Tail effects in the third post-Newtonian gravitational wave energy flux of compact binaries in quasi-elliptical orbits

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The far-zone flux of energy contains hereditary (tail) contributions that depend on the entire past history of the source. Using the multipolar post-Minkowskian wave generation formalism, we propose and implement a semianalytical method in the frequency domain to compute these contributions from the inspiral phase of a binary system of compact objects moving in quasi-elliptical orbits up to third post-Newtonian (3PN) order. The method explicitly uses the quasi-Keplerian representation of elliptical orbits at 1PN order and exploits the doubly periodic nature of the motion to average the 3PN fluxes over the binary's orbit. Together with the instantaneous (nontail) contributions evaluated in a companion paper, it provides crucial inputs for the construction of ready-to-use templates for compact binaries moving on quasi-elliptic orbits, an interesting class of sources for the ground-based gravitational-wave detectors such as LIGO and Virgo, as well as space-based detectors like LISA.

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I. INTRODUCTION

The gravitational-wave (GW) energy flux from a system of two point masses in elliptic motion in the leading quadrupolar approximation (Newtonian order) was first obtained by Peters and Mathews [1,2]. Using the first post-Newtonian (1PN) order quasi-Keplerian (QK) representation of the binary's orbit [3], Blanchet and Schäfer [4] computed the 1PN corrections to the above result (confirming earlier work by Wagoner and Will [5]).[1] Using the generalized quasi-Keplerian representation of the 2PN motion [6–8], Gopakumar and Iyer [9] extended these results to 2PN order and computed the "secular" evolution of orbital elements under 2PN gravitational radiation reaction (4.5PN terms in the equations of motion). These constitute one of the basic inputs for gravitational-wave phasing of binaries in quasi-eccentric orbits in the adiabatic approximation. All these works above relate to the *instantaneous* terms in the phasing of gravitational waves.

The multipole moments describing GWs emitted by an isolated system do not evolve independently. They couple to each other and with themselves, giving rise to nonlinear physical effects. Consequently, starting at relative 1.5PN order, the above instantaneous terms in the flux must be supplemented by the contributions arising from these nonlinear multipole interactions. The leading multipole interaction is between the mass quadrupole moment I_{ij} and the mass monopole M or Arnowitt, Deser, and Misner (ADM) mass. It is associated with the nonlinear effect of tails at order 1.5PN, and is physically due to the backscatter of linear waves from the space-time curvature generated by the total mass M. Tails imply a nonlocality in time since they are described as integrals depending on the history of the source from the remote past to the current retarded time. They are thus appropriately referred to as *hereditary* contributions by Blanchet & Damour [10,11]—terms nonlocal in time depending on the dynamics of the system in its entire past [11]. The most detailed study of tails in this context [12,13] is based on the multipolar post-Minkowskian formalism [14,15]. Up to 3PN order the hereditary terms comprise the dominant quadratic-order tails, the quadratic-order memory integral [11,16–19], and the cubic-order tails. The latter *cubic* "monopole-monopole-quadrupole" interaction can be called "tails of tails" of GWs (see [12,13] for earlier references to the general topic of tails). In this paper we set up a general theoretical framework to compute the hereditary contributions for binaries moving in elliptical orbits and apply it to evaluate *all* the tail contributions contained in the 3PN accurate GW energy flux.

For the instantaneous terms in the energy flux, explicit closed-form analytical expressions can be given in terms of dynamical variables related to relative velocity and relative separation. Consequently, these expressions can be conveniently averaged in the time domain over an orbit using their quasi-Keplerian representation. For the hereditary contributions, on the other hand, one can only write down formal analytical expressions as integrals over the past. More explicit expressions in terms of the dynamical variables *a priori* require a model of the binary's orbital evolution in the past to implement the integration. In

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[1] As usual, the nPN order refers to the post-Newtonian terms of order $(v/c)^{2n}$ where v denotes the typical binary's orbital velocity and c is the speed of light.

general, one can show [11] that the past influence of tails decreases with some kernel $\propto 1/(t-t')^2$ where t is the current time and t' the integration time in the past. Thus the "remote-past" contribution to the tail integrals is negligible. More precisely, it was shown [20] that the contribution due to the past of the tail integral is $\mathcal{O}(\xi_{\rm rad}\ln\xi_{\rm rad})$ where $\xi_{\rm rad} \equiv \dot{\omega}/\omega^2$ is the adiabatic parameter associated with the binary's inspiral due to radiation reaction, which is of order 2.5PN. Consequently, the tail integrals may be evaluated using standard integrals for a *fixed* nondecaying circular orbit, and errors due to inspiral by gravitation radiation reaction are at least 4PN order [20].

In the circular-orbit case, with the above simplified model of binary inspiral one can work directly in the time domain. For instance, the hereditary terms in the flux were computed up to 3.5PN [12,13] while those in the GW polarizations could be obtained up to 2.5PN [19,21]. In the elliptic orbit case, on the other hand, the situation is more involved. Even after using the quasi-Keplerian parametrization, one cannot perform the integrals in the time domain (as for the circular-orbit case), since the multipole moments have a more complicated dependence on time and the integrals are not analytically solvable in simple closed forms. By working in the Fourier domain, Ref. [20] computed the hereditary tail terms at 1.5PN for elliptical orbits using the lowest order Keplerian representation.

In the present investigation we tackle the terms at orders 2.5PN and 3PN and we need to go beyond the (Newtonian) Keplerian representation of the orbit to a 1PN quasi-Keplerian representation. Here we encounter two further complications. First, the 1PN parametrization of the binary [3] involves three kinds of eccentricities (e_r, e_t, and e_ϕ). More seriously, at 1PN order the periastron precession effect appears in the problem and one has to contend with two time scales: the orbital time scale and the periastron precession time scale. These new features are to be properly accounted for in the calculations to extend the Fourier method of Ref. [20]. This strategy has been proposed and used earlier in computing the instantaneous terms in the GW polarizations from binaries on elliptical orbits [22–24]. We shall adapt these features here to treat the more involved hereditary contribution to the total energy flux.

Following Ref. [20], we express all the multipole moments needed for the hereditary computation at Newtonian order as discrete Fourier series in the mean anomaly of motion ℓ. However, for the quadrupole moment I_{ij} needed beyond the lowest Newtonian order, the "doubly periodic" nature of the motion needs to be crucially incorporated. The evaluation of the Fourier coefficients is done numerically based on a series of combinations of Bessel functions. All tail terms at 2.5PN and 3PN are computed to provide the "enhancement factors" (functions of eccentricity playing a role similar to the classic Peters & Mathews [1] enhancement factor) for binaries in elliptical orbits at the 2.5PN and 3PN orders. The present work extends results for hereditary contributions at 1.5PN [20] for elliptical orbits to 2.5PN and 3PN orders. The 3PN hereditary contributions comprise the tail-of-tail terms and are also extensions of [12,13] for circular orbits to the elliptical case.[2]

Combining the *hereditary* contributions computed in this paper with the *instantaneous* contributions computed in the companion paper [29] will yield the complete 3PN energy flux, generalizing the circular-orbit results at 2.5PN [30] and 3PN [31–33] to the elliptical orbit case. The final expressions represent GWs from a binary evolving adiabatically under gravitational radiation reaction, including precisely up to 3PN order the effects of eccentricity and periastron precession during epochs of inspiral when the orbital parameters are essentially constant over a few orbital revolutions. It thus represents the first input to go towards the full *quasi-elliptical* case, namely, the evolution of the binary in an elliptical orbit under gravitational radiation reaction.

Recently, Damour, Gopakumar, and Iyer [24] proposed an analytic method based on an improved "method of variation of constants" to construct high accuracy templates for the GW signals from the inspiral phase of compact binaries moving in quasi-elliptical orbits. The three time scales, respectively, related to orbital motion, orbital precession, and radiation reaction, are handled without the usual approximation of assuming adiabaticity relative to the radiation-reaction time scale. The explicit results of the above treatment [24] relate to "Newtonian" radiation reaction (2.5PN terms in the equations of motion). It leads to post-adiabatic (fast) oscillations resulting in amplitude corrections at order 2.5PN beyond the secular terms. More recently this work has been extended [34] to 1PN radiation reaction (3.5PN terms in the equations of motion).[3]

The paper is organized as follows: In Sec. II we review the solution of the equations of motion of compact binaries and discuss its important properties relevant for this present work. Section III provides the Fourier decomposition of multipole moments and its use in averaging the energy flux. Section IV provides the expressions for all the tail contributions whose numerical implementation is elaborated in Sec. V. The complete 3PN contributions are exhibited in Sec. VI together with relevant checks. The paper ends with an appendix listing the Fourier coefficients of the required Newtonian moments in terms of the Bessel functions.

[2]Recall that tails are not just mathematical curiosities in general relativity but facets that should show up in the GW signals of inspiralling compact binaries and be decoded by the detectors Virgo/LIGO and LISA [25–28].

[3]For circular orbits, *secular* evolution of the phase, computed in the adiabatic approximation, is known up to 3.5PN order [32,33].

II. SOLUTION OF THE EQUATIONS OF MOTION OF COMPACT BINARIES

A. Doubly periodic structure of the solution

In this work and the next one [29], we shall often need to use the explicit solution for the motion of nonspinning compact binary systems in the post-Newtonian (PN) approximation. We review here the relevant material we need, which includes the general ''doubly periodic'' structure of the PN solution, and the quasi-Keplerian representation of the 1PN binary motion by means of different types of eccentricities. We closely follow the works [3,22,35].

The equations of motion of a compact binary system up to the 3PN order admit, when neglecting the radiation-reaction term at the 2.5PN order, ten first integrals of the motion corresponding to the conservation of energy, angular and linear momenta, and position of the center of mass [36,37]. When restricted to the frame of the center of mass, the equations admit four first integrals associated with the energy E and the angular momentum vector $\mathbf{J}$, given at 3PN order by Eqs. (4.8)–(4.9) of Ref. [38].

The motion takes place in the plane orthogonal to $\mathbf{J}$. Denoting by $r = |\mathbf{x}|$ the binary's orbital separation in that plane, and by $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$ the relative velocity, we find that E and $\mathbf{J}$ are functions of r, $\dot{r}^2$, v^2, and $\mathbf{x}\times\mathbf{v}$ (we are employing for definiteness the harmonic coordinate system of [38][4]), and depend on the total mass $m = m_1 + m_2$ and reduced mass $\mu = m_1 m_2/m$. We adopt polar coordinates r, ϕ in the orbital plane, and express E and the norm $J = |\mathbf{J}|$, thanks to $v^2 = \dot{r}^2 + r^2\dot{\phi}^2$, as some explicit functions of r, $\dot{r}^2$, and $\dot{\phi}$. The latter functions can be inverted (by means of straightforward PN iteration) to give $\dot{r}^2$ and $\dot{\phi}$ in terms of r and the constants of motion E and J. Hence,

$$\dot{r}^2 = \mathcal{R}[r; E, J], \tag{2.1a}$$

$$\dot{\phi} = \mathcal{G}[r; E, J], \tag{2.1b}$$

where the functions $\mathcal{R}$ and $\mathcal{G}$ denote certain polynomials in $1/r$, the degree of which depends on the PN approximation in question (it is seventh degree for both $\mathcal{R}$ and $\mathcal{G}$ at 3PN order [39]). The various coefficients of the powers of $1/r$ are themselves polynomials in E and J, and also, of course, depend on m and the dimensionless reduced mass ratio $\nu \equiv \mu/m$. In the case of bounded ellipticlike motion, one can prove [22] that the function $\mathcal{R}$ admits two real roots, r_P and r_A such that $r_P < r_A$, which admit some nonzero finite Newtonian limits when $c \to \infty$, and represent, respectively, the radii of the orbit's periastron and apastron. The other roots tend to zero when $c \to \infty$.

We are considering a given binary's orbital configuration, fully specified by some given values of the integrals of motion E and J. We no longer indicate the dependence on E and J which is always implicit in what follows. The binary's orbital period, or time of return to the periastron, is obtained by integrating the radial motion as

$$P = 2\int_{r_P}^{r_A} \frac{dr}{\sqrt{\mathcal{R}[r]}}. \tag{2.2}$$

We introduce the fractional angle (i.e. the angle divided by 2π) of the advance of the periastron per orbital revolution,

$$K = \frac{1}{\pi}\int_{r_P}^{r_A} dr \frac{\mathcal{G}[r]}{\sqrt{\mathcal{R}[r]}}, \tag{2.3}$$

which is such that the precession of the periastron per period is given by $\Delta\phi = 2\pi(K-1)$. As K tends to 1 in the limit $c\to\infty$ (as is easily checked from the Newtonian limit), it is often convenient to pose $k \equiv K - 1$, which will then entirely describe the *relativistic precession*.

Let us define the mean anomaly ℓ and the mean motion n by

$$\ell = n(t - t_P), \tag{2.4a}$$

$$n = \frac{2\pi}{P}. \tag{2.4b}$$

Here t_P denotes the instant of passage to the periastron. For a given value of the mean anomaly ℓ, the orbital separation r is obtained by inversion of the integral equation

$$\ell = n\int_{r_P}^{r} \frac{dr'}{\sqrt{\mathcal{R}[r']}}. \tag{2.5}$$

This defines the function $r(\ell)$ which is a periodic function in ℓ with period 2π. The orbital phase ϕ is then obtained in terms of the mean anomaly ℓ by integrating the angular motion as

$$\phi = \phi_P + \frac{1}{n}\int_0^{\ell} d\ell' \mathcal{G}[r(\ell')], \tag{2.6}$$

where ϕ_P denotes the value of the phase at the instant t_P. In the particular case of a circular orbit, $r =$ const, the phase evolves linearly with time, $\dot{\phi} = \mathcal{G}[r] = \omega$, where ω is the orbital frequency of the circular orbit given by

$$\omega = Kn = (1 + k)n. \tag{2.7}$$

In the general case of a noncircular orbit it is convenient to keep the definition of $\omega = Kn$ (which will notably be very useful in the next work [29]) and to explicitly introduce the linearly growing part of the orbital phase (2.6) by writing it in the form

$$\phi = \phi_P + \omega(t - t_P) + W(\ell) = \phi_P + K\ell + W(\ell). \tag{2.8}$$

Here $W(\ell)$ denotes a certain function which is periodic in ℓ (hence, periodic in time with period P). According to (2.6) this function is given in terms of the mean anomaly ℓ by

[4] All calculations in this paper will be done at the relative 1PN order, and at that order there is no difference between the harmonic and ADM coordinates.

$$W(\ell) = \frac{1}{n}\int_0^{\ell} d\ell'[\mathcal{G}[r(\ell')] - \omega]. \tag{2.9}$$

Finally, the decomposition (2.8) exhibits clearly the doubly periodic nature of the binary motion, in terms of the mean anomaly ℓ with period 2π, and in terms of the periastron advance $K\ell$ with period $2\pi K$.[5] It may be noted that in Refs. [23,24] the notation λ is used; it corresponds to $\lambda = K\ell$ and will also occasionally be used here.

B. Quasi-Keplerian representation of the motion of compact binaries

In the following we shall also use the explicit solution of the motion at 1PN order, in the form due to Damour & Deruelle [3]. The solution is given in parametric form in terms of the eccentric anomaly u. Then the radius r and mean anomaly ℓ are expressed as

$$r = a_r(1 - e_r \cos u), \tag{2.10a}$$

$$\ell = u - e_t \sin u. \tag{2.10b}$$

The phase angle ϕ is given by (the additive constant $\phi_{\rm P}$ is, for convenience, set equal to zero)

$$\phi = KV, \tag{2.11}$$

where the true anomaly V is defined by[6]

$$V = 2\arctan\left[\left(\frac{1+e_\phi}{1-e_\phi}\right)^{1/2}\tan\frac{u}{2}\right]. \tag{2.12}$$

In the above, K is the periastron advance given in general terms by Eq. (2.3), and a_r is the semimajor axis of the orbit. Note that there are, in this parametrization at 1PN order, three kinds of eccentricities—e_r, e_t, and e_ϕ (labeled after the coordinates r, t, and ϕ). All these eccentricities differ from one another by 1PN terms, while the advance of the periastron per orbital revolution appears also starting at the 1PN order. Because of these features, this representation is referred to as the "quasi-Keplerian" parametrization for the 1PN orbital motion of the binary. The periodic function W of Eq. (2.9) now reads

$$W = K(V - \ell). \tag{2.13}$$

To close the above solution we need to know the explicit dependence of the orbital elements in terms of the 1PN conserved energy E and angular momentum J in the center-of-mass frame (taken, as usual, per unit of the reduced mass μ). This is given in Ref. [3]. Note that the semimajor axis a_r and mean motion n depend at 1PN order only on the constant of energy through

$$a_r = -\frac{Gm}{2E}\left\{1 + \left(\frac{7}{2} - \frac{\nu}{2}\right)\frac{E}{c^2}\right\}, \tag{2.14a}$$

$$n = \frac{(-2E)^{3/2}}{Gm}\left\{1 + \left(\frac{15}{4} - \frac{\nu}{4}\right)\frac{E}{c^2}\right\}. \tag{2.14b}$$

Posing $h \equiv J/(Gm)$, the 1PN periastron precession simply reads[7]

$$K = 1 + \frac{3}{c^2h^2}, \tag{2.15}$$

while the three different eccentricities are given by

$$e_r = \left\{1 + 2Eh^2\left[1 + \left(-\frac{15}{2} + \frac{5}{2}\nu\right)\frac{E}{c^2} + \frac{-6+\nu}{c^2h^2}\right]\right\}^{1/2}, \tag{2.16a}$$

$$e_t = \left\{1 + 2Eh^2\left[1 + \left(\frac{17}{2} - \frac{7}{2}\nu\right)\frac{E}{c^2} + \frac{2-2\nu}{c^2h^2}\right]\right\}^{1/2}, \tag{2.16b}$$

$$e_\phi = \left\{1 + 2Eh^2\left[1 + \left(-\frac{15}{2} + \frac{\nu}{2}\right)\frac{E}{c^2} - \frac{6}{c^2h^2}\right]\right\}^{1/2}. \tag{2.16c}$$

Notice the following simple ratios (valid at 1PN order):

$$\frac{e_t}{e_r} = 1 + (8 - 3\nu)\frac{E}{c^2}, \tag{2.17a}$$

$$\frac{e_t}{e_\phi} = 1 + (8 - 2\nu)\frac{E}{c^2}, \tag{2.17b}$$

$$\frac{e_r}{e_\phi} = 1 + \nu\frac{E}{c^2}. \tag{2.17c}$$

In the following paper [29] we shall need and use the explicit solution of the generalized QK binary motion up to 3PN order.

III. FOURIER DECOMPOSITION OF THE BINARY'S MULTIPOLE MOMENTS

A. Peters & Mathews derivation of the Newtonian energy flux

The method we shall use in this paper is exemplified by the computation of the averaged energy flux of compact binaries at Newtonian order using a Fourier decomposition of the Keplerian motion [1]. The GW energy flux, say

$$\mathcal{F} \equiv \left(\frac{d\mathcal{E}}{dt}\right)^{\rm GW} \equiv \left(\int d\Omega \frac{d\mathcal{E}}{dtd\Omega}\right)^{\rm GW}, \tag{3.1}$$

where $\mathcal{E}$ is the energy carried in the gravitational waves, reduces at Newtonian order to the standard Einstein quadrupole formula[8]

[5]Recall that, though standard, the term doubly periodic may mislead a bit in that the motion in physical space is not periodic in general. The radial motion $r(t)$ is periodic with period P while the angular motion $\phi(t)$ is periodic (modulo 2π) with period P/k where $k = K - 1$. Only when the two periods are commensurable, i.e. when $k = 1/N$ where N is a natural integer, is the motion periodic in physical space (with period NP).

[6]We have denoted the true anomaly by V rather than by the symbol v of earlier papers to avoid conflict with the relative speed v.

[7]Thus it is sometimes useful to define $k' = k/3$ which reduces to $1/(c^2h^2)$ at 1PN order.

[8]From now on we set $c = 1$ and $G = 1$.

$$\mathcal{F}^{(\mathrm{N})} = \frac{1}{5} \overset{(3)}{I}{}^{(\mathrm{N})}_{ij}(t) \overset{(3)}{I}{}^{(\mathrm{N})}_{ij}(t), \tag{3.2}$$

where (N) means the Newtonian limit, the superscript (n) refers to differentiation with respect to time n times, and $I_{ij}^{(\mathrm{N})}$ is the symmetric-trace-free (STF) quadrupole moment at Newtonian order given by

$$I_{ij}^{(\mathrm{N})} = \mu x^{\langle i} x^{j\rangle}. \tag{3.3}$$

Here x^i is the binary's orbital separation, and the angular brackets around indices indicate the STF projection: $x^{\langle i}x^{j\rangle} \equiv x^i x^j - \frac{1}{3}\delta^{ij} r^2$. Peters and Mathews [1] obtained the expression of the (averaged) Newtonian flux for compact binaries on eccentric orbits by two methods. The first method was to take directly the average in time of Eq. (3.2) using the expression (3.3) computed for the Keplerian ellipse; the second method was to decompose the components of the quadrupole moment into discrete Fourier series using the known Fourier decomposition of the Keplerian motion (the two methods, as expected, agreed on the result).

In the second method the quadrupole moment, which is a periodic function of time at Newtonian order, is thus decomposed into the Fourier series

$$I_{ij}^{(\mathrm{N})}(t) = \sum_{p=-\infty}^{+\infty} {}_{(p)}I_{ij}^{(\mathrm{N})} e^{\mathrm{i}p\ell}, \tag{3.4a}$$

$$\text{with } {}_{(p)}I_{ij}^{(\mathrm{N})} = \int_0^{2\pi} \frac{d\ell}{2\pi} I_{ij}^{(\mathrm{N})} e^{-\mathrm{i}p\ell}, \tag{3.4b}$$

where ℓ is the mean anomaly of the binary motion, Eq. (2.4). Since $I_{ij}^{(\mathrm{N})}$ is real, the Fourier discrete coefficients satisfy ${}_{(p)}I_{ij}^{(\mathrm{N})} = {}_{(-p)}I_{ij}^{(\mathrm{N})*}$ ($*$ denotes the complex conjugate). Inserting Eqs. (3.4) into (3.2) we obtain

$$\mathcal{F}^{(\mathrm{N})} = \frac{1}{5} \sum_{p=-\infty}^{+\infty} \sum_{q=-\infty}^{+\infty} (\mathrm{i}pn)^3 (\mathrm{i}qn)^3 {}_{(p)}I_{ij}^{(\mathrm{N})} {}_{(q)}I_{ij}^{(\mathrm{N})} e^{\mathrm{i}(p+q)\ell}. \tag{3.5}$$

Next we perform an average over one period P which means the average over $\ell = n(t - t_{\mathrm{P}})$ which is easily performed with the formula

$$\langle e^{\mathrm{i}p\ell}\rangle \equiv \int_0^{2\pi} \frac{d\ell}{2\pi} e^{\mathrm{i}p\ell} = \delta_{p,0}. \tag{3.6}$$

This immediately yields the averaged energy flux in the form of the Fourier series

$$\langle \mathcal{F}^{(\mathrm{N})}\rangle = \frac{2}{5} \sum_{p=1}^{+\infty} (pn)^6 |{}_{(p)}I_{ij}^{(\mathrm{N})}|^2. \tag{3.7}$$

Using dimensional analysis (and the known circular-orbit limit) this flux is necessarily of the form

$$\langle \mathcal{F}^{(\mathrm{N})}\rangle = \frac{32}{5} \nu^2 \left(\frac{m}{a}\right)^5 f(e), \tag{3.8}$$

where $\nu = \mu/m$ and a is the semimajor axis of the Newtonian orbit, and the function $f(e)$ is a dimensionless function depending only on the binary's eccentricity e. The coefficient in front of (3.8) is chosen in such a way that $f(e)$ reduces to 1 for circular orbits, i.e. when $e = 0$. Thus we have

$$f(e) = \frac{1}{16\mu^2 a^4} \sum_{p=1}^{+\infty} p^6 |{}_{(p)}I_{ij}^{(\mathrm{N})}|^2. \tag{3.9}$$

The Fourier coefficients of the quadrupole moment are explicitly given by Eqs. (A3) in the Appendix below. Remarkably this function admits an algebraically closed-form expression, crucial for the timing of the binary pulsar PSR 1913 + 16 [40], and given by

$$f(e) = \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1-e^2)^{7/2}}. \tag{3.10}$$

The function $f(e)$ is the Peters & Mathews [1] "enhancement" function, so designated because in the case of the binary pulsar, which has eccentricity $e = 0.617\cdots$, it enhances the effect of the orbital $\dot{P}$ by a factor ~ 11.843. The proof that the series (3.9) can be summed up to yield the closed-form expression (3.10) is given in the Appendix of [1]. Of course Eq. (3.10) is in full agreement with the direct computation of the average performed in the time domain [1], i.e.

$$f(e) = \frac{1}{32\mu^2 a^4 n^6} \langle \overset{(3)}{I}{}^{(\mathrm{N})}_{ij} \overset{(3)}{I}{}^{(\mathrm{N})}_{ij}\rangle. \tag{3.11}$$

The method of decomposing the Newtonian moment of compact binaries as discrete Fourier series was used in Ref. [20] to compute the tail at the dominant 1.5PN order. To extend this result we need to be more systematic about the Fourier decomposition of the (not necessarily Newtonian) source multipole moments.

B. General structure of the Fourier decomposition

The two sets of source-type multipole moments of the compact binary system are denoted by $I_L(t)$ and $J_L(t)$ following Ref. [41]. Here the multi-index notation means $L \equiv i_1 i_2 \cdots i_l$, where l is the number of indices or multipolarity (which is not to be confused with the mean anomaly ℓ). In this section we investigate the structure of the mass and current moments I_L and, say, J_{L-1} (where $L - 1 \equiv i_1 i_2 \cdots i_{l-1}$ is chosen in the current moment for convenience rather than L), at any PN order and for a compact binary system moving on a general noncircular orbit.[9] Their general structure can be written as

[9]However the intrinsic spins of the compact objects are neglected, so the motion takes place in a fixed orbital plane.

$$I_L(t) = \sum_{k=0}^{l} \mathcal{F}_k[r, \dot{r}, v^2] x^{\langle i_1\cdots i_k} v^{i_{k+1}\cdots i_l\rangle}, \tag{3.12a}$$

$$J_{L-1}(t) = \sum_{k=0}^{l-2} \mathcal{G}_k[r, \dot{r}, v^2] x^{\langle i_1\cdots i_k} v^{i_{k+1}\cdots i_{l-2}} \varepsilon^{i_{l-1}\rangle ab} x^a v^b, \tag{3.12b}$$

where ε^{iab} is the Levi-Civita symbol (such that $\varepsilon^{123} = 1$), where $x^i = y_1^i - y_2^i$ and $v^i = dx^i/dt = v_1^i - v_2^i$ denote the relative position and ordinary velocity of the two bodies (in a harmonic coordinate system). In (3.12) we pose, for instance, $x^{i_1\cdots i_k} \equiv x^{i_1}\cdots x^{i_k}$, and the angular brackets surrounding indices refer to the usual STF projection with respect to those indices.

Using polar coordinates r, ϕ in the orbital plane (as in Sec. 4), the above introduced coefficients $\mathcal{F}_k$ and $\mathcal{G}_k$ depend on the masses and on r, $\dot{r}$ and $v^2 = \dot{r}^2 + r^2\dot{\phi}^2$. For quasi-elliptic motion we can explicitly factorize out the dependence on the orbital phase ϕ by inserting $x = r\cos\phi$, $y = r\sin\phi$, and $v_x = \dot{r}\cos\phi - r\dot{\phi}\sin\phi$, $v_y = \dot{r}\sin\phi + r\dot{\phi}\cos\phi$. Furthermore, using the explicit solution of the motion (Sec. II B), we can express r, $\dot{r}$, and v^2, and hence the $\mathcal{F}_k$'s and $\mathcal{G}_k$'s, as periodic functions of the mean anomaly $\ell = n(t - t_P)$, where $n = 2\pi/P$. We then find that the above general structure of the multipole moments can be expressed in terms of the phase angle ϕ, as the following finite sum over some "magnetic-type" index m ranging from $-l$ to $+l$,

$$I_L(t) = \sum_{m=-l}^{l} {}_{(m)}\mathcal{A}_L(\ell) e^{\mathrm{i}m\phi}, \tag{3.13a}$$

$$J_{L-1}(t) = \sum_{m=-l}^{l} {}_{(m)}\mathcal{B}_{L-1}(\ell) e^{\mathrm{i}m\phi}, \tag{3.13b}$$

involving some coefficients ${}_{(m)}\mathcal{A}_L$ and ${}_{(m)}\mathcal{B}_{L-1}$ depending on the mean anomaly ℓ and which are complex ($\in \mathbb{C}$). (Some of these coefficients could be vanishing in particular cases.) The point, for our purpose, is that these coefficients are *periodic* functions of ℓ with period 2π. As we can see, the structure of the mass and current moments I_L and J_{L-1} is basically the same, but their coefficients ${}_{(m)}\mathcal{A}_L$ and ${}_{(m)}\mathcal{B}_{L-1}$ will have a different parity, because of the Levi-Civita symbol entering the current moment J_{L-1}.

To proceed further, let us exploit the doubly periodic nature of the dynamics in the two variables $\lambda \equiv K\ell$ and ℓ (as reviewed in Sec. 4). The phase is given in full generality by Eq. (2.8) where we recall that $W(\ell)$ is periodic in ℓ. In the following it will be more convenient to single out in the expression of the phase the purely relativistic precession of the periastron, namely, $\lambda - \ell = k\ell$ where $k = K - 1$. Inserting the expression of the phase variable into Eqs. (3.13) yields many factors which do modify the coefficients of (3.13), but in such a way that they remain periodic in ℓ. Hence we can write

$$I_L(t) = \sum_{m=-l}^{l} {}_{(m)}\mathcal{I}_L(\ell) e^{\mathrm{i}mk\ell}, \tag{3.14a}$$

$$J_{L-1}(t) = \sum_{m=-l}^{l} {}_{(m)}\mathcal{J}_{L-1}(\ell) e^{\mathrm{i}mk\ell}, \tag{3.14b}$$

where the coefficients ${}_{(m)}\mathcal{I}_L(\ell)$ and ${}_{(m)}\mathcal{J}_{L-1}(\ell)$ are 2π periodic. Finally, this makes it possible to use a discrete Fourier series expansion in the interval $\ell \in [0, 2\pi]$ for each of these coefficients, namely,

$${}_{(m)}\mathcal{I}_L(\ell) = \sum_{p=-\infty}^{+\infty} {}_{(p,m)}\mathcal{I}_L e^{\mathrm{i}p\ell}, \tag{3.15a}$$

$${}_{(m)}\mathcal{J}_{L-1}(\ell) = \sum_{p=-\infty}^{+\infty} {}_{(p,m)}\mathcal{J}_{L-1} e^{\mathrm{i}p\ell}, \tag{3.15b}$$

with inverse relations given by

$${}_{(p,m)}\mathcal{I}_L = \int_0^{2\pi} \frac{d\ell}{2\pi} {}_{(m)}\mathcal{I}_L(\ell) e^{-\mathrm{i}p\ell}, \tag{3.16a}$$

$${}_{(p,m)}\mathcal{J}_{L-1} = \int_0^{2\pi} \frac{d\ell}{2\pi} {}_{(m)}\mathcal{J}_{L-1}(\ell) e^{-\mathrm{i}p\ell}. \tag{3.16b}$$

This leads then to the following final decompositions of the multipole moments,

$$I_L(t) = \sum_{p=-\infty}^{+\infty} \sum_{m=-l}^{l} {}_{(p,m)}\mathcal{I}_L e^{\mathrm{i}(p+mk)\ell}, \tag{3.17a}$$

$$J_{L-1}(t) = \sum_{p=-\infty}^{+\infty} \sum_{m=-l}^{l} {}_{(p,m)}\mathcal{J}_{L-1} e^{\mathrm{i}(p+mk)\ell}. \tag{3.17b}$$

Obviously, since the moments I_L and J_{L-1} are real, their Fourier coefficients must satisfy ${}_{(p,m)}\mathcal{I}_L = {}_{(-p,-m)}\mathcal{I}_L^*$ and ${}_{(p,m)}\mathcal{J}_{L-1} = {}_{(-p,-m)}\mathcal{J}_{L-1}^*$.

The previous decompositions were general, but it is still useful to introduce a special notation for the particular case of the Newtonian (N) order, for which the relativistic precession k tends to zero. In this case we recover the usual periodic Fourier decomposition of the moments [generalizing Eqs. (3.4)], with only one Fourier summation over the index p, so that

$$I_L^{(\mathrm{N})}(t) = \sum_{p=-\infty}^{+\infty} {}_{(p)}\mathcal{I}_L^{(\mathrm{N})} e^{\mathrm{i}p\ell}, \tag{3.18a}$$

$$J_{L-1}^{(\mathrm{N})}(t) = \sum_{p=-\infty}^{+\infty} {}_{(p)}\mathcal{J}_{L-1}^{(\mathrm{N})} e^{\mathrm{i}p\ell}. \tag{3.18b}$$

The Newtonian Fourier coefficients are equal to the sums over m of the doubly periodic Fourier coefficients in Eqs. (3.17) when taken in the Newtonian limit, namely,

$$ {}_{(p)}I_L^{(\mathrm{N})} = \sum_{m=-l}^{l} {}_{(p,m)}I_L^{(\mathrm{N})}, \tag{3.19a} $$

$$ {}_{(p)}\mathcal{J}_{L-1}^{(\mathrm{N})} = \sum_{m=-l}^{l} {}_{(p,m)}\mathcal{J}_{L-1}^{(\mathrm{N})}. \tag{3.19b} $$

IV. TAIL CONTRIBUTIONS IN THE FLUX OF COMPACT BINARIES

The technique of the previous section is applied to the computation of the tail integrals in the energy flux of compact binaries. Although the computations are effectively done up to the 3PN level, the method we propose could, in principle, be implemented at any PN order.

Expression of the tail integrals in the 3PN energy flux

As reviewed in the Introduction, the first hereditary term in the energy flux $\mathcal{F}$ occurs at the 1.5PN order and is due to GW tails caused by interaction between the mass quadrupole moment and the total ADM mass. At the 3PN order, three kinds of hereditary terms appear: (1) The tails caused by quadratic nonlinear interaction between higher-order multipole moments with the mass; (2) the "tails of tails" due to the cubic nonlinear interaction between the tail itself and the mass; (3) a particular "tail-squared" term arising from self-interaction of the tail.[10]

In the equations to follow, we list the expressions for all these hereditary tail terms. They are given as nonlocal integrals over the source multipole moments of the system $I_{ij}(t), I_{ijk}(t), \ldots$ and $J_{ij}(t), \ldots$, where we use the specific definition of the PN source moments given in Ref. [41]. Thus the energy flux $\mathcal{F}$ defined by Eq. (3.1) can be split at 3PN order into

$$ \mathcal{F}^{(3\mathrm{PN})} = \mathcal{F}_{\mathrm{inst}} + \mathcal{F}_{\mathrm{hered}}, \tag{4.1} $$

where the "instantaneous" part, which depends on the source moments at the same instant (say t), reduces at the Newtonian order to the Einstein quadrupole moment flux $\mathcal{F}^{(\mathrm{N})}$ given by Eq. (3.2). On the other hand, the "hereditary" part reads

$$ \mathcal{F}_{\mathrm{hered}} = \mathcal{F}_{\mathrm{tail}} + \mathcal{F}_{\mathrm{tail(tail)}} + \mathcal{F}_{(\mathrm{tail})^2}, \tag{4.2} $$

where the quadratic-order tail integrals are explicitly given by (see Ref. [31])[11]

$$ \begin{aligned} \mathcal{F}_{\mathrm{tail}} &= \frac{4M}{5} I_{ij}^{(3)}(t)\int_0^{+\infty} d\tau I_{ij}^{(5)}(t-\tau)\left[\ln\left(\frac{\tau}{2r_0}\right)+\frac{11}{12}\right] \\ &\quad + \frac{4M}{189} I_{ijk}^{(4)}(t)\int_0^{+\infty} d\tau I_{ijk}^{(6)}(t-\tau)\left[\ln\left(\frac{\tau}{2r_0}\right)+\frac{97}{60}\right] \\ &\quad + \frac{64M}{45} J_{ij}^{(3)}(t)\int_0^{+\infty} d\tau J_{ij}^{(5)}(t-\tau)\left[\ln\left(\frac{\tau}{2r_0}\right)+\frac{7}{6}\right], \end{aligned} \tag{4.3} $$

while the cubic-order tails (proportional to M^2) are

$$ \begin{aligned} \mathcal{F}_{\mathrm{tail(tail)}} &= \frac{4M^2}{5} I_{ij}^{(3)}(t)\int_0^{+\infty} d\tau I_{ij}^{(6)}(t-\tau)\left[\ln^2\left(\frac{\tau}{2r_0}\right) \right. \\ &\quad \left. + \frac{57}{70}\ln\left(\frac{\tau}{2r_0}\right)+\frac{124\,627}{44\,100}\right], \end{aligned} \tag{4.4a} $$

$$ \mathcal{F}_{(\mathrm{tail})^2} = \frac{4M^2}{5}\left(\int_0^{+\infty} d\tau I_{ij}^{(5)}(t-\tau)\left[\ln\left(\frac{\tau}{2r_0}\right)+\frac{11}{12}\right]\right)^2. \tag{4.4b} $$

In these expressions recall that M is the *conserved* mass monopole or total ADM mass of the source. The first term in (4.3) is the dominant tail at order 1.5PN while the second and third represent the subdominant tails both appearing at order 2.5PN. The higher-order tails are not given since they are at least at 3.5PN order (see [12] for their expressions). The two cubic-order tails given in Eqs. (4.4) are both at 3PN order.

The constant r_0 scaling the logarithms in the above tail integrals has been defined to match with the choice made in the computation of tails of tails in Ref. [12]. This is the length scale appearing within the regularization factor $(r/r_0)^B$ used in the multipolar moment formalism valid for general sources [41]. Note that r_0 is a freely specifiable constant entering the relation between the retarded time in radiative coordinates [used in Eqs. (4.3) and (4.4)] and the corresponding time in harmonic coordinates. Hence r_0 merely relates the origins of time in the two coordinate systems and is unobservable.

We shall compute all the tail and tail-of-tail terms (4.3) and (4.4) [i.e. up to the 3PN order] averaged over the mean anomaly ℓ. Together with the instantaneous terms reported in the next paper [29], we shall obtain the complete expression of the 3PN energy flux. It is clear from Eqs. (4.3) and (4.4) that all the terms necessitate an evaluation at the relative Newtonian order *except* the mass-type quadrupolar tail term—first term in (4.3)—which must crucially include the 1PN corrections. We start with all the terms required at relative Newtonian order and then tackle the more difficult 1PN quadrupolar tail term.

B. Tails at relative Newtonian order

As a warm up, we consider the mass-type quadrupolar tail term in the energy flux, the first term in Eq. (4.3), but

[10]Recall that the hereditary character of the nonlinear memory integral [11,16–19] is that of a time antiderivative in the waveform (i.e. the radiative moments). Thus the nonlinear memory becomes instantaneous in the energy flux, which is made out of time derivatives of the radiative moments, and will be included into the instantaneous terms computed in [29].

[11]For convenience, we do not indicate the neglected PN terms, e.g. $\mathcal{O}(c^{-n})$. All equations are valid through the aimed 3PN precision. In the companion paper [29] we shall restore all powers of $1/c$ (and G).

given simply at the relative Newtonian order, namely,[12]

$$\langle \mathcal{F}^{(\mathrm{N})}_{\text{mass quad}} \rangle_{\text{tail}} = \left\langle \frac{4M}{5} \overset{(3)}{I}{}^{(\mathrm{N})}_{ij}(t) \int_0^{+\infty} d\tau \overset{(5)}{I}{}^{(\mathrm{N})}_{ij}(t-\tau) \times \left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{11}{12} \right] \right\rangle, \tag{4.5}$$

where the brackets $\langle\rangle$ refer to the average over the mean anomaly ℓ as defined by Eq. (3.6). The term (4.5) was already computed using a Fourier series at Newtonian order in Ref. [20]; note that the method of [20] is valid only for periodic motion and thus is applicable only at the Newtonian level. In this section we recover the Newtonian result of [20].

The Fourier decomposition of the Newtonian quadrupole moment was already given in general form by Eqs. (3.4). We insert that decomposition into the flux (4.5), and we evaluate the tail integral by using the fact that, if $\ell(t) = n(t - t_{\mathrm{P}})$ corresponds to the current time t, then clearly $\ell(t-\tau) = \ell(t) - n\tau$ corresponds to the retarded time $t-\tau$. Next we perform the average over the current value $\ell(t)$ with the help of the formula (3.6). The result is

$$\langle \mathcal{F}^{(\mathrm{N})}_{\text{mass quad}} \rangle_{\text{tail}} = -\frac{4M}{5} \sum_{p=-\infty}^{+\infty} (pn)^8 |_{(p)} \mathcal{I}^{(\mathrm{N})}_{ij}|^2 \times \int_0^{+\infty} d\tau e^{\mathrm{i}pn\tau} \left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{11}{12} \right]. \tag{4.6}$$

It remains to handle the last factor in (4.6) which is the tail integral in the Fourier domain, and which is computed using the closed-form formula

$$\int_0^{+\infty} d\tau e^{\mathrm{i}\sigma\tau} \ln\left(\frac{\tau}{2r_0}\right) = -\frac{1}{\sigma} \left[\frac{\pi}{2} \mathrm{sign}(\sigma) + \mathrm{i}(\ln(2|\sigma|r_0) + C) \right], \tag{4.7}$$

where $\sigma \equiv pn$, $\mathrm{sign}(\sigma) = \pm 1$, and $C = 0.577\cdots$ denotes the Euler constant. Inserting Eq. (4.7) into (4.6), we check that the imaginary parts cancel out, and the result reduces to

$$\langle \mathcal{F}^{(\mathrm{N})}_{\text{mass quad}} \rangle_{\text{tail}} = \frac{4\pi M}{5} \sum_{p=1}^{+\infty} (pn)^7 |_{(p)} \mathcal{I}^{(\mathrm{N})}_{ij}|^2. \tag{4.8}$$

Observe that the range of p's corresponds to positive frequencies only. Equation (4.8) agrees with the result of [20] and can interestingly be compared with the expression of the Newtonian part of the averaged flux (quadrupole formula) as given by Eq. (3.7). Although Eq. (4.8) is expressed in terms of the relatively simple Fourier series (4.8) (unlike for the case of the 1PN quadrupole tail in Sec. IV D which will turn out to be substantially more intricate), it has to be left in this form since no analytic closed-form expression can be found for the infinite sum of these Fourier components [20]. This is in contrast with the quadrupolar Newtonian flux (3.7) which does admit a closed-form expression [recall Eq. (3.10)]. In Sec. V we shall further proceed following Ref. [20] by expressing Eq. (4.8) in terms of a new enhancement factor depending on the eccentricity and which will be computed numerically.

Let us stress that the result (4.8) and all similar results derived below are "exact" only in a PN sense. Indeed we have formally replaced inside the tail integral the orbit of the binary at any earlier time $t-\tau$ by its orbit at the current time t, thereby neglecting the effect of the binary's adiabatic evolution by radiation reaction in the past. As a result there should be a remainder term in (4.8), given by the order of magnitude of the adiabatic parameter $\xi_{\mathrm{rad}} \equiv \dot{\omega}/\omega^2$ associated with the binary's inspiral by radiation reaction. Indeed, we know [11,20] that the replacement of the current motion inside the tail integral is valid only modulo some remainder $\mathcal{O}(\xi_{\mathrm{rad}})$ or, rather, $\mathcal{O}(\xi_{\mathrm{rad}} \ln \xi_{\mathrm{rad}})$. In terms of a PN expansion such remainder brings a correction of relative 2.5PN order which is always negligible here (indeed the tails are themselves at 1.5PN order so the total error due the neglect of the influence of the past in the tails is 4PN).

The other tail integrals, given by the second and third terms in Eq. (4.3), are evaluated in exactly the same way. With the PN accuracy of the present calculation these integrals are truly Newtonian, so the mass octupole moment I_{ijk} and current quadrupole moment J_{ij} are required at Newtonian order only. For simplicity, we do not add a superscript (N) to indicate this because there can be no confusion with other results. We thus need to evaluate the time-averaged fluxes

$$\langle \mathcal{F}_{\text{mass oct}} \rangle_{\text{tail}} = \left\langle \frac{4M}{189} I^{(4)}_{ijk}(t) \int_0^{+\infty} d\tau I^{(6)}_{ijk}(t-\tau) \times \left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{97}{60} \right] \right\rangle, \tag{4.9a}$$

$$\langle \mathcal{F}_{\text{curr quad}} \rangle_{\text{tail}} = \left\langle \frac{64M}{45} J^{(3)}_{ij}(t) \int_0^{+\infty} d\tau J^{(5)}_{ij}(t-\tau) \times \left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{7}{6} \right] \right\rangle. \tag{4.9b}$$

Inserting the Fourier decomposition of the moments, performing the average using Eq. (3.6), and using the integration formula (4.7) immediately results in

$$\langle \mathcal{F}_{\text{mass oct}} \rangle_{\text{tail}} = \frac{4\pi M}{189} \sum_{p=1}^{+\infty} (pn)^9 |_{(p)} \mathcal{I}_{ijk}|^2, \tag{4.10a}$$

$$\langle \mathcal{F}_{\text{curr quad}} \rangle_{\text{tail}} = \frac{64\pi M}{45} \sum_{p=1}^{+\infty} (pn)^7 |_{(p)} \mathcal{J}_{ij}|^2. \tag{4.10b}$$

In Sec. V we shall have to provide some numerical plots for the eccentricity-dependent enhancement factors associated

[12]We shall compute this term at 1PN relative order in Sec. IV D.

with Eqs. (4.10), since they cannot be computed analytically.

C. Tails of tails and tails squared

We have seen that at the 3PN order (i.e. 1.5PN beyond the dominant tail) the first cubic nonlinear interaction, between the quadrupole moment I_{ij} and two mass monopole factors M, appears. Following Eqs. (4.4) we thus have to compute the "tail-of-tail" contribution,

$$\langle \mathcal{F}_{\text{tail(tail)}} \rangle = \left\langle \frac{4M^2}{5} I^{(3)}_{ij}(t) \int_0^{+\infty} d\tau I^{(6)}_{ij}(t-\tau)\left[\ln^2\left(\frac{\tau}{2r_0}\right) + \frac{57}{70}\ln\left(\frac{\tau}{2r_0}\right) + \frac{124\,627}{44\,100}\right]\right\rangle, \tag{4.11}$$

and the so-called "tail-squared" one,

$$\langle \mathcal{F}_{(\text{tail})^2} \rangle = \left\langle \frac{4M^2}{5}\left(\int_0^{+\infty} d\tau I^{(5)}_{ij}(t-\tau)\left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{11}{12}\right]\right)^2\right\rangle. \tag{4.12}$$

Both contributions are evaluated at relative Newtonian order, inserting the Fourier decomposition of the Newtonian quadrupole moment (3.4) [suppressing the notation (N) for simplicity]. The new feature with respect to the previous computation is the occurrence of a logarithm *squared* in the tail-of-tail integral (4.11). The integration formula required to deal with this term is [compare with Eq. (4.7)]

$$\int_0^{+\infty} d\tau e^{\mathrm{i}\sigma\tau} \ln^2\left(\frac{\tau}{2r_0}\right) = \frac{\mathrm{i}}{\sigma}\left\{\frac{\pi^2}{6} - \left[\frac{\pi}{2}\operatorname{sign}(\sigma) + \mathrm{i}(\ln(2|\sigma|r_0) + C)\right]^2\right\}, \tag{4.13}$$

and with this formula, together with (4.7), we obtain the result

$$\langle \mathcal{F}_{\text{tail(tail)}} \rangle = \frac{4M^2}{5}\sum_{p=1}^{+\infty}(pn)^8 |{}_{(p)}\mathcal{I}^{(\mathrm{N})}_{ij}|^2 \left\{\frac{\pi^2}{6} - 2(\ln(2pnr_0) + C)^2 + \frac{57}{35}(\ln(2pnr_0) + C) - \frac{124\,627}{22\,050}\right\}. \tag{4.14}$$

On the other hand, the tail-squared term is readily computed with (4.7) and found to be

$$\langle \mathcal{F}_{(\text{tail})^2} \rangle = \frac{4M^2}{5}\sum_{p=1}^{+\infty}(pn)^8 |{}_{(p)}\mathcal{I}^{(\mathrm{N})}_{ij}|^2 \times\left\{\frac{\pi^2}{2} + 2\left(\ln(2pnr_0) + C - \frac{11}{12}\right)^2\right\}. \tag{4.15}$$

Summing up the two results (4.14) and (4.15) we finally obtain

$$\langle \mathcal{F}_{\text{tail(tail)}+(\text{tail})^2} \rangle = \frac{4M^2}{5}\sum_{p=1}^{+\infty}(pn)^8 |{}_{(p)}\mathcal{I}^{(\mathrm{N})}_{ij}|^2 \left\{\frac{2\pi^2}{3} - \frac{214}{105} \times \ln(2pnr_0) - \frac{214}{105}C - \frac{116\,761}{29\,400}\right\}. \tag{4.16}$$

As we can see, the contribution from logarithms *squared* has canceled out between the two terms (4.14) and (4.15). Such cancellation is in fact known to occur for general sources [12]. We observe also that the result (4.16) still depends on the arbitrary length scale r_0. It will be important to trace out the fate of this constant and check that the complete energy flux we obtain at the end (including all the instantaneous contributions computed in [29]) is independent of r_0.

D. The mass quadrupole tail at 1PN order

Let us now tackle the computation of the mass quadrupole tail at the relative 1PN order, namely,

$$\langle \mathcal{F}_{\text{mass quad}} \rangle_{\text{tail}} = \left\langle \frac{4M}{5} I^{(3)}_{ij}(t) \int_0^{+\infty} d\tau I^{(5)}_{ij}(t-\tau) \times\left[\ln\left(\frac{\tau}{2r_0}\right) + \frac{11}{12}\right]\right\rangle. \tag{4.17}$$

At the 1PN order (and similarly at any higher PN orders), we must take care of the doubly periodic structure of the solution of the motion [Sec. IIA], and decompose the multipole moments according to the general formulas (3.17). So the 1PN mass quadrupole moment I_{ij} entering Eq. (4.17) is decomposed as

$$I_{ij}(t) = \sum_{p=-\infty}^{+\infty}\sum_{m=-2}^{2} {}_{(p,m)}\mathcal{I}_{ij} e^{\mathrm{i}(p+mk)\ell}, \tag{4.18}$$

with doubly indexed Fourier coefficients ${}_{(p,m)}\mathcal{I}_{ij}$ which are valid through order 1PN. We can be more precise and notice that the harmonics for which $m = \pm 1$ are zero at the 1PN order, so that

$$I_{ij}(t) = \sum_{p=-\infty}^{+\infty}\{{}_{(p,-2)}\mathcal{I}_{ij} e^{\mathrm{i}(p-2k)\ell} + {}_{(p,0)}\mathcal{I}_{ij} e^{\mathrm{i}p\ell} + {}_{(p,2)}\mathcal{I}_{ij} e^{\mathrm{i}(p+2k)\ell}\}, \tag{4.19}$$

but in the following it is more convenient to work with the general decomposition (4.18), keeping in mind that the terms with $m = \pm 1$ are absent. As before we insert (4.18) into (4.17) to obtain [after neglecting 2.5PN radiation-reaction terms $\mathcal{O}(\xi_{\text{rad}})$]

$$\langle\mathcal{F}_{\text{mass quad}}\rangle_{\text{tail}} = \frac{4M}{5}\sum_{p,p';m,m'} n^8(p+mk)^3(p' + m'k)^5{}_{(p,m)}\mathcal{I}_{ij(p',m')}\mathcal{I}_{ij}\langle e^{\mathrm{i}(p+p'+(m+m')k)\ell}\rangle \times \int_0^{+\infty} d\tau e^{-\mathrm{i}(p'+m'k)n\tau}\left[\ln\left(\frac{\tau}{2r_0}\right)+\frac{11}{12}\right], \tag{4.20}$$

where the summations range from $-\infty$ to $+\infty$ for p and p', and from -2 to 2 for m and m'. Evidently, the factors $(p + mk)^3$ and $(p' + m'k)^5$ come from the time derivatives of the quadrupole moment. We have explicitly left the last two factors in (4.20) as they are, namely, the average over ℓ of an elementary "doubly periodic" complex exponential, and the Fourier transform of the tail integral.

The expression (4.20) is to be worked out at the 1PN order. Since the relativistic advance of the periastron k is already a small 1PN quantity, the first thing to do is to evaluate (4.20) at *linear* order in k [i.e., neglecting $\mathcal{O}(k^2)$ which is at least 2PN]. Afterwards, we shall insert the explicit expressions for the 1PN quadrupole moment and ADM mass. We provide here the necessary formulas for performing the linear-order expansion in k of the last two factors in (4.20). The average we perform is over the orbital period (time to return to the periastron) and so is defined by

$$\langle e^{\mathrm{i}(p+mk)\ell}\rangle \equiv \int_0^{2\pi}\frac{d\ell}{2\pi}e^{\mathrm{i}(p+mk)\ell}. \tag{4.21}$$

Using the fact that $mk \ll 1$ since we are in the limit where $k \to 0$ (hence $p + mk$ is never an integer unless $k = 0$), we readily find

$$\langle e^{\mathrm{i}(p+mk)\ell}\rangle = \begin{Bmatrix} \frac{m}{p}k & \text{if } p \neq 0 \\ 1 + \mathrm{i}\pi mk & \text{if } p = 0 \end{Bmatrix} + \mathcal{O}(k^2). \tag{4.22}$$

This result depends only on whether p is zero or not, and is true for any integer m, except that when $m = 0$ the result (4.22) becomes exact as there is no remainder term $\mathcal{O}(k^2)$ in this case.

On the other hand, to compute the tail integral given by the last factor in Eq. (4.20), we expand it at first order in k, obtaining thereby

$$\int_0^{+\infty} d\tau e^{\mathrm{i}(p+mk)n\tau}\ln\left(\frac{\tau}{2r_0}\right) = \left(1-\frac{mk}{p}\right) \times \int_0^{+\infty} d\tau e^{\mathrm{i}pn\tau}\ln\left(\frac{\tau}{2r_0}\right) - \mathrm{i}\frac{mk}{p^2 n} + \mathcal{O}(k^2), \tag{4.23}$$

and we apply for the remaining integral in (4.23) the formula (4.7).

With Eqs. (4.22) and (4.23) in hand we can explicitly work out the tail expression (4.20) at first order in k (the extension to higher order in k would, in principle, be straightforward). The result will be left in the form of the multiple Fourier series (4.20), into which the results (4.22) and (4.23) have been inserted (we do not try to give a more explicit form for this result, which is given by a complicated MATHEMATICA expression). In the next section we shall reexpress this series in terms of some elementary enhancement functions, which will finally be evaluated numerically.

V. NUMERICAL CALCULATION OF THE TAIL INTEGRALS

A. Definition of the eccentricity enhancement factors

We define here some functions of the eccentricity by certain Fourier series of the components of the *Newtonian* multipole moments $I_L^{(\mathrm{N})}$ and $J_{L-1}^{(\mathrm{N})}$ for a Keplerian ellipse with eccentricity e, semimajor axis a, and frequency $n = 2\pi/P$ (such that Kepler's law $n^2a^3 = m$ holds at Newtonian order). In the frame of the center of mass we have $I_L^{(\mathrm{N})} = \mu s_l(\nu)x^{\langle L\rangle}$ and $J_{L-1}^{(\mathrm{N})} = \mu s_l(\nu)x^{\langle L-2}\varepsilon^{i_{l-1}\rangle ab}x^a v^b$ where $\mu = m_1m_2/m = \nu m$. Here we pose $s_l(\nu) \equiv X_2^{l-1} + (-)^l X_1^{l-1}$, where $X_1 \equiv \frac{m_1}{m} = \frac{1}{2}(1+\sqrt{1-4\nu})$ and $X_2 \equiv \frac{m_2}{m} = \frac{1}{2}(1-\sqrt{1-4\nu})$. Let us rescale the latter Newtonian moments in order to make them dimensionless by posing

$$I_L^{(\mathrm{N})} \equiv \mu a^l s_l(\nu)\hat{I}_L, \tag{5.1a}$$

$$J_{L-1}^{(\mathrm{N})} \equiv \mu a^l n s_l(\nu)\hat{J}_{L-1}. \tag{5.1b}$$

Our first enhancement function is of course the Peters & Mathews [1] function, which we have already expressed in Eq. (3.9) as a Fourier series [and which turns out to admit the analytically closed form (3.10)]. In terms of the Fourier components of the rescaled quadrupole moment $\hat{I}_{ij}$, this series reads

$$f(e) = \frac{1}{16}\sum_{p=1}^{+\infty} p^6|_{(p)}\hat{\mathcal{I}}_{ij}|^2, \tag{5.2}$$

and is such that the averaged energy flux of compact binaries at the Newtonian order reads

$$\langle\mathcal{F}^{(\mathrm{N})}\rangle = \frac{32}{5}\nu^2x^5 f(e), \tag{5.3}$$

where we have defined, for future convenience, the frequency-related PN parameter $x = (m\omega)^{2/3}$ where ω is the binary's orbital frequency defined for general orbits by Eq. (2.7). Note that in Eq. (5.3), which is Newtonian, we can approximate ω by n (hence x reduces to m/a).

Next, we define several other enhancement functions of the eccentricity, which will permit us to usefully parametrize the tail terms at Newtonian order. First we pose

$$\varphi(e) = \frac{1}{32}\sum_{p=1}^{+\infty} p^7|_{(p)}\hat{\mathcal{I}}_{ij}|^2. \tag{5.4}$$

Like for $f(e)$, this function is defined in such a way that it

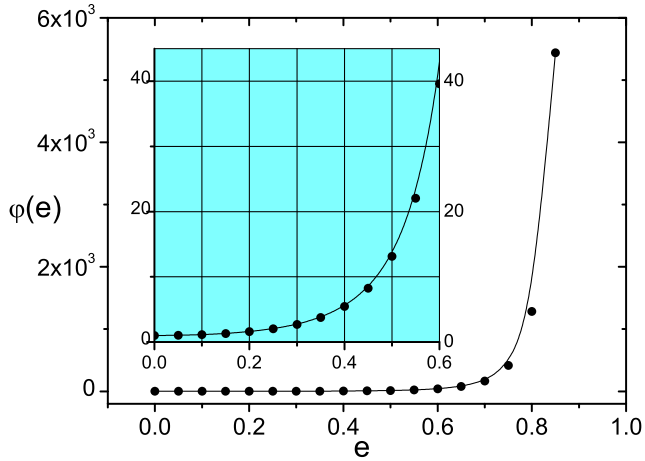

FIG. 1 (color online). Variation of $\varphi(e)$ with the eccentricity e. The function $\varphi(e)$ agrees with the numerical calculation of Ref. [20] modulo a trivial rescaling with $f(e)$. The inset graph is a zoom of the function (which looks like a straight horizontal line in the main graph) at a smaller scale. The dots represent the numerical computation, and the solid line is a fit to the numerical points. In the circular-orbit limit we have $\varphi(0) = 1$.

tends to 1 in the circular-orbit limit, when $e \to 0$. However, unlike for $f(e)$, it does not admit a closed-form expression, and will have to be left in the form of a Fourier series. The function $\varphi(e)$ parametrizes the mass quadrupole tail at Newtonian order, in the sense that we have, from Eq. (4.8),

$$\langle \mathcal{F}^{(\mathrm{N})}_{\text{mass quad}} \rangle = \frac{32}{5}\nu^2 x^5 [4\pi x^{3/2}\varphi(e)]. \tag{5.5}$$

For circular orbits, $\varphi(0) = 1$ and we recognize the coefficient 4π of the 1.5PN tail term ($\propto x^{3/2}$) as computed numerically in Ref. [42] and analytically in Refs. [20,43]. The function $\varphi(e)$ has already been computed numerically from its Fourier series (5.4) in Ref. [20]. Here we show the plot of $\varphi(e)$ in Fig. 1 (see Sec. V B for details on the numerical computation).[13]

We next proceed similarly for the 2.5PN mass octupole and current quadrupole tails. We pose

$$\beta(e) = \frac{20}{49\,209} \sum_{p=1}^{+\infty} p^9 |_{(p)}\hat{\mathcal{I}}_{ijk}|^2, \tag{5.6a}$$

$$\gamma(e) = 4 \sum_{p=1}^{+\infty} p^7 |_{(p)}\hat{\mathcal{J}}_{ij}|^2. \tag{5.6b}$$

Again these functions tend to 1 when $e \to 0$ (as will be checked later) and most probably do not admit any closed-form expressions. With their help these tail terms ($\propto x^{5/2}$) of Eqs. (4.9) read

$$\langle \mathcal{F}_{\text{mass oct}} \rangle_{\text{tail}} = \frac{32}{5}\nu^2 x^5 \left[\frac{16\,403}{2016}\pi(1-4\nu)x^{5/2}\beta(e)\right], \tag{5.7}$$

$$\langle \mathcal{F}_{\text{curr quad}} \rangle_{\text{tail}} = \frac{32}{5}\nu^2 x^5 \left[\frac{\pi}{18}(1-4\nu)x^{5/2}\gamma(e)\right]. \tag{5.8}$$

The numerical graphs of the functions $\beta(e)$ and $\gamma(e)$ are shown in Fig. 2.

Two further enhancement factors are then introduced to parametrize the tail-of-tail and tail-squared integrals (which are Newtonian with the present approximation). The first of these functions looks very much like the Peters & Mathews function $f(e)$, Eq. (5.2), in the sense that its Fourier series involves *even* powers of the modes p. Namely, we define

$$F(e) = \frac{1}{64} \sum_{p=1}^{+\infty} p^8 |_{(p)}\hat{\mathcal{I}}_{ij}|^2. \tag{5.9}$$

Thanks to this even power $\propto p^8$, we find that $F(e)$ can also be computed as an average performed in the time domain similar to the one of Eq. (3.11) for $f(e)$. Namely, we easily verify that

$$F(e) = \frac{1}{128 n^8} \langle \hat{I}^{(4)}_{ij} \hat{I}^{(4)}_{ij} \rangle, \tag{5.10}$$

which can straightforwardly be computed in the time domain with the result that $F(e)$ admits, like for $f(e)$, an analytic closed form which is readily obtained as

$$F(e) = \frac{1 + \frac{85}{6}e^2 + \frac{5171}{192}e^4 + \frac{1751}{192}e^6 + \frac{297}{1024}e^8}{(1-e^2)^{13/2}}. \tag{5.11}$$

On the other hand, we shall need to introduce a function whose Fourier transform differs from the one of $F(e)$ by the presence of the *logarithm* of modes, namely,

$$\chi(e) = \frac{1}{64} \sum_{p=1}^{+\infty} p^8 \ln\left(\frac{p}{2}\right) |_{(p)}\hat{\mathcal{I}}_{ij}|^2. \tag{5.12}$$

One can be convinced that very likely $\chi(e)$ does not admit any analytic form [hence we name it using the Greek alphabet—in contrast to $f(e)$ and $F(e)$]. Note that $\chi(e)$ has been exceptionally defined in such a way that it *vanishes* when $e \to 0$. This is easily checked since in the circular-orbit limit (and at Newtonian order) the quadrupole moment $I^{(\mathrm{N})}_{ij}$ possesses only one harmonic corresponding to $p = 2$, which due to the log term reduces $\chi(e)$ to zero in this case. In Fig. 3 we show the numerical plot of the function $\chi(e)$ [and also the one for $F(e)$].

With those definitions we find that the sum of tail-of-tail and tail-squared contributions obtained in Eq. (4.16) reads

[13] Note that our notation is different from the one in [20]; the function $\varphi_{\mathrm{BS}}(e)$ there is related to our definition by $\varphi_{\mathrm{BS}}(e) = \varphi(e)/f(e)$. In the present work it is better not to divide the various functions by the Peters & Mathews function $f(e)$ entering the Newtonian approximation.

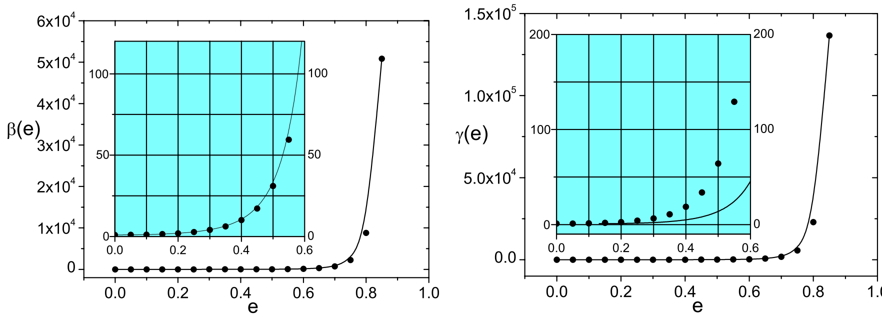

FIG. 2 (color online). Variation of $\beta(e)$ (left panel) and $\gamma(e)$ (right panel) with the eccentricity e. In the circular-orbit limit we have $\beta(0) = \gamma(0) = 1$.

$$\langle \mathcal{F}_{\mathrm{tail(tail)+(tail)^2}} \rangle = \frac{32}{5} \nu^2 x^8 \biggl\{ \biggl[-\frac{116\,761}{3675} + \frac{16}{3}\pi^2 - \frac{1712}{105} C - \frac{1712}{105} \ln(4\omega r_0) \biggr] F(e) - \frac{1712}{105} \chi(e) \biggr\}. \tag{5.13}$$

The circular-orbit limit can be immediately read off from this expression and seen to agree with Eq. (5.9) in Ref. [12] or Eq. (12.7) in Ref. [31].

Finally, we provide the result in the case of the mass quadrupole tail at 1PN order. We have seen in Sec. IV D that the calculation in this case is much more involved, as the Fourier series (4.20) contains several summations, and depends on the intermediate results (4.22) and (4.23). In addition the computation must take into account the 1PN relativistic correction in the mass quadrupole moment and ADM mass; these are provided in Eqs. (5.17) and (5.18) below. We find that there probably is no simple way [i.e. no simple-looking Fourier series like, for instance, (5.12)] for expressing the new enhancement functions of eccentricity which appear at the 1PN order. However, one can check beforehand that the 1PN term is a linear function of the symmetric mass ratio ν; hence, we must introduce two enhancement functions, denoted below α and θ. As before, we normalize these functions so that $\alpha(0) = 1$ and $\theta(0) = 1$. We have [extending Eq. (5.5) at the 1PN order]

$$\langle \mathcal{F}_{\mathrm{mass\ quad}} \rangle_{\mathrm{tail}} = \frac{32}{5} \nu^2 x^{13/2} \biggl\{ 4\pi\varphi(e_t) + \pi x \biggl[-\frac{428}{21}\alpha(e_t) + \frac{178}{21}\nu\theta(e_t) \biggr] \biggr\}. \tag{5.14}$$

This equation provides the definition of the two enhancement functions α and θ, and we resort to the MATHEMATICA computation to obtain them as complicated Fourier decompositions, which will then be directly computed numerically using the method outlined in Sec. V B. Notice that since we are at the 1PN level we must use a specific definition for the eccentricity, and we adopted here the "time" eccentricity e_t entering the Kepler equation (2.10b) in Sec. II B. At the 1PN order the other eccentricities are related to it by Eqs. (2.17). On the other hand, the frequency-related PN parameter, given by

$$x = (m\omega)^{2/3}, \tag{5.15}$$

crucially includes the 1PN relativistic correction coming from the periastron advance $K = 1 + k$, through the definition $\omega = nK$ of Sec. II A. All the 1PN corrections arising from the formulas (4.22) and (4.23), the multipole moments M and I_{ij}, the use of the time eccentricity e_t, and the specific PN variable x, are incorporated in a MATHEMATICA program dealing with the decomposition (4.20) and used to obtain (5.14). The behavior of the enhancement functions $\alpha(e)$ and $\theta(e)$ is given in Fig. 4.

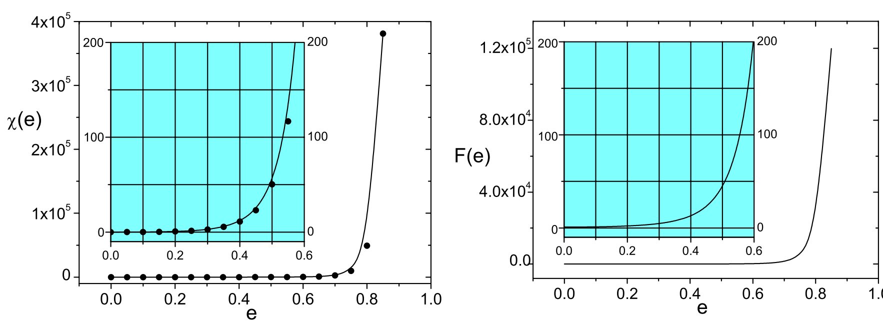

FIG. 3 (color online). Variation of $\chi(e)$ (left panel) and $F(e)$ (right panel) with the eccentricity e. In the right panel, the exact expression of $F(e)$ given by Eq. (5.11) is used. In the circular-orbit limit we have $\chi(0) = 0$ and $F(0) = 1$.

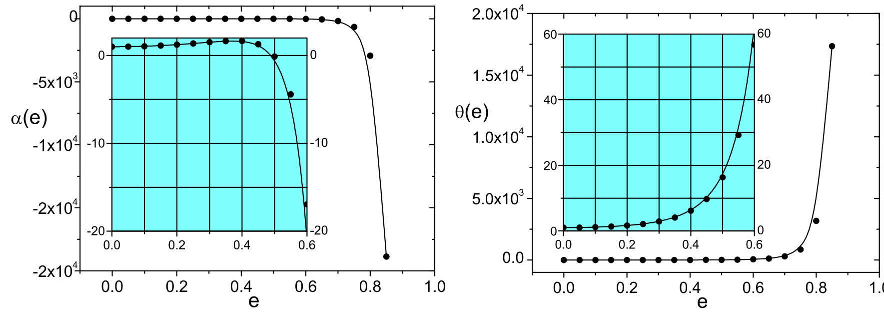

FIG. 4 (color online). Variation of $\alpha(e)$ (left panel) and $\theta(e)$ (right panel) with the eccentricity e. In the circular-orbit limit we have $\alpha(0) = \theta(0) = 1$.

B. Numerical evaluation of the Fourier coefficients

We now describe the numerical implementation of the procedure for the computation of the Fourier coefficients of the multipole moments that lead to the numerical plots of the previous section. We focus the discussion on the computation of the crucial coefficients ${}_{(p,m)}I_{ij}$ at 1PN order which are more difficult to obtain. The mass quadrupole moment with 1PN accuracy is given by [compare with the general structure (3.12a)]

$$I_{ij} = \mu\left\{1 + \left[v^2\left(\frac{29}{42} - \frac{29}{14}\nu\right) + \frac{m}{r}\left(-\frac{5}{7} + \frac{8}{7}\nu\right)\right]x^{\langle i}x^{j\rangle} + \left(\frac{11}{21} - \frac{11}{7}\nu\right)r^2 v^{\langle i}v^{j\rangle} + \left(-\frac{4}{7} + \frac{12}{7}\nu\right)r\dot{r}x^{\langle i}v^{j\rangle}\right\}, \tag{5.16}$$

where x^i and $v^i = dx^i/dt$ are the relative position and velocity in harmonic coordinates, and $r = |x^i|$ (like in Sec. II B). Equation (5.16) is valid for nonspinning compact binaries on an arbitrary quasi-Keplerian orbit in the center-of-mass frame (see e.g. [44]). Since we investigate tails with 1PN relative accuracy, we need also the relation of the ADM mass M to the total mass $m = m_1 + m_2$ at 1PN order,

$$M = m\left[1 + \nu\left(\frac{v^2}{2} - \frac{m}{r}\right)\right]. \tag{5.17}$$

Using the quasi-Keplerian representation of the motion [Sec. II B], the dependence of I_{ij} on x^i, v^i, r, v^2, and $\dot{r}$ can be parametrized in terms of the eccentric anomaly u. However, as explained previously, we require $I_{ij}(\ell)$ in the time domain to proceed. The steps of our numerical implementation scheme can be summarized as follows:

(1) To begin with, each component of the 1PN mass quadrupole is expressed in terms of the quasi-Keplerian parameters using Eqs. (2.10), (2.11), and (2.12). The components of the mass quadrupole are now functions of the eccentric anomaly u, and are parametrized by the mean motion n and by one of the eccentricities which is chosen to be e_t—the time eccentricity in Kepler's equation (2.10b).[14]

(2) We next invert, numerically, the equation for the mean anomaly $\ell = u - e_t \sin u$ to obtain the function $u(\ell)$. This can be done either by using the series representation in terms of Bessel functions,

$$u = \ell + 2\sum_{s=1}^{+\infty}\frac{1}{s}J_s(se_t)\sin(s\ell), \tag{5.18}$$

or numerically by finding the root of $\ell = u - e_t \sin u$. The latter is a more efficient and more accurate method and we employed it in this work (we used the FINDROOT routine in MATHEMATICA). In this case we generated a table of 20 000 points of u and ℓ between 0 and 2π (for each value of e_t). The above inversion enables us to reexpress all functions of the eccentric anomaly u as functions of the mean anomaly ℓ. If required, a more accurate implementation for solving Kepler's equation along the lines of [45] can be used in the future.

(3) One needs to be careful in dealing with the u dependence of V in Eq. (2.12) to avoid the discontinuity there. To this end it is best to use

$$V(u) = u + 2\arctan\left(\frac{\beta_\phi \sin u}{1 - \beta_\phi \cos u}\right), \tag{5.19}$$

where $\beta_\phi \equiv [1 - (1 - e_\phi^2)^{1/2}]/e_\phi$. By this process, we thus have in hand the Fourier coefficients ${}_{(m)}I_{ij}(\ell)$ defined in Eq. (3.14a) as explicit (numerical) functions of ℓ.

(4) Recall that these functions also have a dependence on the mass ratio ν and the PN parameter x defined by $(m\omega)^{2/3}$ where $\omega = nK$. To avoid assuming numerical values for ν and x and hence to preserve the full generality of the result, we split the function ${}_{(m)}I_{ij}$ into

[14]The semimajor axis a_r and the other eccentricities e_r and e_ϕ are deduced from n and e_t using Eqs. (2.14), (2.15), (2.16), and (2.17).

$$ {}_{(m)}I_{ij}(\ell, e_t, \nu, x) = {}_{(m)}\mathcal{I}^{00}_{ij}(\ell, e_t) + x[{}_{(m)}\mathcal{I}^{10}_{ij}(\ell, e_t) + \nu_{(m)}\mathcal{I}^{11}_{ij}(\ell, e_t)]. \quad (5.20) $$

Notice that we have neglected the terms higher than 1PN in writing the above expression. Now the various ${}_{(m)}\mathcal{I}^{ab}_{ij}$ are only functions of ℓ and e_t. We evaluate the Fourier coefficients of these terms separately in the next step of the procedure.

(5) For a fixed value of e_t, we can straightforwardly get the plot of ${}_{(m)}\mathcal{I}^{00}_{ij}$ versus ℓ. Equivalently, one can also write the Fourier decomposition of ${}_{(m)}\mathcal{I}^{00}_{ij}(\ell)$ as

$$ {}_{(m)}\mathcal{I}^{00}_{ij}(\ell) = \sum_{p=-\infty}^{+\infty} {}_{(p,m)}\mathcal{I}^{00}_{ij} e^{\mathrm{i}p\ell}. \quad (5.21) $$

Now we seek a numerical fit to Eq. (5.20), in powers of $e^{ip\ell}$, to extract out the coefficients ${}_{(p,m)}\mathcal{I}^{00}_{ij}$. We do the same for different values of e_t and for ${}_{(p,m)}\mathcal{I}^{10}_{ij}$ and ${}_{(p,m)}\mathcal{I}^{11}_{ij}$.

(6) The fitting procedure mentioned above can be implemented either starting with the STF moment I_{ij} or the non-STF projected one. The expressions will be different in these two cases, as for the first case the zz component of the moment is not equal to zero by definition [since $I_{zz} = -(I_{xx} + I_{yy})$] whereas for the latter case the zz component is zero due to planar motion. This provides a simple algebraic check on the numerical calculation.

(7) Instead of using the basic multipole moment as the starting function (e.g. I_{ij}), we find that using the leading time derivative (i.e. $I^{(3)}_{ij}$) improves the numerical convergence of the results because one deals with lower derivatives of the basic function. This is very helpful for higher values of eccentricity.

(8) Substituting the Fourier coefficients into Eq. (4.20), one can generate the numerical values of the averaged energy flux $\langle \mathcal{F}_{\text{mass quad}} \rangle$ for the different values of e_t, and hence get the numerical values of the enhancement functions and, most importantly, of the 1PN ones $\alpha(e_t)$ and $\theta(e_t)$ defined by (5.14). The plots of these functions reported in Sec. VA readily follow.

We have just described the procedure for the most difficult 1PN quadrupole tail yielding the computation of $\alpha(e_t)$ and $\theta(e_t)$. This procedure is quite general, and provides a method which could be extended to higher PN orders. However, at the Newtonian order it is in fact much more efficient to make use of the well-known Fourier decomposition of the Keplerian motion. Using this we can derive the components of the multipole moments (at Newtonian order) as a series of combinations of Bessel functions. Then it is a very simple matter to compute numerically the associated Newtonian enhancement functions [namely, the functions $\varphi(e)$, $\beta(e)$, $\gamma(e)$, and $\chi(e)$ defined in Sec. VA]. For the convenience of the reader, we give in the Appendix all the expressions for each of the components of the required Newtonian moments [$I^{(\mathrm{N})}_{ij}$, $I^{(\mathrm{N})}_{ijk}$, and $J^{(\mathrm{N})}_{ij}$] as a series of Bessel functions. We have used them to compute numerically the functions $\varphi(e)$, $\beta(e)$, $\gamma(e)$, and $\chi(e)$.[15]

VI. THE HEREDITARY CONTRIBUTION TO THE 3PN ENERGY FLUX

A. Final expression of the tail terms

Based on the treatment outlined above of a numerical scheme for the computation of the orbital average of the hereditary part of the energy flux up to 3PN, we finally provide the complete results for the numerical plots of the dimensionless enhancement factors. It is convenient for the final presentation to redefine in a minor way the "elementary" enhancement functions of Sec. VA, which were directly given by simple Fourier decompositions. Let us choose

$$ \psi(e) \equiv \frac{13\,696}{8191}\alpha(e) - \frac{16\,403}{24\,573}\beta(e) - \frac{112}{24\,573}\gamma(e), \quad (6.1a) $$

$$ \zeta(e) \equiv -\frac{1424}{4081}\theta(e) + \frac{16\,403}{12\,243}\beta(e) + \frac{16}{1749}\gamma(e), \quad (6.1b) $$

$$ \kappa(e) \equiv F(e) + \frac{59\,920}{116\,761}\chi(e). \quad (6.1c) $$

Considering thus the 1.5PN and 2.5PN terms, composed of tails, and the 3PN terms, composed of the tail-of-tail and the tail-squared terms, the total hereditary contribution to the energy flux (4.2) when averaged over ℓ (and normalized to the Newtonian value for circular orbits) finally reads

$$ \langle \mathcal{F}_{\text{hered}} \rangle = \frac{32}{5}\nu^2 x^5 \left\{ 4\pi x^{3/2}\varphi(e_t) + \pi x^{5/2}\left[-\frac{8191}{672}\psi(e_t) - \frac{583}{24}\nu\zeta(e_t) \right] + x^3\left[-\frac{116\,761}{3675}\kappa(e_t) + \left[\frac{16}{3}\pi^2 - \frac{1712}{105}C - \frac{1712}{105}\ln(4\omega r_0) \right] F(e_t) \right] \right\}. \quad (6.2) $$

In this result all the enhancement functions reduce to 1 in the circular case, when $e_t = 0$, so the circular limit is immediately deduced from inspection of Eq. (6.2), and is seen to be in complete agreement with Refs. [12,31]. The function $F(e_t)$ is known analytically, and we recall here its expression,

$$ F(e_t) = \frac{1 + \frac{85}{6}e_t^2 + \frac{5171}{192}e_t^4 + \frac{1751}{192}e_t^6 + \frac{297}{1024}e_t^8}{(1 - e_t^2)^{13/2}}. \quad (6.3) $$

[15]On the other hand, for the Newtonian tail terms, we could proceed exactly in the same way as for the 1PN term, following steps 1–8. We have verified that both methods agree well.

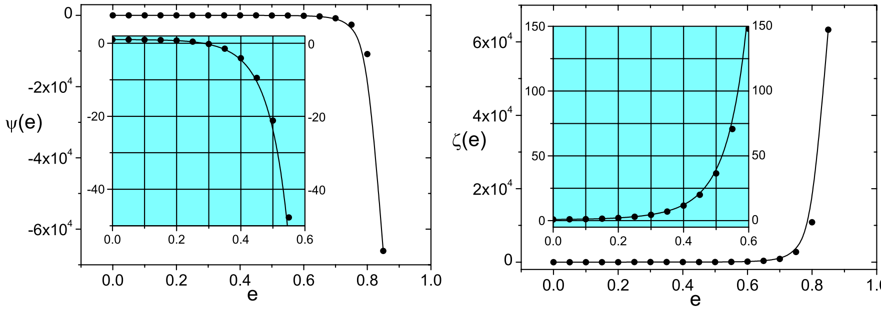

FIG. 5 (color online). Variation of $\psi(e)$ (left panel) and $\zeta(e)$ (right panel) with the eccentricity e. The inset graph is a zoom of the function (which looks like a straight horizontal line in the main graph) at a smaller scale. The dots represent the numerical computation, and the solid line is a fit to the numerical points. In the circular-orbit limit we have $\psi(0) = \zeta(0) = 1$.

However, the other enhancement functions $\varphi(e_t)$, $\psi(e_t)$, $\zeta(e_t)$, and $\kappa(e_t)$ in Eq. (6.2) (very likely) do not admit any analytic closed-form expressions. We have explained in Sec. V B the details of the numerical calculation of these functions. We now present the numerical plots of the final functions $\psi(e_t)$, $\zeta(e_t)$, and $\kappa(e_t)$ in Figs. 5 and 6 as functions of the eccentricity e_t [recall that the function $\varphi(e_t)$ has already been given in Fig. 1].[16]

As seen from Eq. (6.2) the final result depends on the constant r_0 at the 3PN order. Let us understand in a bit more detail the occurrence of this constant. We first recall from Ref. [12] that the dependence on the constant r_0 of the *radiative* quadrupole moment at infinity, say U_{ij}, arises precisely at the 3PN order, and comes exclusively from the contribution of tails of tails (i.e. the cubic multipole interaction $M^2 \times I_{ij}$). It is explicitly given by

$$U_{ij}(t) = I_{ij}^{(2)}(t) + \cdots + \frac{214}{105} M^2 I_{ij}^{(4)}(t) \ln r_0 + \cdots, \quad (6.4)$$

in which we have indicated that U_{ij} simply reduces to the second time derivative of I_{ij} at the Newtonian order, and where we show the only term which depends on the constant r_0; such a term appears at 3PN order and turns out to be proportional to the fourth time derivative of I_{ij}. The dots in Eq. (6.4) denote many terms which do not depend on r_0. From (6.4) it is then trivial to deduce that the corresponding dependence on r_0 of the averaged energy flux at 3PN order must be

$$\begin{aligned}\langle \mathcal{F}^{(3\mathrm{PN})} \rangle &= \frac{1}{5} \langle U_{ij}^{(1)} U_{ij}^{(1)} \rangle + \cdots \\ &= \frac{1}{5} \langle I_{ij}^{(3)} I_{ij}^{(3)} \rangle + \cdots + \frac{428}{525} M^2 \langle I_{ij}^{(3)} I_{ij}^{(5)} \rangle \ln r_0 \\ &\quad + \cdots. \end{aligned} \quad (6.5)$$

Now we can take advantage of the fact that inside the operation of averaging over ℓ [denoted by $\langle\rangle$ and defined by (4.21)] one can freely operate by parts the time derivatives. Hence, we can write that $\langle I_{ij}^{(3)} I_{ij}^{(5)} \rangle = -\langle I_{ij}^{(4)} I_{ij}^{(4)} \rangle$, and so we arrive at the result

$$\begin{aligned}\langle \mathcal{F}^{(3\mathrm{PN})} \rangle &= \frac{1}{5} \langle I_{ij}^{(3)} I_{ij}^{(3)} \rangle + \cdots - \frac{428}{525} M^2 \langle I_{ij}^{(4)} I_{ij}^{(4)} \rangle \ln r_0 \\ &\quad + \cdots. \end{aligned} \quad (6.6)$$

The factor of $\ln r_0$ in Eq. (6.6) looks like a "quadrupole formula" but where the third time derivative of the moment would be replaced by the fourth one. Notice that the above expression has been computed for general radiative-type moments and is true for any PN source, in particular, for a binary system moving on an eccentric orbit. Therefore the dependence on $\ln r_0$ found in (6.6) should perfectly match with the one we have obtained in Eq. (6.2). Thus, comparing with (6.2), one readily infers that the function $F(e_t)$ in the case of an eccentric binary must necessarily be given by the components of the quadrupole moment in the time domain as

$$F(e_t) = \frac{M^2}{128 \nu^2 x^8} \langle I_{ij}^{(4)} I_{ij}^{(4)} \rangle. \quad (6.7)$$

This prediction is perfectly in agreement with our finding for the function $F(e_t)$ in Eq. (5.10) (indeed, since we are at

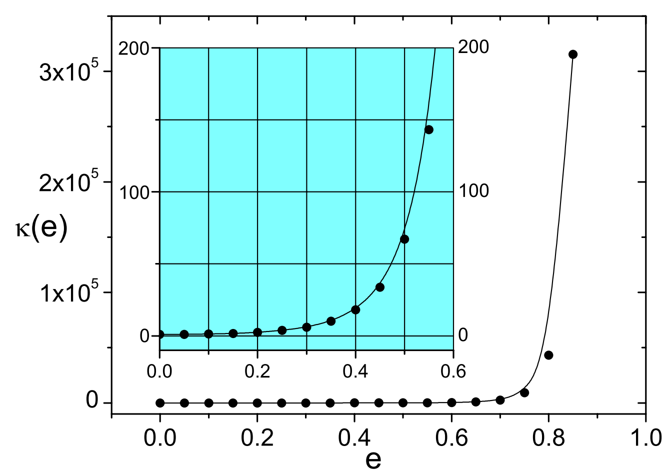

FIG. 6 (color online). Variation of $\kappa(e)$ with the eccentricity e. In the circular-orbit limit we have $\kappa(0) = 1$.

[16] The numerical results used for Figs. 1–6 are available in the form of tables on request from the authors.

leading order, M reduces to m, e_t agrees with e, ω equals n). We have therefore confirmed the correctness of the dependence upon r_0 of Eq. (6.2).

We already know from the study of the circular-orbit case (cf. [31]) that the dependence on r_0 is canceled out with a similar term contained in the expression of the source-type quadrupole moment I_{ij} at 3PN order. This cancellation must in fact be true for general sources, and has been proved on general grounds in Ref. [12]. It will therefore give an interesting check of our calculations when we show in the companion paper [29] that the cancellation of r_0 occurs for general eccentric orbits.

To finish, let us provide here the expressions of our final enhancement functions at the first order in e_t^2 when $e_t \to 0$. These expansions will be useful in the following paper [29], when we compare the perturbative limit of the complete energy flux at 3PN order (including all instantaneous terms) with the result of black-hole perturbations. Note that those expansions are obtained analytically. For the functions which are Newtonian, we can either use the Fourier coefficients in the Appendix and expand them at first order in e_t^2, or follow the general procedure explained in Sec. V B for the relevant moments but expanding Eq. (5.18) to only first order in e_t^2, namely,

$$u = \ell + e_t \sin\ell + \frac{e_t^2}{2}\sin 2\ell + \mathcal{O}(e_t^3). \tag{6.8}$$

Concerning the two 1PN functions [$\psi(e_t)$ and $\zeta(e_t)$], on the other hand, we obtain them directly using the latter procedure. We find

$$\varphi(e_t) = 1 + \frac{2335}{192} e_t^2 + \mathcal{O}(e_t^4), \tag{6.9a}$$

$$\psi(e_t) = 1 - \frac{22\,988}{8191} e_t^2 + \mathcal{O}(e_t^4), \tag{6.9b}$$

$$\zeta(e_t) = 1 + \frac{1\,011\,565}{48\,972} e_t^2 + \mathcal{O}(e_t^4), \tag{6.9c}$$

$$\kappa(e_t) = 1 + \left(\frac{62}{3} - \frac{4\,613\,840}{350\,283}\ln 2 + \frac{24\,570\,945}{1\,868\,176}\ln 3\right) \times e_t^2 + \mathcal{O}(e_t^4), \tag{6.9d}$$

and of course [since this is immediately deduced from Eq. (6.3)]

$$F(e_t) = 1 + \frac{62}{3} e_t^2 + \mathcal{O}(e_t^4). \tag{6.10}$$

We have checked that the numerical results of Figs. 1, 5, and 6 agree well with Eqs. (6.9) in the limit of small eccentricities.

B. Conclusion and future directions

The far-zone flux of energy contains hereditary contributions that depend on the entire past history of the source. Using the GW generation formalism consisting of a multipolar post-Minkowskian expansion with matching to a PN source, we have proposed and implemented a semianalytical method to compute the hereditary contributions from the inspiral phase of a binary system of compact objects moving on quasi-elliptical orbits up to 3PN order. The method explicitly uses the 1PN quasi-Keplerian representation of elliptical orbits and exploits the doubly periodic nature of the motion to average the fluxes over the binary's orbit. Together with the instantaneous contributions evaluated in the next paper [29], it provides crucial inputs for the construction of ready-to-use templates for binaries moving on eccentric orbits, an interesting class of sources for the ground-based gravitational-wave detectors LIGO/Virgo and especially space-based detectors like LISA.

The extension of these methods to compute the hereditary terms in the 3PN angular momentum flux and 2PN linear momentum flux is the next step required to proceed towards the above goal and is currently under investigation. The extension to compute the 3.5PN terms for elliptical orbits is currently not possible due to some as yet uncalculated terms in the generation formalism at this order for general orbits. It would also require the use of the 2PN generalized quasi-Keplerian representation for some of the leading multipole moments.

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APPENDIX: FOURIER COEFFICIENTS OF THE MULTIPOLE MOMENTS

In this appendix we provide the expressions of the Fourier coefficients of the *Newtonian* multipole moments in terms of combinations of Bessel functions. We decompose the components of the moments as Fourier series,

$$I_L^{(\mathrm{N})}(t) = \sum_{p=-\infty}^{+\infty} {}_{(p)}\mathcal{I}_L^{(\mathrm{N})} e^{\mathrm{i}p\ell}, \tag{A1a}$$

$$J_{L-1}^{(\mathrm{N})}(t) = \sum_{p=-\infty}^{+\infty} {}_{(p)}\mathcal{J}_{L-1}^{(\mathrm{N})} e^{\mathrm{i}p\ell}, \tag{A1b}$$

where the Fourier coefficients can be obtained by evaluating the following integrals:

$${}_{(p)}\mathcal{I}_L^{(\mathrm{N})} = \frac{1}{2\pi}\int_0^{2\pi} d\ell I_L^{(\mathrm{N})}(t) e^{-\mathrm{i}p\ell}, \tag{A2a}$$

$${}_{(p)}\mathcal{J}_{L-1}^{(\mathrm{N})} = \frac{1}{2\pi}\int_0^{2\pi} d\ell J_{L-1}^{(\mathrm{N})}(t) e^{-\mathrm{i}p\ell}. \tag{A2b}$$

For the mass quadrupole moment at Newtonian order we have[17]

$$
{}_{(p)}I^{(\mathrm{N})}_{xx} = \left(\frac{1}{6}+\frac{3}{2}e_t^2\right)J_p(pe_t) + \left(-\frac{7}{8}e_t-\frac{3}{8}e_t^3\right)(J_{p-1}(pe_t)+J_{p+1}(pe_t)) + \left(\frac{1}{4}+\frac{1}{4}e_t^2\right)(J_{p-2}(pe_t)+J_{p+2}(pe_t)) + \left(-\frac{1}{8}e_t+\frac{1}{24}e_t^3\right)(J_{p-3}(pe_t)+J_{p+3}(pe_t)), \tag{A3a}
$$

$$
{}_{(p)}I^{(\mathrm{N})}_{xy} = -\mathrm{i}\sqrt{1-e_t^2}\left\{\frac{5}{8}e_t(-J_{p-1}(pe_t)+J_{p+1}(pe_t)) + \left(-\frac{1}{4}-\frac{1}{4}e_t^2\right)(J_{p+2}(pe_t)-J_{p-2}(pe_t)) + \frac{1}{8}e_t(J_{p+3}(pe_t)-J_{p-3}(pe_t))\right\}, \tag{A3b}
$$

$$
{}_{(p)}I^{(\mathrm{N})}_{yy} = \left(\frac{1}{6}-e_t^2\right)J_p(pe_t) + \left(\frac{3}{8}e_t+\frac{1}{4}e_t^3\right)(J_{p-1}(pe_t)+J_{p+1}(pe_t)) - \frac{1}{4}(J_{p-2}(pe_t)+J_{p+2}(pe_t)) + \left(\frac{1}{8}e_t-\frac{1}{12}e_t^3\right)(J_{p-3}(pe_t)+J_{p+3}(pe_t)), \tag{A3c}
$$

$$
{}_{(p)}I^{(\mathrm{N})}_{zz} = \left(-\frac{1}{3}-\frac{1}{2}e_t^2\right)J_p(pe_t) + \left(\frac{1}{2}e_t+\frac{1}{8}e_t^3\right)(J_{p-1}(pe_t)+J_{p+1}(pe_t)) - \frac{1}{4}e_t^2(J_{p-2}(pe_t)+J_{p+2}(pe_t)) + \frac{1}{24}e_t^3(J_{p-3}(pe_t)+J_{p+3}(pe_t)). \tag{A3d}
$$

For the mass octupole moment we find

$$
{}_{(p)}I^{(\mathrm{N})}_{xxx} = -\left\{\left(\frac{3}{8}e_t+\frac{11}{8}e_t^3\right)J_p(pe_t) + \left(-\frac{3}{40}-\frac{21}{20}e_t^2-\frac{11}{40}e_t^4\right)(J_{p-1}(pe_t)+J_{p+1}(pe_t)) + \left(\frac{11}{20}e_t+\frac{3}{20}e_t^3\right)(J_{p-2}(pe_t)+J_{p+2}(pe_t)) + \left(-\frac{1}{8}-\frac{3}{20}e_t^2+\frac{3}{40}e_t^4\right)(J_{p-3}(pe_t)+J_{p+3}(pe_t)) + \left(\frac{1}{16}e_t-\frac{3}{80}e_t^3\right)(J_{p-4}(pe_t)+J_{p+4}(pe_t))\right\}, \tag{A4a}
$$

$$
{}_{(p)}I^{(\mathrm{N})}_{xxy} = \mathrm{i}\sqrt{1-e_t^2}\left\{\left(\frac{1}{40}+\frac{27}{40}e_t^2\right)(J_{p+1}(pe_t)-J_{p-1}(pe_t)) + \left(-\frac{19}{40}e_t-\frac{9}{40}e_t^3\right)(J_{p+2}(pe_t)-J_{p-2}(pe_t)) + \left(\frac{1}{8}+\frac{7}{40}e_t^2\right)(J_{p+3}(pe_t)-J_{p-3}(pe_t)) + \left(-\frac{1}{16}e_t+\frac{1}{80}e_t^3\right)(J_{p+4}(pe_t)-J_{p-4}(pe_t))\right\}, \tag{A4b}
$$

$$
{}_{(p)}I^{(\mathrm{N})}_{xyy} = -\left\{\left(\frac{1}{8}e_t-e_t^3\right)J_p(pe_t) + \left(-\frac{1}{40}+\frac{21}{40}e_t^2+\frac{1}{5}e_t^4\right)(J_{p-1}(pe_t)+J_{p+1}(pe_t)) + \left(-\frac{2}{5}e_t+\frac{1}{20}e_t^3\right)(J_{p-2}(pe_t) + J_{p+2}(pe_t)) + \left(\frac{1}{8}+\frac{3}{40}e_t^2-\frac{1}{10}e_t^4\right)(J_{p-3}(pe_t)+J_{p+3}(pe_t)) + \left(-\frac{1}{16}e_t+\frac{1}{20}e_t^3\right)(J_{p-4}(pe_t)+J_{p+4}(pe_t))\right\}, \tag{A4c}
$$

$$
{}_{(p)}I^{(\mathrm{N})}_{yyy} = \mathrm{i}\sqrt{1-e_t^2}\left\{\left(\frac{3}{40}-\frac{3}{5}e_t^2\right)(-J_{p-1}(pe_t)+J_{p+1}(pe_t)) + \left(\frac{13}{40}e_t+\frac{1}{5}e_t^3\right)(-J_{p-2}(pe_t)+J_{p+2}(pe_t)) + \left(-\frac{1}{8}-\frac{1}{10}e_t^2\right)(-J_{p-3}(pe_t)+J_{p+3}(pe_t)) + \left(\frac{1}{16}e_t-\frac{1}{40}e_t^3\right)(-J_{p-4}(pe_t)+J_{p+4}(pe_t))\right\}, \tag{A4d}
$$

$$
{}_{(p)}I^{(\mathrm{N})}_{zzx} = -\left\{\left(-\frac{1}{2}e_t-\frac{3}{8}e_t^3\right)J_p(pe_t) + \left(\frac{1}{10}+\frac{21}{40}e_t^2+\frac{3}{40}e_t^4\right)(J_{p-1}(pe_t)+J_{p+1}(pe_t)) + \left(-\frac{3}{20}e_t-\frac{1}{5}e_t^3\right)(J_{p-2}(pe_t) + J_{p+2}(pe_t)) + \left(\frac{3}{40}e_t^2+\frac{1}{40}e_t^4\right)(J_{p-3}(pe_t)+J_{p+3}(pe_t)) - \frac{1}{80}e_t^3(J_{p-4}(pe_t)+J_{p+4}(pe_t))\right\}, \tag{A4e}
$$

$$
{}_{(p)}I^{(\mathrm{N})}_{zzy} = \mathrm{i}\sqrt{1-e_t^2}\left\{\left(-\frac{1}{10}-\frac{3}{40}e_t^2\right)(-J_{p-1}(pe_t)+J_{p+1}(pe_t)) + \left(\frac{3}{20}e_t+\frac{1}{40}e_t^3\right)(-J_{p-2}(pe_t)+J_{p+2}(pe_t)) - \frac{3}{40}e_t^2(-J_{p-3}(pe_t)+J_{p+3}(pe_t)) + \frac{1}{80}e_t^3(-J_{p-4}(pe_t)+J_{p+4}(pe_t))\right\}. \tag{A4f}
$$

Finally, for the current quadrupole moment,

[17]Note that the Fourier coefficients we provide are for *normalized* multipole moments as defined in Eqs. (5.1a) and (5.1b).

$$ {}_{(p)}\mathcal{J}_{xz}^{(\mathrm{N})} = -\frac{1}{4}\sqrt{1-e_t^2}\bigg\{3e_tJ_p(pe_t) - \frac{1}{4}(1+e_t^2)(J_{p-1}(pe_t)+J_{p+1}(pe_t)) + \frac{1}{8}e_t(J_{p-2}(pe_t)+J_{p+2}(pe_t))\bigg\}, \tag{A5a} $$

$$ {}_{(p)}\mathcal{J}_{yz}^{(\mathrm{N})} = \frac{\mathrm{i}}{4}(1-e_t^2)\bigg\{(J_{p+1}(pe_t) - J_{p-1}(pe_t)) - \frac{1}{2}e_t(J_{p+2}(pe_t) - J_{p-2}(pe_t))\bigg\}. \tag{A5b} $$

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