcal{N}_2^3 + 5\mathcal{N}_1\mathcal{N}_2^4 + 3\mathcal{N}_2^5\,] \\ & + N_S\,[\,8(\mathcal{N}_1-\mathcal{N}_2)^6 + 124\mathcal{N}_1^2\mathcal{N}_2^2(\mathcal{N}_1-\mathcal{N}_2)^2 + 32\mathcal{N}_1\mathcal{N}_2(\mathcal{N}_1^2-\mathcal{N}_2^2)^2 + 8\mathcal{N}_1^3\mathcal{N}_2^3\,]\end{aligned} \qquad (2.43)$$