CHAPTER 3

HOMOGENEOUS SPHERICAL MODELS

3.1 Construction of the models

We briefly discuss stationary models of homogeneous (ie. uniform density) spheres and then go on to generalize these models to time dependent states. A homogeneous, stationary, self gravitating fluid sphere corresponds to a polytrope of index zero which is not realisable as a collisionless stellar system with isotropic velocities (Vandervoort 1980). A distribution function (f_0) that is a function of energy alone implies isotropic velocites and therefore will not describe a uniform sphere. One alternative is to seek a function of both E and L^2 ($\underline{L}$ is the angular momentum per unit mass = $\underline{r} \times \underline{V}$). Another is to introduce some rotation by requiring f_0 to be 4 function of E, L^2 and L_z. Two functions thatdescribe a uniform sphere are (see Fridman & Polyachenko 1984 - hereafter FP- and references therein)

$$f_0 \propto \left(\frac{L^2}{2R_0^2} + \varphi(R_0) - E \right)^{-1/2} \tag{3.1a}$$

$$f_0 \propto \delta(V_r)\, \delta\left(V_\perp - V_c(r)\right) \tag{3.1b}$$

where $V_c(r)$ is the circular velocity at r, V_r is the radial velocity and $V_\perp^2 = V^2 - V_r^2$. Any f_0 is a member of a two parameter family of functions characterized by total mass (M_0) and radius (R_0). The different members of such a family can be derived from each other by a change in the units of length and time (of course there could be other continuously varying parameters which modify the distribution function in a less trivial way). The interior gravitational potential is

$$\varphi_0(r) = \omega_0^2 \frac{r^2}{2} \tag{3.2}$$

Since by Poisson's equation

$$\nabla^2 \varphi_0 = 3\omega_0^2 = 4\pi G \rho_0 = \frac{3GM_0}{R_0^3} \tag{3.3}$$

we can (for convenience) use ω_0 and R_0 as parameters instead of M_0 and R_0. We note below some general properties of the s in (3.1). They can we written in the form

$$f_0 = f_0\left(E/\omega_0\ ,\ L^2\ ;\ \omega_0\ ,\ R_0\right) \tag{3.4}$$

Since

$$\rho_0 = \int f_0 \, d^3v \tag{3.5}$$

using (3.3), we have

$$\begin{aligned} \int f_0 \, d^3v &= \frac{3\omega_0^2}{4\pi G} \qquad \text{for } r < R_0 \\ &= 0 \qquad \text{for } r \geqslant R_0 \end{aligned} \tag{3.6}$$

Defining $\underline{v}' = \underline{v}/\sqrt{\omega_0}$ and $\underline{r}' = \underline{r}\sqrt{\omega_0}$, we have

$$\int f_0\left(\frac{v'^2}{2}+\frac{r'^2}{2}\;,\;|\underline{r}'\times\underline{v}'|^2\right)d^3v' = \frac{3\sqrt{\omega_0}}{4\pi G} \quad \text{for } r' < R_0\sqrt{\omega_0}$$

$$= 0 \quad \text{for } r' \geq R_0\sqrt{\omega_0} \qquad (3.7)$$

where we have dropped ω_0 and R_0 in $f_0(\cdots\cdots)$. In later work it is always understood that $f_0(\cdots)$ depends on these.

The time dependent generalization is based on the following simple observation. In (3.7), the expression in the first slot is the energy which is a constant of motion for a particle moving in the potential given in (3.2). The energy is quadratic in $\underline{V}$ and $\underline{r}$ (equivalently $\underline{V}'$ and $\underline{r}'$). When the potential acquires time dependence (while remaining proportional to r^2), the corresponding constant of motion is the sum of the Lewis Invariants (with the same function $\xi(t)$) for motion along x, y and z. Since the Lewis Invariant is a quadratic expression in $\underline{V}$ and _ , replacing energy by this sum of three Lewis invariants in (3.7), should on integration over velocities give uniform density inside a spherical volume. We carry out this, programme explicitly below.

Let us write the gravitational potential of the time dependent sphere as

$$\varphi(r,t) = \omega^2(t)\,\frac{r^2}{2} \qquad (3.8)$$

The Lewis invariant for motion along the x direction is

$$I_x = \frac{x^2}{2\xi^2} + \frac{1}{2}\left(\xi V_x - \dot{\xi} x\right)^2 \tag{3.9a}$$

where

$$\ddot{\xi} + \omega^2(t)\,\xi - \frac{1}{\xi^3} = 0 \tag{3.9b}$$

We have similar expressions for I_y and I_z. Choosing the same function $\xi(t)$ for all of them, and adding, we define

$$\begin{aligned} \mathcal{I} &= I_x + I_y + I_z \\ &= r^2/2\xi^2 + \frac{1}{2}\left|\xi \underline{V} - \dot{\xi}\underline{r}\right|^2 \end{aligned} \tag{3.10}$$

The proposed distribution function for the time dependent sphere is

$$f = f_0\left(\mathcal{I}, L^2\right) \tag{3.11}$$

Notice that ω_0 and R_0 do not have the same physical meaning as in (3.4) but they still define characteristic scales of time and length associated with a given time dependent model. Alternatively they can be regarded as labelling a stationary model which gives birth to a family of time dependent models.

The density

$$\rho = \int f_0\left(\frac{1}{2}\left|\xi\underline{v} - \dot{\xi}\underline{r}\right| + \frac{r^2}{2\xi^2},\ \left|\underline{r}\times\underline{v}\right|^2\right) d^3v$$

$$= \frac{1}{\xi^3}\int f_0\left(\frac{u^2}{2} + \frac{a^2}{2},\ \left|\underline{a}\times\underline{u}\right|^2\right) d^3u \tag{3.12}$$

where

$$\underline{u} = \xi \underline{V} - \dot{\xi} \underline{r} \qquad (3.13)$$

$$\underline{a} = \underline{r}/\xi$$

From (3.7), (3.12) and (3.13)

$$\rho(r,t) = \frac{3\sqrt{\omega_0}}{4\pi G \xi^3} \quad \text{for} \quad r < R_0\sqrt{\omega_0}\,\xi$$

$$= 0 \qquad \text{for} \quad r \geq R_0\sqrt{\omega_0}\,\xi \qquad (3.14)$$

We see that the radius of the sphere is $\propto \xi(t)$. The strength of the gravitational potential

$$\omega^2(t) = \sqrt{\omega_0}/\xi^3 \qquad (3.15)$$

Therefore, from (3.9b) and (3.15), ξ satisfies

$$\ddot{\xi} + \frac{\sqrt{\omega_0}}{\xi^2} - \frac{1}{\xi^3} = 0 \qquad (3.16)$$

We note

(i) the spatially bound solutions of (3.16) are time periodic, while others eventually expand to infinity. These, as we shall see below correspond to spheres with total energy negative and positive respectively.

(ii) Choosing ξ □ $1/\sqrt{\omega_0}$ corresponds to the stationary mode1 described by (3.1a). Small oscillations about this stationary configuration (easily computed from (3.16)) occur with angular frequency ω_0 .

3.2 Some properties of homogeneous time dependent spheres

The mass (M_0), potential energy (W_0) and the kinetic energy (T_0) are easily calculated for the stationary sphere

(3.4).

$$M_0 = \frac{\omega_0^2 R_0^3}{G}$$

$$W_0 = -\frac{3}{5}\,\frac{\omega_0^4 R_0^5}{G} \qquad (3.17)$$

$$T = -\frac{W_0}{2}$$

In a similar manner various parameters for the time dependent model described by (3.11) can be directly calculated.

Mass = $M = M_0$

Radius = $R(t) = R_0\sqrt{\omega_0}\,\xi(t)$

Potential energy = $W = W_0/\sqrt{\omega_0}\,\xi(t)$

Mean velocity at $\underline{r}$ = $\underline{v}_m(\underline{r},t) = \underline{r}\left(\frac{\dot{\xi}}{\xi}\right)$

Bulk kinetic energy = $T_b = \frac{1}{2}\int \rho\, v_m^2\, d^3r = \left(\frac{\dot{\xi}}{\omega_0}\right)^2 T_0$

Heat (kinetic energy associated with local peculiar velocities)

$$= T_h = \frac{1}{2}\int |\underline{v} - \underline{v}_m|^2 f\, d^3v\, d^3r = \frac{T_0}{\omega_0 \xi^2}$$

Total kinetic energy = $T = T_b + T_h$

The second mass moment about the origin is

$$I = \int \rho\, r^2\, d^3r = \frac{2\xi^2}{\omega_0} T_0 \qquad (3.18)$$

The **Virial** theorem (see eg. BT) is

$$\frac{1}{2}\ddot{I} = 2T + W \qquad (3.19)$$

Using the formulae for I, T_b, T_h and W from (3.18) in (3.19) we have

$$\ddot{\xi} + \frac{\sqrt{\omega_0}}{\xi^2} - \frac{1}{\xi^3} = 0 \qquad (3.16')$$

which is identical to (3.16) governing radial oscillations. This connection with the Virial theorem allows easy interpretation of the terms occurring in (3.16). The term $(-1/\xi^3)$ resembles the (repulsive) centrifugal force occurring in the radial equation for a central force problem. It is clearly proportional to (T_h/ξ) and it makes good physical sense that the sphere is hottest when maximally compressed. The heat makes it bounce back. The total energy

$$E = T + W = \frac{2T_0}{\omega_0}\left(\frac{\dot{\xi}^2}{2} - \frac{\sqrt{\omega_0}}{\xi} + \frac{1}{2\xi^2}\right) \qquad (3.20)$$

Equation (3.16) admits a first integral

$$\lambda = \frac{\dot{\xi}^2}{2} - \frac{\sqrt{\omega_0}}{\xi} + \frac{1}{2\xi^2} \qquad (3.21)$$

Therefore

$$E = \frac{2T_0}{\omega_0}\lambda \qquad (3.22)$$

when $\lambda \geq 0$ the solutions to (3.16) are unbounded as $t \to \infty$ while for $\lambda < 0$ the solutions are bounded and time periodic. As noted earlier, we see that these correspond to positive and negative total energies respectively.

3.3 Oscillations of homogeneous spheroids

We use the method used for spheres to construct models of oscillating uniform density spheroids. The method is applicable because the interior potential of a homogeneous spheroid is quadratic in the spatial coordinates. A spheroid with axes (a_1, a_2, a_3 with $a_1 = a_2$) along the x, y and z directions respectively and mass density ρ_0 has an interior potential (see eg: Chandrasekhar 1969).

$$\varphi_0 = \Omega_0^2 \frac{r^2}{2} + \omega_0^2 \frac{z^2}{2} \quad \text{where} \quad r^2 = x^2 + y^2 \tag{3.23}$$

where

$$\Omega_0^2 = 2\pi G \rho_0 A_1(m)$$
$$\omega_0^2 = 2\pi G \rho_0 A_3(m) \tag{3.24}$$

and axis ratio $m = a_3/a_1$

$$A_1(m) = \left(\frac{m}{1-m^2}\right) h(m) + 1 - \frac{1}{1-m^2} \tag{3.25}$$

$$h(m) = (1-m^2)^{-1/2} \sin^{-1}\left((1-m^2)^{1/2}\right) \quad \text{for } m < 1$$
$$= (m^2-1)^{-1/2} \ln\left(m + (m^2-1)^{1/2}\right) \quad \text{for } m > 1 \tag{3.26}$$

Since φ_0 and ρ_0 satisfy $\nabla^2 \varphi_0 = 4\pi G \rho_0$, we have the following relation between A_1 and A_3:

$$A_3(m) = 2 - 2A_1(m) \tag{3.27}$$

Some examples of phase space distributions functions for stationary uniform spheroids are Freeman's rotating model which is cold in the plane of rotation and Polyachenko's hot model (see e.g. FP for details and references).

We first explain the general strategy for constructing time dependent spheroids and then go to details. The distribution functions of these models depends on the following integrals of motion

$$E_\perp = \frac{V_x^2}{2} + \frac{V_y^2}{2} + \Omega_0^2 \frac{r^2}{2}$$

$$E_z = \frac{V_z^2}{2} + \omega_0^2 \frac{z^2}{2} \tag{3.28}$$

$$L = x V_y - y V_x$$

The time dependent model is constructed from the static distribution function by replacing $E_\perp/\Omega_0$ and E_z/ω_0 by the corresponding Lewis invariants (see equation 3.32 below). The axes are proportional to $\xi(t)$ and $\eta(t)$ (see equation 3.37) which from the construction of the Lewis invariants satisfy equations (3.34). Self consistency eliminates the explicit time dependence in these equations since the force constants $\Omega^2(t)$ and $\omega^2(t)$ are functions of ξ and η from (3.24). We get a pair of autonomous coupled second order ordinary differential equations for ξ and η (equation 3.39).

Uniform spheroids can be grouped into families with three parameters determining the mass M, and axes a_1, a_2. For convenience we choose the parameters to be Ω_0, ω_0,

a_1. Then, the density ρ_0 is determined from Poisson's equation.

$$4\pi G\rho_0 = 2\Omega_0^2 + \omega_0^2$$

while a_3 is determined by equation (3.24)

$$A_1(a_3/a_1) = \Omega_0^2/2\pi G\rho_0$$

and

$$M = \frac{4\pi}{3}\rho_0 a_1^2 a_3$$

We write the distribution function for any stationary uniform spheroid in the form (with Ω_0, ω_0 and a_1 as parameters)

$$f_0 = f_0\left(\frac{E_\perp}{\Omega_0}, \frac{E_z}{\omega_0}, L\right) \tag{3.29}$$

The density $\rho_0 = \int f_0\, d^3V = \frac{1}{4\pi G}\nabla^2\varphi_0$

Therefore,

$$\int f_0\left(\frac{V_x^2}{2\Omega_0} + \frac{V_y^2}{2\Omega_0} + \Omega_0\frac{r^2}{2}, \frac{V_z^2}{2\omega_0} + \omega_0\frac{z^2}{2}, xV_y - yV_x\right)d^3V.$$

$$= \frac{1}{4\pi G}\left(2\Omega_0^2 + \omega_0^2\right) \quad \text{when } \frac{r^2}{a_1^2} + \frac{z^2}{a_3^2} \le 1$$

$$= 0 \qquad \text{otherwise}$$

Defining

$$X = \sqrt{\Omega_0}\,x, \quad Y = \sqrt{\Omega_0}\,y, \quad Z = \sqrt{\omega_0}\,z$$

$$U_X = V_x/\sqrt{\Omega_0}, \quad U_Y = V_y/\sqrt{\Omega_0}, \quad U_Z = V_z/\sqrt{\omega_0} \tag{3.30}$$

we have

$$\int f_0\left(\frac{U_X^2}{2}+\frac{U_Y^2}{2}+\frac{X^2+Y^2}{2},\ \frac{U_Z^2}{2}+\frac{Z^2}{2},\ XU_Y-YU_X\right)d^3U$$

$$=\frac{2\Omega_0^2+\omega_0^2}{4\pi G\Omega_0\sqrt{\omega_0}} \quad \text{when} \quad \frac{X^2+Y^2}{\Omega_0 a_1^2}+\frac{Z^2}{\omega_0 a_3^2}\leq 1 \qquad (3.31)$$

$$= 0 \qquad \text{otherwise}$$

The time dependent model is constructed from the stationary distribution function by replacing $E_\perp/\Omega_0$ and E_z/ω_0 respectively by the corresponding Lewis invariants

$$I_1 = \frac{r^2}{2\xi^2}+\frac{1}{2}\left|\xi \underline{V_\perp}-\dot{\xi}\underline{r}\right|^2 \quad \text{where} \quad \underline{r}=(x,y,0), \quad \underline{V_\perp}=(V_x,V_y,0) \qquad (3.32)$$

$$I_3 = \frac{z^2}{2\eta^2}+\frac{1}{2}\left(\eta V_z-\dot{\eta}z\right)^2$$

The interior potential is

$$\varphi(r,z,t) = \Omega^2(t)\frac{r^2}{2}+\omega^2(t)\frac{z^2}{2} \qquad (3.33)$$

where $\Omega^2(t)$ and $\omega^2(t)$ are determined from the density $\rho(r,z,t)$ and the time dependent axis ratio by formulas similar to equation (3.24).

ξ, η are any solutions to

$$\ddot{\xi}+\Omega^2(t)\xi-\frac{1}{\xi^3}=0 \quad ; \ \xi>0$$

$$\ddot{\eta}+\omega^2(t)\eta-\frac{1}{\eta^3}=0 \quad ; \ \eta>0 \qquad (3.34)$$

The time dependent distribution function is

$$f = f_0(I_1, I_3, L)$$

The density is

$$\rho = \int f \, d^3v$$

Therefore

$$\rho = \int f_0\left(\frac{r^2}{2\xi^2} + \frac{1}{2}|\xi \underline{v_\perp} - \dot{\xi}\underline{r}|^2, \frac{z^2}{2\eta^2} + \frac{1}{2}(\eta v_z - \dot{\eta} z)^2, x v_y - y v_x\right) d^3v$$

Defining

$$X' = x/\xi \quad , Y' = y/\xi \quad , Z' = z/\eta$$

$$U'_X = \xi v_x \quad , U'_Y = \xi v_y \quad , U'_Z = \eta v_z$$

$$\rho = \frac{1}{\xi^2\eta}\int f_0\left(\frac{X'^2 + Y'^2}{2} + \frac{U_X'^2 + U_Y'^2}{2}, \frac{Z'^2}{2} + \frac{U_Z'^2}{2}, X'U'_Y - Y'U'_X\right) d^3U' \quad (3.35)$$

From (3.31) and (3.35)

$$\rho = \frac{1}{\xi^2\eta}\frac{(2\Omega_0^2 + \omega_0^2)}{4\pi G \Omega_0 \sqrt{\omega_0}} \quad \text{when} \quad \frac{X'^2 + Y'^2}{\Omega_0 a_1^2} + \frac{Z'^2}{\omega_0 a_3^2} \leq 1 \quad (3.36)$$

$$= 0 \qquad \text{otherwise}$$

Therefore the new axes are

$$b_1 = b_2 = \sqrt{\Omega_0}\, a_1 \xi$$

$$b_3 = \sqrt{\omega_0}\, a_3 \eta \quad (3.37)$$

and the axis ratio $u = b_3/b_1$

From (3.24)

$$\Omega^2 = 2\pi G \rho A_1(u)$$

$$\omega^2 = 2\pi G \rho A_3(u) \quad (3.38)$$

writing $\qquad \dfrac{2\Omega_0^2 + \omega_0^2}{2\Omega_0\sqrt{\omega_0}} \quad , \; K = \sqrt{\dfrac{\omega_0}{\Omega_0}} \; \dfrac{a_3}{a_1}$

and using (3.36), (3.37) and (3.38) in equations (3.34) we get the following equations governing the behaviour of the axes of the homogeneous spheroid

$$\ddot{\xi} + \frac{\varepsilon}{\xi\eta} A_1\left(\frac{K\eta}{\xi}\right) - \frac{1}{\xi^3} = 0$$
$$\ddot{\eta} + \frac{\varepsilon}{\xi^2} A_3\left(\frac{K\eta}{\xi}\right) - \frac{1}{\eta^3} = 0 \qquad (3.39)$$

3.4 Some properties of the oscillations of uniform spheroids

3.4a Hamiltonian formulation

In this Section we point out some general features of solutions to (3.39). Firstly, they can be recast in Hamiltonian form by a simple scaling.

$$x = \xi \quad , \quad y = K\eta \qquad (3.40)$$

The axes

$$b_1 = \sqrt{\Omega_0}\, a_1 x \quad , \quad b_3 = \sqrt{\Omega_0}\, a_1 y \qquad (3.41)$$

while the axis ratio (ellipticity) is

$$u = b_3/b_1 = y/x \qquad (3.42)$$

Introducing a function

$$g(u) = \frac{h(u) - u}{1 - u^2} \qquad (3.43)$$

we can express A_1 and A_3 in terms of g

$$A_1(u) = u g(u)$$
$$A_3(u) = 2 - 2ug(u) \qquad (3.44)$$

For convenience we shall use constants α and β in place of and K

$$\alpha = K^4 - \frac{\omega_0^2}{\Omega_0^2}\left(\frac{a_3}{a_1}\right)^4 \qquad (3.45)$$

$$\beta = \xi K = \sqrt{\Omega_0} / g(a_3/a_1)$$

$$\alpha, \beta > 0$$

Written in terms of x, y, α and β, equations (3.39) read

$$\ddot{x} + \frac{\beta}{x^2} g - \frac{1}{x^3} = 0 \qquad (3.46)$$

$$\ddot{y} + \frac{2\beta}{x^2}(1 - ug) - \frac{\alpha}{y^3} = 0$$

To see that (3.46) can be derived from a Hamiltonian, we need the following identity for g

$$(1-u^2)\frac{dg}{du} = 3ug - 2 \qquad (3.47)$$

Using (3.47) **it** is easy to verify that (3.46) is generated by the Hamiltonian (with parameters α and β in the potential V)

$$\mathcal{H} = \frac{p_x^2 + p_y^2}{2} + V(x,y) \qquad (3.48)$$

where $$V = \frac{1}{2x^2} + \frac{\alpha}{4y^2} - \frac{\beta}{x}\left[u - (1-u^2)g\right]$$

An immediate consequence is that $\mathcal{H}$ is a constant of motion. Direct calculations using the Virial theorem and (3.46) show that

$$\mathcal{H} = \frac{5E}{2M\Omega_0 a_1^2} \tag{3.49}$$

where M and E are total mass and energy of the model. The time-independent Hamiltonian structure of (3.46) guarantees that the oscillations do not damp asymptotically. Another advantage is that we can use Poincare's method of the surface of section (see e.g. Lichtenberg and Lieberman 1983) to understand the nature of these oscillations.

For every allowed value of α and β, V has a minimum whose location (x_0, y_1) is determined by $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0$. Equivalently, with $u_0 = y_0/x_0$,

$$\alpha = \frac{2u_0^3}{g(u_0)}\left[1 - u_0 g(u_0)\right] \tag{3.50}$$

$$\beta = 1/x_0 g(u_0)$$

The solutions turn out to be unique; $(\alpha,\beta) \Leftrightarrow (x_0, y_0)$. The minimum in V corresponds to a stationary spheroid with axis ratio u_0. From (3.50) it is clear that u_0 depends only on α implying that α alone determines the ellipticity of the underlying stationary model. Note that while α is dimensionless, β has dimensions of $(\text{time})^{1/2}$ and hence can be set to any convenient value by a choice of units. u_0 is an increasing function of α implying that prolateness of the stationary mode1 increases with increasing α. For α = 1, the stationary model is a sphere. Figure 3.1 shows a contour plot of $\mathcal{V}$ for α = 1 and β = 3/2. The minimum is at $x_0 = 1$, $y_0 = 1$ where $\mathcal{V}$ = -0.75. Away from the minimum and toward the coordinate axes, $\mathcal{V}$ rises indefinitely. The contours of $\mathcal{V}$ for other values of α and β are topologically similar.

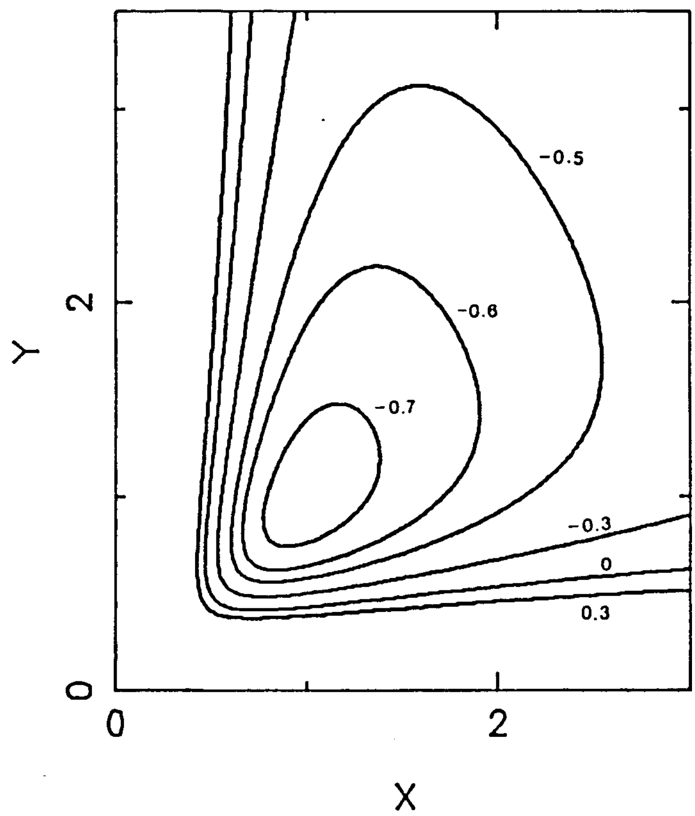

FIGURE 3.1. Level surfaces of $\mathcal{V}(x,y)$ for $\alpha = 1$, $\beta = 3/2$

3.4b A preliminary study of Orbits

(i) Variation of Ellipticity: **It** is clear from (3.49) that $\mathcal{H}$ 70 implies that $E > 0$. Stellar systems with positive total energy eventually disperse to infinity. Therefore (x,y) should asymptotically increase without bound for $\mathcal{H} > 0$. One such orbit is shown in Figure 3.2 for $\mathcal{H} = 0{\cdot}1$. The orbit can be interpreted as a collapse from infinity followed by bounce and expansion back to infinity at a different ellipticity.

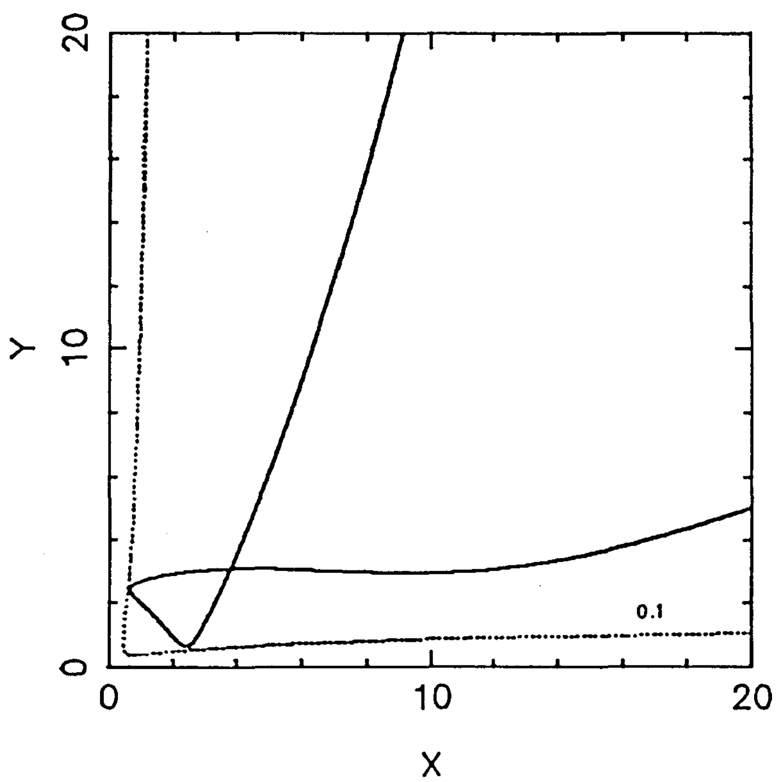

FIGURE 3.2. An unbounded orbit at $\mathcal{H} = 0{\cdot}1$ showing the changes in ellipticity

Under what circumstances will ellipticity be conserved? We should look for conditions under which u is constant while x and y change with time; if $\dot{y}/\dot{x} = y/x$ at some time under what circumstances is $\ddot{y}/\ddot{x} = y/x$? Using (3.46) we require

$$\frac{\dfrac{\alpha}{y^3} - \dfrac{2\beta}{x^2}(1 - ug)}{1/x^3 - \beta g/x^2} = u \qquad (3.51)$$

Since u is assumed to be constant, we can use (3.50) to set

$$\alpha = \frac{2u^3}{g}(1 - ug) \qquad (3.52)$$

Using the identity (3.47) for $g(u)$, we get the condition

$$(1 - u^2)\frac{dg}{du} = 0$$

g is an increasing function of u. Therefore $u = 1$; the only self similar oscillations allowed are spherical oscillations. When $u = 1$, $g = 2/3$ and $\alpha = 1$ (3.46) reduces to

$$\ddot{x} + \frac{(2\beta/3)}{x^2} - \frac{1}{x^3} = 0$$

We determine β from (3.46). Since = 1, $\Omega_0 = \omega_0$ and $a_1 = a_3$ from the very definitions of ω_0 and Ω_0 in terms of a_1 and a_3. Therefore $\beta = \dfrac{\sqrt{\omega_0}}{g(1)} = \dfrac{3}{2}\sqrt{\omega_0}$

Using this we get

$$\ddot{x} + \frac{\sqrt{\omega_0}}{x^2} - \frac{1}{x^3} = 0 \qquad (3.53)$$

which is identical to equation (3.16). Note that while the external appearance of these oscillating spheres is the same as those described by (3.16), the phase space structure is different since f now depends on L_z as well.

(ii) General features: We have solved equations (3.46) numerically for $\alpha = 1$ and $\beta = 3/2$ using a simple first order scheme (update momenta and then coordinates). This has the advantage that it represents an exact symplectic map of the 4-d phase space onto itself. So general properties like Liouville's theorem are preserved to machine accuracy irrespective of step size. The suitability of the scheme was tested on the Toda Hamiltonian (see e.g Lichtenberg and Lieberman 1983) with satisfactory results - no spurious chaos induced by discretisation or round off was found. Figure 3.3 shows a surface of section (p_y versus y at x = 1, $p_x > 0$) for $\mathcal{H} = -0.45$. The (unstable) fixed point on the upper left corner corresponds to oscillations of uniform spheres. For $\mathcal{H} \lesssim -0.45$, this fixed point is stable (although we do not show the section here). At $\mathcal{H} \simeq -0.45$, the oscillating sphere is unstable to spheroidal modes and bifurcates into a quasi-periodically oscillating spheroid. It should be noted that this is not necessarily a general feature of all uniform density oscillating spheres. The class of uniform density spheroids is much more restricted than the class of uniform density spheres. So the present work only allows us to note that the instability occurs for the subset of spheres that are members of a sequence of uniform spheroids.

The stable fixed point on the right hand side with its accompanying islands corresponds to oscillations that are roughly "orthogonal" to the oscillations of spheres (i.e. they represent oscillations of ellipticity). The chain of islands in between is due to orbits trapped near a 5:3 resonance between the two "orthogonal" nonlinearly coupled modes.

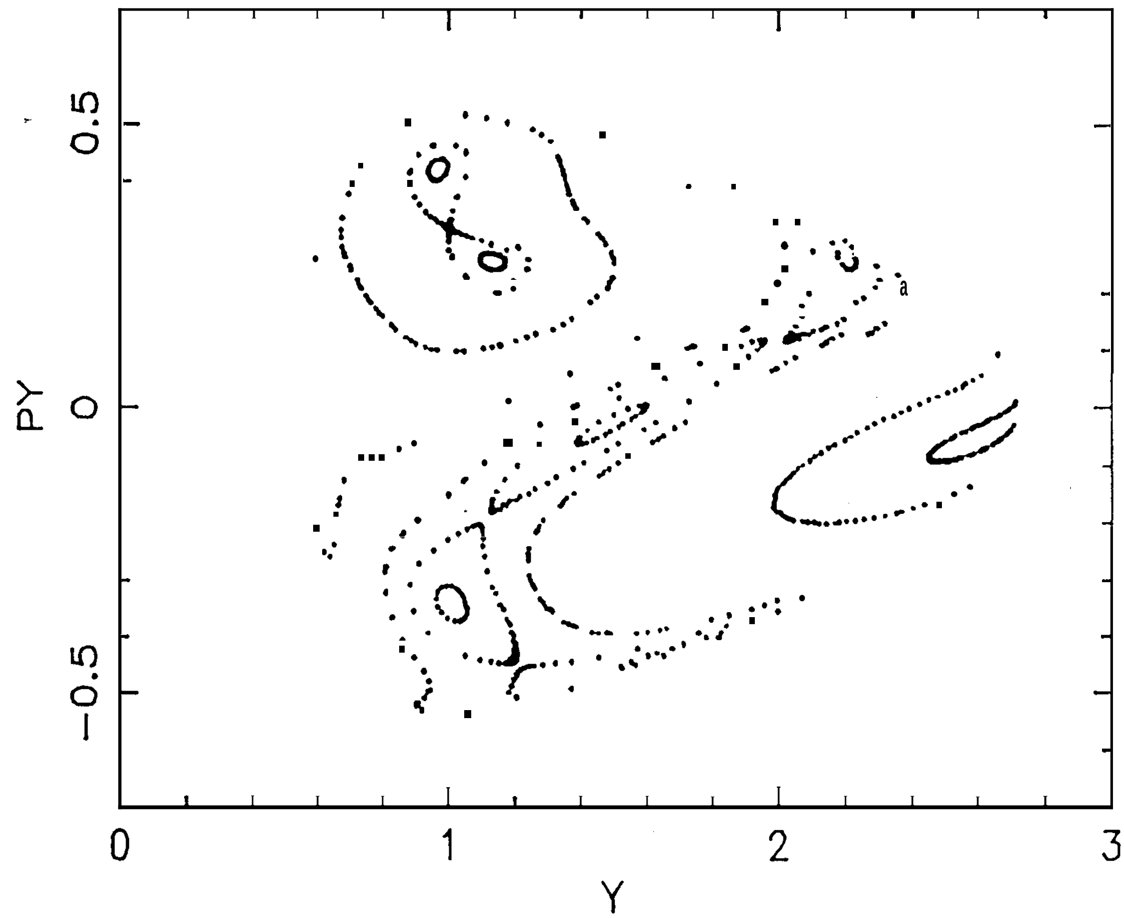

FIGURE 3.3. Surface of section ($x=1, p_x>0$) for $\mathcal{H} = -0.45$

Figure 3.4 is a Section at x = 2.5, $p_x > 0$ for $\mathcal{H}$ = −0.3. At this high value of $\mathcal{H}$, the basic instability of the spheres has given birth to chaotic oscillations of spheroids. The oscillation "orthogonal" to spherical oscillations is still stable and large regions of phase space around this are filled with regular orbits.

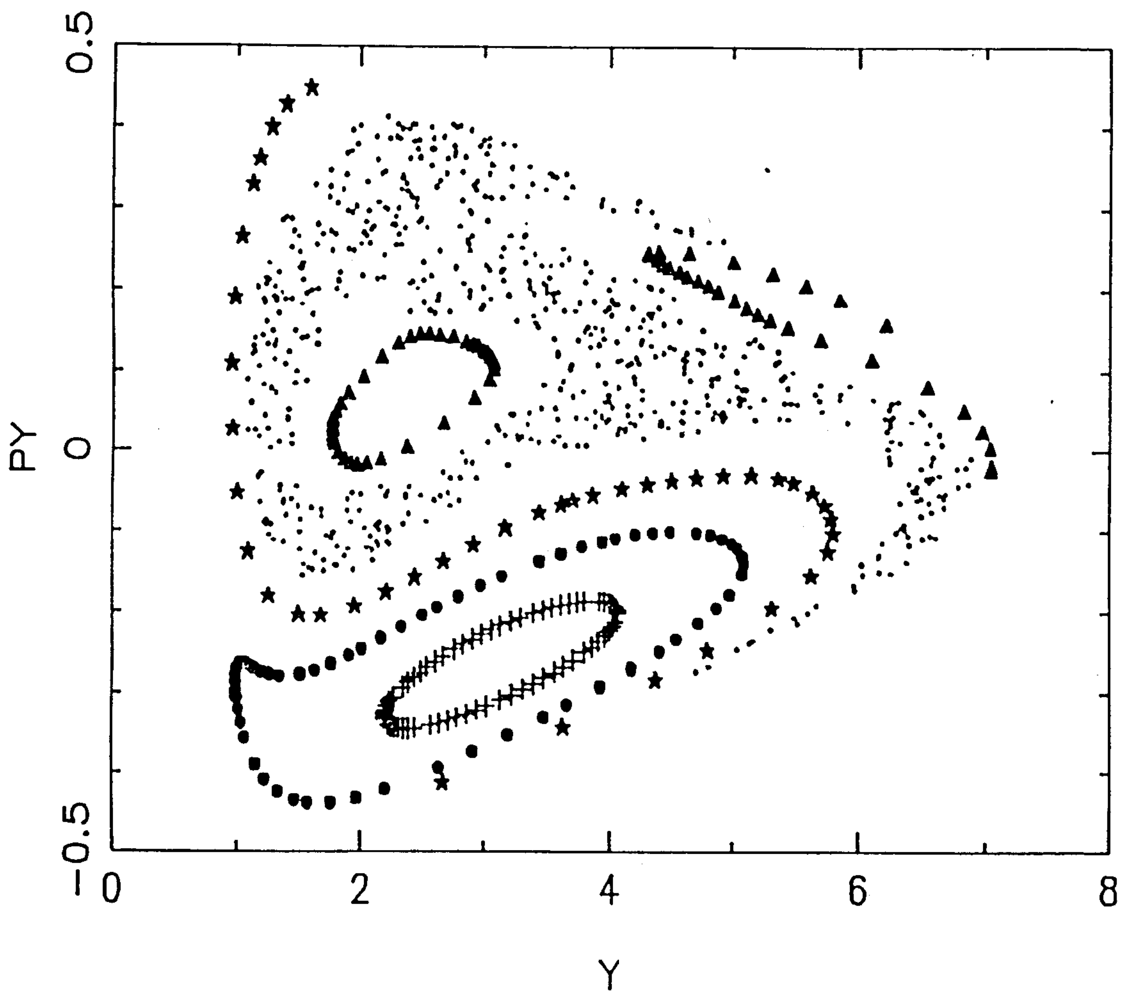

FIGURE 3.4. Surface of section ($x = 2{\cdot}5$, $p_x > 0$) for $\mathcal{H} = -0{\cdot}3$

3.4c Discs and Needles

We briefly discuss the extreme **oblate/prolate** limits of the spheroidal model. These correspond to discs and needles repectively.

(i) Discs: In the stationary model we let $a_3 \rightarrow 0$, while keeping the mass M fixed. The oscillations perpendicular to the disc as described by the time-dependent behaviour of b_3. Since we are looking for solutions that correspond to highly flattened configurations, we let $b_3 \longrightarrow 0$.

When $a_3 \longrightarrow 0$, $m = 3$ $\longrightarrow 0$

Also $g(m) = \pi/2$ and $\Omega_0 = \dfrac{3\pi G M}{4 a_1^3}$ in this limit.

From equation (3.45), $\beta = \sqrt{\Omega_0}/g(m)$ $\quad 2\Omega_0/\pi$

We recall that $b_1 = \sqrt{\Omega_0}\, a_1 x$ and $b_3 = \sqrt{\Omega_0}\, a_1 y$. Since $b_3 \rightarrow 0$, and both $\sqrt{\Omega_0}$ and a_1 are finite, $y \rightarrow 0$. The oscillations in the plane of the disc are described by the first of the equations **(3.46)**.

$$\ddot{x} + \frac{\beta}{x^2}\, g(y/x) - \frac{1}{x^3} = 0$$

Since $y \rightarrow 0$, we replace $g(y/x)$ by $g(0) = \frac{\pi}{2}$ and write

$$\ddot{x} + \frac{\sqrt{\Omega_0}}{x^2} - \frac{1}{x^3} = 0 \qquad (3.54)$$

which governs the oscillations of the radius (b_1) of the disc with surface density $\propto \left(1 - \frac{r^2}{b_1^2}\right)^{1/2}$ obtained by projecting a uniform density spheroid onto its plane of symmetry. What we have is a time dependent generalisation of **Kalnajs'** circular discs. A **still** more general **family** of time dependent discs is discussed in the next chapter.

(ii) Needles: This extremely prolate limit of spheroids corresponds to a stationary model, with $a_1 \to 0$ while the mass of which is kept constant at M. When 4, $\to 0$, $m = \frac{a_3}{a_1} \to \infty$ and asymptotically

$$b_1 = \frac{x}{\sqrt{m}}\left(\frac{3GMa_3}{2}\right)^{1/4} \quad , \quad b_3 = \frac{y}{\sqrt{m}}\left(\frac{3GMa_3}{2}\right)^{1/4}$$

The equilibrium values of x and y are determined by setting $b_1 =$ a, and $b_3 = a_3$:

$$x_0 = \frac{1}{\sqrt{m}}\left(\frac{3GM}{2a_3^3}\right)^{-1/4} \quad , \quad y_0 = \sqrt{m}\left(\frac{3GM}{2a_3^3}\right)^{-1/4}$$

The oscillations of the length of the needle are described by the second of the equations (3.46)

$$\ddot{y} + \frac{2\beta}{x^2}(1 - ug) - \frac{\alpha}{y^3} = 0$$

Since the equilibrium value of $x \to 0$ when $m \to \infty$, we set $u = \frac{y}{x} \simeq m$ in the above equation:

$$\ddot{y} + \frac{2\beta}{y^2}\left[m^2\left(1 - mg(m)\right)\right] - \frac{\alpha}{y^3} = 0 \qquad (3.55)$$

As $m \longrightarrow \infty$

$$\alpha = \frac{\omega_0^2}{\Omega_0^2}\left(\frac{a_3}{a_1}\right)^4 \longrightarrow 2m^2 \ln m$$

$$\beta = \sqrt{\Omega_0}/g(m) \longrightarrow m^{3/2}\left(\frac{3GM}{2a_3^3}\right)^{1/4} \tag{3.56}$$

$$(1-mg)\,m^2 \longrightarrow \ln m$$

$$\Omega_0 \longrightarrow m\left(\frac{3GM}{2a_3^3}\right)^{1/2}$$

Also, since $y \to \infty$, we need to work with $b_3 = \sqrt{\Omega_0}\, a_1 y$. So, with $y = \frac{b_3}{a_1\sqrt{\Omega_0}}$, we have

$$\ddot{b}_3 + 3GM(\ln m)\left[\frac{1}{b_3} - \frac{a_3}{b_3^3}\right] = 0 \tag{3.57}$$

This is the equation of motion for a particle in a potential well that is infinitely deep at $b_3 = a_3$. Strictly speaking the original CBE is not applicable to 2 and lower dimensional systems due to collisional effects (Rybicki 1972). The calculations on discs and needles given above must be regarded as describing systems in which the thickness is finite but small compared to the other dimensions.

3.5 Discussion

There has been earlier work on the oscillations of uniform spheres and spheroids by Chandrasekhar and Elbert (1972, hereafter CE) and Som Sunder and Kochhar (1985, 1986, hereafter SK I and SK II). The approach of CE was to apply the scalar form of the Virial theorem to a sphere. The moment of inertia term and the potential. term could be expressed in terms of the instantaneous radius, a, and the kinetic energy followed from energy conservation. The resulting equation for the variation of a with time is identical to (3.16). Notice that application of the Virial theorem in this manner presupposes the existence of undamped oscillations preserving the uniformity of the sphere. The application of the Lewis Invariant proves this hypothesis by providing the underlying phase space distribution function.

CE also used the tensor virial theorem to study the oscillations of spheroidal systems. There are now two independent kinetic energy terms along the a_1 and a_3 axes, so energy conservation alone is insufficient. CE introduced an additional postulate setting these equal to each other at all times (an algebraic error was rectified by SK I). SK II criticised this assumption as unnatural and instead assumed that the mean streaming velocity in the stellar system was a linear function of the coordinates. This hypothesis was a natural one to make sure that the uniform density and spheroidal shape are preserved as for fluid ellipsoids. This assumption enabled them to derive a pair of coupled equations for a_1 and a_3 which are identical to (3.39). This identity can be understood since the Lewis Invariants I_1 and I_3 (equations 3.32) depend on velocities in the combination $|\xi \underline{v_\perp} - \dot{\xi} \underline{r}|^2$ and $(\eta v_z - \dot{\eta} z)^2$ respectively. When the distribution function depends on velocities through I_1 and I_3, it is clear that the mean value of the velocities $\underline{v_\perp}$ and v_z are linear functions of $\underline{r}$ and z. In brief, the distribution functions presented in this paper provide underlying detailed dynamical models realising the assumptions of CE and SK II for uniform spheres and spheroids. We know of no way to provide a similar basis for genera1 (eg. Gaussian) density profiles studied in CE. We should also mention (i) that the limiting case of a cold collapsing spheroid has been studied by Lin, Mestel and Shu (1965) and (ii) the work of Louis and Gerhard (1988) who constructed an oscillating non uniform density spherical oscillating model by numerical methods.

The stability of these oscillating solutions is an important question that remains unanswered. If a given oscillating solution is stable, it implies the existence of nearby solutions which do not have precisely uniform density, but share its nonrelaxing properties. We know that the stability of static models depends on the details of the distribution function - there is a trend for hotter models in general to be stable. When the parent static uniform sphere or spheroid is stable, one might expect models with sufficiently small oscillations to be stable as well. The existence of nonrelaxing solutions would probably be missed by numerical codes which directly attack the CBE (eg. White 1986) because of the unavoidable dissipation produced by finite grid size.

CHAPTER 4

GENERALIZED FREEMAN DISCS

In section (3.4c) we briefly discussed a limiting case when the minor axis of oblate (time dependent, uniform density) spheroids was allowed to shrink to zero, while the mass of the spheroid was held constant. The resulting axisymmetric, time dependent disc had surface density $\propto \left(1 - \frac{r^2}{a^2}\right)^{1/2}$. When "a", the radius of the disc is constant in time, the disc reduces to the equilibrium solutions discovered and studied by Kalnajs (1972). In this chapter we shall construct time dependent, collisionless discs that are much more general in that the discs are allowed to be nonaxisymmetric and they can not only oscillate but rotate as well. We begin by taking a quick look at Kalnajs' discs and Freeman's analytic bars which are their nonaxisymmetric generalizations.

4.1 Kalnajs Discs

These are axisymmetric discs which are stationary solutions to the CBE. They are described by a phase space distribution function (f_K) which has three parameters; the disc mass (M), radius (a) and angular (rotation) speed (Ω). The surface density

$$\Sigma(\underline{r}) = \frac{3M}{2\pi a^2}\left(1 - \frac{r^2}{a^2}\right)^{1/2} \tag{4.1}$$

The gravitational potential at any point in the disc due to the surface density in (4.1) is

$$\varphi(\underline{r}) = \frac{1}{2}\Omega_o^2 r^2 \tag{4.2}$$

where

$$\Omega_o^2 = \frac{3\pi G M}{4a^3} \tag{4.3}$$

Using instead of M

$$f_K(E, L_z) = \frac{3M}{4\pi^2 a^3 (\Omega_o^2 - \Omega^2)^{1/2}} \left[(\Omega_o^2 - \Omega^2)a^2 + 2(\Omega L_z - E)\right]^{-1/2} \quad \text{for } [\cdots] > 0$$

$$= 0 \quad \text{for } [\cdots] \leq 0 \tag{4.4}$$

$$E = \frac{V_\theta^2 + V_r^2}{2} + \varphi(r)$$

$$L_z = r V_\theta \tag{4.5}$$

where (r, θ) are polar coordinates and V_θ and V_r are velocities along $\hat{\theta}$ and $\hat{r}$ respectively. It is also clear that $\Omega_o \geqslant \Omega$. We briefly note some important properties

(i) The argument of the radical in (4.4) can be written as

$$(\Omega_o^2 - \Omega^2)(a^2 - r^2) - (V_\theta - \Omega r)^2 - V_r^2$$

Thus

a) The mean velocity of stars at $\underline{r}$ is $r\Omega\hat{\theta}$. Therefore these discs rotate rigidly with angular speed Ω.

b) The distribution of peculiar velocities at any point is isotropic.

(ii) The degree of heat can be varied. Hot Kalnajs discs have $\Omega \ll \Omega_0$ where random motions balance self gravity. Cold discs ($\Omega \simeq \Omega_0$) are supported against self gravity by rotation.

4.2 Freeman's Analytic Bars (or Freeman Discs)

Freeman (1966) attempted to model the bars of spiral galaxies through a generalization of Kalnajs discs. Here, we briefly describe them. Freeman discs are elliptic discs stationary in a frame rotating with constant angular velocity (say $\Omega \hat{z}$). Let (x, y) be coordinates in the rotating frame such that the major axis of the disc is along x and the minor axis is along y. The surface density of a Freeman disc of mass is

$$\Sigma(x,y) = \frac{3M}{2\pi ab}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{1/2}; \quad a \geqslant b \qquad (4.6)$$

The gravitaticnal potential in the interior of the disc is

$$\varphi(x,y) = A^2\frac{x^2}{2} + B^2\frac{y^2}{2} - D \qquad (4.7)$$

where

$$a^2A^2 = \frac{3GM}{k^2a}\left\{F(k) - E(k)\right\}$$

$$b^2B^2 = \frac{3GM}{k^2a}\left\{E(k) - (1-k^2)F(k)\right\}$$

$$D = \frac{3GM}{2a} F(k)$$

$$k^2 = 1 - b^2/a^2 \qquad (4.8)$$

and

$$E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2\eta)^{1/2} d\eta$$

$$F(k) = \int_0^{\pi/2} (1 - k^2 \sin^2\eta)^{-1/2} d\eta \qquad (4.9)$$

are complete elliptic integrals.

The potential in (4.7) gives rise to a force of self gravity per unit mass (acting on a star at x, y) that is a linear function of x and y. Since we are in a rotating frame, in addition to self gravity, Coriolis and centrifugal forces act on the stars in the disc. The Coriolis force is a linear function of V_x and V_y while the centrifugal force is a linear function of Y and y. The net result is that the equations of motion (for a star in the disc) are linear in (x, y, V_x, V_y). Linearity of the equations of motion guarantees the existence of integrals of motion that are quadratic functions of (x, y, V_x, V_y). In particular, the integrals of motion may be chosen to be positive definite. Freeman used these integrals to construct self consistent discs by finding those distribution functions (denoted by "f_{FD}"s ;these are functions of the integrals of motion) that on integration over velocities give the surface density in

(4.6). We note below some properties of the Freeman discs.

(i) The discs may have any axis ratio (b/a), but they should not rotate too rapidly:

$$\Omega < A < B$$

(ii) When $a = b$, they reduce to Kalnajs' discs.

Freeman discs have many other interesting properties and for more information see Hunter (1970, 1974).

4.3 Generalized Freeman Discs (GFDs)

The GFD introduced here is a Freeman disc which writhes and rotates in its plane under the action of its self gravity. *By* "writhe" we mean that the disc can change its axis ratio (b/a) as well as its size. While writhing and rotating the surface density of the disc $\Sigma \propto (1 - q(t))^{1/2}$ where $q(t)$ is a positive definite quadratic form in x and y (which are inertial Cartesian coordinates in the plane of the disc) with time dependent coefficients. The surface density Σ causes the force of self gravity to be linear in x and y, though time dependent. Hence the time dependent transformation from (x, y, V_x, V_y) at time t to (x', y', V_x', V_y') at time t' is linear in the phase space coordinates. Let us write this as

$$Z' = SZ \tag{4.10}$$

where

$$Z = (x, y, v_x, v_y)^T$$
$$Z' = (x', y', v_x', v_y')^T \tag{4.11}$$

(T denotes transpose)

and S is a 4 x 4 time dependent symplectic matrix. By "symplectic" we mean that

$$S^T \omega S = \omega \tag{4.12}$$

where

$$\omega = \begin{bmatrix} 0_{2\times2} & \mathbb{1}_{2\times2} \\ -\mathbb{1}_{2\times2} & 0_{2\times2} \end{bmatrix} \tag{4.13}$$

Let us consider the action of the transformation in (4.10) on a quadratic form

$$I(t) = Z^T Q(t) Z \tag{4.14}$$

where $Q(t)$ is a 4 x 4 symmetric (in general time dependent) matrix.

$$I(t') = Z'^T Q(t') Z'$$
$$= Z^T S^T Q(t') S Z \tag{4.15}$$

If we require that $I(t') = I(t)$ we get

$$S^T Q(t') S = Q(t) \tag{4.16}$$

So, when I is an integral of motion, **(4.16)** determines the time evolution of the coefficients of I. By Jeans' theorem I can be used to construct a phase space distribution function (f_{GFD}) that describes a GFD. I should be chosen to be positive definite, otherwise the level surfaces of I in phase space will not be compact - **it** is easily seen that time evolution as given in **(4.16)** will preserve the positive definiteness of I. If we choose f_{GFD} as

$$f_{GFD} = f_0 (1 - I)^{-1/2} \quad ; f_0 = \text{positive constant} \tag{4.17}$$

then, by integrating f_{GFD} over velocities (V_x and V_y) **it** is straightforward to check that the surface density so obtained is $\propto (1 - q(t))^{1/2}$.

4.4 A convenient form of the equations of evolution

Equation **(4.17)** shows that the GFD is described completely by one number f_0 and the ten coefficients of I. Once these quantities are specified at some initial instant of time, evolution is governed by **(4.16)**. Let us rewrite **(4.16)** in terms of $P = Q^{-1}$ because the elements of the matrix P directly give "phase space averages" of products like xy, xx, xV_x etc. Defining

$$\overline{Z_i Z_k} = \frac{1}{M} \int Z_i Z_k f_{GFD}(I)\, d^4 Z \tag{4.18}$$

we show in Appendix B that

$$P_{ik} = 5\, \overline{Z_i Z_k} \tag{4.19}$$

To write the equation of evolution in terms of P, we take the (matrix) inverse of (4.16) and obtain

$$P(t') = S P(t) S^T \tag{4.20}$$

The constant f_0 in (4.17) is proportional to the mass (M) of the disc. P (like Q) is a 4 x 4 symmetric matrix and contains 10 independent elements. We now take M and 10 independent elements of P as the basic variables that describe the time evolution of GFDs. It is clear that M does not change with time and we may choose this to be unity. So the GFD evolves in a 10 dimensional "phase space" spanned by P_{ik}. This time evolution is itself determined by (4.20) where S is required to be that transformation which is generated by the self gravity of the GFD. The proper formulation of the time evolution of a GFD is the (self consistent) infinitesimal version of (4.20). This is written down below.

Meanwhile we can understand some general properties of time evolution (that are independent of self consistency) from (4.20) itself - for example, Liouville's theorem implies that $\det P$ is conserved. In Appendix C we try to understand the general nature of (4.20) for a general (symplectic) S. The further results of this chapter do not directly require these formal developments. Here, we merely note that in Appendix C, it is shown that (4.20) is a Hamiltonian system. Therefore, in common with all finite dimensional Hamiltonian systems, the GFD will also show nonrelaxing behaviour.

4.5 Effecting Self Consistency

We shall now write **(4.20)** in its infinitesimal form. Writing

$$S = 1 + \Delta t . K \qquad \textbf{(4.21)}$$

where Δt is a small time step, we have

$$P(t+\Delta t) = (1 + \Delta t . K) P(t) (1 + \Delta t . K^T)$$

$$= P(t) + \Delta t \{ KP + PK^T \} + O(\Delta t^2)$$

Therefore $$= KP + PK^T \qquad \textbf{(4.22)}$$

We need to work qut K. S itself was defined by **(4.10)** and **(4.11)**. Therefore

$$\dot{Z} = KZ \qquad \textbf{(4.23)}$$

defines K which contains information about the equations of motion for a star in the GFD. The gravitational potential of a GFD with surface density $\Sigma \propto (1-q)^{1/2}$ can be written as

$$\varphi(x, y, t) = \frac{\alpha}{2} x^2 + \beta x y + \gamma \frac{y^2}{2} \qquad \textbf{(4.24)}$$

where α, β and γ are time dependent. The equations of motion for a star are

$$\dot{x} = V_x \qquad \dot{y} = V_y$$

$$\dot{V}_x - -\alpha x - \beta y \qquad \dot{V}_y = -\beta x - \gamma y \qquad \textbf{(4.25)}$$

Therefore, from **(4.23)** and **(4.25)**

$$K = \begin{bmatrix} O_{2\times 2} & \mathbb{1}_{2\times 2} \\ -\mathbb{F}_{2\times 2} & O_{2\times 2} \end{bmatrix} \tag{4.26}$$

where

$$\mathbb{F} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \tag{4.27}$$

contains the "strengths" of the force of self gravity of the GFD. If we express α, β and γ in terms of the elements of P, we would have effected self consistency. This we implement below. To calculate the force of self gravity we need to know the size and orientation of the GFD. This information is contained in

$$P_{11} = 5\,\overline{x^2}$$
$$P_{22} = 5\,\overline{y^2} \tag{4.28}$$
$$P_{12} = P_{21} = 5\,\overline{xy}$$

Let us suppose that the major axis ($=2a$) of the disc is along x' and the minor axis ($=2b$) is along y'. When the major axis makes an angle of θ with respect to the positive x axis we can write

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.29}$$

Therefore

$$\overline{x'^2} = \overline{x^2}\cos^2\theta + \overline{xy}\sin 2\theta + \overline{y^2}\sin^2\theta$$

$$\overline{y'^2} = \overline{x^2}\sin^2\theta - \overline{xy}\sin 2\theta + \overline{y^2}\cos^2\theta \qquad (4.30)$$

$$\overline{x'y'} = -\frac{\overline{x^2}}{2}\sin 2\theta + \overline{xy}\cos 2\theta + \frac{\overline{y^2}}{2}\sin 2\theta$$

Since the disc is symmetrically disposed about the x' and y' axes, $\overline{x'y'} = 0$.

Therefore from (4.30)

$$\tan 2\theta = \frac{2\,\overline{xy}}{\overline{x^2} - \overline{y^2}} = \frac{2P_{12}}{P_{11} - P_{22}} \qquad (4.31)$$

determines θ upto an additive constant of $\pi/2$. The ambiguity in θ is resolved by requiring that the major axis of the disc lies along x'. i.e. by requiring $\overline{x'^2} \geqslant \overline{y'^2}$. We now express $\overline{x'^2}$ and $\overline{y'^2}$ directly in terms of a and b. The surface density of the GFD with mass M is (see (4.6))

$$\Sigma(x', y') = \frac{3M}{2\pi ab}\left(1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right)^{1/2} \qquad (4.32)$$

Then $$\overline{x'^2} = \frac{1}{M}\int x'^2\,\Sigma(x', y')\,dx'dy'$$

$$= a^2/5$$

Similarly $\overline{y'^2} = b^2/5$. Using these values of $\overline{x'^2}$ and $\overline{y'^2}$ in (4.30) and writing $\overline{x^2}$, $\overline{xy}$ and $\overline{y^2}$ in terms of P_{11}, P_{12} and P_{22}, we have

$$a^2 = P_{11}\cos^2\theta + P_{12}\sin 2\theta + P_{22}\sin^2\theta$$
$$b^2 = P_{11}\sin^2\theta - P_{12}\sin 2\theta + P_{22}\cos^2\theta \qquad (4.33)$$

The force per unit mass at a point (x', y') inside the disc is

$$-A^2 x' \hat{e}_{x'} - B^2 y' \hat{e}_{y'} \qquad (4.34)$$

where A^2 and B^2 are determined in terms of a, b and M from (4.8) and (4.9). $\hat{e}_{x'}$ and $\hat{e}_{y'}$ are unit vectors along x' and y'. Resolving the force along the x and y axes, we get the following expressions for α, β and γ (in 4.27):

$$\alpha = A^2\cos^2\theta + B^2\sin^2\theta$$
$$\beta = (A^2 - B^2)\sin\theta\cos\theta \qquad (4.35)$$
$$\gamma = A^2\sin^2\theta + B^2\cos^2\theta$$

The time evolution of the GFD is given by (4.22), together with equations (4.26), (4.27) and (4.35). As shown in Appendix C, the evolution occurs on an 8 dimensional submanifold of the 10 dimensional space spanned by the elements of the matrix P; the reduction in dimension from 10 to 8 is due to the existence of conserved quantities which essentially are (i) the volume in phase space occupied by the GFD and (ii) the 2-area of the plane section of the GFDin phase space that is maximal. In addition, the total energy and angular momentum are conserved. Freeman's bars could only rotate with constant angular velocity and in the rotating ,frame, their size and shape were constant in time. For GFDs the coupling between rotation and oscillation allows for time varying rotation. The rich behaviour implicit in the equations remains to be explored.