Topological phase in two flavor neutrino oscillations

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We show that the phase appearing in neutrino flavor oscillation formulae has a geometric and topological contribution. We identify a topological phase appearing in the two flavor neutrino oscillation formula using Pancharatnam's prescription of quantum collapses between nonorthogonal states. Such quantum collapses appear naturally in the expression for appearance and survival probabilities of neutrinos. Our analysis applies to neutrinos propagating in vacuum or through matter. For the minimal case of two flavors with *CP* conservation, our study shows for the first time that there is a geometric interpretation of the neutrino oscillation formulae for the detection probability of neutrino species.

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I. INTRODUCTION

The phenomenon of neutrino flavor oscillation results from the phase difference acquired by the mass eigenstates due to their time evolution while propagating in vacuum or in matter. The observation of neutrino flavor oscillations in solar, atmospheric, reactor, and accelerator experiments reveal the remarkable fact that the neutrinos exhibit sustained quantum coherence even over astrophysical length scales [1,2]. It is then natural to ask what we can learn about neutrinos from these coherent phases. Here, we address the issue of geometric and topological phases involved in the physics of neutrino oscillations.

On the theoretical front, it is well known that the phenomenon of neutrino oscillations cannot be accommodated within the standard model (SM) of particle physics. Therefore, the experimental observation of neutrino oscillations provides a concrete evidence for the requirement of physics beyond the SM and neutrinos have been an intensive area of research in the past several years.

The study of geometric phases in the context of neutrino oscillations has been carried out in the past by several authors [3–18], but none of the papers seem to provide a unified perspective on the problem taking into account the different *avatars* of geometric phase. It is worthwhile to stress here that one needs to be cautious while interpreting claims in the literature as they crucially depend on which version of the geometric phase one is dealing with. We will first summarize the related literature and then focus on the specific question that we address in this paper. We mostly restrict our attention to the case of two neutrino flavors and the *CP* (*CP* stands for charge-conjugation and parity) conserving situation, which is the minimal scenario for studying the physics of oscillations. We find, in contrast to earlier studies of this problem that the geometric phase appears even in this minimal context.

Let us first review the papers that are connected to Berry's [19] cyclic adiabatic phase. Berry studied phases that appear when the Hamiltonian of a quantum system depends on parameters that are varied slowly and cyclically. Nakagawa [3] followed this work by an elegant paper in which he pointed out that the geometric phase could also arise in systems where adiabatic theorem did not hold. The key point made by Nakagawa was that while for existence of geometric phases, adiabatic condition was not necessary (this was also independently pointed out by Aharonov and Anandan [20]), the adiabatic theorem itself could be most easily understood in terms of geometric arguments. As an application of his general formalism, Nakagawa considered two flavor neutrino oscillations in matter. He concluded that the Berry phase played no role in this situation. The topological phase in the two flavor neutrino case, which is the central result of the present paper, was missed in his work because he restricted himself to a limited region in the parameter (ray) space and did not consider generalizations of the geometric phase that allow for quantum collapse.

Subsequent work on Berry's geometric phase and neutrinos exploited the spin degree of freedom of neutrinos and its interaction with the transverse magnetic field leading to geometric effects and spin flip. Since at that time, spin precession was a plausible solution to the solar-neutrino problem, there is a body of work by several authors on the subject of geometric phase effects in this context, both in the absence and presence of matter and mass-splitting terms [4–11]. However, in the present scenario, spin flavor precession is disfavored as the leading solution to the solar-neutrino problem at 99.86% C.L. [21], which makes it phenomenologically uninteresting. Also, we would like to mention that in the present study, spin plays only a passive role, and we shall not discuss this particular aspect any further.

Naumov [12–15] studied geometric phases for two and three flavor neutrino oscillations taking into account the optic potentials [22] induced by coherent forward scattering of neutrinos against the background matter via SM interactions. The slowly changing parameters in the Hamiltonian were identified as a set of optic potentials

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$q(t)$, which were connected to the refractive indices of neutrinos in a medium. For the naturally existing cyclic cases like spherically symmetric or sandwich-like density profiles, he found that the geometric (or topological) phase was zero for both two and three flavors due to only one of the optic potentials appearing in an *essential* manner in the Hamiltonian. Note that the two terms ''topological'' and ''geometric'' were used interchangeably in Naumov's works. Here, we will make a distinction between the two terms. The topological phase refers to phase factors that are insensitive to small changes in the circuit, while geometric phases are sensitive to such changes.

In a more recent paper, He *et al.* [16] carried out a detailed study of the Berry phase in neutrino oscillations for both two and three flavors, active and sterile mixing, and with inclusion of nonstandard interactions. For the particular case of two flavor oscillations in matter, they claimed that the Berry phase can only appear if nonstandard (R-parity violating supersymmetry) neutrino-matter interactions are taken into account.

All the above papers [3,12–16] claim that the geometric phases do not appear in the oscillation probabilities for the case of two flavor neutrinos with *CP* conservation in vacuum or in matter as long as neutrino-matter interactions are standard, i.e. coherent forward scattering is induced by charged current interaction of electron neutrino (ν_e) with electrons in matter. The above claims can be understood as the necessity of having at least two *essential* parameters in the Hamiltonian to detect curvature. Because of the absence of flavor changing neutral currents in the SM, it turns out that for the case of ordinary electrically neutral matter, even though one has two varying parameters—electron number density (n_e) and neutron number density (n_n), only one of these will appear in an essential way in the Hamiltonian and hence the Berry's geometric phase is expected to be zero. The other parameter n_n just adds a global phase to the time-evolved neutrino flavor state and hence does not affect oscillation. But also it is worth stressing that if both the conditions of having a nontrivial multidimensional parameter space as well as cyclic evolution of the states in parameter space were satisfied, the net geometric phase (resulting from the difference between the geometric phases picked up by the individual mass eigenstates) would have appeared in the formulae for detection probability and hence been observable.

Next, we will briefly review and summarize papers dealing with geometric phases that are generalizations of the Berry phase [20,23,24] in the context of neutrinos [17,18]. Such geometric effects can appear under less restrictive conditions than those required for Berry's version of the geometric phase. In fact such phases can appear even in situations where there are no parameters varying in the Hamiltonian and the evolution is not necessarily cyclic or unitary. Note, however, that in general the geometric phases appearing in transition amplitudes are global phases that do not have any observable consequences. To observe such a phase one needs a split-beam interference experiment in which a beam is spatially separated into two parts that suffer different histories. Such an experiment is hard to design for neutrinos because they interact so weakly and are nearly impossible to deflect or confine. This renders such phases uninteresting as they are not observable as far as neutrinos are concerned. Our aim here is to explore whether there are geometric effects that survive at the level of detection probabilities that are directly measurable quantities.

Blasone *et al.* [17] claimed that Berry's phase was present in the physics of neutrino oscillation in vacuum even for the two flavor *CP* conserving case. Their argument is based on the fact that under Schrödinger evolution, the pure flavor states come back to themselves after one period (T) of oscillation having acquired an overall phase. This overall phase was shown to be a sum of a pure dynamical phase and a part that depended on the mixing angle only and independent of energy and masses of the two mass states (hence, geometric). They called this extra phase the Berry phase. Note that this phase picked up by a neutrino flavor state arises purely due to Schrödinger evolution of the system giving a closed loop in the Hilbert space but not due to any slowly varying parameters leading to adiabatic evolution of the Hamiltonian itself. Hence, strictly speaking it is the Aharonov-Anandan cyclic phase [20] that generalized Berry's adiabatic phase to situations where the adiabaticity constraint did not apply and only the cyclic condition is met. Also, we should note that since the phase obtained was a global phase at the amplitude level, it does not appear in measurable quantities like neutrino appearance or survival probabilities as mentioned above.

After Berry's [19] seminal paper on this subject, Ramaseshan and Nityananda [25] pointed out that Berry's phase had a connection with the phase obtained by Pancharatnam [23] in the fifties in his study on interference of polarized light. These insights were carried over to the ray space of quantum mechanics by Samuel and Bhandari [24]. They showed that the two seemingly different geometric phases obtained by Berry and Pancharatnam (appearing under different sets of conditions) could be described in a unified framework. They also pointed out that geometric phases are not restricted to unitary, cyclic, and adiabatic evolution [19] of a quantum system and can appear in an even more general context that allows for quantum collapses, which occur during measurements. Following this line of thought, Wang *et al.* [18] extended the study of Blasone *et al.* [17] to obtain noncyclic geometric phases for two and three flavor neutrinos in vacuum. Their claim can be understood as follows. Consider the Schrödinger evolution of a quantum state over an arbitrary time period from $\tau = 0$ to τ. Now this open loop (noncyclic) Schrödinger evolution of a quantum state over a time τ can be closed by a collapse of the time-evolved quantum state at τ onto the original state at $\tau = 0$ by the shorter geodesic curve joining the two states in the ray space [24]. The phase associated with the complex number ($re^{i\beta}$) representing the inner product of the original state vector and the time-evolved state vector (with the dynamical phase removed) has a pure geometric origin. This noncyclic geometric phase was evaluated by Wang *et al.* [18] for both the two and three flavor cases. But, again note that this phase will be unobservable as it only appears at the level of amplitude.

The main purpose of the present work is to establish that Pancharatnam's phase does appear in detection probabilities and hence is directly observable. For the simplest case of two flavors in vacuum or in constant density matter (restricting to SM interactions) with *CP* conservation, we obtain a Pancharatnam phase of π, and this leads to an elegant geometric interpretation of the neutrino oscillation formulae. We also make a direct connection of this phase with the Herzberg and Longuet-Higgins topological phase [26] in molecular physics. We show that the Pancharatnam phase of π remains even in the presence of slowly varying matter density, and this can be ascribed to the topological nature of this phase. Inclusion of *CP* violation can change the topological nature of the phase and make it a path-dependent geometric phase.

Although one should do a full three flavor analysis for a complete treatment, we work in an effective two flavor approximation that is fairly justified [27,28] due to the smallness of Θ_{13} and hierarchy of mass splittings ($|\delta m^2_{21}/\delta m^2_{32}| << 1$) and in addition on matter interactions being standard.[1] In many physical situations, observations depend on mainly one mixing and one mass squared splitting. Conventionally, Θ_{12} and δm^2_{21} describe oscillations of solar neutrinos, while Θ_{23} and δm^2_{32} are used to describe atmospheric neutrinos. The mixing angle Θ_{13} gives small effects on both solar and atmospheric neutrinos. Working with only two flavors is of course advantageous as the results obtained are physically more transparent and can be visualized in analogous situations in optics and the Poincaré sphere can be used as a calculational tool to study the system.

For the ease of visualization of the phenomena of oscillations, in the past several authors have discussed simple pictorial depiction of neutrino oscillations in terms of precession of a (pseudo) spin vector in three-dimensional space in a variety of contexts for the case of two neutrino flavors [30–37]. Below we give a brief account of the papers dealing with geometric representation of neutrino flavor oscillations. Harris and Stodolsky [30] addressed the question of a unified treatment of generic two-state systems (including particle mixing involving two neutrino types) in media using density matrices. It was shown that the equation of motion for the polarization vector represented the precession of polarization vector about a vector representing an effective magnetic field (which could result from the mass terms in vacuum or matter terms). Kim *et al.* [31] discussed the analogy of solar-neutrino oscillations with that of precession of electron spin in a time-dependent magnetic field. They applied this picture in the limit of adiabatic approximation. Stodolsky [32] described the evolution of a statistical ensemble (neutrinos from supernovae or in the early Universe) applying the density matrix approach [30] and showed that oscillations in presence of mixing and matter interactions in a thermal environment could be viewed in terms of precession. Kim *et al.* [33] derived the geometric picture for two and three flavor neutrinos and applied it to nonadiabatic as well as adiabatic cases. Thomson and McKellar [34] treated the case of neutrino background giving rise to nonlinear feedback terms in the equation of motion for polarization vectors and gave a pictorial representation for the same. Enqvist *et al.* [35] describe visualization of oscillations of a thermal neutrino ensemble of the early Universe. The geometrical representation in wave packet treatment of oscillations was discussed by Giunti *et al.* [36]. As in optics, the Poincaré sphere is a convenient tool for visualizations and calculations pertaining to neutrino oscillations, particularly in looking for geometric effects.

This paper is organized as follows: In Sec. II, we develop an analogy between the neutrino flavor states and polarized states in optics since such a mapping allows for a convenient visualization of geometric effects. We then go on to show in Sec. III that the Pancharatnam phase does appear in the detection probabilities of neutrino species in the two flavor neutrino system in vacuum and also in matter. We conclude with a discussion of our key result and future directions in Sec. IV. Throughout we set $\hbar = c = 1$.

II. CORRESPONDENCE BETWEEN TWO FLAVOR NEUTRINOS AND POLARIZATION STATES IN OPTICS

Since the concept of Pancharatnam's phase was developed in the context of optics, it is worthwhile to first develop a correspondence between the mathematics of two flavor neutrino states and polarization states in optics. Let us first recall the conditions under which the two flavor neutrinos and polarization states in optics can be analyzed within an unified framework.

A. Two flavor neutrinos

In the ultrarelativistic limit, the Dirac equation for two flavor neutrinos (antineutrinos) can be reduced to a Schrödinger form [22,38] written in terms of a two-component vector of positive (negative) energy probability amplitude. This is analogous to Maxwell's equations re-

[1] It turns out that in the presence of nonstandard interactions during propagation, it is possible to do the analysis with only two flavors for the case of solar neutrinos, while a complete three flavor analysis is needed for the case of the atmospheric neutrinos [29].

ducing to the linear Schrödinger form for the polarization states in optics in the paraxial limit [39].

The two neutrino flavor states can be mapped to a two-level quantum system with distinct energy eigenvalues, $E_i \simeq p + m_i^2/2p$ in the ultrarelativistic limit along with the assumption of equal fixed momenta (or energy) [37,40]. In the presence of matter, the relativistic dispersion relation $E_i = f(p, m_i)$ gets modified due to the neutrino-matter interactions (in an electrically neutral homogeneous medium) leading to

$$E_{i=\mp} = \left(p + \frac{m_1^2 + m_2^2}{4p} + \frac{V_C}{2} + V_N\right) \mp \frac{1}{2} \times \sqrt{(\omega \sin 2\Theta)^2 + (V_C - \omega \cos 2\Theta)^2}, \quad (1)$$

where $\omega = \delta m^2/2p$ with mass splitting $\delta m^2 = m_2^2 - m_1^2$ and $p \simeq E$ being the fixed momentum (energy) of the neutrino. Θ is the mixing angle in vacuum. $V_C = \sqrt{2} G_F n_e = 7.6 \times 10^{-14} Y_e \rho$ eV and $V_N = -\sqrt{2} G_F n_n/2 = -3.8 \times 10^{-14} Y_n \rho$ eV are the respective effective potentials due to coherent forward scattering of neutrinos with electrons (via charged current interactions) and neutrons (via neutral current interactions). $G_F = 1.16637 \times 10^{-5}$ GeV^{-2} parameterizes the weak interaction strength (Fermi constant). V_C and V_N depend on the electron (n_e) and neutron (n_n) number densities (in units of cm^{-3}). $n_{e/n} = \rho Y_{e/n} N_{\rm Avo}$, where ρ is the mass density in g cm^{-3}, $Y_{e/n}$ is the relative electron (neutron) number density, and its value is roughly $\sim$0.5 for Earth matter, and $N_{\rm Avo}$ is the Avogadro's number. Setting $V_C = V_N = 0$, we recover the vacuum case.

Note the fact that although there are two densities n_e and n_n appearing in the eigenvalues, it is only n_e that appears in a nontrivial way (through V_C) in the flavor Hamiltonian

$$\mathbb{H}_\nu = \left(p + \frac{m_1^2 + m_2^2}{4p} + \frac{V_C}{2} + V_N\right)\mathbb{I} + \frac{1}{2} \times \begin{pmatrix} V_C - \omega \cos 2\Theta & \omega \sin 2\Theta \\ \omega \sin 2\Theta & -(V_C - \omega \cos 2\Theta) \end{pmatrix}. \quad (2)$$

The above Hamiltonian [Eq. (2)] also describes an inhomogeneous medium provided the scale of variation of matter induced potential V_C is slow compared to the scale of the order of $\hbar/(E_+ - E_-)$), hence ensuring no transitions between the mass eigenstates. This defines the adiabaticity condition [37,40]. As neutrinos traverse a density gradient, at a particular value of n_e the diagonal elements of $\mathbb{H}_\nu$ can vanish causing an interchange of flavors irrespective of the value of the vacuum mixing angle Θ. This phenomenon of resonant conversion in matter is known as the Mikheyev-Smirnov-Wolfenstein (MSW) effect [41,42].

The off-diagonal form of the Hamiltonian in flavor basis (both in vacuum and matter) leads to flavor oscillations of neutrinos, which is the only mechanism that mixes the neutrinos of different generations or flavors while preserving the lepton number (note that the absence of flavor changing neutral currents prevents any flavor change within the SM). Also note that the matter term appears in diagonal elements only so in the absence of vacuum mixing, neutrinos of different flavors cannot mix. The term proportional to the identity gives an overall phase to each of the mass eigenstates and hence does not affect oscillations. This corresponds to the gauge freedom of any state of a two-level quantum system [3].

In the next subsection, we describe the polarized states in optics in the language of quantum mechanics.

B. Polarized states in optics

Polarization optics is mathematically identical to the evolution of a two-state quantum system. In a helicity basis for polarized light, we can write $|\Re\rangle$ and $|\mathfrak{L}\rangle$ representing right and left circular polarizations. A general polarized light beam $|\Psi\rangle$ can then be expanded in this basis as $|\Psi\rangle = \alpha|\Re\rangle + \beta|\mathfrak{L}\rangle$, where $|\alpha|^2 + |\beta|^2 = N$, the intensity of the beam of polarized light. We can parameterize an arbitrary state of polarized light by

$$|\Psi\rangle = \sqrt{N} \exp\{i\eta\} \begin{pmatrix} \cos(\theta/2)\exp(-i\phi/2) \\ \sin(\theta/2)\exp(i\phi/2) \end{pmatrix}, \quad (3)$$

where N is the total intensity, which is normalized to unity, and the angles θ and ϕ (where $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$) describe the state of polarization of the beam, represented on the two-dimensional unit sphere ($\mathbb{S}^2$) called the Poincaré sphere. Orthogonal polarization states are antipodal points of the sphere. η is the overall phase of the beam. The states on the sphere are defined modulo this overall phase of η and represent the ray space [43]. The north pole ($\theta = 0$) represents right circular light and the south pole ($\theta = \pi$) represents left circular light. States on the equator ($\theta = \pi/2$) represent linear polarizations. Any other point on the surface of the sphere represents elliptic polarization. The Poincaré sphere is a useful device to visualize the changes in the state of polarization of a light beam traversing through a medium.

The mapping between the polarized states and a two-level quantum system originates from the following fact. Neglecting absorption effects,[2] the effect of different media can be encoded in terms of 2×2 Hermitian matrix (Hamiltonian). The time evolution of optical states in a medium is governed by a Schrödinger-like equation with the medium represented by the most general form of Hamiltonian for a two-level system given by

$$\mathbb{H} = \mathcal{A}\sigma_x + \mathcal{B}\sigma_y + \mathcal{C}\sigma_z + \mathcal{D}\mathbb{I}, \quad (4)$$

where, the coefficients of the three traceless Pauli matrices,

[2]The incoherent scattering cross section for neutrinos (10^{-44} cm^2 for 1 MeV neutrinos impinging on target of mass 1 MeV) is extremely small as compared to photons in a medium.

$\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are responsible for generating rotations of incident optical states about x, y, z axes on the Poincaré sphere. $\mathcal{D}$ just adds an overall phase that can be absorbed in a redefinition of the state. Hence, given an arbitrary medium, it can be represented by a Hamiltonian as mentioned above, and the eigenstates of the Hamiltonian represent those optical states that do not suffer any change (when incident on such a medium) in their state of polarization except for picking up an overall phase shift. The polarization of any other state (other than the eigenstates) incident on this medium will undergo a periodic change. On the Poincaré sphere this can be visualized as a rotation of the incident state vector about the axis defined by a line joining the two eigenstates of the Hamiltonian. Mathematically, these unitary rotations on the Poincaré sphere are generated by $e^{-i\mathbb{H}t}$. This is identical to unitary time evolution generated by the Hamiltonian of the quantum states in the Hilbert space. The quantum-mechanical analogue of the Poincaré sphere is the Blöch sphere, which geometrically represents the space of pure states of a two-level quantum system.

Nonvanishing values of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ simultaneously parameterize the effect of an elliptically birefringent medium. Circular (linear) birefringence are special cases where the conditions $\mathcal{A}$, $\mathcal{B} = 0$ and $\mathcal{C}$, $\mathcal{D} \neq 0$ ($\mathcal{B}$, $\mathcal{C} = 0$ and $\mathcal{A}$, $\mathcal{D} \neq 0$) are satisfied.

C. Neutrinos and optics analogy

We can now describe the isomorphism between neutrino states and polarized states in optics. The complete set of states for two flavor neutrino system can be represented on the Poincaré sphere just like the optical states as depicted in Fig. 1. For convenience we define a new coordinate ϑ, which goes from $0 \rightarrow 2\pi$ as we traverse the unit great circle in the $x - z$ plane. In terms of the old coordinates, the points θ, $\phi = 0$ are now labeled by $\vartheta = \theta$, and the points θ, $\phi = \pi$ are labeled by $\vartheta = 2\pi - \theta$. If we assume that the flavor states are the north and south poles of the Poincaré sphere, then the mass eigenstates are represented by the two antipodal points lying on an axis making an angle $2\Theta = \vartheta$ with respect to the polar axis. States on the equator coincide with the mass eigenstates for the special case of maximal mixing ($\Theta = \vartheta/2 = \pi/4$), which corresponds to complete flavor conversion (MSW effect). Geometrically, the MSW effect can be viewed as rotation about an equatorial axis, rotating the north pole into the south pole.

Ignoring the term proportional to the Identity, the neutrino Hamiltonian [Eq. (2)] both in vacuum or matter can be recast in exactly the same form given by [see Eq. (4)]

$$\mathbb{H}_\nu = \frac{\omega}{2}[(\sin\vartheta)\sigma_x - (\cos\vartheta)\sigma_z], \quad (5)$$

where $\omega = \delta m^2/2p$ and the mixing angle Θ is replaced

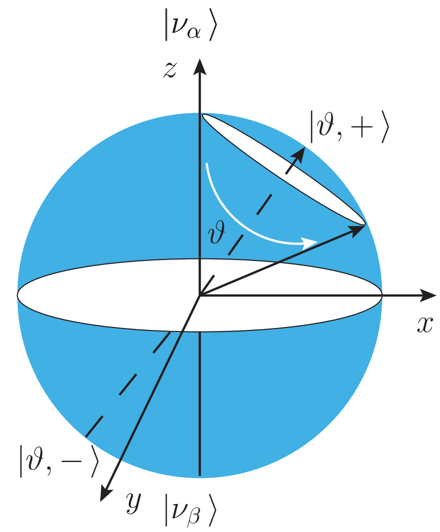

FIG. 1 (color online). Neutrino states on the Poincaré sphere. The flavor states $|\nu_\alpha\rangle$ and $|\nu_\beta\rangle$ are the two antipodal points on the z axis, while $|\vartheta, \pm\rangle$ correspond to the mass (energy) eigenstates lying on an axis making an angle ϑ with respect to the z axis.

by $\vartheta/2$.[3] Comparing the two Hamiltonians [Eqs. (4) and (5)] we see that the neutrino Hamiltonian represents a medium with elliptic birefringence. And neutrino oscillations can be viewed as the neutrino flavor state precessing [37] about the line joining the mass eigenstates (analogous to elliptic axis) induced by the time-evolution operator $e^{-i\mathbb{H}_\nu t}$ on the Poincaré sphere. In the language of neutrino optics, both vacuum and matter exhibit elliptic birefringence property with different elliptic axes.

The absence of flavor changing neutral currents in the SM gives rise to a real form of the Hamiltonian ($\mathcal{B} = 0$), and it corresponds to a CP-conserving situation. The eigenvectors (also called mass eigenstates) of Eq. (5) are given by

$$|\vartheta, +\rangle = \begin{pmatrix} \cos(\vartheta/2) \\ \sin(\vartheta/2) \end{pmatrix} \quad \text{and} \quad |\vartheta, -\rangle = \begin{pmatrix} -\sin(\vartheta/2) \\ \cos(\vartheta/2) \end{pmatrix}. \quad (6)$$

Note that states $|\vartheta, +\rangle$ and $|\vartheta, -\rangle$ are orthogonal antipodal points on the Poincaré sphere, which always lie on the great circle formed by the intersection of the $x - z$ plane with the Poincaré sphere. Mass eigenstates lying outside the $x - z$ plane imply CP violation. This fact has very interesting consequences for the physics of geometric phases in CP nonconserving situations [44].

[3]In defining the Poincaré sphere, it is useful to work with half angles $\vartheta/2$ as it allows for a mapping of the entire set of states on to a two-dimensional sphere $\mathbb{S}_2$ as ϑ changes from 0 to 4π.

III. PANCHARATNAM'S PHASE IN THE TWO FLAVOR NEUTRINO SYSTEM

The Pancharatnam phase:- We give a brief introduction to the idea of Pancharatnam's phase in quantum-mechanical language along the lines of Refs. [24,43,45]. Given any two nonorthogonal states $|\mathfrak{A}\rangle$ and $|\mathfrak{B}\rangle$ in the Hilbert space describing a system, a notion of geometric parallelism between the two states can be drawn from the inner product $\langle\mathfrak{A}|\mathfrak{B}\rangle$. The two states are said to be parallel (in phase) if $\langle\mathfrak{A}|\mathfrak{B}\rangle$ is real and positive, which defines *the Pancharatnam connection (or rule)*. Geometrically, it implies that the norm of the vector sum of the two states $||(|\mathfrak{A}\rangle+|\mathfrak{B}\rangle)||^2=\langle\mathfrak{A}|\mathfrak{A}\rangle+\langle\mathfrak{B}|\mathfrak{B}\rangle+2|\langle\mathfrak{A}|\mathfrak{B}\rangle|\cos(\mathrm{ph}\langle\mathfrak{A}|\mathfrak{B}\rangle)$ is maximum. Physically, it implies that if we let the two states interfere with each other the resulting state will have maximum probability (intensity). Note that if $|\mathfrak{A}\rangle$ is in phase with $|\mathfrak{B}\rangle$, and $|\mathfrak{B}\rangle$ is in phase with $|\mathfrak{C}\rangle$, then $|\mathfrak{C}\rangle$ is not necessarily in phase with state $|\mathfrak{A}\rangle$. The phase difference between the states $|\mathfrak{C}\rangle$ and $|\mathfrak{A}\rangle$ is *the Pancharatnam phase*, and it is equal to half the solid angle Ω subtended by the geodesic triangle $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ on the Poincaré sphere for a two-level system at its center. In general, for an n-level system, the space of states is given by $\mathcal{CP}^{n-1}$ ($\mathcal{CP}$ stands for complex projective), which reduces to the Poincaré sphere ($\mathbb{S}^2$) for a two-level system ($n = 2$). Nonintegrability of Pancharatnam's connection follows from the nontransitivity of the rule.

Pancharatnam's phase reflects the curvature of projective Hilbert space (ray space) and is independent of any parameterization or slow variation. Thus, it can also appear in situations where the Hamiltonian is constant in time. All one needs is that the state has a nontrivial trajectory on the Poincaré sphere. This condition is met naturally for neutrinos since they are produced and detected as flavor states (which are not the stationary mass eigenstates) and hence they automatically explore the curvature of the ray space (Poincaré sphere) under the Schrödinger time evolution. Furthermore, note the fact that Schrödinger evolution (possibly) interrupted by measurements can lead to Pancharatnam's phase. If we take any state and subject it to multiple quantum collapses (such that consecutive collapses are between nonorthogonal states) and bring it back to itself, then the resulting state is given by $|\mathfrak{A}\rangle\langle\mathfrak{A}|\mathfrak{C}\rangle\times\langle\mathfrak{C}|\mathfrak{B}\rangle\langle\mathfrak{B}|\mathfrak{A}\rangle$, where the phase of the complex number $\langle\mathfrak{A}|\mathfrak{C}\rangle\langle\mathfrak{C}|\mathfrak{B}\rangle\langle\mathfrak{B}|\mathfrak{A}\rangle$ is given by $\Omega/2$.

The Herzberg and Longuet-Higgins phase and CP-conserving neutrino Hamiltonian:- Let us reexamine the form of the neutrino Hamiltonian given by Eq. (5) and the eigenvectors given by Eq. (6). Note that the eigenvectors depend only on a single parameter ϑ and satisfy

$$\begin{aligned}|\vartheta,\pm\rangle &= \mp|\vartheta+\pi,\mp\rangle = -|\vartheta+2\pi,\pm\rangle\\ &= \pm|\vartheta+3\pi,\mp\rangle = |\vartheta+4\pi,\pm\rangle. \end{aligned} \tag{7}$$

The minus sign picked up by both the mass eigenstates as we change ϑ from $0\to2\pi$ is precisely *the Herzberg and Longuet-Higgins phase* [26,46] of π, which was first obtained in the context of molecular physics in 1963. So, we note that just by looking at the form of the Hamiltonian for neutrino system, we should expect the Herzberg and Longuet-Higgins phase to appear. Also, note that the space of rays for the real neutrino Hamiltonian is the great circle ($\mathbb{S}^1$) lying on the $x-z$ plane of the Poincaré sphere (Fig. 1) and global structure of the eigenvectors is a *Möbius band*. The variation of ϑ results in parallel transport of the mass eigenstates (with dynamical phase removed) following the parallel transport rule along ϑ,

$$\Im m\langle\vartheta_{\mp}|\frac{d}{d\vartheta}|\vartheta_{\mp}\rangle = 0. \tag{8}$$

This parallel transport rule (formally referred to as natural connection) has an anholonomy defined on the Möbius band and this leads to the topological phase of π. The topological phase factor ß depends on the vector potential A_ϑ given by

$$ß = \oint A_\vartheta d\vartheta = \oint \Im m\langle\vartheta_{\mp}|\frac{d}{d\vartheta}|\vartheta_{\mp}\rangle d\vartheta. \tag{9}$$

This vector potential A_ϑ is nonintegrable, and this is the anholonomy of the connection. Physically, this corresponds to half a unit of magnetic flux piercing the origin of the $x-z$ plane, encircling which leads to this topological phase. And, the origin of the circle is connected to the null Hamiltonian (i.e. all elements are zero), which corresponds to the degeneracy point.

Naively speaking, one would think that this phase will be impossible to access for neutrinos because we do not have a handle on the mixing angle $\vartheta/2$ to be varied in a controlled way from $\vartheta=0\to2\pi$. The key point to understand here is the fact that as long as we carry out a quantum evolution of a state in a closed loop enclosing the point of singularity (degeneracy point, origin of the Poincaré sphere), which can be achieved either via adiabatic variation of ϑ or via Schrödinger evolution interrupted by collapses, one will always get this phase. However, note that in the former case, the amplitude of the initial state undergoing evolution does not change but in the latter case, it diminishes. In what follows, we will show that the transition probability for neutrinos actually does carry imprints of such a topological phase, which can be explicitly derived using Pancharatnam's prescription. We then show that the phase of π actually appears there and is in fact observed by all the experiments carried out so far.

The topological phase in two flavor neutrino oscillations (invoking collapses and adiabatic evolution):- In what follows, we consider the most general situation, i.e. neutrinos are traversing through matter with slowly varying density (i.e. ϑ is a slowly varying parameter changing from ϑ_1 to ϑ_2). Vacuum or constant density matter will be special cases where ϑ is a constant.

In order to see the effect of geometric phases, usually one performs a split-beam experiment. In the case of optics, one separates a beam into two parts in space, and each part traverses a different path. Finally, the beams are recombined to observe the relative phase shift as they interfere. In optics, the reflective and refractive property of the medium is exploited to make devices like mirrors and lenses, which facilitates designing of such experiments in the laboratory. In the case of neutrinos, such a procedure is not possible owing to the fact that the refractive index is extremely small ($n_{\rm refr} - 1 \simeq 10^{-19}$ for neutrinos of energy 1 MeV in ordinary matter). Treating the Sun (with density $\rho = 150\ {\rm g\,cm^{-3}}$ in the core) as a spherical lens for a neutrino beam of energy 10 MeV passing through it, one gets the focal length to be around $10^{18} R_\odot$ [22], which is about 10^5 times the size of our galaxy. Spatially split-beam interference experiments with neutrinos are clearly impossible. However, the fact that neutrinos are produced and detected as flavor states allows us to think of the time evolution of neutrinos as a split-beam experiment in energy space as depicted in Fig. 2.

Let us consider a neutrino created as a flavor state $|\nu_\alpha\rangle$ (for example, neutrinos produced inside the Sun are predominantly in the electron neutrino flavor state, $|\nu_e\rangle$) and detected as another flavor state, $|\nu_\beta\rangle$ ($|\nu_\beta\rangle$ can either be a $|\nu_e\rangle$, i.e. survival of the same electron neutrino flavor or a $|\nu_\mu\rangle$, i.e. appearance of muon neutrino flavor), then

$$|\nu_\alpha\rangle = \nu_{\alpha+}|\vartheta_1, +\rangle + \nu_{\alpha-}|\vartheta_1, -\rangle, \tag{10}$$

where $|\vartheta_1, \pm\rangle$ are the eigenstates of $\mathbb{H}_\nu(\vartheta_1)$. Now we consider an adiabatic evolution of the mass eigenstates from $|\vartheta_1, \pm\rangle$ to $|\vartheta_2, \pm\rangle$ due to a slow enough variation of background density such that no mixing between the two eigenstates is ensured under time evolution, and $|\vartheta_1, \pm\rangle$ evolves to

$$|\vartheta_1, \pm\rangle \to e^{-i\mathcal{D}_\pm}|\vartheta_2, \pm\rangle$$

$$\text{with } \mathcal{D}_\pm = \pm\frac{1}{2}\int_0^t \sqrt{(\omega\sin\vartheta)^2 + (V_C - \omega\cos\vartheta)^2}\,dt' + \int_0^t\left(p + \frac{m_1^2 + m_2^2}{4p} + \frac{V_C}{2} + V_N\right)dt', \tag{11}$$

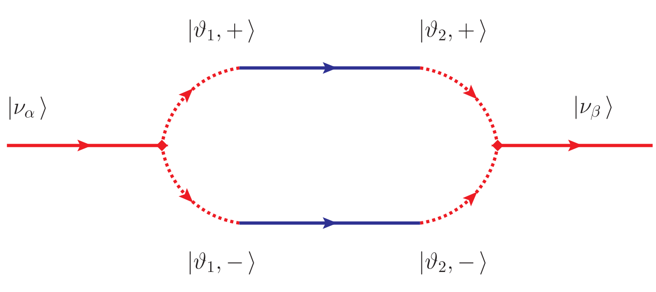

FIG. 2 (color online). Schematic of a split-beam experiment for neutrinos in energy space. $|\nu_\alpha\rangle$ and $|\nu_\beta\rangle$ are the two flavor states, while $|\vartheta_1, \pm\rangle$ and also $|\vartheta_2, \pm\rangle$ correspond to two sets of mass (energy) eigenstates. $|\vartheta_1, \pm\rangle$ are adiabatically evolved to states $|\vartheta_2, \pm\rangle$, respectively (upon removing the dynamical phase).

as the dynamical phases, relevant both for the vacuum case ($V_C = V_N = 0$) and in the presence of varying matter density profile and t is the time of flight of the neutrino. The quantities that depend on time (or distance) are V_C and V_N defined earlier [see Eq. (1)]. Note that the states $|\vartheta_1, \pm\rangle$ are $|\vartheta_2, \pm\rangle$ are connected via parallel transport rule [Eq. (8)] on the Poincaré sphere. The two time-evolved states $e^{-i\mathcal{D}_\pm}|\vartheta_2, \pm\rangle$ are finally recombined to form a flavor state at the detector.

In order to see this explicitly, let us proceed as follows: The amplitude for the transition between states $\nu_\alpha \to \nu_\beta$ is given by

$$\mathcal{A}(\nu_\alpha \to \nu_\beta) = \langle\nu_\beta|\mathcal{U}|\nu_\alpha\rangle, \tag{12}$$

where $\mathcal{U}$ is the unitary evolution operator given by

$$\mathcal{U} = e^{-i\mathcal{D}_+}|\vartheta_2, +\rangle\langle\vartheta_1, +| + e^{-i\mathcal{D}_-}|\vartheta_2, -\rangle\langle\vartheta_1, -|. \tag{13}$$

Inserting two complete sets of states in the amplitude,

$$\begin{aligned}\mathcal{A}(\nu_\alpha \to \nu_\beta) &= \sum_{i,j=-}^{+}\langle\nu_\beta|\vartheta_2, i\rangle\langle\vartheta_2, i|\mathcal{U}|\vartheta_1, j\rangle\langle\vartheta_1, j|\nu_\alpha\rangle \\ &= \langle\nu_\beta|\vartheta_2, +\rangle\langle\vartheta_2, +|\mathcal{U}|\vartheta_1, +\rangle\langle\vartheta_1, +|\nu_\alpha\rangle \\ &\quad + \langle\nu_\beta|\vartheta_2, -\rangle\langle\vartheta_2, -|\mathcal{U}|\vartheta_1, -\rangle\langle\vartheta_1, -|\nu_\alpha\rangle.\end{aligned} \tag{14}$$

Note that the cross terms do not contribute in the adiabatic limit. Upon substituting Eq. (13) in Eq. (14), we get

$$\begin{aligned}\mathcal{A}(\nu_\alpha \to \nu_\beta) &= e^{-i\mathcal{D}_+}\langle\nu_\beta|\vartheta_2, +\rangle\langle\vartheta_1, +|\nu_\alpha\rangle \\ &\quad + e^{-i\mathcal{D}_-}\langle\nu_\beta|\vartheta_2, -\rangle\langle\vartheta_1, -|\nu_\alpha\rangle.\end{aligned} \tag{15}$$

Then the probability for flavor transition $\nu_\alpha \to \nu_\beta$ is given by

$$\begin{aligned}\mathcal{P}(\nu_\alpha \to \nu_\beta) &= |\mathcal{A}(\nu_\alpha \to \nu_\beta)|^2 \\ &= \langle\nu_\alpha|\vartheta_1, +\rangle\langle\vartheta_2, +|\nu_\beta\rangle\langle\nu_\beta|\vartheta_2, +\rangle\langle\vartheta_1, +|\nu_\alpha\rangle \\ &\quad + \langle\nu_\alpha|\vartheta_1, -\rangle\langle\vartheta_2, -|\nu_\beta\rangle\langle\nu_\beta|\vartheta_2, -\rangle \\ &\quad \times\langle\vartheta_1, -|\nu_\alpha\rangle + [\langle\nu_\alpha|\vartheta_1, -\rangle e^{i\mathcal{D}_-} \\ &\quad \times\langle\vartheta_2, -|\nu_\beta\rangle\langle\nu_\beta|\vartheta_2, +\rangle e^{-i\mathcal{D}_+} \\ &\quad \times\langle\vartheta_1, +|\nu_\alpha\rangle + \text{c.c.}].\end{aligned} \tag{16}$$

The cross term in Eq. (16) is related to the interference term resulting from the two path interferometer depicted in Fig. 2. Upon dropping the dynamical phase, we have $\langle\nu_\alpha|\vartheta_1, -\rangle\langle\vartheta_2, -|\nu_\beta\rangle\langle\nu_\beta|\vartheta_2, +\rangle\langle\vartheta_1, +|\nu_\alpha\rangle$, which can be viewed as a series of closed loop quantum collapses with intermediate adiabatic evolutions given by $|\nu_\alpha\rangle \to |\vartheta_1, +\rangle \to |\vartheta_2, +\rangle \to |\nu_\beta\rangle \to |\vartheta_2, -\rangle \to |\vartheta_1, -\rangle \to |\nu_\alpha\rangle$, which essentially covers a great circle in the $x - z$ plane as is shown in Fig. 3(a). This closed trajectory subtends a solid angle of $\Omega = 2\pi$ at the center of the great circle.

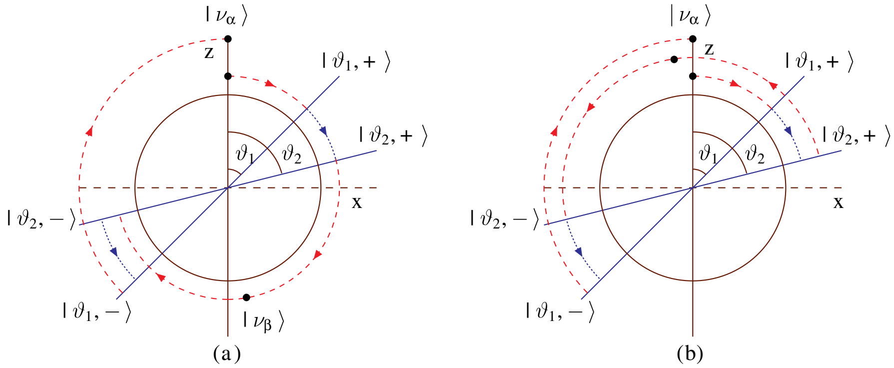

FIG. 3 (color online). Two representative cases depicting the collapse processes (dashed red lines) with intermediate adiabatic evolutions upon removing the dynamical phase (dotted blue lines) on the great circle ($\mathbb{S}^1$) arising due to the cross term $\langle\nu_\alpha|\vartheta_1,-\rangle\times\langle\vartheta_2,-|\nu_\beta\rangle\langle\nu_\beta|\vartheta_2,+\rangle\langle\vartheta_1,+|\nu_\alpha\rangle$ in the probability. The initial flavor state $|\nu_\alpha\rangle$ is on the positive z axis, while final flavor state $\nu_\beta\rangle$ is not necessarily its antipodal point. The two sets of mass eigenstates are antipodal points on two axes making angles ϑ_1 and ϑ_2, respectively with respect to the z axis. Case (a) corresponds to appearance probability $[\mathcal{P}(\nu_\alpha\to\nu_\beta)]$ for which we get a cyclic loop in ϑ space. (b) The collapse processes for survival probability $[\mathcal{P}(\nu_\alpha\to\nu_\alpha)]$ does not enclose any loop.

Hence, without any further calculation, we can immediately predict that the phase of the interference term will be π (half the solid angle) due to Pancharatnam's prescription. On the circle, each of the individual collapse processes that essentially projects a state with given angle ϑ to another state with different angle ϑ' can be thought of as an infinite series of infinitesimally close collapses between states defined as $|\vartheta\rangle$ and $|\vartheta+\delta\vartheta\rangle$ as far as geometric phases are concerned. The entire closed loop of collapses with intermediate adiabatic evolutions mentioned above can be viewed as a smooth variation of ϑ from $0\to 2\pi$ in the limit $\delta\vartheta\to 0$ hence making a direct connection to the Herzberg and Longuet-Higgins phase mentioned above. Nonetheless, we must note that the evolution of a state is unitary under infinitesimal collapses ($\delta\vartheta\to 0$ limit), while it is nonunitary under finite collapses leading to a loss in intensity (probability). But the geometric phase of the evolving state remains unaltered for the two cases mentioned above.

For the case when $\alpha=\beta$, i.e. survival probability, it is easy to see that the collapses do not form a closed loop enclosing the origin and therefore the interference term will not pick up any phase. This case is depicted in Fig. 3(b).

In a simpler situation when ϑ does not change, i.e. the case of vacuum or constant density matter, the number of states will be fewer (in the absence of variation of density, $|\vartheta_1,\pm\rangle$ is the same as $|\vartheta_2,\pm\rangle$) and the collapses are given by $|\nu_\alpha\rangle\to|\vartheta_1,+\rangle\to|\nu_\beta\rangle\to|\vartheta_1,-\rangle\to|\nu_\alpha\rangle$. As long as the collapses lead to closed loop encircling the origin, we will obtain this topological phase. So this phase of π appears whether we consider vacuum and/or ordinary matter with constant density or with slowly changing (but noncyclic) electron number density. This is due to the topological character of this phase, which will be preserved as long as we have CP-conserving (real) Hamiltonian and states are always lying on a great circle in the $x-z$ plane in the Poincaré sphere.

Next we write down an explicit expression for the observable quantities, i.e. appearance and survival probabilities for two neutrino flavors. Using the general expression obtained in Eq. (16), the appearance probability for transition $\nu_e\to\nu_\mu$ is given by[4]

$$\begin{aligned}\mathcal{P}(\nu_e\to\nu_\mu) &= \mathbb{U}^\star_{e+}(\Theta_1)\mathbb{U}_{\mu+}(\Theta_2)\mathbb{U}^\star_{\mu+}(\Theta_2)\mathbb{U}_{e+}(\Theta_1)\\ &\quad+\mathbb{U}^\star_{e-}(\Theta_1)\mathbb{U}_{\mu-}(\Theta_2)\mathbb{U}^\star_{\mu-}(\Theta_2)\mathbb{U}_{e-}(\Theta_1)\\ &\quad+[\mathbb{U}^\star_{e-}(\Theta_1)e^{i\mathcal{D}_-}\mathbb{U}_{\mu-}(\Theta_2)\mathbb{U}^\star_{\mu+}(\Theta_2)e^{-i\mathcal{D}_+}\\ &\quad\times\mathbb{U}_{e+}(\Theta_1)+\text{c.c.}]. \end{aligned}\tag{17}$$

Note that the matrix $\mathbb{U}(\Theta)$ is the lepton mixing matrix (defined in a basis where the charged lepton mass matrix is diagonal). It is also referred to as the Pontecorvo-Maki-Nakagawa-Sakata matrix [47,48] and connects the flavor states to the mass eigenstates. For the 2×2 case, it is a real orthogonal rotation matrix given by

$$\mathbb{U}(\Theta)=\begin{pmatrix}\cos\Theta & \sin\Theta\\ -\sin\Theta & \cos\Theta\end{pmatrix}. \tag{18}$$

Substituting the elements of $\mathbb{U}(\Theta)$ we get

$$\begin{aligned}\mathcal{P}(\nu_e\to\nu_\mu) &= \cos^2\Theta_1\sin^2\Theta_2+\sin^2\Theta_1\cos^2\Theta_2\\ &\quad+[2\cos(\mathcal{D}_+-\mathcal{D}_-)](-\sin\Theta_1)\\ &\quad\times\cos\Theta_2\sin\Theta_2\cos\Theta_1. \end{aligned}\tag{19}$$

[4]In order to connect with the standard expressions used in neutrino literature, we shall revert to Θ instead of $\vartheta/2$.

We note that there are four inner products appearing in the interference term in the final expression for the probability out of which the first three inner products, viz., $\langle\vartheta_1, +|\nu_e\rangle = \mathbb{U}_{e+}(\Theta_1) = \cos\Theta_1 > 0$, $\langle\nu_\mu|\vartheta_2, +\rangle = \mathbb{U}^\star_{\mu+}(\Theta_2) = \sin\Theta_2 > 0$ and $\langle\vartheta_2, -|\nu_\mu\rangle = \mathbb{U}_{\mu-}(\Theta_2) = \cos\Theta_2 > 0$ clearly implying that these states are mutually parallel to each other in pairs according to Pancharatnam's rule, which is to have the inner product of any two states real and positive, while the last one $\langle\nu_e|\vartheta_1, -\rangle = \mathbb{U}^\star_{e-}(\Theta_1) = -\sin\Theta_1 < 0$ by Pancharatnam's rule has $|\nu_e\rangle$ antiparallel to $|\vartheta_1, -\rangle$, since the physically allowed values for the mixing angles Θ_1 and Θ_2 are within the interval $[0, \pi/2]$ for $\delta m^2 > 0$ [40]. (On the Poincaré sphere, the corresponding ϑ_1 and ϑ_2 can take values between $[0, \pi]$.) The minus sign appearing in the interference term is thus the Pancharatnam's phase of π appearing in the neutrino oscillation formula (see Fig. 3(a)).

If in a hypothetical situation, for some range of parameters Θ_1 and Θ_2, the first three of the inner products are real and negative (i.e. the states are aligned antiparallel to each other or completely out of phase), while the fourth inner product is real and positive (the states are in phase) then also we will have this minus sign. The nontransitivity also holds here leading to the nontrivial topological phase of π. This situation where the inner product becomes real and negative defines an ''antiparallel'' rule (in the same spirit in which Pancharatnam defined his rule of two states being ''in phase or parallel'') would correspond to the norm of the vector sum of the two states being at its minimum value. Physically, this implies the interference of the two given states would be destructive and the resulting state will have minimum intensity or a dark fringe in optics.

The existence of Pancharatnam's phase of π can be simply connected to the fact that the mixing matrix $\mathbb{U}(\Theta)$ matrix for two flavors is an orthogonal rotation matrix parameterized by the mixing angle Θ of which one element has a negative sign. Thus, this phase is *built into* the structure of $\mathbb{U}(\Theta)$ matrix.

The survival probability is given by

$$\mathcal{P}(\nu_e \to \nu_e) = \cos^2\Theta_1\cos^2\Theta_2 + \sin^2\Theta_1\sin^2\Theta_2 + [2\cos(\mathcal{D}_+ - \mathcal{D}_-)]\sin\Theta_1\cos\Theta_2\sin\Theta_2 \times \cos\Theta_1. \tag{20}$$

Note that in the case of survival probability, the cross term does not pick up any nonzero topological phase, and geometrically this is exactly what we had expected from Fig. 3(b). The loop in ϑ space is open in this case, and this is what leads to this result. The topological phase of the interference term in survival probability is zero, while it is π in the case of the appearance probability, and this fact is in accord with unitarity.

The above expressions [Eqs. (19) and (20)] reduce to the standard results [27,37,40,49] for vacuum if we substitute $\Theta_1 = \Theta_2 = \Theta$,

$$\mathcal{P}(\nu_e \to \nu_\mu) = \sin^2 2\Theta\sin^2\frac{\delta m^2 l}{4E} \quad \text{and}$$
$$\mathcal{P}(\nu_e \to \nu_e) = 1 - \sin^2 2\Theta\sin^2\frac{\delta m^2 l}{4E}, \tag{21}$$

where in the ultrarelativistic limit, we can use $t \simeq l$ and $p \simeq E$ leading to $\mathcal{D}_\pm = \pm\delta m^2 l/2E$ [see Eq. (11)] for the vacuum case ($V_C = V_N = 0$). In constant density matter, the quantities Θ and δm^2 in Eq. (21) are replaced by their respective renormalized values in matter, Θ^m and $(\delta m^2)^m$ but the form of the expression will remain the same. Hence, our result is consistent with the standard neutrino oscillation formulation, and it provides a clear geometric interpretation of the phenomenon of neutrino oscillations. More precisely, the standard result for neutrino oscillations is in fact a realization of the Pancharatnam topological phase.

IV. DISCUSSION

As mentioned in the introduction, the existing work on the subject of geometric phases in neutrino oscillations led to the widespread belief that the two flavor neutrino oscillation formulae in *CP* conserving situations were devoid of any geometric or topological phase component. Appearance of the cyclic Berry phase was dismissed on the grounds of not having any time-varying parameter in vacuum and having only one *essential* parameter (thereby enclosing no area) in the case of normal matter [3,12–16]. Concerning the appearance of the general geometric phase in the two flavor neutrino case for propagation in vacuum, there are claims reporting its appearance [17,18]. But, it should be noted that such terms appeared only at amplitude level and as argued earlier, a phase appearing in the amplitude can be observed only via a split-beam experiment, which is not feasible to design in the case of neutrinos.

In this paper, we have examined the minimal case of two flavor neutrino oscillations and *CP* conservation. Contrary to all existing claims in the literature concerning the geometric or topological phase in two flavor neutrino oscillation probabilities, our study provides the first clear prediction that a topological phase of π exists at the probability level even in the minimal case of *CP* conservation. We show that it is inherently present in the physics of neutrino oscillations via the structure of the Pontecorvo-Maki-Nakagawa-Sakata neutrino mixing matrix. This existence of this topological phase is linked to the presence of a flux line of strength π at the origin of ray space, which is connected to the degeneracy point associated with the null Hamiltonian.

Pancharatnam's idea is quite useful in terms of predictive power as it allows for a clear visualization of the appearance of such a phase due to geometric effects without doing any algebra. Our prescription is general as it contains effects due to collapses and also due to adiabatic evolution. In the absence of either of these, one would get the same phase. So no matter what the details are, as long as the singular (degeneracy) point is enclosed by a cyclic loop (in the space of rays) as ϑ is varied from $0 \rightarrow 2\pi$, we will get this phase, and this is due to its topological robustness. The adiabatic and collapse processes both conspire in such a fashion that the net phase would always be π. This does not happen for geometric phases.

The topological phase obtained in this paper is a consequence of anholonomy, which can arise in situations even when there is no curvature. The most striking example of this is the Aharonov-Bohm effect [46]. To experience the effect of anholonomy, the main requirement is to encircle the singular point, this fact was exploited by Herzberg and Longuet-Higgins in pointing out the topological phase in molecular physics. On the other hand, for Berry's phase to appear, a net curvature is a must which is fulfilled by having at least two essential parameters in the Hamiltonian varying cyclically. This is an important distinction between the geometric phases as obtained by Herzberg and Longuet-Higgins and by Berry.

If we consider mixed flavor states[5] instead of the pure flavor states, there will be a greater number of physical situations (or, possible diagrams for the interference term like the ones shown in Fig. 3 for pure flavor states) that can be explored to see if one encircles the singular point or not. A mixed state corresponds to a general point on the surface of the Poincaré sphere like an elliptically polarized state in optics. If the mixed states are such that they lie on the $x - z$ plane, it will always lead to the same quantized topological phase of π. But, for a general mixed state lying anywhere else on the Poincaré sphere, the phase will be geometric in nature.

It might be a nontrivial task to extend our geometrical interpretation to the case of three neutrinos flavors because it will involve a higher dimensional sphere (the ray space is $\mathcal{CP}^2$ for the three level quantum system).

It is natural to ask what happens when we invoke *CP* violation. In vacuum, *CP* violation cannot be induced in the two flavor case as a consequence of *CPT* invariance and unitarity [28]. However, matter with constant or varying density can induce *CP* violation via the coherent forward scattering of neutrinos with background matter. If we introduce *CP* violation induced by background matter with constant density [28], we still expect to get the same phase of π as we have two pairs of orthogonal states that will always lie on a great circle. If the density is varying slowly (adiabatic condition holds), then the intermediate states (connected by adiabatic evolution) will be lifted from the great circle, hence resulting in a path-dependent solid angle, and the phase will be geometric [44].

[5]Here, mixed state refers to a superposition of pure flavor states and should not be confused with the mixed states in the density matrix language, which are not pure.

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[1] M. C. Gonzalez-Garcia and M. Maltoni, Phys. Rep. **460**, 1 (2008).
[2] R. Z. Funchal, contribution to Neutrino 08, May 27, Christchurch, New Zealand (unpublished).
[3] N. Nakagawa, Ann. Phys. (N.Y.) **179**, 145 (1987).
[4] J. Vidal and J. Wudka, Phys. Lett. B **249**, 473 (1990).
[5] C. Aneziris and J. Schechter, Int. J. Mod. Phys. A **6**, 2375 (1991).
[6] C. Aneziris and J. Schechter, Phys. Rev. D **45**, 1053 (1992).
[7] A. Y. Smirnov, Pis'ma Zh. Eksp. Teor. Fiz. **53**, 280 (1991).
[8] A. Y. Smirnov, Phys. Lett. B **260**, 161 (1991).
[9] E. K. Akhmedov, A. Y. Smirnov, and P. I. Krastev, Z. Phys. C **52**, 701 (1991).
[10] M. M. Guzzo and J. Bellandi, Phys. Lett. B **294**, 243 (1992).
[11] V. M. Aquino, J. Bellandi, and M. M. Guzzo, Phys. Scr. **54**, 328 (1996).
[12] V. A. Naumov, JETP Lett. **54**, 185 (1991).
[13] V. A. Naumov, Sov. Phys. JETP **74**, 1 (1992).
[14] V. A. Naumov, Int. J. Mod. Phys. D **1**, 379 (1992).
[15] V. A. Naumov, Phys. Lett. B **323**, 351 (1994).
[16] X.-G. He, X.-Q. Li, B. H. J. McKellar, and Y. Zhang, Phys. Rev. D **72**, 053012 (2005).
[17] M. Blasone, P. A. Henning, and G. Vitiello, Phys. Lett. B **466**, 262 (1999).
[18] X.-B. Wang, L. C. Kwek, Y. Liu, and C. H. Oh, Phys. Rev. D **63**, 053003 (2001).
[19] M. V. Berry, Proc. R. Soc. A **392**, 45 (1984).
[20] Y. Aharonov and J. Anandan, Phys. Rev. Lett. **58**, 1593 (1987).
[21] O. G. Miranda, J. Math. Phys. (N.Y.) **37**, 121 (2006).
[22] G. G. Raffelt, *Stars as Laboratories for Fundamental Physics: The Astrophysics of Neutrinos, Axions, and Other Weakly Interacting Particles* (University of Chicago Press, Chicago, 1996).
[23] S. Pancharatnam, Proc. Indian Acad. Sci. A **44**, 247 (1956).
[24] J. Samuel and R. Bhandari, Phys. Rev. Lett. **60**, 2339 (1988).
[25] S. Ramaseshan and R. Nityananda, Curr. Sci. **55**, 1225 (1986).

[26] G. Herzberg and H. C. Longuet-Higgins, Discuss. Faraday Soc. **35**, 77 (1963).
[27] T. K. Kuo and J. Pantaleone, Rev. Mod. Phys. **61**, 937 (1989).
[28] E. K. Akhmedov, Phys. Scr. **T121**, 65 (2005).
[29] A. Bandyopadhyay *et al.* (ISS Physics Working Group), arXiv:0710.4947.
[30] R. A. Harris and L. Stodolsky, Phys. Lett. **116B**, 464 (1982).
[31] C. W. Kim, W. K. Sze, and S. Nussinov, Phys. Rev. D **35**, 4014 (1987).
[32] L. Stodolsky, Phys. Rev. D **36**, 2273 (1987).
[33] C. W. Kim, J. Kim, and W. K. Sze, Phys. Rev. D **37**, 1072 (1988).
[34] M. J. Thompson and B. H. J. McKellar, Phys. Lett. B **259**, 113 (1991).
[35] K. Enqvist, K. Kainulainen, and J. Maalampi, Nucl. Phys. **B349**, 754 (1991).
[36] C. Giunti, C. W. Kim, and U. W. Lee, Phys. Lett. B **274**, 87 (1992).
[37] C. W. Kim and A. Pevsner, *Neutrinos in Physics and Astrophysics* (Harwood Academic Publishers, Chur, Switzerland, 1993).
[38] A. Halprin, Phys. Rev. D **34**, 3462 (1986).
[39] N. Mukunda, R. Simon, and E. C. G. Sudarshan, J. Opt. Soc. Am. A **2**, 1291 (1985).
[40] M. C. Gonzalez-Garcia and Y. Nir, Rev. Mod. Phys. **75**, 345 (2003).
[41] L. Wolfenstein, Phys. Rev. D **17**, 2369 (1978).
[42] S. P. Mikheyev and A. Y. Smirnov, Sov. J. Nucl. Phys. **42**, 913 (1985).
[43] J. Samuel, Pramana **48**, 959 (1997).
[44] P. Mehta (work in progress).
[45] M. V. Berry, J. Mod. Opt. **34**, 1401 (1987).
[46] A. Shapere and F. Wilczek, *Geometric Phases in Physics* (World Scientific, Singapore, 1989).
[47] B. Pontecorvo, Sov. Phys. JETP **7**, 172 (1958).
[48] Z. Maki, M. Nakagawa, and S. Sakata, Prog. Theor. Phys. **28**, 870 (1962).
[49] J. D. Walecka, *Introduction To Modern Physics: Theoretical Foundations* (World Scientific, Singapore, 2008).