CHAPTER 1

SNR for the Bispectrum: scaling laws

1.1 Introduction

In this chapter we present the scaling dependence of SNR for the bispectrum and phase reconstruction based on it on the average number of speckles and the average numher of photons per speckle. We deal only with a typical bispectrum element and not the special near axis elements. We shall discuss these special near axis/origin regions later in chapter 5. Let $I(X)$ be the focal plane intensity distribution which is convolution of the source structure $S(X)$ and the system (telescope+atmosphere) response (PSF) $R(X)$. Equivalently, in Fourier representation

$$I_u = R_u S_u \qquad\qquad I_u = \int d^2x \, e^{iux} I(x) \tag{1.1}$$

where we use u,v... etc for spatial frequencies. As mentioned before, due to atmospheric turbulence R is random in u and in time. If one exposes images longer than about ten milliseconds then the average response $\langle R \rangle$ is significantly different from zero only for spatial frequencies less than about one arcsecond^{-1}. The power spectrum method described in the introductory chapter does not contain any phase information. To reconstruct the phases of the object Fourier transform is important and Weigelt (1977) has proposed the use of the so called bispectrum

$$\langle I_u I_v I_{-u-v} \rangle \tag{1.2}$$

a third order statistics in the intensities, which is nonzero right upto the diffraction limit of the telescope. This method discussed in detail by Bartlet et.al. (1984), has been applied to astronomical imaging (Lohman et.al. 1983). The triple product in

Eq 1.2 is a double Fourier transform of the focal plane triple correlation

$$T(y,z) = \int d^2x \; I(x)\, I(x+y)\, I(x+z) \,. \tag{1.3}$$

In practice, one measures the triple correlation of the image of the object and that for a neighbouring point source to get the object triple correlation

$$S_u S_v S_{-u-v} = |S_u|\,|S_v|\,|S_{-u-v}| \; exp\{i(\theta_u + \theta_v - \theta_{u+v})\} \tag{1.4}$$

from which the object phases are recursively solved for. One has to assume values'for two of the phases at the begining of the recursive algorithm. This corresponds to an overall shift in the position of the source. Apart from this, given enough SNR, the object structure can be determined uniquely.

In Fig 1.1 we show the bispectrum for a point source. The full bispectrum is four dimensional: only a planar cross section is shown. We note three features. The central feature corresponds

Fig 1.1

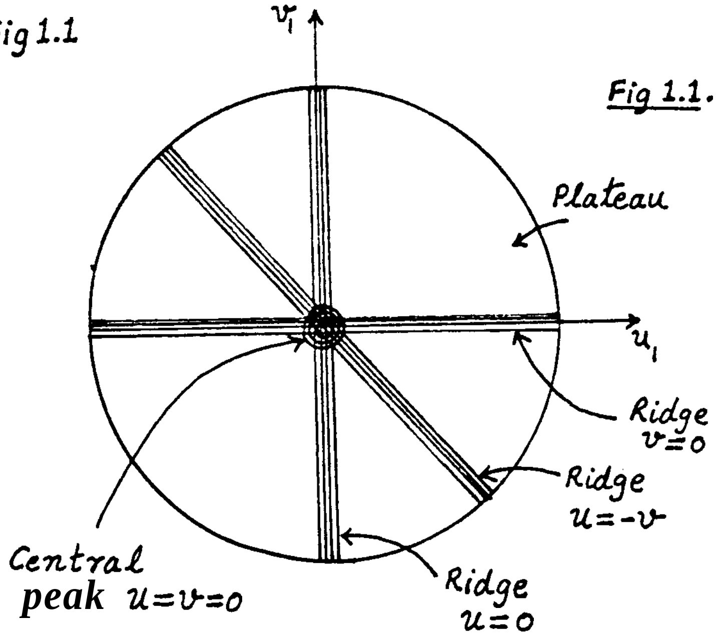

Fig 1.1. Bispectrum for a point source shown schematically. Only the cross section in the u_1, v_1 plane is shown. The four dimensional properties of various features are given.

to the case when all the three spatial frequencies entering the triple correlation are near zero. In terms of the focal plane intensity this would mean that we are dealing with (almost) the cube of the total flux. This is highly correlated object and is rather peaky. Then there are three correlation ridges. Note that for these ridges the spatial frequency of one of the three Fourier components is near zero while the other two are conjugates. This is basically a power spectrum and shows as an enhancement in the triple correlation. The parallel with the power spectrum, however, is not exact. Because of the third Fourier component this feature is quite capable of yielding phase information. And finally there is the plateau region which comes about because intensities at two points within a speckle are correlated. The strength of this feature is related to the per speckle photon count and is therefore much weaker than the central feature which is related to the entire flux. However, there are a large number of spekles and one may expect the contribution due to this plateau region to dominate as far' as phase information is concerned. In this chapter, a prelude to the chapter 5, we are mainly concerned with the SNR at one statistically independent region in the bispectrum and the SNR for phase determination by combining information from all such regions. In chapter 5 we deal with phase information from the near axes/origin regions of the bispectrum (to be precise, with focal plane analogues of these features).

We emphasize that in this chapter we are dealing with Fourier representation of the focal plane. In the chapters that follow we work mainly with focal plane correlations. These two domains have reciprocal relationship. For example, the central peak in one domain corresponds to the plateau region in the other domain. The ridges in one domain correspond to ridges in the other domain.

Wirnitzer (1985) has calculated the SNR for the bispectrum as well as for phase reconstruction for general light levels. His results, like our results in this chapter, deal only with the plateau region. For bright sources and for one frame of data his results can be summarized as follows

$$SNR_{BISPECTRUM} \sim 1 \tag{1.5}$$

$$SNR_{PHASE} \sim N_S^{1/2} \tag{1.6}$$

where N_S is the average number of spekles per frame. Note that the above SNR for the bispectrum is of the same order as that for the power spectrum:[3]

$$SNR_{POWER} \sim 1 \quad \text{high flux } \mathcal{N} > 1 \tag{1.7}$$

$$\sim \mathcal{N} \quad \text{loo flux } \mathcal{N} < 1 \tag{1.8}$$

Paradoxically it appears that in so random a phenomenon as speckles the phase of a Fourier component is better determined than the amplitude. This paradox is removed below.

1.2 SNR for bispectrum at high light levels

In this section we use an extremely simplified picture of the speckle phenomenon to estimate the SNR for the bispectrum in the wave limit. The wave limit holds when the sources are so bright that wave theory is entirely adequate to describe the intensity fluctuations. In particular, the noise due to the fluctuations in the number of detected photons is negligible when compared to the noise given by the wave picture. The system response R(X) can be considered to contain N_s speckles, each with roughly the diffraction limited size, spread over an area about 1" wide (the seeing disk). The number of speckles is about $100D^2$ where D is the telescope diameter in meter. The intensity of each speckle is random but in the following we shall regard it as a constant say I_0 . The positions X_i's of the speckles can be considered random and statistically independent of each other. Thus for the system response we use

$$R(x) = I_0 \sum_i \delta(x - x_i) \quad ; \quad R_u = R_0 \sum_i e^{iux_i} \tag{1.9}$$

We see that the system response function in the Fourier domain is just a sum of N_s uncorrelated complex numbers. Because of the assumption of sufficient randomness of the speckle positions the average of all these numbers is zero. Consider the power spectrun in this picture

$$R_u R_{-u} = R_0^2 \{ N_s + \sum_{i \neq j} exp\{ iu(x_i - x_j) \} \tag{1.10}$$

The power spectrum is seen to be made of N_s deterministic terns and $\sim N_s^2$ random terms with zero average. It is then clear that for the average only the N_s deterministic terms contribute while for the average of the square of the power spectrum the contribution from random terms dominates. This gives us the high flux SNR for one power spectrum point

$$SNR_{POWER} \sim 1 \tag{1.11}$$

This result is well-known in the literature. Now consider the bispectrum which in our picture is given by

$$R_u R_v R_{-u-v} R_o^{-3} = \sum_i 1 + \sum_{j \neq l} e^{i(u+v)(x_j - x_l)} + \sum_{j \neq l} e^{iu(x_j - x_l)} + \sum_{j \neq l} e^{iv(x_j - x_l)} + \sum_{j,k,l \text{ all distinct}} e^{i[u(x_j - x_l) + v(x_k - x_l)]} \tag{1.12}$$

Note that the triple product $R_u R_v R_{-u-v}$ contains N_S deterministic terms and $N_S^3 - N_S$ random terms with zero mean. It follows that the average of the bispectrum is

$$\langle R_u R_v R_{-u-v} \rangle = R_o^3 N_S = N_S \mathcal{N}^3 \tag{1.12}$$

Now consider the modulus of the bispectrum

$$R_o^{-6} |R_u R_v R_{-u-v}|^2 = \{N_S + [(N_S^3 - N_S) \text{ random terms}]\}\{N_S + [(N_S^3 - N_S) \text{ random terms}]^*\}$$
$$= N_S^2 + N_S^3 - N_S + \text{cross terms with nonzero phases.}$$

The cross terms vanish on averaging because they have nonzero phase factors. Thus

$$\langle |R_u R_v R_{-u-v}|^2 \rangle \sim R_o^6 N_S^3 \qquad \text{as } N_S \gg 1 \tag{1.13}$$

This gives us the SNR for a general bispectrum element

$$SNR_{BISPECTRUM} \sim N_S^{-1/2} \tag{1.14}$$

The physical reason for this dependance on N_S is that the signal comes from those events for which the points at which intensities are correlated lie within a speckle size of each other. In the present picture of the PSF information is carried by coincident speckles. In the case of double correlation there are N_S pairs of coincident speckles while the total number of pairs is N_S^2. In the case of triple correlation the number of coincident triplet of speckles is still N_S but the total number of triplets possible has gone up to N_S^3. This explains the poorer SNR for one bispectrum element. In general as one goes for larger and larger ordered correlations the SNR for one element in the frequency domain will go down. However, the bispectrum is just an intermediate step. The relevant quantity is the phase and it is well-known that the bispectrum elements store phase information redundantly. The number of statistically independent regions (the speckles) in the image plane, and therfore in the spatial frequency domain, is N_S. The number of such regions is N_S^2 for the bispectrum because it is a function of two spatial frequencies. This redundancy in bispectrum is expected to lead to an improvement by a factor $N_S^{1/2}$ for reconstructiong individual phase values from the bispectrum (Wirnitzer 1985). With our estimate for the SNR for the bispectrum but Wirnitzer's improvement factor (with which we agree) we get the SNR for phase reconstruction as

$$SNR_{PHASE} \sim 1 \tag{1.15}$$

Note that this is of the same order as that for the amplitude.

1.3 SNR for bispectrum at low light levels

At low light levels (fainter thatabout 13'th magnitude) a speckle receives less than a photon per exposure (10 ms, 100 A bandwidth) and one must also consider the noise due to the photonic nature of light. This involves two steps: 1) Since photon noise introduces bias terms dominant at low light levels one starts with unbiased estimators for the bispectrum which when averaged over the Poisson statistics obeyed by the photons gives the classical bispectrum for that realization of the atmosphere. Note that by this construction the average, even for low light levels, is going to be the same as in the wave limit. 2) Calculate the variance for such an unbiased estimator considering both the atmospheric and photonic noise. Wirnitzer starting with the correct unbiased estimator gets the right N_0^3 leading noise term for low light levels. Here, N_0 is the total number of photon counts in an exposure $N_0 = N_S \mathcal{N}$. The error in the previous calculations is traced to erroneous estimation of the bispectrum average itself. Thus the previous overestimate continues to be so even in the low flux limit. Our results for the low flux SNR for the bispectrum and the phase reconstruction are obtained by using our average Eq 1.12 and the low flux variance

$$\text{low flux} \qquad SNR_{BISPECTRUM} \sim N_S^{-1/2} \mathcal{N}^{3/2} \tag{1.16}$$

$$SNR_{PHASE} \sim \mathcal{N}^{3/2} \tag{1.17}$$

Note that so long as the photon count per speckle is less than unity we get the SNR for phase determination less than the SNR for power spectrum.

The previous overestimate is by a factor of $N_S^{1/2}$. For a 5 m telescope and 1" seeing the average number of speckles is about 2500. The previous over estimate is by a factor of 50. A factor of 50 in flux means a magnitude difference of about 4.5. Idealizations of the PSF similar to ones used in this chapter have been used before and are known to give the correct scalings with $\mathcal{N}$ and N_S.

The high flux SNR was presented in the ll'th meeting of the Astronomical Society of India. The low flux SNR was presented at the NOAO-ESO conference [11] where it became clear that similar conclusio was reached (independently) by other groups: Ayers et al ; Hofmann; Nakajima.