APPENDIX A

SHELL MODELS OF INHOMOGENEOUS SPHERICAL SYSTEMS

These models are time dependent generalizations of the stationary model of equations (3.1b). For convenience we write down the distribution function again

$$f_0 \propto \delta(V_r)\,\delta(V_\perp - V_c(r)) \quad \text{for} \quad r < R_0$$
$$= 0 \quad \text{for} \quad r \geq R_0 \qquad \text{(A.1)}$$

This stationary model has uniform density within radius equal to R_0 and it consists of stars moving on circular orbits. The gravitational force on a star at radius r is $\frac{GM(r)}{r^2} \propto r$. Balancing the centrifugal force, V_c^2/r, against gravity we see that $V_c(r) \propto r$. So we imagine that the system is made of concentric shells of stars each of which moves in a circle about the centre. What happens if all the stars in a shell of radius r (for brevity S_r) are, at some instant of time, given some (say, outward) radial velocity? The motion of the stars in S_r is not affected if the mass $[M(r)]$ in the shells enclosed is lumped into a point mass at the centre. The outer shells do not affect the motion of the stars in S_r anyway, so for the present let us ignore them. When all the stars in S_r are given some radial velocity, they will (if the radial velocity is not too large!) oscillate between $r_>$ and where $r_< < r < r_>$. The motion of every star of S_r is a Keplerian ellipse with apocentre $r_>$ and pericentre $r_<$.

We have got one shell oscillating periodically. To

keep it oscillating, it has been asssumed that it does not not intersect other shells (the mass enclosed by the shell in the course of its oscillations should not vary with time). This can be achieved by making the other shells oscillate too!

If a shells oscillate with eaual periods and in phase, it looks as if we can prevent shells from crossing. To do this, it is necessary (though not sufficient) that the $r_<$ and $r_>$ of a shell should both be greater than $r_<$ and $r_>$ of shells enclosed and at the same time are less than the $r_<$ and $r_>$ of shells outside. The semimajor axis of a particle in a shell moving on a Keplerian ellipse is given by

$$a = \frac{r_< + r_>}{2} \qquad \text{(A.2)}$$

So a orders the shells.

Since we require that the periods of oscillation of all shells are equal,

$$(\text{Period})^2 = \frac{4\pi^2}{G}\frac{a^3}{M(a)} = \text{constant} \qquad \text{(A.3)}$$

Hence
$$M(a) \propto \frac{4\pi^2}{G} a^3 \qquad \text{(A.4)}$$

Even if at some instant of time all stars are at their apocentres (and hence will be at their pericentres after half a period) we have not yet ensured that shells will not cross at any time. To make sure that shells do not cross, we need

to follow the motion of a shell for half a period from apocentre to pericentre (the other half of is similar).

Choosing a period of oscillation to be 2π units of time, we require that

$$\left.\frac{\partial r}{\partial a}\right|_{t=\text{const}} > 0 \quad \text{for} \quad 0 \leq t \leq \pi \tag{A.5}$$

Since the time variation of the radius of a shell is identical to the time variation of the radial coordinate in the Kepler problem, we shall use the standard parameterization of and t in terms of the eccentric anomaly η.

$$r = a(1 - e\cos\eta) \tag{A.6a}$$

$$t = \eta - e\sin\eta \tag{A.6b}$$

e is in general a function of a. Since all orbits are closed

$$0 \leq e(a) \quad 1 \tag{A.7}$$

To express the requirement in (A.5), we first need to see how η varies with a when t is held fixed. From (A.6b)

$$dt = d\eta(1 - e\cos\eta) - \sin\eta\, de = 0$$

Therefore

$$d\eta = \frac{\sin\eta\, de}{1 - e\cos\eta} \tag{A.8}$$

From (A.6a)

$$dr = da(1 - e\cos\eta) + a(e\sin\eta\, d\eta - de.\cos\eta) \quad \text{(A.9)}$$

Using (A.8) in (A.9) and with $e' = \frac{de}{da}$ and $x = \cos\eta$ we rewrite (A.5) as

$$\left.\frac{\partial r}{\partial a}\right|_{t=\text{const}} = \frac{(1-ex)^2 + ae'(e-x)}{(1-ex)} \quad \text{(A.10)}$$

The denominator is always positive. So we only need require that the numerator be positive for all $x \in [-1, 1]$:

$$N(x) = (1-ex)^2 + ae'(e-x) > 0 \text{ for } x \in [-1, 1] \quad \text{(A.11)}$$

This is a quadratic expression in x , which gives us constraints on the allowed functional forms of $e(a)$.

We summarize the models briefly and work out two examples.

The models are described implicitly by the equations

$$M(a) = \frac{a^3}{G_1} \quad \text{for } a \le a_{max} \quad \text{(A.12a)}$$

$$r = a(1 - e\cos\eta) \quad \text{(A.12b)}$$

$$t = \eta - e\sin\eta \quad \text{(A.12c)}$$

The allowed functional forms of $e(a)$ are constrained by (A.11). In principle from (A.12b) and (A.12c), we can express a as a function of r and t. Then

$$M(r,t) = \frac{1}{G}\left\{a(r,t)\right\}^3 \qquad (A.13)$$

Example 1: Let e = constant (i.e. independent of a). Then (A.11) is immediately satisfied.

From (A.12)

$$M(r,t) = \frac{1}{G}\left[\frac{r}{1-e\cos\eta}\right]^3 \qquad (A.14)$$

where η is now a function of t alone. Since $M \propto r^3$ this is a uniform density oscillating sphere such as the ones described in section 3.1.

Example 2:. Let $e = \lambda a$ where λ is a positive constant. We choose $a_{max} = 1$. So $0 < \lambda < 1$. We shall see below that (A.11) places further constraints on λ.

$$N(x) = (1-ex)^2 + e(e-x) > 0 \quad \text{for } x \in [-1,1]$$

where we have chosen to work with e itself instead of λ and

$$\text{i.e. } N(x) = e^2x^2 - 3ex + (1+e^2) > 0 \text{ for } x \in [-1,1]$$

It is clear that $N > 0$ for $x \leq 0$. The roots of $N(x) = 0$ are real. So if we require that the smaller root be greater than 1, then $N(x)$ will be positive for $x \leq 1$. The smaller root

$$\mu = \frac{3 - \sqrt{5-4e^2}}{2e} > 1$$

implies that

$$2e^2 - 3e + 1 > 0$$

This condition is satisfied if $e < 1/2$ or $e > 1$. Since e is always less than 1, we should require that $0 \leq e < 1/2$. The choice $a_{max} = 1$ implies that

$$0 < \lambda < 1/2 \qquad \text{(A.15)}$$

The mode1 is now described by

$$M(a) = a^3/G$$

$$r = a(1 - \lambda a \cos\eta) \qquad \text{(A.16)}$$

$$t = \eta - \lambda a \sin\eta$$

The density is $$\rho(r,t) = \frac{1}{4\pi r^2} \frac{\partial M}{\partial r}$$

It is not easy to express a as a function of r and t. But we can see what the system looks like at

(i) Maximum contraction $(\eta = 0)$

Then $$r = a(1 - \lambda a) \qquad \text{(A.17)}$$

and $$t = 0$$

Since $a_{max} = 1$ the radius of the system is $r_{max} = (1 - \lambda)$. Solving for a from (A.17) we have

$$a = \frac{1 - \sqrt{1 - 4\lambda r}}{2\lambda}$$

Therefore

$$M(r, t=0) = \frac{1}{G_1}\left[\frac{1-\sqrt{1-4\lambda r}}{2\lambda}\right]^3 \qquad \text{(A.18)}$$

and

$$\rho(r, t=0) = \frac{\left(1-\sqrt{1-4\lambda r}\right)^2}{16\pi\lambda^2 G_1 r^2\sqrt{1-4\lambda r}}$$

(ii) Maximum expansion ($\eta = \pi$)

Then $t = \pi$

$$r = a(1+\lambda a) \qquad \text{(A.19)}$$

Since $a_{max} = 1$ the radius of the system is $r_{max} = (1+\lambda)$.

Solving for a from (A.19) we have

$$a = \frac{\sqrt{1+4\lambda r}-1}{2\lambda}$$

Therefore

$$M(r, t=\pi) = \frac{1}{G_1}\left[\frac{\sqrt{1+4\lambda r}-1}{2\lambda}\right]^3$$

$$\rho(r, t=\pi) = \frac{\left(\sqrt{1+4\lambda r}-1\right)^2}{16\pi\lambda^2 G_1 r^2\sqrt{4\lambda r+1}} \qquad \text{(A.20)}$$

The ρ corresponding to (A.18) and (A.20) are shown in the figures A1 and A2 below for $\lambda = \frac{1}{4}$ and $G_1 = \frac{3}{\pi}$.

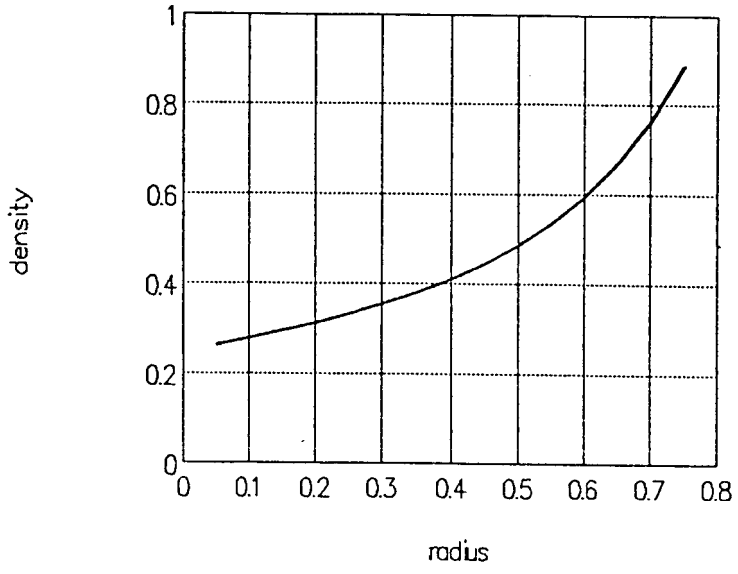

FIGURE A1. ρ at maximum contraction for $\lambda = \frac{1}{4}$, $G = \frac{3}{\pi}$

$r_{max} = 0.75$

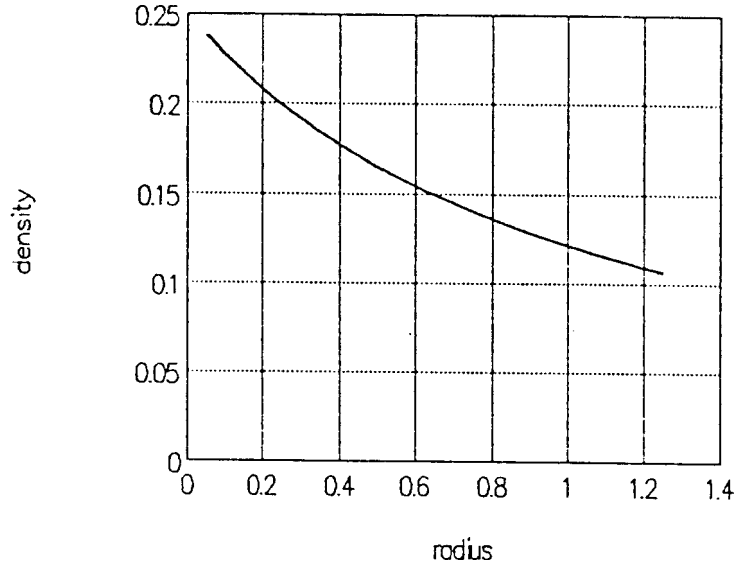

FIGURE A2. ρ at maximum expansion for $\lambda = \frac{1}{4}$, $G = \frac{3}{\pi}$

$r_{max} = 1.25$

Discussion

All models constructed in this appendix have finite size $-M(a) \propto a^3$ implies that for finite total mass $= M(a_{max})$, a_{max} is finite.

While it is not difficult to construct a variety of models, it should be noted that we have not used Jeans' theorem at all. The gravitational potential $\varphi(r,t)$ is nonlinear and time dependent. Other than angular momentum (which is conserved due to spherical symmetry) the potential may not admit global constants of motion. The models have been constructed by selectively populating periodic elliptical orbits of $\varphi(r,t)$. The nature of the unpopulated orbits might reveal interesting dynamics, and throw some light on what could happen if the models were slightly perturbed.

APPENDIX B

PROOF OF EQUATION (4.19)

By definition

$$\overline{Z_i Z_j} = \frac{1}{M}\int Z_i Z_j f_{GFD}\, d^4Z$$

$$= \frac{f_0}{M}\int Z_i Z_j (1-I)^{-1/2}\, d^4Z \qquad (B.1)$$

where $$I = Z^T Q Z \qquad (B.2)$$

Since $Z_i Z_j = \frac{\partial I}{\partial Q_{ij}}$ (where Q_{ij} is the element of Q in the i^{th} row and the j^{th} column) we have

$$\overline{Z_i Z_j} = -\frac{2f_0}{M}\frac{\partial}{\partial Q_{ij}}\int (1-I)^{1/2}\, d^4Z \qquad (B.3)$$

Q is a positive definite symmetric 4x4 matrix. Let us denote its eigenvalues by λ_1^2, λ_2^2, λ_3^2, λ_4^2. We can diagonalize Q (and hence I) by performing a special ($\det Q = 1$) orthogonal transformation, $\mathcal{O}$:

$$W = \mathcal{O} Z$$

$$I = Z^T Q Z = W^T(\mathcal{O}^T Q \mathcal{O})W = \sum_{i=1}^{4} \lambda_i^2 W_i^2 \qquad (B.4)$$

$\det Q = 1$ implies $d^4Z = d^4W$. Therefore

$$\overline{Z_i Z_j} = -\frac{2f_0}{M}\frac{\partial}{\partial Q_{ij}}\int\left\{1-\left(\sum_{i=1}^{4}\lambda_i^2 W_i^2\right)\right\}^{1/2} d^4W$$

$$= -\frac{2f_0}{M}\frac{\partial}{\partial Q_{ij}}\left(\frac{1}{\sqrt{\det Q}}\right)\int(1-u^2)^{1/2}\,d^4u$$

$$= \frac{k_1 f_0}{M}(\det Q)^{-3/2}\frac{\partial}{\partial Q_{ij}}[\det Q] \qquad \text{(B.5)}$$

where $$k_1 = \int(1-u^2)^{1/2}d^4u = \frac{4}{15}\pi^2 \qquad \text{(B.6)}$$

Defining $P = Q^{-1}$ we have

$$P_{ij} = C_{ji}/\det Q$$

where C_{ij} is the "cofactor" of Q_{ij}. Therefore

$$\frac{\partial}{\partial Q_{ij}}[\det Q] = C_{ij} = (\det Q)P_{ji} = (\det Q)P_{ij} \qquad \text{(B.7)}$$

From *(8.5)* and (8.7)

$$\overline{Z_i Z_j} = \frac{k_1 f_0}{M\sqrt{\det Q}}\,P_{ij} \qquad \text{(B.8)}$$

The mass

$$M = \int f\,d^4Z = f_0\int(1-I)^{-1/2}d^4Z$$

$$= f_0\int\left\{1-\left(\sum_{i=1}^{4}\lambda_i^2 W_i^2\right)\right\}^{-1/2}d^4W$$

(B.9)

$$= \frac{f_o}{\sqrt{\det Q}} k_2$$

where $$k_2 = \int (1-u^2)^{-1/2} d^4u = \frac{4\pi^2}{3} \qquad \text{(B. 10)}$$

Using (B.9) and (B.10) in (B.8) we get

$$\overline{Z_i Z_j} = \frac{k_1}{k_2} P_{ij} = \frac{1}{5} P_{ij} \qquad \text{(B.11)}$$

APPENDIX C

A HAMILTONIAN BASIS FOR THE CBE
AND
AN APPLICATION TO THE DYNAMICS OF A GFD

Morrison (1980) and Morrison and Greene (1980) have shown that many non disspative field equations governing the behaviour of fluids and plasmas can be written in Hamiltonian form. In particular the Vlasov-Poisson equations for collisionless plasmas falls in this class of field equations. Since the CBE for stellar systems is identical to the Vlasov equation plus the Poisson equation with a sign change, the Hamiltonian description carries over. Morrison (1982) and Littlejohn (1982) present very readable accounts of these developments.

The relevant field that evolves according to the CBE is the distribution function f. We shall say that a Hamiltonian structure for the CBE has been found if it can be written as

$$\dot{f} = [f, \mathcal{H}] \tag{C1}$$

where $\mathcal{H}$, the "Hamiltonian" is a functional of f. The expression on the right hand side is the Poisson Bracket (PB) between f and $\mathcal{H}$. The PB is any binary operation between two functionals giving a third functional obeying the usual

rules of antisymmetry, Leibnitz rule and the Jacobi indentity (for more details see eg. Arnold 1978). A PB is called singular if there exists a functional that gives zero PB with all other functionals. By (C1), this means that the functional is conserved under the action of any $\mathcal{H}$ whatsoever. For the CBE it turns out that there are an infinity of such conserved functionals. For example, any

$$\mathcal{C}[f] = \int C(f)\, d^3x\, d^3v$$

is such a conserved functional. These, in principle, have to be eliminated before the PB becomes nondegenerate.

We give a nonrigorous derivation of the PB for the CBE below. Let us imagine that f represents a large number (N) of point particles each of mass equal to 1/N. Let the position of the p^{th} particle in (6-dimensional) phase space be denoted by

$$\mathcal{Z}^{(P)} = \left(x^{(P)}, y^{(P)}, z^{(P)}, V_x^{(P)}, V_y^{(P)}, V_z^{(P)}\right)^T \qquad (C2)$$

and let

$$\mathcal{Z} = \left(x, y, z, V_x, V_y, V_z\right)^T \qquad (C3)$$

denote an arbitrary point in phase space. Then

$$= \frac{1}{N} \sum_{P=1}^{N} \delta\left(\mathcal{Z} - \mathcal{Z}^{(P)}\right) \qquad (C4)$$

where

$$\delta(Z - Z^{(P)}) = \delta(x - x^{(P)}) \ldots\ldots \delta(V_z - V_z^{(P)}) \qquad \text{(C5)}$$

For two functions

$$F_1 = F_1(Z^{(1)}, Z^{(2)}, \ldots\ldots, Z^{(N)})$$

$$F_2 = F_2(Z^{(1)}, Z^{(2)}, \ldots\ldots, Z^{(N)}) \qquad \text{(C6)}$$

the PB is

$$[F_1, F_2] = \sum_{P=1}^{N} \sum_{i,k=1}^{6} \left(N \frac{\partial F_1}{\partial Z_i^{(P)}} \frac{\partial F_2}{\partial Z_k^{(P)}} \right) \qquad \text{(C7)}$$

where Ω is the 6 x 6 matrix

$$\Omega = \begin{pmatrix} 0_{3\times3} & \mathbb{1}_{3\times3} \\ -\mathbb{1}_{3\times3} & 0_{3\times3} \end{pmatrix} \qquad \text{(C8)}$$

The factor N appears in "( )" of equation (C7) because $\frac{\partial}{\partial(\text{momentum})} = N \frac{\partial}{\partial(\text{velocity})}$. When F_1 and F_2 are symmetric in their arguments, they may be thought of as functionals of f and we may write $F_1 = \mathcal{F}_1[f]$ and $F_2 = \mathcal{F}_2[f]$. Functional derivatives are defined by the functional Taylor series

$$\mathcal{F}_1[f] = \mathcal{F}_1[0] + \int f(Z) \left\{ \frac{\delta \mathcal{F}_1}{\delta f} \right\} d^6Z + \text{-----} \qquad \text{(C9)}$$

Therefore (with $\mathcal{F}_1[0]$ set to zero)

$$F_1 = \frac{1}{N} \sum_{P=1}^{N} \left(\frac{\delta \mathcal{F}_1}{\delta f} \right)_{z = z^{(P)}}$$

(C10)

$$\frac{\partial F_1}{\partial z_i^{(P)}} = \frac{1}{N} \frac{\partial}{\partial z_i^{(P)}} \left(\frac{\delta \mathcal{F}_1}{\delta f} \right)$$

Similar results are true for $\tilde{\$}$. Using (C10) in (C7) we get

$$[F_1, F_2] = \frac{1}{N} \sum_{P=1}^{N} \left\{ \sum_{i,k=1}^{6} \frac{\partial}{\partial z_i^{(P)}} \left(\frac{\delta \mathcal{F}_1}{\delta f} \right) \frac{\partial}{\partial z_k^{(P)}} \left(\frac{\delta \mathcal{F}_2}{\delta f} \right) \right\}_{z = z^{(P)}}$$

$$= \int f(z) \left\{ \sum_{i,k=1}^{6} \frac{\partial}{\partial z_i^{(P)}} \left(\frac{\delta \mathcal{F}_1}{\delta f} \right) \frac{\partial}{\partial z_k^{(P)}} \left(\frac{\delta \mathcal{F}_2}{\delta f} \right) \right\}_{z = z^{(P)}} d^6 z$$

The expression in $\{\ \}''$ is the ordinary PB between $\frac{\delta \mathcal{F}_1}{\delta f}$ and $\frac{\delta \mathcal{F}_2}{\delta f}$ which are functions on single particle (6-D) phase space. On this phase space we take $\{x, V_x\} = 1$. Therefore

$$[\mathcal{F}_1, \mathcal{F}_2] \overset{def}{=\!=} [F_1, F_2] = \int f(z) \left\{ \frac{\delta \mathcal{F}_1}{\delta f}, \frac{\delta \mathcal{F}_2}{\delta f} \right\} d^6 z \qquad \text{(C11)}$$

This agrees with Morrison (1982).

Given f, the total energy of the stellar system is

$$\mathcal{H} = \frac{1}{2} \int f v^2 \, d^6 z - \frac{G}{2} \int \frac{f(z) f(z') \, d^6 z \, d^6 z'}{|x' - x|} \qquad \text{(C12)}$$

Then $$\frac{\delta \mathcal{H}}{\delta f} = \frac{v^2}{2} + \varphi \qquad \text{(C13a)}$$

where $$\varphi = -G \int \frac{f(x', v')\, d^6 z'}{|x' - x|} \qquad \text{(C13b)}$$

To now see that the CBE can be written in Hamiltonian form, let us define a functional

$$S_{z_0}[f] = f(z_0) \qquad \text{(C14)}$$

Then

$$\frac{\delta S_{z_0}}{\delta f} = \delta(z - z_0) \qquad \text{(C15)}$$

Using (C13) and (C15) in (C11) we have

$$[S_{z_0}, \mathcal{H}] = -\underline{v}_0 \cdot \frac{\partial f}{\partial \underline{x}_0} + \frac{\partial \varphi}{\partial \underline{x}_0} \cdot \frac{\partial f}{\partial \underline{v}_0} \qquad \text{(C16)}$$

Therefore (dropping the nought in the subscript)

$$\dot{f} = [f, \mathcal{H}] \qquad \text{(C17)}$$

We show below that the restriction of the PB defined in (C11) to the space of functions describing GFDs

$$f_{GFD} \propto (1-I)^{-1/2} \qquad \left\{ \begin{matrix} \text{see equations} \\ 4.14 \ \& \ 4.17 \end{matrix} \right\} \qquad \text{(C18)}$$

gives us a PB on the 10 dimensional space of the matrix P. Let Γ be the space of all real symmetric 4 x 4 matrices. P (which in addition is positive definite) belongs to Γ. For any $H \in \Gamma$, we define a function on Γ:

$$h(P) \propto \int (Z^T H Z) f_{GFD} \, d^4Z$$

The right hand side is proportional to $Tr(PH)$. Ignoring overall constants we define

$$h(P) = Tr(PH) \qquad \text{(C19)}$$

Two such functions $h(P)$ and $e(P)$ are also functionals on the space of functions of the form (C18). Their PB is

$$[e, h] = \int \left\{ \frac{\delta e}{\delta f_{GFD}}, \frac{\delta h}{\delta f_{GFD}} \right\} f_{GFD} \, d^4Z$$

$$\propto Tr(PE\omega H - PH\omega E)$$

Again ignoring an overall constant we define

$$[e, h] = Tr(PE\omega H - PH\omega E) \qquad \text{(C20)}$$

where

$$\omega = \begin{pmatrix} O_{2\times 2} & \mathbb{I}_{2\times 2} \\ -\mathbb{I}_{2\times 2} & O_{2\times 2} \end{pmatrix} \tag{C21}$$

We now show that the equation of motion for P (equation (4.22))

$$\dot{P} = KP + PK^{T}$$

can be written in Hamiltonian form. We note that K is of the form

$$K = \omega H \quad ; \quad H \in \Gamma \tag{C22}$$

Therefore

$$\dot{P} = \omega H P - P H \omega \tag{C23}$$

Let us define 10 real symmetric 4 x 4 matrices indexed by an unordered pair of superscripts (a, b):

$$M^{ab}_{ij} = \frac{\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja}}{2} \tag{C24}$$

These select the elements P_{ab} from P:

$$P_{ab} = Tr(PM^{ab}) \tag{C25}$$

Multiplying (C23) by M^{ab} and taking the trace we get

$$\dot{P}_{ab} = Tr\left(PM^{ab}\omega H - PH\omega M^{ab}\right)$$

which from (C20) and (C25) now reads

$$\dot{P}_{ab} = \left[P_{ab}, h\right] \tag{C26}$$

The equations of motion for the **GFD** are now in Hamiltonian form. The PB between P_{ab} and P_{cd} is easily calculated to be

$$\left[P_{ab}, P_{cd}\right] = \frac{1}{2}\left\{P_{db}\omega_{ac} + P_{ca}\omega_{bd} + P_{da}\omega_{bc} + P_{cb}\omega_{ad}\right\} \tag{C27}$$

These are identical to the basic Lie bracket relations for the elements of the Lie Algebra of the group Sp(4,R). We derived (C27) by defining a Morrison type PB for functions (on Γ) of the special form (C19). Having got the fundamental PB in (C27), PBs between more general functions on Γ can be calculated using the usual rules for manipulating PBs.

From (C27) it is clear that the P_{ab} are not canonical variables. Hence (C26) is in nonstandard form. Also the PB is degenerate. This follows from the existence of functions on Γ that are conserved for <u>any</u> h whatsoever (in group theory language, a function on the Lie Algebra that commutes with every other function is called a Casimir function). We

can construct conserved functions most easily by studying the properties of the evolution equations for P written in the "finite form".

$$P_t = S_t P_o S_t^T \tag{C28}$$

For completeness the derivation of several useful properties of $P\omega$ and its eigenvalues is given below (also see Arnold 1978).

(i) The eigenvalues of $P\omega$ are constants of motion

Proof: An eigenvalue, λ_t, obeys $|\lambda_t \mathbb{I} - P_t\omega| = 0$

(C29)

$$|\lambda_t \mathbb{I} - P_t\omega| = |\lambda_t \mathbb{I} - S_t P_o S_t^T \omega|$$
$$= |S_t^{-1}(\lambda_t \mathbb{I}) S_t - P_o\omega| = |\lambda_t \mathbb{I} - P_o\omega|$$

Therefore $\lambda_t = \lambda_o$ □ (C30)

We have also shown that the entire characteristic equation (C29) is conserved. Writing (C29) as

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0 \tag{C31}$$

we note that a_1, a_2, a_3 and a_4 are also constants.

(ii) If λ is an eigenvalue so is $-\lambda$

Proof: $|\lambda \mathbb{I} - P\omega| = 0 \Rightarrow |(\lambda \mathbb{I} - P\omega)^T| = 0$

$$\Rightarrow |\lambda \mathbb{I} + \omega P| = 0 \quad \text{Since} \quad \omega^2 = -\mathbb{I}$$

$$|\lambda \omega - P| = 0 \Rightarrow |\lambda \mathbb{I} + P\omega| = 0 \qquad \square$$

Hence only the terms with even powers of λ (a_2 and a_3) in (C31) are nonzero. These coefficients are sums of products of the eigenvalues and they are related to the traces of various powers of $P\omega$. So we can write our Casimirs as

$$\mathcal{F}_1 = Tr\,(P\omega)^2$$

$$\mathcal{F}_2 = Tr\,(P\omega)^4 \qquad \text{(C32)}$$

Since (C29) is of degree 4, higher powers of $P\omega$ can be expressed in terms of lower powers (by the Cayley-Hamilton theorem - we thank C.S.Shukre for pointing this out to us). So two independent Casimirs are $\mathcal{F}_1$ and $\mathcal{F}_2$ which are conserved for <u>any</u> S_t. Therefore the evolution of P is restricted to an 8 dimensional submanifold of Γ. We could have anticipated the existence of two conserved quantities from the existence of two Poincare invariants for the Hamiltonian evolution on the (x, y, v_x, v_y) space. For evolution under the action of generic potentials Poincare invariants other than the phase volume do not give global conserved quantities on Γ - harmonic potentials are exceptional. The Poincare invariants for a GFD are the volume occupied by the GFD in phase space and the 2-area of the maximum plane section through the GFD.

(iii) The eigenvalues of P_ω are pure imaginary. We give a nonrigorous proof of this. Let us consider the linear dynamical system

$$\dot{Z} = P_\omega Z$$

This conserves $\mathcal{G} = Z^T \omega P_\omega Z$. It is easy to see that when P is positive definite ωP_ω is negative definite. So the level surfaces of $\mathcal{G}$ are compact implying that $Z^T Z$ remains bounded. We know that if λ is an eigenvalue of P_ω, so are λ^*, $-\lambda$ and $-\lambda^*$. If λ is not pure imaginary two of these four will have positive real parts implying that for some initial Z, $Z^T Z$ will grow without bound.
So the eigenvalues of P_ω are of the form

a) , , , ($\alpha \neq \beta$, the nondegenerate case)

or of the form

b) $i\alpha$, $-i\alpha$ ($\alpha = \beta$, the degenerate case)

The 8 dimensional surfaces on which the P_s are constrained can be labelled either by (α, β) or by $(\mathcal{P}_1, \mathcal{P}_2)$.
The implication for the dynamics of GFDs - even when they are tidally perturbed - is that two configurations which have different (α, β) or $(\mathcal{P}_1, \mathcal{P}_2)$ cannot be deformed into one another through symplectic means.

Our starting point in the above analysis was the PB structure of Morrison and Greene. In fact we could have started with (C28) without worrying about the underlying

applications to galactic dynamics. In some recent work in quantum mechanics and paraxial optics properties of equations which are related to (C28) are studied (see eg. Mukunda et al. and references therein for a different perspective). This approach emphasises the group theory aspect wherein (C28) is viewed as the action of the group Sp(4,R) - in general Sp(2n,R) - on its Lie Algebra. A theorem due to Kirillov, Kostant and Souriau (see eg. Chu 1974) guarantees the existence of a symplectic structure in more general cases.

It is interesting that the two constants α and β associated with the oscillating spheroids of chapter 3 can be simply interpreted in the framework given here. The motion of particles along the z axis of the spheroid is governed by a time dependent oscillator equation, and the projected "area" of the model in the $z - V_z$ plane is conserved. There is a similar conserved quantity associated with the $x - V_x$ (or $y - V_y$) plane. For a time independent oscillator this phase area is clearly proportional to $\Omega_0 a^2$ where Ω_0 is the frequency and a the maximum amplitude. Once these two constants are computed for a given static model, they retain the same values for any oscillating model derived from the static one by Hamiltonian evolution in phase space. A look at equation (3.45) shows that β is essentially one of these invariant areas and α is their ratio.

APPENDIX D

NUMERICAL SCHEME USED FOR GFDs

We outline the basic features of the numerical scheme used to solve (4.22) self consistently. We recall that

$$\dot{P} = KP + PK^T \tag{D.1}$$

where

$$K = \begin{pmatrix} O_{2\times 2} & \mathbb{I}_{2\times 2} \\ -\mathbb{F} & O_{2\times 2} \end{pmatrix} \tag{D.2}$$

$\mathbb{F}$ is the 2 x 2 matrix given by (4.27) and (4.35). depends on , , P_{12} , P_{22} and this' is what makes (D.1) nonlinear and self consistent. When (D.1) is to be solved numerically, one has to use a finite difference scheme. We use the "finite" version of the evolution equations as given in (4.20)

$$P(t_2) = S P(t_1) S^T \tag{D.3}$$

instead of writing down a naive finite difference approximation to (D.1) itself. The advantage of evolving P by (D.3) is that symplectic properties (like conservation of $Tr(P\omega)^2$, $Tr(P\omega)^4$) are automatically preserved upto machine accuracy. S is that symplectic matrix that takes $Z(t_1)$ to $Z(t_2)$:

$$\mathcal{Z}(t_2) = S\,\mathcal{Z}(t_1) \tag{D.4}$$

This is just the evolution equation (in finite difference form) for two coupled harmonic oscillations. A simple leap frog scheme (update coordinates and then momenta with forces computed from the new coordinates) will guarantee preservation of the symplectic nature of the equations as well as numerical accuracy to second order in Δt, the time step. The matrix that updates coordinates is

$$\mathcal{S}_x = \begin{bmatrix} \mathbb{I}_{2\times 2} & \Delta t\,\mathbb{I}_{2\times 2} \\ \mathbb{O}_{2\times 2} & \mathbb{I}_{2\times 2} \end{bmatrix} \tag{D.5}$$

while the matrix that updates momenta is

$$\mathcal{S}_v = \begin{bmatrix} \mathbb{I}_{2\times 2} & \mathbb{O}_{2\times 2} \\ -\Delta t\,\mathbb{F} & \mathbb{I} \end{bmatrix} \tag{D.6}$$

We write

$$S = \mathcal{S}_{v\frac{1}{2}} \left(\overbrace{\mathcal{S}_x \mathcal{S}_v}^{1} \cdots\cdots \overbrace{\mathcal{S}_x \mathcal{S}_v}^{N} \right) \mathcal{S}_{x\frac{1}{2}} \tag{D.7}$$

where the time interval $(t_2 - t_1)$ has been subdivided into N equal intervals

$$\Delta t = \frac{t_2 - t_1}{N} \tag{D.7}$$

$\mathcal{S}_{x\frac{1}{2}}$ and $\mathcal{S}_{v\frac{1}{2}}$ are updates over times $\frac{\Delta t}{2}$; this is required by any leap frog scheme. Since we are dealing with a self

consistent problem, δ_V at each stage is calculated from the present value of P.

The addition of the tidal field of a perturber just adds terms to and the whole scheme outlined above goes through.

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