Chapter 3

2.5PN GW polarizations from inspiralling compact binaries on circular orbits: Hereditary terms and the complete waveform

3.1 Introduction

One of the key assumptions one often makes in describing a physical system is its localizability in space and time. This facilitates very accurate predictions about the evolution of the system as if it were (i) isolated from other sources located far away in space and that (ii) its present state is uncorrelated with its states at remote earlier epochs. General relativity necessitates spacetime nonlocality with its two key features: finite propagation velocity of physical effects and infinite nonlinearity (see Ref [56] for a more detailed physical description of this effect).

If one uses a multipolar description of gravitational fields, it can be shown that evolution of the individual multipoles are *not* independent but each multipole couples with other multipoles (including themselves) [56, 57, 58, 59] via nonlinear Einstein's field equations. Since we use the MPM formalism to compute the gravitational waveform h_{ij} and the corresponding polarizations, there will be nonlocal contributions to the waveform and polarizations. Computation of such nonlocal contributions to the two independent gravitational wave polarizations from an isolated compact binary system is the topic of discussion of this chapter [1]. We call these nonlocal contributions 'hereditary terms'.

Hereditary terms can be of different types. For instance, the hereditary term entering the

[1]Henceforth nonlocal always refers to nonlocal in *time* since we deal with isolated sources.

expression for gravitational waveform include:

1. *The GW tails*: They are due to the nonlinear interaction between the mass quadrupole M_{ij} and the total ADM mass energy content of the spacetime M [56, 57]. Physically, this effect can be visualized as the backscattering of the linear waves (described by I_{ij}) off the constant spacetime curvature generated by the mass energy M. This can be viewed as a part of the gravitational field propagating inside the light cone (*e.g.* [56]). This is the leading hereditary effect which first appears at 1.5PN in the waveform [2]. Since tail integrals appear at 1.5PN for different radiative multipoles, tail effect due to higher multipoles occur at higher PN orders (see Eq. (3.3)) .

2. *Nonlinear memory term*: This is one of the most interesting components in the 2.5PN GW polarizations presented in this chapter. It is due to the re-radiation of stress-energy distribution of linear waves [155, 156, 157, 57]. As we shall discuss in this chapter, there are oscillatory and non-oscillatory parts constituting the memory effect.

3. *Tails of tails*: These are contributions to the GW polarizations due to the cubic interaction of quadrupole moment M_{ij} with two monopoles M [58, 59]. This is a 3PN effect irrelevant for the computation of the present 2.5PN polarization. However this is one of the most important new features one would encounter in calculating the 3PN polarization.

Besides these three, there is an additional *tail-square effect* in the computation of GW fluxes which is due to the square of the 1.5PN term in the waveform [59].

The 1.5PN tail contribution was calculated independently by Blanchet and Schäfer [108] and Wiseman [141]. Blanchet, Damour and Iyer calculated the tail contribution at 2PN in the waveform [140] arising from the mass octupole and current quadrupole. Blanchet in Ref [98] computed the 2.5PN tail contribution to the GW luminosity. Later, Blanchet evaluated the 'tails of tails' and ' tail-squared' terms at the 3PN order in the luminosity which was used in Refs [143, 99] to calculate the 3.5PN GW energy flux and the associated 3.5PN GW phasing. This phasing formula of Ref [99] is being used to analyse the data of LIGO by different research groups across the world.

This chapter deals with the explicit computation of the 'tails' and 'memory' terms in the complete 2.5PN GW polarizations. While we follow a procedure similar to Blanchet-Schäfer [108] for the calculation of the tails, we compute newly for the first time an explicit closed-form expression for the 2.5PN memory contribution.

[2] Recall that 1.5PN here is relative to the leading Newtonian order.

3.2 The 2.5PN gravitational waveform

We discuss the individual hereditary contributions to the GW polarizations at 2.5PN order in this section. Based on the expressions for the mass and current type radiative moments (Eqs (2.3)-(2.4)) and using Eq. (2.1) for the waveform, the total waveform is given by

$$h^{\rm TT}_{km} = (h^{\rm TT}_{km})_{\rm inst} + (h^{\rm TT}_{km})_{\rm hered}\,. \tag{3.1}$$

The hereditary terms up to 2.5PN consists of 'tails' and 'memory' contributions.

$$(h^{\rm TT}_{km})_{\rm hered} = (h^{\rm TT}_{km})_{\rm tail} + (h^{\rm TT}_{km})_{\rm memory} + \mathcal{O}(6)\,. \tag{3.2}$$

The structure of these terms and the necessary inputs needed for their evaluation are discussed in the next section.

3.2.1 The hereditary contributions to the waveform

Concerning hereditary parts, we have the tail integrals (dominantly 1.5PN) which read as

$$\begin{aligned}(h^{\rm TT}_{km})_{\rm tail} = \frac{2G}{c^4R}\mathcal{P}_{ijkm}\frac{2G\,M}{c^3}\int_{-\infty}^{T_R} dV \Bigg\{ &\left[\ln\left(\frac{T_R-V}{2b}\right)+\frac{11}{12}\right] I^{(4)}_{ij}(V) \\ &+\frac{N_a}{3c}\left[\ln\left(\frac{T_R-V}{2b}\right)+\frac{97}{60}\right] I^{(5)}_{ija}(V) \\ &+\frac{4N_b}{3c}\left[\ln\left(\frac{T_R-V}{2b}\right)+\frac{7}{6}\right] \varepsilon_{ab(i}J^{(4)}_{j)a}(V) \\ &+\frac{N_{ab}}{12c^2}\left[\ln\left(\frac{T_R-V}{2b}\right)+\frac{59}{30}\right] I^{(6)}_{ijab}(V) \\ &+\frac{N_{bc}}{2c^2}\left[\ln\left(\frac{T_R-V}{2b}\right)+\frac{5}{3}\right] \varepsilon_{ab(i}J^{(5)}_{j)ac}(V)\Bigg\}\,,\end{aligned} \tag{3.3}$$

and the non-linear memory integral (which is purely of order 2.5PN) given by

$$(h^{\rm TT}_{km})_{\rm memory} = \frac{2G}{c^4R}\mathcal{P}_{ijkm}\frac{G}{c^5}\int_{-\infty}^{T_R} dV \left\{-\frac{2}{7}I^{(3)}_{a<i}(V)I^{(3)}_{j>a}(V) + \frac{N_{ab}}{30}I^{(3)}_{<ij}(V)I^{(3)}_{ab>}(V)\right\}\,. \tag{3.4}$$

The latter expression is in complete agreement with the results of [156, 57, 58].

3.2.2 Source multipole moments required at the 2.5PN order

Unlike the instantaneous part, the hereditary calculations require source multipole moments with much lesser accuracy. The highest accuracy moment needed is the 1PN mass quadrupole given in Ref [107]. The relevant multipole moments are rewritten below up to the required accuracy. We need,

$$
\begin{aligned}
I_{ij} &= \nu\, m\, \mathrm{STF}_{ij} \left\{ x^{ij} \left[1 + \gamma \left(-\frac{1}{42} - \frac{13}{14}\nu \right) \right] \right. \\
&\quad \left. + \frac{r^2}{c^2} v^{ij} \left[\frac{11}{21} - \frac{11}{7}\nu + \gamma \left(\frac{1607}{378} - \frac{1681}{378}\nu + \frac{229}{378}\nu^2 \right) \right] \right\} + \mathcal{O}(4) \ , \qquad (3.5a) \\
I_{ijk} &= \nu\, m\, (X_2 - X_1)\, \mathrm{STF}_{ijk} \left\{ x^{ijk} \right\} , \qquad (3.5b) \\
I_{ijkl} &= \nu\, m\, \mathrm{STF}_{ijkl} \left\{ x^{ijkl} \left[1 - 3\nu \right] \right\} . \qquad (3.5c)
\end{aligned}
$$

Further, the current moments are given by,

$$
\begin{aligned}
J_{ij} &= \nu\, m\, (X_2 - X_1)\, \mathrm{STF}_{ij} \left\{ \varepsilon_{abi} x^{ja} v^b \right\} , \qquad (3.6a) \\
J_{ijk} &= \nu\, m\, \mathrm{STF}_{ijk} \left\{ \varepsilon_{kab} x^{aij} v^b \left[1 - 3\nu \right] \right\} . \qquad (3.6b)
\end{aligned}
$$

[We recall that $X_1 = \frac{m_1}{m}$, $X_2 = \frac{m_2}{m}$, and $\nu = X_1 X_2$; the PN parameter γ is defined by (2.17); the STF projection is mentioned explicitly in front of each term.]

With all the latter source moments valid for a specific matter system (compact binary in circular orbit) the gravitational waveform is fully specified up to the 2.5PN order. The only other input which we need is the 1PN equation of motion of Ref [159]

$$
\frac{d\mathbf{v}}{dt} = -\omega^2\, \mathbf{x} + \mathcal{O}(4) \, , \qquad (3.7)
$$

where

$$
\omega^2 = \frac{G\, m}{r^3} \left\{ 1 + \left[-3 + \nu \right] \gamma + \mathcal{O}(4) \right\} \, . \qquad (3.8)
$$

Using the above given inputs, we describe the computations of the tails and memory integrals in what follows. What is different is that unlike for the evaluation of the instantaneous part where one only requires the equation of motion of the binary, the hereditary calculation, involving an integration over the binary's entire past, requires a model for the binary's orbit.

3.3 Computation of the hereditary terms at the 2.5PN order

We now come to the computation of the hereditary terms, *i.e.* terms made up of integrals extending over all the past history of the non-stationary source, from $-\infty$ in the past up to $T_R = T - R/c$. In the following we shall refer to T_R as the *current* time – the one at which the observation of the radiation field occurs. As we have seen in Sec. 2.2.2, at the 2.5PN order the waveform contains two types of hereditary terms: tail integrals, given by Eq. (3.3), and the non-linear memory integral (3.4).[3]

Evidently, in order to evaluate the hereditary terms, one must take into account the fact that the binary's orbit will have evolved, under gravitational radiation reaction, from early time to today. However we shall show, following Refs. [108, 141], the tails can basically be computed using the binary's *current* dynamics at time T_R, *i.e.* a circular orbit travelled at the current orbital frequency $\omega(T_R)$ (this will be true modulo negligible 4PN terms in the waveform). Concerning the memory integral however, the result in general depends on the details of the model of the binary evolution in the entire past.

3.3.1 Model for adiabatic inspiral in the remote past

In this chapter we adopt a simplified model of binary's past evolution in which the orbit decays adiabatically because of gravitational radiation damping according to the standard quadrupolar (*i.e.* Newtonian) approximation. We shall justify later that such a model is sufficient for our purpose – because we shall have to take into account the PN corrections in the tails only at the current epoch. The orbit will be assumed to remain circular, apart from the gradual inspiral, at any time $V < T_R$. We shall ignore any astrophysical (non-gravitational) processes such as the binary's formation by capture process in some dense stellar cluster, the successive supernova explosions and associated core collapses leading to the formation of the two compact objects, etc.

Let us recall the expressions of the binary's orbital parameters as explicit functions of time V in the quadrupolar circular-orbit approximation [139]. The orbital separation $r(V)$ evolves according to a power law, namely

$$r(V) = 4\left[\frac{G^3 m^3 \nu}{5\,c^5}(T_c - V)\right]^{1/4}, \tag{3.9}$$

where T_c denotes the coalescence instant, at which the two bodies merge together and the orbital frequency formally tends to infinity. The factor $1/c^5$ therein represents the 2.5PN order of radiation reaction. We assume that our current detection of the binary takes place before the coalescence instant, $T_R < T_c$, in a regime where the binary inspiral is *adiabatic* and the approximation valid.[4]

[3]Notice that in the gravitational-wave *flux* (in contrast to the waveform), the non-linear memory integral is instantaneous – it is made of a simple time anti-derivative and the flux depends on the time-derivative of the waveform.

[4]The formal PN order of the time interval left till coalescence is the inverse of the order of radiation-reaction, $T_c - T_R = \mathcal{O}(c^5) = \mathcal{O}(-5)$.

The orbital frequency ω in this model ($\omega = 2\pi/P$, where P is the orbital period), is related at any time to the orbital separation (3.9) by Kepler's (Newtonian) law $G\,m = r^3\,\omega^2$. [Again we shall justify later our use of a Newtonian model for the early-time inspiral]. Hence,

$$\omega(V) = \frac{1}{8}\left[\frac{G^{5/3}m^{5/3}\nu}{5\,c^5}(T_c - V)\right]^{-3/8} . \tag{3.10}$$

Instead of ω it is convenient to make use of the traditional frequency-related post-Newtonian parameter $x \equiv (G\,m\,\omega/c^3)^{2/3}$ already considered in (2.21). It is given by

$$x(V) = \frac{1}{4}\left[\frac{\nu\,c^3}{5\,G\,m}(T_c - V)\right]^{-1/4} . \tag{3.11}$$

The orbital phase $\phi = \int \omega\, dt = \frac{c^3}{G\,m} \int x^{3/2} dt$, namely the angle between the binary's separation and the ascending node $\mathcal{N}$, reads as

$$\phi(V) = \phi_c - \frac{1}{\nu}\left[\frac{\nu\,c^3}{5\,G\,m}(T_c - V)\right]^{5/8} , \tag{3.12}$$

where ϕ_c is the value of the phase at the coalescence instant.

The latter expressions are inserted into the various hereditary terms, and integrated from $-\infty$ in the past up to now. In order to better understand the structure of the integrals, it is advisable to re-express the above quantities (3.9)–(3.12) in terms of their values at the *current* time T_R. A simple way to achieve this is to introduce, following [108], the new time-related variable

$$y \equiv \frac{T_R - V}{T_c - T_R} , \tag{3.13}$$

and to make use of the power-law dependence in time of Eqs. (3.9)–(3.12). This leads immediately, for the orbital radius $r(V)$ and similarly for $\omega(V)$ and $x(V)$, to

$$r(V) = r(T_R)(1 + y)^{1/4} , \tag{3.14}$$

where $r(T_R)$ refers to the current value of the radius. For the orbital phase we get

$$\phi(V) = \phi(T_R) - \frac{1}{\nu}\left[\frac{\nu\,c^3}{5\,G\,m}(T_c - T_R)\right]^{5/8} \left[(1 + y)^{5/8} - 1\right] . \tag{3.15}$$

The latter form is however not the one we are looking for. Instead, we want to make explicit the fact that the phase difference between T_R and some early time V will become larger when the inspiral rate gets slower, *i.e.* when the *relative* change of the orbital frequency ω in one corresponding period P becomes smaller.

To this end we introduce a dimensionless "adiabatic parameter" associated with the inspiral rate *at the current time* T_R. This is properly defined as the ratio between the current period and the time left till coalescence. We adopt the definition [5]

$$\xi(T_R) \equiv \frac{1}{(T_c - T_R)\,\omega(T_R)}\,. \tag{3.16}$$

The adiabatic parameter ξ is of the order 2.5PN. Written in terms of the PN variable x defined by (3.11), it reads

$$\xi(T_R) = \frac{256\,\nu}{5}\,x^{5/2}(T_R)\,. \tag{3.17}$$

Now Eq. (3.15) can be expressed with the help of $\xi(T_R)$ in the more interesting form

$$\phi(V) = \phi(T_R) - \frac{8}{5\,\xi(T_R)}\Big[(1+y)^{5/8} - 1\Big]\,, \tag{3.18}$$

which makes clear that the phase difference $\Delta\phi = \phi(T_R) - \phi(V)$, which is 2π times the number of orbital cycles between V and T_R, tends to infinity when $\xi(T_R) \to 0$ on any "remote-past" time interval for which y is bounded from below, for instance $y > 1$. What is important is that (3.18) depends on the *current* value of the adiabatic parameter, so we shall be able to compute the hereditary integrals in the relevant limit where $\xi(T_R) \to 0$, appropriate to the current adiabatic regime. Notice that, at recent time, when $V \to T_R$ or equivalently $y \to 0$, we have

$$\phi(V) = \phi(T_R) - \frac{y}{\xi(T_R)} + \mathcal{O}\Big(y^2\Big)\,, \tag{3.19}$$

which is of course the same as the Taylor expansion

$$\phi(V) = \phi(T_R) - (T_R - V)\,\omega(T_R) + \mathcal{O}\Big[(T_R - V)^2\Big]\,. \tag{3.20}$$

3.3.2 The nonlinear memory integral

We tackle the computation of the novel hereditary term at the 2.5PN order, namely the nonlinear memory integral given by (3.4). As we shall see the computation boils down to the evaluation of only two "elementary" hereditary integrals, below denoted $I(T_R)$ and $J(T_R)$. A third type of elementary integral, $K(T_R)$, will be necessary to compute the tail integrals in Sec. 3.3.3.

The two wave polarisations corresponding to Eq. (3.4), calculated with our conventions and notation explained after (2.31), are readily obtained from the Newtonian approximation

[5] We have $\xi = \frac{8}{3}\,\dot{\omega}/\omega^2$ so our definition agrees with the actual relative frequency change $\propto \dot{\omega}/\omega^2$ in one period. It is also equivalent to the one adopted in [108]: $\xi = \xi_{\mathrm{BS}}/\pi$.

to the quadrupole moment I_{ij} [first term in (3.5)], and cast into the form

$$\begin{aligned}(h_+)_{\text{memory}}(T_R) &= \frac{2G}{c^4R}\frac{G^4m^5\nu^2\sin^2 i}{c^5}\int_{-\infty}^{T_R}\frac{dV}{r^5(V)}\left\{-\frac{12}{5}+\frac{2}{15}\sin^2 i\right.\\ &\qquad\left.+\left(\frac{4}{15}-\frac{2}{15}\sin^2 i\right)\cos[4\phi(V)]\right\}, \quad (3.21a)\\ (h_\times)_{\text{memory}}(T_R) &= \frac{2G}{c^4R}\frac{G^4m^5\nu^2\sin^2 i}{c^5}\int_{-\infty}^{T_R}\frac{dV}{r^5(V)}\left\{\frac{4\cos i}{15}\sin[4\phi(V)]\right\}. \quad (3.21b)\end{aligned}$$

In our model of binary evolution the radius of the orbit, $r(V)$, and orbital phase, $\phi(V)$, are given by Eqs. (3.9) and (3.12), at any time such that $V < T_R < T_c$. Because $r(V) \propto (T_c - V)^{1/4}$ we see that the integrals in (3.21) are perfectly well-defined, and in fact absolutely convergent at the bound $V \to -\infty$. There are two distinct types of terms in (3.21). A term, present only in the *plus* polarisation (3.21a), is independent of the orbital phase ϕ, and given by a steadily varying function of time, having an amplitude increasing like some power law but without any oscillating behaviour. This "steadily increasing" term is specifically responsible for the memory effect. The other terms, present in both polarisations, oscillate with time like some sine or cosine of the phase, in addition of having a steadily increasing maximal amplitude.

Consider first the steadily growing, non-oscillating term. Its computation simply relies, as clear from (3.21a), on the single elementary integral

$$I(T_R) \equiv \frac{(G\,m)^4}{c^7}\int_{-\infty}^{T_R}\frac{dV}{r^5(V)}\,, \qquad (3.22)$$

where we find convenient to factorize out an appropriate coefficient in order to make it dimensionless. The calculation of (3.22) is easily done directly, but it is useful to perform our change of variable (3.13), as an exercise to prepare the treatment of the (somewhat less easy) oscillating terms. Thus, we write in a first stage

$$I(T_R) = \frac{(G\,m)^4}{c^7}\,\frac{T_c - T_R}{r^5(T_R)}\int_0^{+\infty}\frac{dy}{(1+y)^{5/4}}\,. \qquad (3.23)$$

The factor in front is best expressed in terms of the dimensionless PN parameter $x(T_R)$, and of course the remaining integral is trivially integrated. We get

$$I(T_R) = \frac{5}{256\,\nu}\,x(T_R)\int_0^{+\infty}\frac{dy}{(1+y)^{5/4}} = \frac{5}{64\,\nu}\,x(T_R)\,. \qquad (3.24)$$

With this result we obtain the steadily-increasing or memory term in Eq. (3.21a). However, as its name indicates, this term keeps a "memory" of the past activity of the system. As a test of the numerical influence of the binary's past history on Eq. (3.24), let us suppose that the binary was created *ex nihilo* at some finite initial time T_0 in a circular orbit state. This premise is not very realistic – we should more realistically assume *e.g.* a binary capture in some stellar cluster and/or consider an initially eccentric orbit – but it should give an estimate of how sensitive is the memory term on initial conditions. In this crude model we have to consider the integral $I_0(T_R)$ extending from T_0 up to T_R. We find that the ratio of $I_0(T_R)$ and our earlier model $I(T_R)$ is

$$\frac{I_0(T_R)}{I(T_R)} = 1 - \left(\frac{T_c - T_R}{T_c - T_0}\right)^{1/4} = 1 - \frac{r(T_R)}{r(T_0)} . \tag{3.25}$$

Let us choose the observation time T_R such that the binary is visible by the VIRGO/LIGO detectors. At the entry of the detectors' frequency bandwidth, say $\omega_{\rm seismic} \simeq 30\,{\rm Hz}$, we obtain $T_c - T_R \simeq 10^3\,{\rm s}$ and $r(T_R) \simeq 700\,{\rm km}$ in the case of two neutron stars ($m = 2.8\,M_\odot$). For a binary system initially formed on an orbit of the size of the Sun, $r(T_0) \simeq 10^6\,{\rm km}$ (corresponding to $T_c - T_0 \simeq 10^8\,{\rm yr}$), we find that the fractional difference between our two models I_0 and I amounts to about 10^{-3}. For an initial orbit of the size of a white dwarf, $r(T_0) \simeq 10^4\,{\rm km}$, the fractional difference is of the order of 10% – rather large indeed. So we conclude that indeed the memory term in (3.21a), depends rather severely on detailed assumptions concerning the past evolution of the binary system. We shall have to keep this feature in mind when we present our final results for this term.

Turn next our attention to the phase-dependent, oscillating terms in Eqs. (3.21). Clearly these terms are obtained once we know the elementary integral

$$J(T_R) \equiv \frac{(G\,m)^4}{c^7} \int_{-\infty}^{T_R} dV\, \frac{e^{4i\phi(V)}}{r^5(V)} . \tag{3.26}$$

Inserting (3.14) and (3.18) into it we are led to the form [which exactly parallels (3.24)]

$$J(T_R) = \frac{5}{256\,\nu}\, x(T_R)\, e^{4i\phi(T_R)} \int_0^{+\infty} \frac{dy}{(1+y)^{5/4}}\, e^{-\frac{32i}{5\xi(T_R)}\left[(1+y)^{5/8}-1\right]} . \tag{3.27}$$

We shall compute this integral in the form of an approximation series, valid in the adiabatic limit $\xi(T_R) \to 0$. The easiest way to obtain successive approximations is to integrate by parts. We obtain

$$\int_0^{+\infty} \frac{dy}{(1+y)^{5/4}} e^{-\frac{32i}{5\xi}\left[(1+y)^{5/8}-1\right]} = \frac{\xi}{4i}\left\{1 - \frac{7}{8}\int_0^{+\infty} \frac{dy}{(1+y)^{15/8}} e^{-\frac{32i}{5\xi}\left[(1+y)^{5/8}-1\right]}\right\} . \tag{3.28}$$

A further integration by parts shows that the integral in the curly brackets of (3.28) is itself of order ξ, so we have the result [6]

$$\int_0^{+\infty} \frac{dy}{(1+y)^{5/4}} e^{-\frac{32i}{5\xi}\left[(1+y)^{5/8}-1\right]} = \frac{\xi}{4i}\Big\{1+\mathcal{O}(\xi)\Big\}. \tag{3.29}$$

A standard way to understand it is to remark that, when y is different from zero, the phase of the integrand of (3.29) oscillates very rapidly when $\xi \to 0$, so the integral is made of a sum of alternatively positive and negative terms and is essentially zero. Consequently, the value of the integral is essentially given by the contribution due to the bound at $y = 0$, which can be approximated by

$$\int_0 dy\, e^{-\frac{4iy}{\xi}} = \frac{\xi}{4i}\,. \tag{3.30}$$

Because $\xi(T_R)$ is of order 2.5PN the result (3.29) is sufficient for the control of the 2.5PN waveform, thus our elementary integral reads, with the required precision,

$$J(T_R) = x^{7/2}(T_R)\,\frac{e^{4i\phi(T_R)}}{4i}\Big\{1+\mathcal{O}(\xi)\Big\}. \tag{3.31}$$

We find that the "oscillating" integral $J(T_R)$ is an order 2.5PN smaller than the "steadily growing" or "memory" integral $I(T_R)$. This can be interpreted by saying that the cumulative (secular) effect of the integration over the whole binary inspiral in $I(T_R)$ is comparable to the inverse of the order of radiation reaction forces – a quite natural result. By contrast the oscillations in J, due to the sequence of orbital cycles in the entire life of the binary system, compensate (more or less) each other yielding a net result which is 2.5PN smaller than for I. Furthermore, the argument leading to the evaluation of (3.30) shows that J, contrarily to I, is quite insensitive to the details of the binary's past evolution.

Substituting Eqs. (3.24) and (3.31) into (3.21) we finally obtain the hereditary memory-type contributions to the polarisation waveforms as

$$(h_+)_{\text{memory}} = \frac{2\,G\,m\,\nu\,x}{c^2R}\sin^2 i\left\{-\frac{17+\cos^2 i}{96}+\frac{\nu}{30}x^{5/2}(1+\cos^2 i)\sin(4\phi)\right\}, \tag{3.32a}$$

$$(h_\times)_{\text{memory}} = \frac{2\,G\,m\,\nu\,x}{c^2R}\sin^2 i\left\{-\frac{\nu}{15}x^{5/2}\cos i\,\cos(4\phi)\right\}, \tag{3.32b}$$

where of course all quantities correspond to the current time T_R. The phase-independent term is the non-linear memory or Christodoulou effect [155, 156, 157, 57, 58] in the case

[6] By successive integration by parts one generates the asymptotic series (divergent for any value of ξ)

$$\int_0^{+\infty} \frac{dy}{(1+y)^{5/4}} e^{-\frac{32i}{5\xi}\left[(1+y)^{5/8}-1\right]} \sim -8\sum_{n=1}^{+\infty}(5n-3)(5n-8)\cdots(12)(7)\left(\frac{i\xi}{32}\right)^n,$$

where we have used the standard notation $\sim$ for equalities valid in the sense of asymptotic series.

of inspiralling compact binaries. Our calculation, leading to a factor $\propto \sin^2 i(17 + \cos^2 i)$, agrees with the result of Wiseman and Will [156].[7] The non-linear memory term *stricto-sensu* affects the plus polarisation but not the cross polarisation, for which it is exactly *zero*. As we have said, it represents a part of the waveform whose amplitude grows with time, but which is nearly constant over one orbital period. It is therefore essentially a *zero-frequency* (DC) effect, which has rather poor observational consequences in the case of the LIGO-VIRGO detectors, whose frequency bandwidth is always limited from below by $\omega_{\text{seismic}} > 0$. In addition, we know that detecting and analyzing the ICBs relies essentially on monitoring the phase evolution, which in turn is determined by the total gravitational-wave flux. But we have already noticed that the non-linear memory integral (3.4) is instantaneous in the flux (and in fact it does not contribute to the phase in the circular orbit case [98]). It thus seems that the net cumulative "memory" change in the waveform of ICBs is hardly detectable. On the other hand, the frequency-dependent terms found in Eq. (3.32) form an integral part of the 2.5PN waveform.

An important comment is in order. As we have seen the memory effect, *i.e.* the DC term in Eq. (3.32a), is "Newtonian" because the cumulative integration over the binary's past just compensates the formal 2.5PN order of the hereditary integral. Thus we expect that some formal PN terms strictly higher that 2.5PN will actually contribute to the 2.5PN waveform *via* a similar cumulative integration. For instance at the 3.5PN level there will be a memory-type integral in the radiative quadrupole moment U_{ij}, which is quadratic in the mass octupole moment (of the symbolic form $I_{ab<i} \times I_{j>ab}$). After integration over the past using our model of adiabatic inspiral, we expect that the "steadily-growing" part of the integral should yield a DC contribution to the waveform at the relative 1PN order. In the present chapter we do not consider such higher-order post-Newtonian DC contributions, and leave their computation to future work.

3.3.3 Gravitational-wave tails

The tails up to 2.5PN order are given by Eq. (3.3). Because of the logarithmic kernel they involve, the tails are more complicated than simple time anti-derivatives, and they constitute a crucial part of both the waveform and the energy flux, and in particular of the orbital phase with important observational consequences (for a review see [44]).

The computation of tails reduces to the computation of a new type of "elementary" integral, differing from $J(T_R)$ given in (3.26) by the presence of an extra logarithmic factor in

[7] The difference of a factor of -2 of between their and our result is here probably due to a different convention for the polarisation waveforms.

the integrand, and given by [8]

$$K(T_R) \equiv \frac{(G\,m)^4}{c^7} \int_{-\infty}^{T_R} dV\, \frac{e^{4i\phi(V)}}{r^5(V)}\, \ln\left(\frac{T_R - V}{T_c - T_R}\right). \tag{3.33}$$

For convenience we have inserted into the logarithm of (3.33) the constant time scale $T_c - T_R$ instead of the normalization $2b$ more appropriate for tails [or, for instance, $2b\,e^{-11/12}$, see (3.3)], but we can do this at the price of adding another term which will be proportional to $J(T_R)$ already computed in Sec. 3.3.2.

With the help of the y-variable (3.13) we transform the latter integral into

$$K(T_R) = \frac{5}{256\,\nu}\, x(T_R)\, e^{4i\phi(T_R)} \int_0^{+\infty} \frac{dy\, \ln y}{(1+y)^{5/4}}\, e^{-\frac{32i}{5\xi(T_R)}\left[(1+y)^{5/8}-1\right]}. \tag{3.34}$$

Because of the factor $\ln y$ we are not able to directly integrate by parts as we did to compute $J(T_R)$. Instead we must split the integral into some "recent-past" contribution, say $y \in]0,1[$, and the "remote-past" one, $y \in]1,+\infty[$. In the remote-past integral, whose lower bound at $y = 1$ allows for differentiating the factor $\ln y$, we perform the integrations by parts in a way similar to (3.28). This leads to

$$\int_1^{+\infty} \frac{dy\, \ln y}{(1+y)^{5/4}}\, e^{-\frac{32i}{5\xi}\left[(1+y)^{5/8}-1\right]} = \mathcal{O}\left(\xi^2\right), \tag{3.35}$$

which is found to be of the order of $\mathcal{O}\left(\xi^2\right)$ instead of $\mathcal{O}(\xi)$ in Eq. (3.29). This is a consequence of the $\ln y$ which is zero at the bound $y = 1$, and thus kills the all-integrated term. Hence we deduce from (3.35) that the contribution from the remote past in the tail integrals is in fact quite small. The details concerning the remote-past activity of the source are negligible when computing the tails. More precisely, because $\xi = \mathcal{O}(5)$, we can check that terms such as (3.35) do not contribute to the waveform before the 4PN order, so the tail integrals can be legitimately approximated, with 2.5PN accuracy, by their "recent-past" history (in agreement with the findings of [108, 141]).

Now, in the recent-past integral, *i.e.* $y \in]0,1[$, we are allowed to replace the integrand by its equivalent when $y \to 0$, modulo terms of the same magnitude as (3.35). This fact has been proved rigorously in the Appendix B of [108]. Here we shall not reproduce the proof

[8] Actually in order to describe the tails we should consider the more general integrals

$$K_{n,p}(T_R) = \frac{(G\,m)^{p-1}}{c^{2p-3}} \int_{-\infty}^{T_R} dV\, \frac{e^{in\phi(V)}}{r^p(V)}\, \ln\left(\frac{T_R - V}{T_c - T_R}\right).$$

The dominant tails at the 1.5PN order correspond to $p = 4$ and $n = 2$, the tails at the 2PN order have $p = 9/2$ and $n = 1,3$, and those at the 2.5PN order have $p = 5$ and $n = 2,4$. Here we deal with the particular case $p = 5$, $n = 4$ because the calculation parallels the one of $I(T_R)$ and $J(T_R)$ in Sec. 3.3.2. The other cases are treated in a similar way.

but simply state the end result, which reads

$$\int_0^1 \frac{dy\, \ln y}{(1+y)^{5/4}}\, e^{-\frac{32i}{5\xi}\left[(1+y)^{5/8}-1\right]} = \int_0^1 dy\, \ln y\, e^{-\frac{4iy}{\xi}} + \mathcal{O}\left(\xi^2 \ln \xi\right) . \tag{3.36}$$

As we see, the remainder is $\mathcal{O}\left(\xi^2 \ln \xi\right)$, instead of being merely $\mathcal{O}\left(\xi^2\right)$ as in Eq. (3.35). The integral in the R.H.S. of (3.36) gives the main contribution to the tail integral (the only one to be taken into account up to very high 4PN order). It is easily computed by using a standard formula,[9]

$$\int_0^1 dy\, \ln y\, e^{-\frac{4iy}{\xi}} = \frac{\xi}{4i}\left[\frac{\pi}{2i} - \ln\left(\frac{4}{\xi}\right) - C\right] + \mathcal{O}\left(\xi^2\right) , \tag{3.37}$$

where $C = 0.577\cdots$ is the Euler constant.

Finally the required result is

$$K(T_R) = x^{7/2}(T_R)\, \frac{e^{4i\phi(T_R)}}{4i}\left\{\frac{\pi}{2i} - \ln\left(\frac{4}{\xi(T_R)}\right) - C + \mathcal{O}\left(\xi \ln \xi\right)\right\} , \tag{3.38}$$

where it is crucial that all the binary's parameters are evaluated at the current time T_R. It is interesting to compare (3.38) with our earlier result for $J(T_R)$ given by (3.31), in order to see the effect of adding a logarithmic-type kernel into an oscillating, phase-dependent integral. Eq. (3.38), together with its trivial extension to $K_{n,p}$, is used for the computation of all the tails in the waveform at the 2.5PN order (and it could in fact be used up to the order 3.5PN included). Actually we still have to justify this because during the derivation of (3.38) we employed a *Newtonian* model for the binary's inspiral in the past, *e.g.* we assumed the Kepler law $G\,m = r^3(V)\,\omega^2(V)$ at any time $V < T_R$. In the case of the memory term (3.4) this is okay because it needs to be evaluated with Newtonian accuracy. But in the case of tails, Eq. (3.3), the dominant effect it at the 1.5PN order, so we have to take into account a 1PN relative correction in order to control the 2.5PN waveform. Nevertheless, our model of Newtonian inspiral in the past *is* compatible with taking into account 1PN effects, for the basic reason that for tails, the past behaviour of the source is negligible, so the 1PN effects have only to be included into the *current* values of the parameters, *i.e.* $x(T_R)$, $\phi(T_R)$ and $\xi(T_R)$, in Eq. (3.38).

To see more precisely how this works, suppose that we want to replace the "Newtonian"

[9] For any real number ϵ we have

$$\epsilon \int_0^1 dy\, \ln y\, e^{i\epsilon y} + i \int_1^{+\infty} \frac{dy}{y}\, e^{i\epsilon y} = -\frac{\pi}{2}\,\mathrm{sgn}(\epsilon) - i\left(\ln|\epsilon| + C\right) ,$$

where $\mathrm{sgn}(\epsilon)$ and $|\epsilon|$ denote the sign and absolute value of ϵ.

inspiral (3.9) by the more accurate 1PN law,

$$r(V) = 4\left[\frac{G^3m^3\nu}{5c^5}(T_c - V)\right]^{1/4} + \frac{G\,m\,\eta}{c^2}\,, \tag{3.39}$$

where $\eta \equiv -\frac{1751}{1008} - \frac{7}{12}\nu$ (see *e.g.* [44]). Substituting the variable y we obtain the 1PN equivalent of (3.14) as

$$r(V) = r(T_R)(1+y)^{1/4}\Big\{1 + \eta\Big[(1+y)^{-1/4} - 1\Big]x(T_R)\Big\}\,, \tag{3.40}$$

where we parametrized the 1PN correction term by means of the variable x evaluated at *current* time T_R (we consistently neglect higher PN terms). For the orbital phase we get

$$\phi(V) = \phi(T_R) - \frac{8}{5\,\xi(T_R)}\Big\{\Big[(1+y)^{5/8} - 1\Big]\Big(1 + \zeta\,x(T_R)\Big) + \tau\Big[(1+y)^{3/8} - 1\Big]x(T_R)\Big\}\,, \tag{3.41}$$

where $\zeta = -\frac{743}{672} - \frac{11}{8}\nu$, $\tau = \frac{3715}{2016} + \frac{55}{24}\nu$. We insert those expressions into our basic integral (3.33), and split it into recent past and remote past contributions. Exactly like in (3.35) we can prove that the remote past of that integral, for which $y \in]1, +\infty[$, is negligible – of the order of $\mathcal{O}(\xi^2)$. Next, in the recent-past integral, $y \in]0, 1[$, we expand at the 1PN order [*i.e.* $x(T_R) \to 0$], to obtain some 1PN correction term, with respect to the previous calculation, of the form

$$\int_0^1 \frac{dy\,\ln y}{(1+y)^{5/4}}\Big[(1+y)^{\alpha} - 1\Big]\,e^{-\frac{32i}{5\xi}\left[(1+y)^{5/8}-1\right]}\,, \tag{3.42}$$

where α can take the values $-\frac{1}{4}$, $\frac{3}{8}$ or $\frac{5}{8}$. Now the point is that this integral, like the remote-past one, is *also* of the order of $\mathcal{O}(\xi^2)$ or, rather, $\mathcal{O}(\xi^2 \ln\xi)$. Indeed, the new factor $(1+y)^{\alpha} - 1$ in the integrand of (3.42) is crucial in that it adds (after taking the equivalent when $y \to 0$) an extra factor y, and we have thus to treat the following equivalent,

$$\int_0^1 dy\,y\,\ln y\,e^{-\frac{4iy}{\xi}} = \mathcal{O}\Big(\xi^2\ln\xi\Big)\,, \tag{3.43}$$

which is *smaller* by a factor $\xi = \mathcal{O}(5)$ than the integral (3.37), as easily seen by integrating by parts. This means that the order of magnitude of the correction induced by our more sophisticated 1PN model for inspiral in the past is negligible. In conclusion, even at the relative PN order, one can use Eq. (3.38) for computing the tails, but of course the current values of the binary's orbital parameters $x(T_R)$, $\phi(T_R)$ and $\xi(T_R)$ must consistently include their relevant PN corrections. This is what we do in the present chapter, following the computation in Refs. [98, 59] of the higher-order tails up to relative 2PN order (*i.e.* 3.5PN beyond quadrupolar radiation).

Finally our results for the 2.5PN-accurate tail terms are as follows. It is convenient, following [101], to introduce, in place of the "natural" constant time-scale b entering the tails and defined by (2.6), a constant *frequency*-scale ω_0 given by

$$\omega_0 \equiv \frac{e^{\frac{11}{12}-C}}{4\,b}\,. \tag{3.44}$$

Like b, it can be chosen at will, for instance to be equal, as suggested in [101], to the seismic cut-off frequency of some interferometric detector, $\omega_0 = \omega_{\text{seismic}}$. Then we find that all the dependence of the tails in the *logarithm* of the orbital frequency, *i.e.* the terms involving $\ln\omega$ and coming from the logarithm present in the R.H.S. of (3.37), can be factorized out, up to the 2.5PN order, in the way

$$(h_{+,\times})_{\text{tail}} = (k_{+,\times})_{\text{tail}} - 2\,x^{3/2}\left[1-\frac{\nu}{2}x\right]\frac{\partial h_{+,\times}}{\partial\phi}\,\ln\left(\frac{\omega}{\omega_0}\right), \tag{3.45}$$

where all the dependence upon $\ln(\omega/\omega_0)$ is given as indicated [*i.e.* the $(k_{+,\times})_{\text{tail}}$'s are independent of $\ln(\omega/\omega_0)$]. In the above expression the factor $1-\frac{\nu}{2}x$ comes from the relation between the total ADM mass M and the binary's rest mass $m = m_1 + m_2$, namely $M = m\left[1-\frac{\nu}{2}x\right]$. Because we are computing the tails with 1PN relative precision, this means that the factor of $\ln(\omega/\omega_0)$, namely $\partial h_{+,\times}/\partial\phi$ and therefore also $h_{+,\times}$ itself, is given at the relative 1PN order. The existence of this structure implies an elegant formulation of the 2.5PN waveform in terms of a new phase variable ψ given by Eq. (3.48) below. The phase ψ was already introduced in [101], and we have shown here that it is also valid, interestingly enough, for tails at the relative 1PN order. The "main" tails contributions are then given, up to 2.5PN order, by

$$\begin{aligned}
(k_+)_{\text{tail}} &= \frac{2\,G\,m\,\nu\,x}{c^2R}\Big\{-2\pi x^{3/2}(1+c_i^2)\cos 2\phi \\
&\quad +\frac{s_i}{40}\frac{\delta m}{m}x^2\Big[\Big(11+7c_i^2+10(5+c_i^2)\ln 2\Big)\sin\phi \\
&\qquad -5\pi(5+c_i^2)\cos\phi - 27[7-10\ln(3/2)](1+c_i^2)\sin 3\phi \\
&\qquad +135\pi(1+c_i^2)\cos 3\phi\Big] \\
&\quad +x^{5/2}\Big[\frac{\pi}{3}\Big(19+9c_i^2-2c_i^4+\nu(-16+14c_i^2+6c_i^4)\Big)\cos 2\phi \\
&\qquad +\frac{1}{5}\Big(-9+14c_i^2+7c_i^4+\nu(27-42c_i^2-21c_i^4)\Big)\sin 2\phi \\
&\qquad -\frac{16\pi}{3}(1-c_i^4)(1-3\nu)\cos 4\phi \\
&\qquad +\frac{8}{15}(1-c_i^4)(1-3\nu)(21-20\ln 2)\sin 4\phi\Big]\Big\}\,,
\end{aligned} \tag{3.46a}$$

$$(k_\times)_{\text{tail}} = \frac{2\,G\,m\,\nu\,x}{c^2R}\Big\{-4\pi x^{3/2}c_i\sin 2\phi$$

$$
\begin{aligned}
&-\frac{3\,s_i\,c_i}{20}\frac{\delta m}{m}x^2\,[(3+10\ln 2)\cos\phi+5\pi\sin\phi\\
&\qquad -9[7-10\ln(3/2)]\cos 3\phi-45\pi\sin 3\phi]\\
&+x^{5/2}\left[\frac{2\pi}{3}c_i\left(13+4\,s_i^2+\nu(2-12\,s_i^2)\right)\sin 2\phi\right.\\
&\qquad+\frac{2}{5}c_i(1-3\nu)\left(-6+11\,s_i^2\right)\cos 2\phi\\
&\qquad-\frac{32\pi}{3}c_i\,s_i^2(1-3\nu)\sin 4\phi\\
&\qquad\left.\left.+\frac{16}{15}c_i\,s_i^2(1-3\nu)\,(-21+20\ln 2)\cos 4\phi\right]\right\}\,. \qquad (3.46b)
\end{aligned}
$$

Here c_i and s_i denote the cosine and sine of the inclination angle i, and $\delta m = m_1 - m_2$ is the mass difference (so that $\delta m/m = X_1 - X_2$). Up to the 2PN order, we have agreement with the results of [101].

3.4 Results for the 2.5PN polarisation waveforms

3.4.1 The complete plus and cross polarisations

We have computed all the five different contributions to the waveform contained in Eq. (3.1). This together with the results of [140] provides a complete 2.5PN accurate waveform for the circular orbit case. In this Section, from the 2.5PN waveform, we present the result for the two gravitational-wave polarisations, extending a similar analysis at the 2PN order in Ref. [101].

The polarisations corresponding to the instantaneous terms are computed using Eqs. (2.29) and (2.30), while those corresponding to the hereditary terms where obtained in Eqs. (3.32) and (3.45)–(3.46). As in the earlier work [101], these polarisations are represented in terms of the gauge invariant parameter $x \equiv (Gm\omega/c^3)^{2/3}$, where ω represents the orbital frequency of the circular orbit, accurate up to 2.5PN order. This requires the relation between γ and x, which has already been given in Eq. (2.20). The final form of the 2.5PN polarisations may now be written as,

$$
h_{+,\times}=\frac{2\,G\,m\,\nu\,x}{c^2\,R}\left\{H^{(0)}_{+,\times}+x^{1/2}\,H^{(1/2)}_{+,\times}+x\,H^{(1)}_{+,\times}+x^{3/2}\,H^{(3/2)}_{+,\times}+x^2\,H^{(2)}_{+,\times}+x^{5/2}\,H^{(5/2)}_{+,\times}\right\}\,. \qquad (3.47)
$$

In particular, we shall recover the 2PN results of [101]. However, for the comparison we have to employ the same phase variable as in [101], which means introducing an auxiliary phase variable ψ, shifted away from the actual orbital phase ϕ we have used up to now, by Eq. (5) of [101]. Furthermore, the phase ψ given in [101] is *a priori* adequate up to only the 2PN order, but we have proved it to be also correct at the higher 2.5PN order. Indeed, the motivation for the shift $\phi \longrightarrow \psi$ is to "remove" all the logarithms of the frequency [*i.e.* $\ln\omega$ or, rather, $\ln(\omega/\omega_0)$] in the two polarisation waveforms and to absorb them into the definition of the new phase angle ψ. As a result, the two polarisation waveforms, when expressed in terms of ψ instead of ϕ, become substantially simpler. From Eq. (3.45) we see that if we re-express the waveform by means of the phase [101]

$$\psi = \phi - 2x^{3/2}\left[1 - \frac{\nu}{2}x\right]\ln\left(\frac{\omega}{\omega_0}\right), \tag{3.48}$$

we are able to move all the $\ln\omega$-terms into the phase angle. Notice that the possibility of this move is interesting because it shows that in fact the $\ln\omega$-terms, which were originally computed as some modification of the wave *amplitude* at orders 1.5PN, 2PN and 2.5PN, now appear as a modulation of the *phase* of the wave at the relative orders 4PN, 4.5PN and 5PN. The reason is that the lowest-order phase evolution is at the inverse of the order of radiation-reaction, *i.e.* $c^5 = \mathcal{O}(-5)$, so as usual there is a difference of 2.5PN order between amplitude and phase. This shows therefore that the modification of the phase in (3.48) is presently negligible (it is of the same order of magnitude as unknown 4PN terms in the orbital phase evolution when it is given as a function of time). It could be ignored in practice, but it is probably better to keep it as it stands the definition of templates of ICBs. The phase shift (3.48) corresponds to some spreading of the different frequency components of the wave, *i.e.* the "wave packets" composing it, along the line of sight from the source to the detector (see [108] for a discussion).

With this above choice of the phase variable, the same as in [101], all terms up to 2PN match with those listed in Eqs. (3) and (4) of [101], though we recast them in a slightly different form for our convenience to present the 2.5PN terms. We find,

$$H_+^{(0)} = -(1+c_i^2)\cos 2\psi - \frac{1}{96}\, s_i^2\,(17+c_i^2)\,, \tag{3.49a}$$

$$H_+^{(0.5)} = -\, s_i\frac{\delta m}{m}\left[\cos\psi\left(\frac{5}{8}+\frac{1}{8}\,c_i^2\right) - \cos 3\psi\left(\frac{9}{8}+\frac{9}{8}\,c_i^2\right)\right], \tag{3.49b}$$

$$\begin{aligned} H_+^{(1)} = {}& \cos 2\psi\left[\frac{19}{6}+\frac{3}{2}\,c_i^2-\frac{1}{3}\,c_i^4+\nu\left(-\frac{19}{6}+\frac{11}{6}\,c_i^2+c_i^4\right)\right] \\ & -\cos 4\psi\left[\frac{4}{3}\,s_i^2(1+c_i^2)(1-3\nu)\right], \end{aligned} \tag{3.49c}$$

$$\begin{aligned} H_+^{(1.5)} = {}& s_i\frac{\delta m}{m}\cos\psi\left[\frac{19}{64}+\frac{5}{16}\,c_i^2-\frac{1}{192}\,c_i^4+\nu\left(-\frac{49}{96}+\frac{1}{8}\,c_i^2+\frac{1}{96}\,c_i^4\right)\right] \\ & +\cos 2\psi\left[-2\pi(1+c_i^2)\right] \\ & + s_i\frac{\delta m}{m}\cos 3\psi\left[-\frac{657}{128}-\frac{45}{16}\,c_i^2+\frac{81}{128}\,c_i^4\right. \end{aligned}$$

$$
\begin{aligned}
& + \nu \left(\frac{225}{64} - \frac{9}{8} c_i^2 - \frac{81}{64} c_i^4 \right) \Big] \\
& + s_i \frac{\delta m}{m} \cos 5\psi \left[\frac{625}{384} s_i^2 (1 + c_i^2)(1 - 2\nu) \right] , \qquad (3.49d)
\end{aligned}
$$

$$
\begin{aligned}
H_+^{(2)} \;=\;& \pi\, s_i \frac{\delta m}{m} \cos\psi \left[-\frac{5}{8} - \frac{1}{8} c_i^2 \right] \\
& + \cos 2\psi \Big[\frac{11}{60} + \frac{33}{10} c_i^2 + \frac{29}{24} c_i^4 - \frac{1}{24} c_i^6 \\
& \quad + \nu \left(\frac{353}{36} - 3\, c_i^2 - \frac{251}{72} c_i^4 + \frac{5}{24} c_i^6 \right) \\
& \quad + \nu^2 \left(-\frac{49}{12} + \frac{9}{2} c_i^2 - \frac{7}{24} c_i^4 - \frac{5}{24} c_i^6 \right) \Big] \\
& + \pi\, s_i \frac{\delta m}{m} \cos 3\psi \left[\frac{27}{8} (1 + c_i^2) \right] \\
& + \cos 4\psi \Big[\frac{118}{15} - \frac{16}{5} c_i^2 - \frac{86}{15} c_i^4 + \frac{16}{15} c_i^6 \\
& \quad + \nu \left(-\frac{262}{9} + 16\, c_i^2 + \frac{166}{9} c_i^4 - \frac{16}{3} c_i^6 \right) \\
& \quad + \nu^2 \left(14 - 16\, c_i^2 - \frac{10}{3} c_i^4 + \frac{16}{3} c_i^6 \right) \Big] \\
& + \cos 6\psi \left[-\frac{81}{40} s_i^4 (1 + c_i^2) \left(1 - 5\nu + 5\nu^2 \right) \right] \\
& + s_i \frac{\delta m}{m} \sin\psi \left[\frac{11}{40} + \frac{5 \ln 2}{4} + c_i^2 \left(\frac{7}{40} + \frac{\ln 2}{4} \right) \right] \\
& + s_i \frac{\delta m}{m} \sin 3\psi \left[\left(-\frac{189}{40} + \frac{27}{4} \ln(3/2) \right) (1 + c_i^2) \right] , \qquad (3.49e)
\end{aligned}
$$

$$
\begin{aligned}
H_\times^{(0)} \;=\;& -2 c_i \sin 2\psi , \qquad (3.50a) \\
H_\times^{(0.5)} \;=\;& s_i c_i \frac{\delta m}{m} \left[-\frac{3}{4} \sin\psi + \frac{9}{4} \sin 3\psi \right] , \qquad (3.50b) \\
H_\times^{(1)} \;=\;& c_i \sin 2\psi \left[\frac{17}{3} - \frac{4}{3} c_i^2 + \nu \left(-\frac{13}{3} + 4\, c_i^2 \right) \right] \\
& + c_i\, s_i^2 \sin 4\psi \left[-\frac{8}{3} (1 - 3\nu) \right] , \qquad (3.50c) \\
H_\times^{(1.5)} \;=\;& s_i c_i \frac{\delta m}{m} \sin\psi \left[\frac{21}{32} - \frac{5}{96} c_i^2 + \nu \left(-\frac{23}{48} + \frac{5}{48} c_i^2 \right) \right] \\
& - 4\pi\, c_i \sin 2\psi \\
& + s_i c_i \frac{\delta m}{m} \sin 3\psi \left[-\frac{603}{64} + \frac{135}{64} c_i^2 + \nu \left(\frac{171}{32} - \frac{135}{32} c_i^2 \right) \right]
\end{aligned}
$$

$$+ s_i c_i \frac{\delta m}{m} \sin 5\psi \left[\frac{625}{192}(1-2\nu)\, s_i^2\right], \tag{3.50d}$$

$$\begin{aligned}
H_\times^{(2)} = \; & s_i c_i \frac{\delta m}{m} \cos\psi \left[-\frac{9}{20} - \frac{3}{2}\ln 2\right] \\
& + s_i c_i \frac{\delta m}{m} \cos 3\psi \left[\frac{189}{20} - \frac{27}{2}\ln(3/2)\right] \\
& - s_i c_i \frac{\delta m}{m} \left[\frac{3\pi}{4}\right] \sin\psi \\
& + c_i \sin 2\psi \left[\frac{17}{15} + \frac{113}{30} c_i^2 - \frac{1}{4} c_i^4 \right. \\
& \quad + \nu\left(\frac{143}{9} - \frac{245}{18} c_i^2 + \frac{5}{4} c_i^4\right) \\
& \quad \left. + \nu^2\left(-\frac{14}{3} + \frac{35}{6} c_i^2 - \frac{5}{4} c_i^4\right)\right] \\
& + s_i c_i \frac{\delta m}{m} \sin 3\psi \left[\frac{27\pi}{4}\right] \\
& + c_i \sin 4\psi \left[\frac{44}{3} - \frac{268}{15} c_i^2 + \frac{16}{5} c_i^4 \right. \\
& \quad + \nu\left(-\frac{476}{9} + \frac{620}{9} c_i^2 - 16\, c_i^4\right) \\
& \quad \left. + \nu^2\left(\frac{68}{3} - \frac{116}{3} c_i^2 + 16\, c_i^4\right)\right] \\
& + c_i \sin 6\psi \left[-\frac{81}{20} s_i^4 (1 - 5\nu + 5\nu^2)\right].
\end{aligned} \tag{3.50e}$$

Notice a difference with the results of [101], in that we have included the specific effect of *non-linear memory* the polarization waveform at the *Newtonian* order, *c.f.* the term proportional to $s_i^2(17 + c_i^2)$ in $H_+^{(0)}$ given by (3.49a) above. This is consistent with the order of magnitude of this effect, calculated in Sec. 3.3.2. However, beware of the fact that the memory effect is rather sensitive to the details of the entire time-evolution of the binary prior to the current detection, so the zero-frequency (DC) term we have included (3.49a) may change depending on the binary's past history (see our discussion in Sec. 3.3.2). Nevertheless, we feel that it is a good point to include the "Newtonian" non-linear memory effect, exactly as it is given in Eq. (3.49a), for the detection and analysis of ICBs.[10]

The purely 2.5PN contributions, in the plus and cross polarisations, constitute, together

[10] We already remarked that we have not computed the DC terms possibly present in the higher-order harmonics of the 2.5PN waveform.

with the memory term in (3.49a), the final result of this chapter. They read as,

$$
\begin{aligned}
H_+^{(2.5)} = {} & s_i \frac{\delta m}{m} \cos\psi \left[\frac{1771}{5120} - \frac{1667}{5120} c_i^2 + \frac{217}{9216} c_i^4 - \frac{1}{9216} c_i^6 \right. \\
& + \nu \left(\frac{681}{256} + \frac{13}{768} c_i^2 - \frac{35}{768} c_i^4 + \frac{1}{2304} c_i^6 \right) \\
& \left. + \nu^2 \left(-\frac{3451}{9216} + \frac{673}{3072} c_i^2 - \frac{5}{9216} c_i^4 - \frac{1}{3072} c_i^6 \right) \right] \\
& + \pi \cos 2\psi \left[\frac{19}{3} + 3 c_i^2 - \frac{2}{3} c_i^4 + \nu \left(-\frac{16}{3} + \frac{14}{3} c_i^2 + 2 c_i^4 \right) \right] \\
& + s_i \frac{\delta m}{m} \cos 3\psi \left[\frac{3537}{1024} - \frac{22977}{5120} c_i^2 - \frac{15309}{5120} c_i^4 + \frac{729}{5120} c_i^6 \right. \\
& + \nu \left(-\frac{23829}{1280} + \frac{5529}{1280} c_i^2 + \frac{7749}{1280} c_i^4 - \frac{729}{1280} c_i^6 \right) \\
& \left. + \nu^2 \left(\frac{29127}{5120} - \frac{27267}{5120} c_i^2 - \frac{1647}{5120} c_i^4 + \frac{2187}{5120} c_i^6 \right) \right] \\
& + \cos 4\psi \left[-\frac{16\pi}{3} (1 + c_i^2) s_i^2 (1 - 3\nu) \right] \\
& + s_i \frac{\delta m}{m} \cos 5\psi \left[-\frac{108125}{9216} + \frac{40625}{9216} c_i^2 + \frac{83125}{9216} c_i^4 - \frac{15625}{9216} c_i^6 \right. \\
& + \nu \left(\frac{8125}{256} - \frac{40625}{2304} c_i^2 - \frac{48125}{2304} c_i^4 + \frac{15625}{2304} c_i^6 \right) \\
& \left. + \nu^2 \left(-\frac{119375}{9216} + \frac{40625}{3072} c_i^2 + \frac{44375}{9216} c_i^4 - \frac{15625}{3072} c_i^6 \right) \right] \\
& + \frac{\delta m}{m} \cos 7\psi \left[\frac{117649}{46080} s_i^5 (1 + c_i^2)(1 - 4\nu + 3\nu^2) \right] \\
& + \sin 2\psi \left[-\frac{9}{5} + \frac{14}{5} c_i^2 + \frac{7}{5} c_i^4 + \nu \left(\frac{96}{5} - \frac{8}{5} c_i^2 - \frac{28}{5} c_i^4 \right) \right] \\
& + s_i^2 (1 + c_i^2) \sin 4\psi \left[\frac{56}{5} - \frac{32 \ln 2}{3} - \nu \left(\frac{1193}{30} - 32 \ln 2 \right) \right] .
\end{aligned} \tag{3.51}
$$

$$
\begin{aligned}
H_\times^{(2.5)} = {} & \frac{6}{5} s_i^2 c_i \nu \\
& + c_i \cos 2\psi \left[2 - \frac{22}{5} c_i^2 + \nu \left(-\frac{154}{5} + \frac{94}{5} c_i^2 \right) \right] \\
& + c_i s_i^2 \cos 4\psi \left[-\frac{112}{5} + \frac{64}{3} \ln 2 + \nu \left(\frac{1193}{15} - 64 \ln 2 \right) \right] \\
& + s_i c_i \frac{\delta m}{m} \sin\psi \left[-\frac{913}{7680} + \frac{1891}{11520} c_i^2 - \frac{7}{4608} c_i^4 \right.
\end{aligned} \tag{3.52}
$$

$$
\begin{aligned}
&+\nu\left(\frac{1165}{384}-\frac{235}{576}\,c_i^2+\frac{7}{1152}\,c_i^4\right)\\
&+\nu^2\left(-\frac{1301}{4608}+\frac{301}{2304}\,c_i^2-\frac{7}{1536}\,c_i^4\right)\Bigg]\\
&+\pi\,c_i\sin 2\psi\left[\frac{34}{3}-\frac{8}{3}\,c_i^2-\nu\left(\frac{20}{3}-8\,c_i^2\right)\right]\\
&+s_i\,c_i\frac{\delta m}{m}\sin 3\psi\left[\frac{12501}{2560}-\frac{12069}{1280}\,c_i^2+\frac{1701}{2560}\,c_i^4\right.\\
&+\nu\left(-\frac{19581}{640}+\frac{7821}{320}\,c_i^2-\frac{1701}{640}\,c_i^4\right)\\
&\left.+\nu^2\left(\frac{18903}{2560}-\frac{11403}{1280}\,c_i^2+\frac{5103}{2560}\,c_i^4\right)\right]\\
&+s_i^2\,c_i\sin 4\psi\left[-\frac{32\pi}{3}(1-3\nu)\right]\\
&+\frac{\delta m}{m}\,s_i\,c_i\sin 5\psi\left[-\frac{101875}{4608}+\frac{6875}{256}\,c_i^2-\frac{21875}{4608}\,c_i^4\right.\\
&+\nu\left(\frac{66875}{1152}-\frac{44375}{576}\,c_i^2+\frac{21875}{1152}\,c_i^4\right)\\
&\left.+\nu^2\left(-\frac{100625}{4608}+\frac{83125}{2304}\,c_i^2-\frac{21875}{1536}\,c_i^4\right)\right]\\
&+\frac{\delta m}{m}\,s_i^5\,c_i\sin 7\psi\left[\frac{117649}{23040}\left(1-4\nu+3\nu^2\right)\right].
\end{aligned}
$$

Note that the latter cross polarisation contains a zero-frequency term [first term in Eq. (3.52)], which comes from the inst(r) contribution given by (2.30). We employ the same notation as in [101], except that c_i and s_i denote respectively cosine and sine of the inclination angle i (which is defined as the angle between the vector $\mathbf{N}$, along the line of sight from the binary to the detector, and the normal to the orbital plane, chosen to be right handed with respect to the sense of motion, so that $0 \le i \le \pi$). In particular the mass difference reads $\delta m = m_1 - m_2$. Like in [101], our results are in terms of an alternate phase variable ψ related to the actual orbital phase ϕ (namely the angle oriented in the sense of motion between the ascending node $\mathcal{N}$ and direction of body one – *i.e.* $\phi = 0 \mod 2\pi$ when the two bodies lie along $\mathbf{p}$) by a transformation given by Eq. (3.48). We have verified that the plus and cross polarizations (3.51)-(3.52) reduce in the limit $\nu \to 0$ to the result of black hole perturbation theory as given in the Appendix B of Tagoshi and Sasaki [135] (the phase variable used in [135] differs from ours by $\psi_{\rm TS} = \psi + \pi/2 + 2x^{3/2}[\ln 2 - 17/12]$ and we have $\theta_{\rm TS} = \pi - i$).[11].

Equations (3.51) and (3.52), together with (3.49)–(3.50), provide the 2.5PN accurate tem-

[11]We spotted a misprint in the Appendix B of [135], namely the sign of the harmonic coefficient $\zeta^{\times}_{7,3}$ (*i.e.* having $l = 7$, $m = 3$, and corresponding to the cross polarisation) should be changed, so that one should read $\zeta^{\times}_{7,3} = +\frac{729}{10250240}\cos(\theta)(167 + ...)\sin(\theta)(v^5\cos(3\psi) - ...)$.

plate for the ICBs moving on quasi-circular orbits, extending the results of [101] by half a PN order. They are complete except for the possible inclusion of memory-type (zero-frequency or DC) contributions in higher PN amplitudes (1PN and 2PN). These wave polarisations together with the phasing formula of Ref. [99], *i.e.* the crucial time variation of the phase $\phi(t)$, constitute the currently best available templates for the data analysis of ICB for ground based as well as space-borne GW interferometers.

3.4.2 Comments on the 3PN hereditary waveform

In this section we comment on the further inputs needed for the computation of the hereditary terms in the 3PN GW polarization. This complements the list given in Sec 2.3.2 of chapter 2.

Concerning the inst(c) terms, which are the instantaneous terms coming from the difference between the canonical moments M_L, S_L and the general source ones I_L, J_L, $\cdots$ [*c.f.* Eqs. (2.7)–(2.8)], it does not seem simple to even guess their structure. The crucial new input we would need at 3PN order concerns the relation between the *canonical* mass octupole M_{ijk} (and current quadrupole S_{ij}) to the corresponding *source* octupole moments moments I_{ijk} (and current quadrupole J_{ij}) at 2.5PN order, using for instance an analysis similar to the one in [98].

Finally, at 3PN order we would have to extend the present computation of hereditary terms. In the case of quadratic tails, like in (3.3), the computation would probably be straightforward (indeed we have seen in Sec. 3.3.3 that the complications due to the influence of the model of adiabatic inspiral in the past appear only at the 4PN order), but we have also to take into account the tail-of-tail cubic contribution in the mass-quadrupole moment at 3PN order, given in Eq. (4.13) of Ref. [59]. In addition the analysis should be extended to the non-linear memory terms. The complete 3PN waveform and polarisations can be computed only after all the points listed above are addressed.

3.5 Concluding remarks

3.5.1 Summary of the results

Using the multipolar post-Minkowskian formalism, we have computed the expression for the 2.5PN accurate 'plus' and 'cross' polarizations. The 'instantaneous' and the 'hereditary' contributions are computed separately. This together with the 3.5PN accurate phasing formula of Ref [99, 100] provides the most accurate complete waveform for the nonspinning inspiralling compact binaries in circular orbits. For completeness we present below the 3.5PN phasing formula of [99].

$$
\begin{aligned}
\phi &= -\frac{1}{\nu}\left\{\tau^{5/8} + \left(\frac{3715}{8064} + \frac{55}{96}\nu\right)\tau^{3/8} - \frac{3}{4}\pi\tau^{1/4}\right. \\
&+ \left(\frac{9275495}{14450688} + \frac{284875}{258048}\nu + \frac{1855}{2048}\nu^2\right)\tau^{1/8} + \left(-\frac{38645}{172032} + \frac{65}{2048}\nu\right)\pi\ln\left(\frac{\tau}{\tau_0}\right) \\
&+ \left(\frac{831032450749357}{57682522275840} - \frac{53}{40}\pi^2 - \frac{107}{56}C + \frac{107}{448}\ln\left(\frac{\tau}{256}\right)\right. \\
&+ \left.\left[-\frac{123292747421}{4161798144} + \frac{2255}{2048}\pi^2 + \frac{385}{48}\lambda - \frac{55}{16}\theta\right]\nu + \frac{154565}{1835008}\nu^2 - \frac{1179625}{1769472}\nu^3\right)\tau^{-1/8} \\
&+ \left.\left(\frac{188516689}{173408256} + \frac{488825}{516096}\nu - \frac{141769}{516096}\nu^2\right)\pi\tau^{-1/4}\right\}, \qquad (3.53)\\
x &= \frac{1}{4}\tau^{-1/4}\left\{1 + \left(\frac{743}{4032} + \frac{11}{48}\nu\right)\tau^{-1/4} - \frac{1}{5}\pi\tau^{-3/8}\right. \\
&+ \left(\frac{19583}{254016} + \frac{24401}{193536}\nu + \frac{31}{288}\nu^2\right)\tau^{-1/2} + \left(-\frac{11891}{53760} + \frac{109}{1920}\nu\right)\pi\tau^{-5/8} \\
&+ \left(-\frac{10052469856691}{6008596070400} + \frac{1}{6}\pi^2 + \frac{107}{420}C - \frac{107}{3360}\ln\left(\frac{\tau}{256}\right)\right. \\
&+ \left.\left[\frac{15335597827}{3901685760} - \frac{451}{3072}\pi^2 - \frac{77}{72}\lambda + \frac{11}{24}\theta\right]\nu - \frac{15211}{442368}\nu^2 + \frac{25565}{331776}\nu^3\right)\tau^{-3/4} \\
&+ \left.\left(-\frac{113868647}{433520640} - \frac{31821}{143360}\nu + \frac{294941}{3870720}\nu^2\right)\pi\tau^{-7/8}\right\}, \qquad (3.54)
\end{aligned}
$$

where x_0 is determined by the initial conditions, t_c is the time for coalescence and τ is defined by

$$\tau = \frac{\nu c^3}{5Gm}(t_c - t)\,, \qquad (3.55)$$

The ambiguity parameters appearing in the above formula have been fixed and are given by $\lambda = -\frac{1987}{3080} \simeq -0.6451$ and $\theta = -\frac{11831}{9240} \simeq -1.28$ [100].

The present work extends the earlier calculation of Ref [101]. The complete waveform would be useful for data analysis (both detection and parameter estimation) of ground based GW detectors as well as space based ones.

3.5.2 Implications of the full waveform for data analysis: Recent progress

As mentioned in Sec. 2.1, the full waveform incorporating the amplitude corrections from the higher PN order polarizations could be important for many data analysis purposes and one needs to investigate this possibility and implications systematically.

Recently Van den Broeck [125] initiated this by examining the change in SNR of the

GWs when one includes the amplitude corrections coming from polarizations up to 2.5PN computed in this thesis [102]. There are two kinds of effects when one goes beyond the standard restricted waveform approximation. One is the amplitude correction to the dominant harmonic which is twice the orbital frequency and the other is the appearance of higher harmonics. Van den Broeck showed that in the estimation of SNRs the former plays a more important role than the latter. The implications of the full 2.5PN waveform (which means 2.5PN amplitude terms together with the 3.5PN phasing of [99]) in the context of parameter estimation is presently under investigation [126].

3.6 Future Directions

We summarize below some of the applications of the 2.5PN GW polarizations discussed in this chapter and the previous one and possible generalizations of them in the future.

1. The generalization of the present work including the spin-effects will be a valuable asset to the field and would complement the 2.5PN phasing recently computed in Refs [160, 119]. One would need some of the multipole moments to higher accuracy than those provided in Refs [119], the most important ones being J_{ijk} and I_{ijkl} up to 1.5PN.

2. Including the effects of orbital eccentricity in the 2.5PN waveform calculation would be another step forward. This would be generalizing the earlier work by Ref [111]. This will involve also tackling the tail terms in the waveforms at 1.5PN, 2PN and 2.5PN and also the memory terms at 2.5PN. In calculating the tails, one might follow the approach of Refs [115] though tails in the waveforms may prove to be more difficult to handle than those in the fluxes.

3. One should exhaustively analyse the implications of the full 2.5PN waveform computed here beyond that in Ref [125]. This would involve studying the effect of the higher harmonics and amplitude corrections to the leading harmonic in the case of parameter estimation of the binary for the ground based and space based detectors. The recent proposal to test the nonlinear aspects of gravity using GW observations [22, 23] should be revisited with the full waveform. There can be new dimensions to the whole problem due to the presence of $\delta m/m$, which appears in the full waveform since this term is proportional to the difference of the masses whereas the terms in the phase are proportional to the total mass.

4. The measurability of the 'memory' term, which is the most important feature of the hereditary contribution at 2.5PN presented here, should be examined in detail in the

future. Though there have been earlier work along these lines [161, 157, 162], a study focused on the ‘memory’ effect of the inspiral waveform would be interesting, with the explicit expressions we provide for this effect, especially in the LISA context.