Chapter 2

2.5PN gravitational wave polarizations from inspiralling compact binaries on circular orbits: The instantaneous terms

2.1 Introduction

Among the different issues related to waveform modelling discussed in chapter 1, the computation of the higher PN order GW amplitude, or in other words the two independent GW polarizations, to five halves PN order will be the focus of discussion in this chapter and the one to follow. The details of the motivation and the background is dealt with in the next section.

2.1.1 Background

The "chirp" from the inspiral of two compact objects is one of the most plausible GW signals the ground based GW detectors such as LIGO [17], VIRGO [18] and the space based detector LISA [138] would look for. The early inspiral will fall in the sensitivity band of the space-based detectors, where as, the late inspiral will be a good candidate source for the ground-based detectors. Though these GWs are extremely weak and buried deep in the detector noise, the large number of precisely predictable cycles in the detector bandwidth would push the signal up to the level of detection. One can then use the technique of matched filtering first for the detection of GW and later for the estimation of the parameters of the binary. In order to have a good detection, it is extremely important to cross-correlate the detector output with a number of copies of the theoretically predicted signal (corresponding to different signal parameters) which is as precise as possible, and which remains in accurate phase with the signal in the sensitivity bandwidth of the detector. This has made general relativistic modelling of the inspiralling compact binary (ICB) one of the most demanding requirements for GW data analysis [96, 103, 136, 105, 104, 36, 46, 47].

The data analysis of a GW chirp signal has two aspects to it. The first is related to the *detection* of the signal with high enough signal to noise ratio (SNR) and meeting the required statistical criteria. The second aspect on the other hand relates to the process of *parameter estimation* of the binary. These parameters include the masses, spins, distance to the binary, its location and orientation in the sky. It is the latter process that is more important for astrophysical purposes.

2.1.2 Motivation for higher PN order non-restricted waveforms

In matched filtering, one generally tends to use the so-called restricted waveform (RWF) model, where the phase of the wave is modelled to the highest possible post-Newtonian (PN) accuracy retaining the amplitude at its lowest Newtonian order involving the main signal harmonic at twice the orbital frequency [96]. The usual justification for this model is that while filtering the data the phase information is more important for larger overlap between the signal and the template. Though for detection the RWF approximation may be enough [103, 47, 129, 137], the PN corrections to the amplitude will carry useful information helping better parameter estimation.

Recently, studies using the complete waveform, which includes the contribution from higher harmonics besides the dominant one, have shown that it may play a vital role in parameter estimation [124, 123, 121, 122]. The complete waveform carries information which can break the degeneracy of the model, and allow one to estimate the otherwise badly correlated parameters. In the case of a chirping neutron star binary, the masses of the individual stars can be better extracted because of the mass dependence of the higher harmonics [124]. In the case of black hole binaries, whose frequencies are too low to be seen in the detector sensitivity window for long, higher harmonics compensate for the information lost when the signal does not last long enough to be apparent in the data [124]. An independent study [123] about the angular resolution of the space-based LISA-type gravitational wave detectors with a time domain 2PN waveform, showed the importance of including higher PN corrections to the wave amplitude in predicting the angular resolution of the elliptic-plane detector configuration. This is natural since higher order phasing terms carry hardly any new information about the location and orientation of the source whereas the two polarizations introduce terms with explicit dependence on the angular parameters which enables better estimation of the angular resolution of the source.

With this motivation, in the present work we provide the complete 2.5PN accurate 'plus' and 'cross' polarizations of binaries in quasi-circular orbits. We assume that the binary moves in an orbit which is circular. This is the case for most of the binaries which are in the late stages of inspiral since GW radiation reaction would circularize the orbit [95, 139]. Both for ground based and space detectors there could be sources with non-negligible eccentricity and the issues related to the construction of templates for such systems will be addressed in detail in chapter 6. Also, the present analysis concerns non-spinning binaries. Though we do not consider the spinning binaries, its is worth pointing out that the GW polarizations including the spin effects are complete up to 2PN till date [117, 118, 120].

2.1.3 An overview of current calculations of radiation from ICBs

Currently, post-Newtonian (PN) theory provides the most satisfactory description of the dynamics of ICBs and gravitational radiation emitted by them. Starting from the gravitational-wave generation formalism based on the multipolar post-Minkowskian expansions (see the next Section), the gravitational waveform and energy flux at the 2PN order [1] were computed by Blanchet, Damour and Iyer [140]. This incorporated the tail contribution at 1.5PN order both in the waveform and in the energy flux; the polarisation states corresponding to the 1.5PN waveform were calculated in Ref. [141] (note that some algebraic errors in this reference are corrected in [101]). The 2PN results have been independently obtained using a direct integration of the relaxed Einstein field equation [67, 97]. The associated polarisation states (*i.e.* the "plus" and "cross" polarisation waveforms) were obtained in Ref. [101]. These works provided accurate theoretical templates which are currently used for data analysis in all the laser interferometric GW detectors like LIGO and VIRGO. Extending the wave-generation formalism, the 2.5PN term in the energy flux, which arises from a subdominant tail effect, was added in Ref. [98]. In the case of binaries moving in quasi-elliptical orbits, the instantaneous parts of the waveform, energy flux and angular momentum flux at 2PN order were computed by Gopakumar and Iyer [111]. The polarisations of the waveform at this order (in the adiabatic approximation) has been obtained more recently [142], and the phasing of binaries in inspiralling eccentric orbits has also been discussed [112].

The extension of the gravitational wave generation formalism to third post-Newtonian order, and the computation of the energy flux up to 3.5PN accuracy, was achieved in [143]. To this order, in addition to the "instantaneous" contributions, coming from relativistic corrections in the multipole moments of the source, the results include several effects of tails, and tails generated by tails. But, unlike at the 2PN or 2.5PN order, where the calculation is free from ambiguities, at the 3PN order the incompleteness of the Hadamard self-field regularisation leads to some undetermined constants in the mass quadrupole moment of point particle binaries (we comment more on this below). On the other hand, the computation of

[1] As usual the nPN order refers either to the terms $\sim 1/c^{2n}$ in the equations of motion, with respect to the usual Newtonian acceleration, or in the radiation field, *relative* to the standard quadrupolar waveform.

the binary's flux crucially requires the 3PN equations of motion (EOM). These were obtained earlier by two independent calculations, one based on the ADM Hamiltonian formalism of general relativity [144, 145, 146, 147], the other on the direct 3PN iteration of the Einstein field equations in harmonic coordinates [148, 149, 150, 151]. Both approaches lead to an undetermined constant parameter at 3PN when using a Hadamard regularisation, but this constant has now been fixed using a dimensional regularisation [76, 73]. An independent method [152, 153, 154], using surface integrals together with a strong field point particle limit, has yielded results for the 3PN EOM in agreement with those of the first two methods. In particular, the EOM so obtained are independent of any ambiguity parameter, and consistent with the end result of dimensional regularisation [76, 73]. The conserved 3PN energy is thus uniquely determined, and consequently the 3.5PN energy flux, together with the usual energy balance argument leads to the expression for the evolution for orbital phase and frequency under GW radiation reaction at the relative 3.5PN order [99].

In the present work, we provide the gravitational waveform from ICBs to still higher accuracy, namely 2.5PN, which should in consequence be useful for future improved studies in GW data analysis, for both LIGO-type and LISA-type detectors. We shall include in the 2.5PN waveform instantaneous as well as "hereditary" terms, exactly as they are predicted by general relativity, completing therefore the 2.5PN generation problem for binaries moving in quasi-circular orbits initiated in [98]. Using the waveform we next obtain the two "plus" and "cross" GW polarisations at 2.5PN extending the results of [101]. We shall verify that the 2.5PN wave form is in perfect agreement, in the test-mass limit for one of the bodies, with the result of linear black hole perturbations [135].

2.2 The 2.5PN gravitational waveform

2.2.1 Waveform as a functional of multipole moments

In an appropriate radiative coordinate system $X^\mu = (cT, X^i)$, the transverse-traceless (TT) projection of the deviation of the metric of an isolated body from flat metric defines the asymptotic waveform h^{TT}_{km} (lower-case Latin indices take the values $1, 2, 3$). The leading-order $1/R$ part of h^{TT}_{km} (where $R = |\mathbf{X}|$ is the distance to the body) can be uniquely decomposed [61] into its *radiative* multipole contributions introduced in Section 1.7.1. Furthermore, the PN order of the asymptotic waveform scales with the multipolar order l. Hence, at any PN order only a finite number of multipoles is required, and we have, with 2.5PN accuracy,

$$
\begin{aligned}
h^{\mathrm{TT}}_{km} = \frac{2G}{c^4R}\mathcal{P}_{ijkm}(\mathbf{N})\Big\{&U_{ij}\\
&+\frac{1}{c}\left[\frac{1}{3}N_aU_{ija}+\frac{4}{3}\varepsilon_{ab(i}V_{j)a}N_b\right]\\
&+\frac{1}{c^2}\left[\frac{1}{12}N_{ab}U_{ijab}+\frac{1}{2}\varepsilon_{ab(i}V_{j)ac}N_{bc}\right]\\
&+\frac{1}{c^3}\left[\frac{1}{60}N_{abc}U_{ijabc}+\frac{2}{15}\varepsilon_{ab(i}V_{j)acd}N_{bcd}\right]\\
&+\frac{1}{c^4}\left[\frac{1}{360}N_{abcd}U_{ijabcd}+\frac{1}{36}\varepsilon_{ab(i}V_{j)acde}N_{bcde}\right]\\
&+\frac{1}{c^5}\left[\frac{1}{2520}N_{abcde}U_{ijabcde}+\frac{1}{210}\varepsilon_{ab(i}V_{j)acdef}N_{bcdef}\right]+\mathcal{O}(6)\Big\}\,. \qquad (2.1)
\end{aligned}
$$

The U_L's and V_L's (with $L = ij\cdots$ a multi-index composed of l indices) appearing in the above waveform are respectively called the mass-type and the current-type radiative multipole moments (see discussion in Section 1.7). They are functions of the retarded time $T_R \equiv T - R/c$ in radiative coordinates, $U_L(T_R)$ and $V_L(T_R)$. We denote by $\mathbf{N} \equiv \mathbf{X}/R$ the unit vector pointing along the direction of the source located at distance R from the detector. A product of components of $\mathbf{N} = (N_i)_{i=1,2,3}$ is generally denoted $N_L \equiv N_iN_j\cdots$. The Levi-Civita antisymmetric symbol reads ε_{abi}, such that $\varepsilon_{123} = +1$. The operator $\mathcal{P}_{ijkm}$ represents the usual TT algebraic projector which is given by

$$
\begin{aligned}
\mathcal{P}_{ijkm} &= \frac{1}{2}(\mathcal{P}_{ik}\mathcal{P}_{jm}+\mathcal{P}_{im}\mathcal{P}_{jk}-\mathcal{P}_{ij}\mathcal{P}_{km})\,, && (2.2a)\\
\mathcal{P}_{ij} &\equiv \delta_{ij} - N_iN_j\,. && (2.2b)
\end{aligned}
$$

Using the MPM formalism, the radiative moments entering Eq. (2.1) can be expressed in terms of the source variables with sufficient accuracy, that is a fractional accuracy of $\mathcal{O}(6) \equiv \mathcal{O}(c^{-6})$ relative to the lowest-order quadrupolar waveform. For this approximation to be complete, one must compute: mass-type radiative quadrupole U_{ij} with 2.5PN accuracy; current-type radiative quadrupole V_{ij} and mass-type radiative octupole U_{ijk} with 2PN accuracy; mass-type hexadecapole U_{ijkl} and current-type octupole V_{ijk} with 1.5PN precision; U_{ijklm} and V_{ijkl} up to 1PN order; U_{ijklmn}, V_{ijklm} at 0.5PN; and finally $U_{ijklmno}$, V_{ijklmn} to Newtonian precision. The relations connecting the radiative moments U_L and V_L to the corresponding "canonical" moments M_L and S_L (see Section 1.7 for a short recall of their meaning) are given as follows [57, 58, 59]. For the mass-type moments we have

$$\begin{aligned}
U_{ij}(T_R) &= M^{(2)}_{ij}(T_R) + \frac{2GM}{c^3}\int_{-\infty}^{T_R} dV \left[\ln\left(\frac{T_R - V}{2b}\right) + \frac{11}{12}\right] M^{(4)}_{ij}(V) \\
&\quad + \frac{G}{c^5}\left\{-\frac{2}{7}\int_{-\infty}^{T_R} dV M^{(3)}_{a<i}(V) M^{(3)}_{j>a}(V)\right. \\
&\qquad \left. + \frac{1}{7}M^{(5)}_{a<i}M_{j>a} - \frac{5}{7}M^{(4)}_{a<i}M^{(1)}_{j>a} - \frac{2}{7}M^{(3)}_{a<i}M^{(2)}_{j>a} + \frac{1}{3}\varepsilon_{ab<i}M^{(4)}_{j>a}S_b\right\} \\
&\quad + \mathcal{O}(6)\,, \qquad \text{(2.3a)}\\
U_{ijk}(T_R) &= M^{(3)}_{ijk}(T_R) + \frac{2GM}{c^3}\int_{-\infty}^{T_R} dV \left[\ln\left(\frac{T_R - V}{2b}\right) + \frac{97}{60}\right] M^{(5)}_{ijk}(V) \\
&\quad + \mathcal{O}(5)\,, \qquad \text{(2.3b)}\\
U_{ijkm}(T_R) &= M^{(4)}_{ijkm}(T_R) + \frac{G}{c^3}\left\{2M\int_{-\infty}^{T_R} dV \left[\ln\left(\frac{T_R - V}{2b}\right) + \frac{59}{30}\right] M^{(6)}_{ijkm}(V)\right. \\
&\qquad + \frac{2}{5}\int_{-\infty}^{T_R} dV M^{(3)}_{<ij}(V) M^{(3)}_{km>}(V) \\
&\qquad \left. - \frac{21}{5}M^{(5)}_{<ij}M_{km>} - \frac{63}{5}M^{(4)}_{<ij}M^{(1)}_{km>} - \frac{102}{5}M^{(3)}_{<ij}M^{(2)}_{km>}\right\} \\
&\quad + \mathcal{O}(4)\,, \qquad \text{(2.3c)}
\end{aligned}$$

where the brackets <> denote the symmetric-trace-free (STF) projection, while, for the necessary current-type moments,

$$\begin{aligned}
V_{ij}(T_R) &= S^{(2)}_{ij}(T_R) + \frac{2GM}{c^3}\int_{-\infty}^{T_R} dV \left[\ln\left(\frac{T_R - V}{2b}\right) + \frac{7}{6}\right] S^{(4)}_{ij}(V) \\
&\quad + \mathcal{O}(5)\,, \qquad \text{(2.4a)}\\
V_{ijk}(T_R) &= S^{(3)}_{ijk}(T_R) + \frac{G}{c^3}\left\{2M\int_{-\infty}^{T_R} dV \left[\ln\left(\frac{T_R - V}{2b}\right) + \frac{5}{3}\right] S^{(5)}_{ijk}(V)\right. \\
&\qquad \left. + \frac{1}{10}\varepsilon_{ab<i}M^{(5)}_{j\underline{a}}M_{k>b} - \frac{1}{2}\varepsilon_{ab<i}M^{(4)}_{j\underline{a}}M^{(1)}_{k>b} - 2S_{<i}M^{(4)}_{jk>}\right\} \\
&\quad + \mathcal{O}(4)\,. \qquad \text{(2.4b)}
\end{aligned}$$

The underlined index $\underline{a}$ means that it should be excluded from the STF projection. For all the other needed moments we are allowed to simply write

$$U_L(T_R) = M^{(l)}_L(T_R) + \mathcal{O}(3)\,, \qquad \text{(2.5a)}$$

$$V_L(T_R) = S^{(l)}_L(T_R) + \mathcal{O}(3)\,. \qquad \text{(2.5b)}$$

In the above formulas, M is the total ADM mass of the binary system, which agrees with the mass monopole moment. The M_L's and S_L's are the mass and current-type canonical source moments, and $M_L^{(p)}$, $S_L^{(p)}$ denote their p-th time derivatives.

The parameter b appearing in the logarithms of Eqs. (2.3) and (2.4) is a freely specifiable constant, having the dimension of time, entering the relation between the retarded time $T_R = T - R/c$ in radiative coordinates and the corresponding one $t - \rho/c$ in harmonic coordinates (where ρ is the distance of the source in harmonic coordinates). More precisely we have

$$T_R = t - \frac{\rho}{c} - \frac{2\,G\,M}{c^3} \ln\left(\frac{\rho}{c\,b}\right) . \tag{2.6}$$

The constant b can be chosen at will because it simply corresponds to a choice of the origin of radiative time with respect to harmonic time.

As recalled in Section 1.7, the "canonical" moments M_L, S_L do not have generic expressions valid for all PN orders in terms of the "source" variables. This is why we now relate the M_L's and S_L's to non-canonical source multipole moments I_L, J_L, W_L, $\cdots$, which admit closed-form expressions in terms of the source's stress-energy tensor. At the 2.5PN order what then remains is to take into account the relation of the 2.5PN canonical mass-type quadrupole moment to the corresponding non-canonical mass quadrupole in a center-of-mass frame. This is given by [98, 143] [2]

$$M_{ij} = I_{ij} + \frac{4G}{c^5}\left[W^{(2)} I_{ij} - W^{(1)} I_{ij}^{(1)}\right] + \mathcal{O}(7), \tag{2.7}$$

where I_{ij} is the (non-canonical) source mass quadrupole, and W denotes the "monopole" corresponding to the set of moments W_L. [We shall need W only at the Newtonian order where it will be given by (2.16b); see Section 2.2.3 for the expressions of all the source moments in the case of circular binary systems]. Note that a formula generalizing Eq. (2.7) to all PN orders (and multipole interactions) is not possible at present and needs to be investigated anew for each specific case. This is why it is more convenient to define the source moments to be I_L and J_L (and the other ones W_L, $\cdots$, Z_L as well, but in view of *e.g.* Eq. (2.7) these appear to be much less important than I_L, J_L) rather than M_L and S_L. For all the other moments needed here, besides the mass quadrupole (2.7), we can write, with the required precision, that M_L agrees with the corresponding I_L and that similarly S_L agrees with J_L. Namely we always have

$$M_L = I_L + \mathcal{O}(5)\,, \tag{2.8a}$$

$$S_L = J_L + \mathcal{O}(5)\,, \tag{2.8b}$$

and we can neglect in the 2.5PN waveform all the remainders in (2.8) except for the case of

[2] The equation (11.7a) in [143] contains a sign error, but with no consequence for any of the results of that reference. The correct sign is reproduced here.

M_{ij} where the required result is provided by Eq. (2.7). Thus, from now on, the waveform will be considered a function of the non-canonical source moments I_L, J_L, and also, of the "auxiliary" moment W appearing in Eq. (2.7).

2.2.2 Structure of the waveform: Instantaneous and hereditary contributions

From Eqs. (2.3)–(2.4) it is clear that the radiative moments contain two types of terms, those which depend on the source moments at a single instant, namely the retarded time $T_R \equiv T - R/c$, referred to as *instantaneous* terms, and the other ones which are sensitive to the entire "past history" of the system, *i.e.* which depend on all previous times ($V < T_R$), and are referred to as the *hereditary* terms.

In this work, we find it convenient to further subclassify the instantaneous terms in the radiative moments into three types based on their structure. The leading instantaneous contribution to the radiative moment U_L (resp. V_L) from the source moment I_L (resp. J_L), is of the form $I_L^{(l)}$ or $J_L^{(l)}$. We refer to these as instantaneous contributions from the *source* moments and denote them by the subscript "inst(s)". Starting at 2.5PN order, additional instantaneous terms arise, of the form $I_{ij}^{(n)} I_{km}^{(p)}$ or $I_{ij}^{(n)} J_k$, in the expressions relating radiative moments to source moments [see Eqs. (2.3)–(2.4)]. We call such additional terms, the instantaneous terms in the *radiative* moment and denote them by the subscript "inst(r)". Thirdly, from Eq. (2.7) we see that the replacement of the canonical moments by the source moments also induces some new terms at the 2.5PN level, of the form $I_{ij}^{(n)} W^{(p)}$. We shall call such supplementary terms the instantaneous terms in the *canonical* moment and denote them by the subscript "inst(c)".

At the 1.5PN order, the hereditary terms are due to the interaction of the mass quadrupole moment with the mass monopole (ADM mass M) and leads to the effect of wave tails [57]. Physically, this effect can be visualized as the backscattering of the linear waves (described by I_{ij}) off the constant spacetime curvature generated by the mass energy M. This can be viewed as a part of the gravitational field propagating inside the light cone (*e.g.* [56]). At higher PN orders there are similar tails due to the interaction between M and higher moments I_{ijk}, J_{ij}, $\cdots$. In addition, at the 3PN order (however negligible for the present study), there is an effect of tails generated by tails, because of the cubic interaction between the quadrupole moment and two mass monopoles, $M \times M \times I_{ij}$ [59]. The hereditary term arising at the 2.5PN order in the radiative quadrupole (2.3a) is different in nature. It is made of the quadrupole-quadrupole interaction, $I_{ij} \times I_{kl}$, and can physically be thought of as due to the re-radiation of the stress-energy tensor of the linearized quadrupolar gravitational waves. It is responsible for the so-called "non-linear memory" or Christodoulou effect [155, 156, 157] (investigated within the present approach in [57, 58]). So far, all these effects are taken into account in the calculation of the waveform up to 2PN order [140] and in the energy flux up to 3.5PN [98, 143, 99]. The two different types of hereditary terms will be denoted by subscripts "tail" and "memory" and will be dealt with separately in the next chapter.

Summarizing, with the present notation, the total 2.5PN waveform may be written as

$$\begin{aligned} h^{\mathrm{TT}}_{km} &= (h^{\mathrm{TT}}_{km})_{\mathrm{inst}} + (h^{\mathrm{TT}}_{km})_{\mathrm{hered}}\,, && (2.9)\\ (h^{\mathrm{TT}}_{km})_{\mathrm{inst}} &= (h^{\mathrm{TT}}_{km})_{\mathrm{inst(s)}} + (h^{\mathrm{TT}}_{km})_{\mathrm{inst(r)}} + (h^{\mathrm{TT}}_{km})_{\mathrm{inst(c)}} + \mathcal{O}(6)\,. && (2.10)\end{aligned}$$

We give each of the above contributions explicitly. The instantaneous part of type (s) reads

$$\begin{aligned}(h^{\mathrm{TT}}_{km})_{\mathrm{inst(s)}} = \frac{2G}{c^4R}\mathcal{P}_{ijkm}\Big\{ & I^{(2)}_{ij}\\ & +\frac{1}{c}\left[\frac{1}{3}N_a I^{(3)}_{ija} + \frac{4}{3}\varepsilon_{ab(i}J^{(2)}_{j)a}N_b\right]\\ & +\frac{1}{c^2}\left[\frac{1}{12}N_{ab} I^{(4)}_{ijab} + \frac{1}{2}\varepsilon_{ab(i}J^{(3)}_{j)ac}N_{bc}\right]\\ & +\frac{1}{c^3}\left[\frac{1}{60}N_{abc} I^{(5)}_{ijabc} + \frac{2}{15}\varepsilon_{ab(i}J^{(4)}_{j)acd}N_{bcd}\right]\\ & +\frac{1}{c^4}\left[\frac{1}{360}N_{abcd} I^{(6)}_{ijabcd} + \frac{1}{36}\varepsilon_{ab(i}J^{(5)}_{j)acde}N_{bcde}\right]\\ & +\frac{1}{c^5}\left[\frac{1}{2520}N_{abcde} I^{(7)}_{ijabcde} + \frac{1}{210}\varepsilon_{ab(i}J^{(6)}_{j)acdef}N_{bcdef}\right]\Big\}\,, \qquad (2.11)\end{aligned}$$

where all the source moments are evaluated at the current time T_R. The type (r) is

$$\begin{aligned}(h^{\mathrm{TT}}_{km})_{\mathrm{inst(r)}} = \frac{2G}{c^4R}\mathcal{P}_{ijkm}\frac{G}{c^5}\Big\{ & \frac{1}{7}I^{(5)}_{a<i}I_{j>a} - \frac{5}{7}I^{(4)}_{a<i}I^{(1)}_{j>a} - \frac{2}{7}I^{(3)}_{a<i}I^{(2)}_{j>a} + \frac{1}{3}\varepsilon_{ab<i}I^{(4)}_{j>a}J_b\\ & +\frac{1}{12}N_{ab}\left[-\frac{21}{5}I^{(5)}_{<ij}I_{ab>} - \frac{63}{5}I^{(4)}_{<ij}I^{(1)}_{ab>} - \frac{102}{5}I^{(3)}_{<ij}I^{(2)}_{ab>}\right]\\ & +\frac{1}{2}N_{bc}\,\varepsilon_{abi}\left[\frac{1}{10}\varepsilon_{pq<j}I^{(5)}_{a\underline{p}}I_{c>q} - \frac{1}{2}\varepsilon_{pq<j}I^{(4)}_{a\underline{p}}I^{(1)}_{c>q} - 2J_{<j}I^{(4)}_{ac>}\right]\Big\}\,. \qquad (2.12)\end{aligned}$$

Apart from two terms involving the source dipole moment J_i or angular momentum, these terms are made of quadrupole-quadrupole couplings coming from U_{ij}, U_{ijk} and V_{ijk} in Eqs. (2.3)–(2.4) and computed in Ref. [58]. Though, using dimensional and parity arguments, their structure can easily be written down, the computation of the numerical coefficients in front of each inst(r) term needs a detailed study. The inst(c) terms refer to the instantaneous terms in the "canonical" moment and can be written down as

$$(h^{\mathrm{TT}}_{km})_{\mathrm{inst(c)}} = \frac{2G}{c^4R}\mathcal{P}_{ijkm}\frac{G}{c^5}\Big\{4\left[W^{(4)}I_{ij} + W^{(3)}I^{(1)}_{ij} - W^{(2)}I^{(2)}_{ij} - W^{(1)}I^{(3)}_{ij}\right]\Big\}\,, \qquad (2.13)$$

where W is the particular "monopole" moment introduced in (2.7). Both the inst(r) and inst(c) terms represent new features of the 2.5PN waveform.

2.2.3 Source multipole moments required at the 2.5PN order

Evidently the above formulas remain empty unless we feed them with the explicit expressions of the source multipole moments, essentially the mass-type I_L and current-type J_L, appropriate for a specific choice of matter model. In the present Section, we list the I_L's and J_L's needed for the 2.5PN accurate waveform in the case of point particles binaries in circular orbits. This is the extension of the list of moments given in Eqs. (4.4) of [140] for the computation of the 2PN accurate waveform. We do not give any details on this calculation because it follows exactly the same techniques as in Ref. [143]. One might note, comparing the moments listed below with those given in [140], that in extending the waveform to the higher order there are two kinds of complications: one is computation of existing moments to higher PN accuracy and the other to compute new higher multipole moments albeit at the lowest Newtonian order. The former is a harder task than the latter and usually requires newer inputs since one is obliged to deal with higher order nonlinearities.

For the computation of the waveform up to 2.5PN order the required mass moments are

$$
\begin{aligned}
I_{ij} &= \nu\, m\, \mathrm{STF}_{ij}\Bigg\{x^{ij}\left[1+\gamma\left(-\frac{1}{42}-\frac{13}{14}\nu\right)+\gamma^2\left(-\frac{461}{1512}-\frac{18395}{1512}\nu-\frac{241}{1512}\nu^2\right)\right]\\
&\quad+\frac{r^2}{c^2}v^{ij}\left[\frac{11}{21}-\frac{11}{7}\nu+\gamma\left(\frac{1607}{378}-\frac{1681}{378}\nu+\frac{229}{378}\nu^2\right)\right]+\frac{48}{7}\,\frac{r}{c}x^i v^j\nu\gamma^2\Bigg\}\\
&\quad+\mathcal{O}(6)\,, &\text{(2.14a)}\\
I_{ijk} &= \nu\, m\,(X_2-X_1)\,\mathrm{STF}_{ijk}\Bigg\{x^{ijk}\left[1-\gamma\nu-\gamma^2\left(\frac{139}{330}+\frac{11923}{660}\nu+\frac{29}{110}\nu^2\right)\right]\\
&\quad+\frac{r^2}{c^2}x^i v^{jk}\left[1-2\nu-\gamma\left(-\frac{1066}{165}+\frac{1433}{330}\nu-\frac{21}{55}\nu^2\right)\right]\Bigg\}\\
&\quad+\mathcal{O}(5)\,, &\text{(2.14b)}\\
I_{ijkl} &= \nu\, m\,\mathrm{STF}_{ijkl}\Bigg\{x^{ijkl}\left[1-3\nu+\gamma\left(\frac{3}{110}-\frac{25}{22}\nu+\frac{69}{22}\nu^2\right)\right]\\
&\quad+\frac{78}{55}\,\frac{r^2}{c^2}\,v^{ij}x^{kl}(1-5\nu+5\nu^2)\Bigg\}+\mathcal{O}(4)\,, &\text{(2.14c)}\\
I_{ijklm} &= \nu\, m\,(X_2-X_1)\,\mathrm{STF}_{ijklm}\Bigg\{x^{ijklm}\left[1-2\nu+\gamma\left(\frac{2}{39}-\frac{47}{39}\nu+\frac{28}{13}\nu^2\right)\right]\\
&\quad+\frac{70}{39}\,\frac{r^2}{c^2}x^{ijk}v^{lm}\left(1-4\nu+3\nu^2\right)\Bigg\}+\mathcal{O}(3)\,, &\text{(2.14d)}\\
I_{ijklmn} &= \nu\, m\,\mathrm{STF}_{ijklmn}\left\{x^{ijklmn}(1-5\nu+5\nu^2)\right\}+\mathcal{O}(2)\,, &\text{(2.14e)}\\
I_{ijklmno} &= \nu\, m\,(X_2-X_1)\,(1-4\nu+3\nu^2)\,\mathrm{STF}_{ijklmno}\left\{x^{ijklmno}\right\}+\mathcal{O}(1)\,. &\text{(2.14f)}
\end{aligned}
$$

Further, the requisite current moments are given by

$$
\begin{aligned}
J_{ij} &= \nu\, m\, (X_2 - X_1)\, \mathrm{STF}_{ij} \left\{ \varepsilon_{abi} x^{ja} v^b \left[1 + \gamma \left(\frac{67}{28} - \frac{2}{7}\nu \right) \right.\right. \\
&\qquad \left.\left. + \gamma^2 \left(\frac{13}{9} - \frac{4651}{252}\nu - \frac{1}{168}\nu^2 \right) \right] \right\} + \mathcal{O}(5)\,, && \text{(2.15a)} \\
J_{ijk} &= \nu\, m\, \mathrm{STF}_{ijk} \left\{ \varepsilon_{kab} x^{aij} v^b \left[1 - 3\nu + \gamma \left(\frac{181}{90} - \frac{109}{18}\nu + \frac{13}{18}\nu^2 \right) \right] \right. \\
&\qquad \left. + \frac{7}{45} \frac{r^2}{c^2} \varepsilon_{kab} x^a v^{bij} (1 - 5\nu + 5\nu^2) \right\} + \mathcal{O}(4)\,, && \text{(2.15b)} \\
J_{ijkl} &= \nu\, m\, (X_2 - X_1)\, \mathrm{STF}_{ijkl} \left\{ \varepsilon_{lab} x^{ijka} v^b \left[1 - 2\nu \right.\right. \\
&\qquad \left.\left. + \gamma \left(\frac{20}{11} - \frac{155}{44}\nu + \frac{5}{11}\nu^2 \right) \right] + \frac{4}{11} \frac{r^2}{c^2} \varepsilon_{lab} x^{ia} v^{jkb} \left(1 - 4\nu + 3\nu^2 \right) \right\} \\
&\qquad + \mathcal{O}(3)\,, && \text{(2.15c)} \\
J_{ijklm} &= \nu\, m\, \mathrm{STF}_{ijklm} \left\{ \varepsilon_{mab} x^{aijkl} v^b \left(1 - 5\nu + 5\nu^2 \right) \right\} + \mathcal{O}(2)\,, && \text{(2.15d)} \\
J_{ijklmn} &= \nu\, m\, (X_2 - X_1)(1 - 4\nu + 3\nu^2)\, \mathrm{STF}_{ijklmn} \left\{ \varepsilon_{nab} x^{aijklm} v^b \right\} + \mathcal{O}(1)\,. && \text{(2.15e)}
\end{aligned}
$$

[We recall that $X_1 = \frac{m_1}{m}$, $X_2 = \frac{m_2}{m}$, and $\nu = X_1 X_2$; the PN parameter γ is defined by (2.17); the STF projection is mentioned explicitly in front of each term.]

In addition, the current dipole J_i in (2.12) is the binary's constant total angular momentum which needs to be given only at Newtonian order: we need also to give the monopolar moment W which appears inside the inst(c) terms of (2.13) and comes from the relation (2.7) between canonical and source quadrupoles. We have

$$
\begin{aligned}
J_i &= \nu\, m\, \varepsilon_{iab} x^a v^b + \mathcal{O}(2)\,, && \text{(2.16a)} \\
W &= \frac{1}{3} \nu\, m\, \mathbf{x}.\mathbf{v} + \mathcal{O}(2)\,. && \text{(2.16b)}
\end{aligned}
$$

With all the latter source moments valid for a specific matter system (compact binary in circular orbit) the gravitational waveform is fully specified up to the 2.5PN order.

2.2.4 Equation of motion of the binary up to 2.5PN

In Section 2.2.3 we have given the list of source multipole moments needed to control the waveform at the 2.5PN order. In this Section we proceed to calculate the instantaneous terms [of types (s), (r) and (c)] in the 2.5PN waveform of circular compact binaries. The first step towards it is the computation of time derivatives of different moments I_L, J_L (and also W) using the binary's EOM up to 2.5PN order. In the present work we will require, for the computation of the time-derivatives of multipole moments, the EOM for the case of circular orbits to 2.5PN accuracy. We denote the PN parameter in harmonic coordinates by

$$\gamma \equiv \frac{Gm}{rc^2}\,, \tag{2.17}$$

where $r = |\mathbf{x}|$ is the binary's separation ($\mathbf{x} \equiv \mathbf{y}_1 - \mathbf{y}_2$ is the vectorial separation between the particles and $\mathbf{v} \equiv \mathbf{v}_1 - \mathbf{v}_2$ the relative velocity and $m = m_1 + m_2$ is the total mass of the binary system. Occasionally we also employ $\delta m = m_1 - m_2$ so that $\frac{\delta m}{m} = X_1 - X_2$. Then the 2.5PN binary's acceleration reads[3]

$$\frac{d\mathbf{v}}{dt} = -\omega^2\,\mathbf{x} - \frac{32\,G^3}{5\,c^5}\,\frac{m^3\,\nu}{r^4}\,\mathbf{v} + \mathcal{O}(6)\,, \tag{2.18}$$

where the explicit 2.5PN term ($\sim 1/c^5$) is the radiation reaction force in the harmonic coordinate system used here. The radiation reaction force plays an important role in our calculation of the waveform and must be consistently included in all replacements of accelerations at 2.5PN order (however the reaction force yields no contribution to the energy flux at 2.5PN order for circular orbits [98]). In Eq. (2.18) the orbital frequency $\omega \equiv 2\pi/P$ (where P is the orbital period) is related to the binary's separation r in harmonic coordinates with 2PN accuracy by [158]

$$\omega^2 = \frac{G\,m}{r^3}\left\{1 + \Big[-3+\nu\Big]\gamma + \left[6 + \frac{41}{4}\nu + \nu^2\right]\gamma^2 + \mathcal{O}(6)\right\}\,. \tag{2.19}$$

In the following we shall also need the inverse of Eq. (2.19), *i.e.* γ in terms of ω, which can conveniently be written in the form

$$\gamma = x\left\{1 + \left[1 - \frac{\nu}{3}\right]x + \left[1 - \frac{65}{12}\nu\right]x^2 + \mathcal{O}(6)\right\}\,, \tag{2.20}$$

in which we have introduced the gauge invariant frequency-dependent parameter

$$x \equiv \left(\frac{G\,m\,\omega}{c^3}\right)^{2/3}\,. \tag{2.21}$$

2.3 2.5PN accurate instantaneous part of the waveform

Starting from the multipole moments listed in Eq. (2.14) and (2.15) one can evaluate their time derivatives using the equation of motion of Eq. (2.18). These time derivatives are then contracted with the unit direction $\mathbf{N}$ and inserted into Eqs. (2.11), (2.12) and (2.13). We can

[3] We systematically use the shorthand $\mathcal{O}(n)$ to mean a small post-Newtonian remainder term of the order of $\mathcal{O}(c^{-n})$.

finally write down the inst(s) waveform schematically as,

$$(h^{\rm TT}_{km})_{\rm inst(s)} = \frac{2G\,\nu\, m}{c^4 R}\,\mathcal{P}_{ijkm}\Big\{\xi^{(0)}_{ij} + (X_2-X_1)\xi^{(1/2)}_{ij} + \xi^{(1)}_{ij} + (X_2-X_1)\,\xi^{(3/2)}_{ij} + \xi^{(2)}_{ij} + (X_2-X_1)\,\xi^{(5/2)}_{ij} + \rho^{(5/2)}_{ij}\Big\}\,. \tag{2.22}$$

The instantaneous terms up to 2PN order were already reported in Eqs. (4.5) of Ref. [140]. We have reproduced them in our present computation. We list them below for completeness.

$$\xi^{(0)}_{ij} = 2(v^{ij} - \frac{Gm}{r}n^{ij})\,, \tag{2.23}$$

$$\xi^{(1/2)}_{ij} = -6\frac{Gm}{r}(\mathbf{N}.\mathbf{n})\frac{n^{(i}v^{j)}}{c} - \frac{(vN)}{c}\left\{\frac{Gm}{r}n^{ij} - 2v^{ij}\right\}\,, \tag{2.24}$$

$$\begin{aligned}\xi^{(1)}_{ij} = {}& \frac{1}{3}(1-3\nu)\Big[(\mathbf{N}.\mathbf{n})^2\gamma\left\{10\frac{Gm}{r}n^{ij} - 14v^{ij}\right\} \\ & -32\,(\mathbf{N}.\mathbf{n})(\mathbf{N}.\mathbf{v})\gamma n^{(i}v^{j)} + \frac{(\mathbf{N}.\mathbf{v})^2}{c^2}\left\{6v^{ij} - 2\frac{Gm}{r}n^{ij}\right\}\Big] \\ & -\gamma v^{ij}\left(\frac{1}{3}+\nu\right) + \gamma\frac{Gm}{r}n^{ij}\left(\frac{19}{3}-\nu\right)\,,\end{aligned} \tag{2.25}$$

$$\begin{aligned}\xi^{(3/2)}_{ij} = {}& (1-2\nu)\Big\{\frac{65}{6}(\mathbf{N}.\mathbf{n})^3\gamma\frac{Gm}{r}\frac{n^{(i}v^{j)}}{c} - \frac{46}{3}(\mathbf{N}.\mathbf{n})(\mathbf{N}.\mathbf{v})^2\frac{\gamma}{c}n^{(i}v^{j)} \\ & +\gamma(\mathbf{N}.\mathbf{n})^2\frac{(\mathbf{N}.\mathbf{v})}{c}\left[-\frac{43}{3}v^{ij} + \frac{37}{4}\frac{Gm}{r}n^{ij}\right] \\ & + \frac{(\mathbf{N}.\mathbf{v})^3}{c^3}\left[-\frac{1}{3}\frac{Gm}{r}n^{ij} + 2v^{ij}\right]\Big\} + (\mathbf{N}.\mathbf{n})\gamma(\frac{95-18\nu}{6})\frac{Gm}{r}\frac{n^{(i}v^{j)}}{c} \\ & +\frac{(\mathbf{N}.\mathbf{v})}{c}\left[-\frac{2}{3}(1+\nu)\gamma v^{ij} + \frac{81-2\nu}{12}\gamma\frac{Gm}{r}n^{ij}\right]\,,\end{aligned} \tag{2.26}$$

$$\begin{aligned}\xi^{(2)}_{ij} = {}& \gamma^2 n^{ij}\Big[-\frac{Gm}{r}(\frac{361+65\nu+45\nu^2}{60}) + (\mathbf{N}.\mathbf{v})^2(\frac{101-295\nu-15\nu^2}{15}) \\ & -\frac{Gm}{r}(\mathbf{N}.\mathbf{n})^2(\frac{309-995\nu+195\nu^2}{15}) + \frac{86}{5}(\mathbf{N}.\mathbf{n})^2(\mathbf{N}.\mathbf{v})^2(1-5\nu+5\nu^2) \\ & -\frac{94}{15}\frac{Gm}{r}(\mathbf{N}.\mathbf{n})^4(1-5\nu+5\nu^2)\Big] \\ & +v^{ij}\Big[-\gamma^2(\frac{419+1325\nu+15\nu^2}{60}) - \gamma\frac{(\mathbf{N}.\mathbf{v})^2}{c^2}(1-3\nu-\nu^2) \\ & +\gamma^2(\mathbf{N}.\mathbf{n})^2(\frac{163-545\nu+135\nu^2}{15}) + 2\frac{(\mathbf{N}.\mathbf{v})^4}{c^4}(1-5\nu+5\nu^2) \\ & +\frac{128}{15}\gamma^2(\mathbf{N}.\mathbf{n})^4(1-5\nu+5\nu^2) - 30\gamma\frac{(\mathbf{N}.\mathbf{v})^2(\mathbf{N}.\mathbf{n})^2}{c^2}(1-5\nu+5\nu^2)\Big] \\ & +n^{(i}v^{j)}\gamma\Big[\gamma(\mathbf{N}.\mathbf{n})(\mathbf{N}.\mathbf{v})(\frac{176-560\nu+80\nu^2}{5}) - 20(\mathbf{N}.\mathbf{n})\frac{(\mathbf{N}.\mathbf{v})^3}{c^2}(1-5\nu+5\nu^2)\end{aligned}$$

$$+\frac{228}{5}\gamma(\mathbf{N}.\mathbf{n})^3(\mathbf{N}.\mathbf{v})(1-5\nu+5\nu^2)\Big]\,. \tag{2.27}$$

At the 2.5PN order, most of the terms vanish when the two masses are equal ($X_1 = X_2$) like at the previous "odd order approximations" 0.5PN and 1.5PN. However there is also an extra contribution, denoted by $\rho_{ij}^{(5/2)}$ in Eq. (2.22). This consists of two parts: $(\rho_{ij}^{(5/2)})_{\mathrm{reac}} = -\frac{64}{5}\,\nu\,\frac{Gm}{r}\frac{\gamma^2}{c}n^{(i}v^{j)}$ which arises directly from the 2.5PN radiation-reaction term in the EOM given by Eq. (2.18); and $(\rho_{ij}^{(5/2)})_{\mathrm{quad}} = -\frac{192}{7}\,\nu\,\frac{Gm}{r}\frac{\gamma^2}{c}n^{(i}v^{j)}$ which comes from the 2.5PN contribution in the mass quadrupole (2.14). We find

$$\rho_{ij}^{(5/2)} = -\frac{1408}{35}\,\nu\,\frac{Gm}{r}\frac{\gamma^2}{c}n^{(i}v^{j)}\,. \tag{2.28}$$

The other contributions follow from a long but straightforward computation starting from the multipole moments listed earlier, and read as

$$\begin{aligned}
\xi_{ij}^{(5/2)} \;=\;& \Bigg\{n^{ij}\Bigg[\frac{\gamma}{c^3}\Bigg\{\left(\frac{1}{3}-\frac{4}{3}\,\nu+\,\nu^2\right)(\mathbf{N}.\mathbf{v})^5\Bigg\}\\
&+\frac{\gamma^2}{c}\Bigg\{\left(\frac{1199}{180}-\frac{539}{45}\,\nu-\frac{101}{60}\,\nu^2\right)(\mathbf{N}.\mathbf{v})^3\\
&\qquad+\left(\frac{263}{10}-\frac{526}{5}\,\nu+\frac{789}{10}\,\nu^2\right)(\mathbf{N}.\mathbf{n})^2(\mathbf{N}.\mathbf{v})^3\Bigg\}\\
&+\frac{Gm}{r}\frac{\gamma^2}{c}\Bigg\{\left(-\frac{263}{72}+\frac{553}{90}\,\nu+\frac{17}{120}\,\nu^2\right)(\mathbf{N}.\mathbf{v})\\
&\qquad+\left(-\frac{757}{12}+\frac{8237}{60}\,\nu-\frac{433}{20}\,\nu^2\right)(\mathbf{N}.\mathbf{n})^2(\mathbf{N}.\mathbf{v})\\
&\qquad+\left(-\frac{2341}{72}+\frac{2341}{18}\,\nu-\frac{2341}{24}\,\nu^2\right)(\mathbf{N}.\mathbf{n})^4(\mathbf{N}.\mathbf{v})\Bigg\}\Bigg]\\
&+n^{(i}v^{j)}\Bigg[\frac{\gamma}{c^3}\Bigg\{\left(-\frac{74}{3}+\frac{296}{3}\,\nu-74\,\nu^2\right)(\mathbf{N}.\mathbf{n})(\mathbf{N}.\mathbf{v})^4\Bigg\}\\
&+\frac{\gamma^2}{c}\Bigg\{\left(\frac{5161}{90}-\frac{5612}{45}\,\nu+\frac{461}{30}\,\nu^2\right)(\mathbf{N}.\mathbf{n})(\mathbf{N}.\mathbf{v})^2\\
&\qquad+\left(\frac{1811}{15}-\frac{7244}{15}\,\nu+\frac{1811}{5}\,\nu^2\right)(\mathbf{N}.\mathbf{n})^3(\mathbf{N}.\mathbf{v})^2\Bigg\}\\
&+\frac{Gm}{r}\frac{\gamma^2}{c}\Bigg\{\left(-\frac{479}{60}+\frac{187}{6}\,\nu-\frac{9}{4}\,\nu^2\right)(\mathbf{N}.\mathbf{n})\\
&\qquad+\left(-\frac{5587}{90}+\frac{1282}{9}\,\nu-\frac{65}{2}\,\nu^2\right)(\mathbf{N}.\mathbf{n})^3\\
&\qquad+\left(-\frac{3787}{180}+\frac{3787}{45}\,\nu-\frac{3787}{60}\,\nu^2\right)(\mathbf{N}.\mathbf{n})^5\Bigg\}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+v^{ij}\Bigg[\frac{1}{c^5}\left(2-8\,\nu+6\,\nu^2\right)(\mathbf{N}.\mathbf{v})^5\\
&\quad+\frac{\gamma}{c^3}\left\{\left(-\frac{4}{3}+\frac{10}{3}\,\nu\right)(\mathbf{N}.\mathbf{v})^3+\left(-\frac{158}{3}+\frac{632}{3}\,\nu-158\,\nu^2\right)(\mathbf{N}.\mathbf{n})^2(\mathbf{N}.\mathbf{v})^3\right\}\\
&\quad+\frac{\gamma^2}{c}\Bigg\{\left(-\frac{536}{45}-\frac{1531}{45}\,\nu-\frac{1}{15}\,\nu^2\right)(\mathbf{N}.\mathbf{v})\\
&\qquad+\left(\frac{1345}{36}-\frac{799}{9}\,\nu+\frac{245}{12}\,\nu^2\right)(\mathbf{N}.\mathbf{n})^2(\mathbf{N}.\mathbf{v})\\
&\qquad+\left(\frac{2833}{60}-\frac{2833}{15}\,\nu+\frac{2833}{20}\,\nu^2\right)(\mathbf{N}.\mathbf{n})^4\,(\mathbf{N}.\mathbf{v})\Bigg\}\Bigg]\Bigg\}\,,
\end{aligned}
\tag{2.29}
$$

where we recall that $\mathbf{n} = \mathbf{x}/r$ and the parameter γ is defined by (2.17). This completes the computation of all the *inst (s)* terms up to 2.5PN.

Next we must compute the instantaneous (r) and (c) parts of the waveform for the compact binaries in circular orbits. These parts are purely of order 2.5PN. The inst(c) part is computed starting from the expression for W in Eq. (2.16b), but it turns out to be zero for circular orbits. We find

$$
\begin{aligned}
(h^{\mathrm{TT}}_{km})_{\mathrm{inst(r)}} &= \frac{2G\,\nu\,m}{c^4R}\,\mathcal{P}_{ijkm}\frac{\nu\,\gamma^2}{c}\Bigg\{-\frac{84}{5}\frac{G\,m}{r}(\mathbf{N}.\mathbf{n})(\mathbf{N}.\mathbf{v})\,n^{ij}+28(\mathbf{N}.\mathbf{n})(\mathbf{N}.\mathbf{v})v^{ij}\\
&\qquad+\frac{G\,m}{r}\left(\frac{64}{35}-\frac{192}{5}(\mathbf{N}.\mathbf{n})^2\right)n^{(i}v^{j)}+16(\mathbf{N}.\mathbf{v})^2n^{(i}v^{j)}\Bigg\}\,, &(2.30a)\\
(h^{\mathrm{TT}}_{km})_{\mathrm{inst(c)}} &= 0\,. &(2.30b)
\end{aligned}
$$

2.3.1 Calculation of ‘plus’ and ‘cross’ polarizations

Given an orthonormal triad $(\mathbf{N},\,\mathbf{p},\,\mathbf{q})$, consisting of the radial direction $\mathbf{N}$ to the observer, and two unit polarisation vectors $\mathbf{p}$ and $\mathbf{q}$, transverse to the direction of propagation, we define the two “plus” and “cross” polarisation waveforms by

$$
h_+ = \frac{1}{2}(p_ip_j - q_iq_j)h^{\mathrm{TT}}_{ij}\,, \tag{2.31a}
$$

$$
h_\times = \frac{1}{2}(p_iq_j + q_ip_j)h^{\mathrm{TT}}_{ij}\,, \tag{2.31b}
$$

in which the projector $\mathcal{P}_{ijkm}$ present in front of the TT waveform may be omitted.

In the case of circular binary systems we shall adopt for $\mathbf{p}$ the vector lying along the intersection of the orbital plane with the plane of the sky in the direction of the “ascending node” $\mathcal{N}$, *i.e.* the point at which the bodies cross the plane of the sky moving toward the detector, and $\mathbf{q} = \mathbf{N}\times\mathbf{p}$. Following the convention of Ref. [101], the unit vector joining

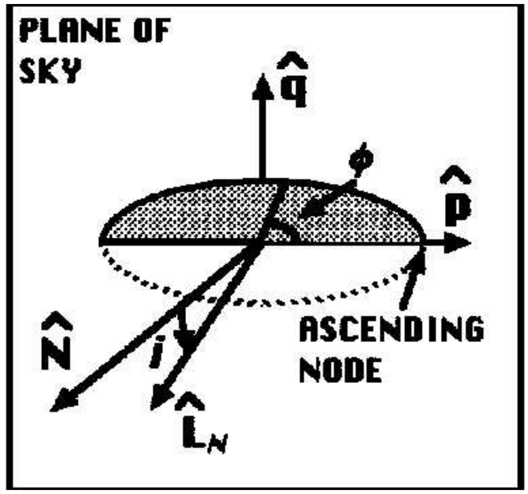

Figure 2.1: The geometry of the binary system. The vectors **p**, **q** and **N** are shown. This figure is adapted from Ref [67].

the particle 2 to the particle 1, *i.e.* $\mathbf{n} = (\mathbf{y}_1 - \mathbf{y}_2)/r$ where $r = |\mathbf{y}_1 - \mathbf{y}_2|$, is given by $\mathbf{n} = \mathbf{p}\cos\phi + (\mathbf{q}\cos i + \mathbf{N}\sin i)\sin\phi$, where i denotes the orbit's inclination angle and ϕ is the orbital phase, namely the angle between the ascending node and the direction of body one.[4] The unit direction of the velocity, *i.e.* $\boldsymbol{\lambda}$ such that $\mathbf{v} = r\omega\boldsymbol{\lambda}$ (for circular orbits), is given by $\boldsymbol{\lambda} = -\mathbf{p}\sin\phi + (\mathbf{q}\cos i + \mathbf{N}\sin i)\cos\phi$ (see Fig. 2.1).

Using Eq. (2.29) and the expressions for the polarizations in Eq. (2.31) we compute the polarization corresponding to the instantaneous part of the waveform. This is only one part of the total 2.5PN polarization since in addition to the instantaneous terms discussed here, the final expression include equally important hereditary contributions to be discussed in the next chapter. The final 2.5PN polarization including the instantaneous and hereditary part is presented at the end of chapter 3.

2.3.2 Comments on the 3PN instantaneous waveform

In Section 2.2.3 we have given the list of source multipole moments needed to control the waveform at the 2.5PN order. The computation of the 3PN waveform obviously requires

[4]The angle ϕ in our convention differs by $\frac{\pi}{2}$ from the same in Refs. [140, 142]. We follow here the convention of BIWW [101], that is related to the BDI one [140, 142] by $\phi_{\mathrm{BDI}} = \phi_{\mathrm{BIWW}} - \frac{\pi}{2}$.

more accurate versions of these moments as well as new moments, which all together would constitute the basis of the computation of the 3PN gravitational wave polarizations. Thus, even though the 3PN mass quadrupole moment [143] and 3PN accurate EOM [144, 145, 148, 149, 146, 147, 150, 151, 76, 73] are available, the present level of accuracy is *not* sufficient enough to compute the 3PN waveform.[5]

The source multipole moments at 3PN order, would yield the control of the inst(s) part of the waveform, as well as the tail terms, but we have to consider also other types of contributions, which are not all under control. The main reason for the present incompleteness at 3PN order is that the instantaneous terms of type (r) and (c), generalizing (2.12)–(2.13) to the 3PN order, are not computed.

Recall that the inst(r) contribution denotes the instantaneous terms in the relations connecting the radiative moments U_L, V_L to the "canonical" moments M_L and S_L (see (2.3)–(2.4)). Though one can guess the structure of these terms at the 3PN order using dimensional and parity arguments, the numerical coefficients in front of each of them require detailed (and generally long and tedious) computation. For instance, in the 3PN waveform we shall need the radiative mass-type octupole moment U_{ijk} at the 2.5PN order, and therefore we have to know what is the remainder term $\mathcal{O}(5)$ in Eq. (2.3b) which we do not know at present (such a calculation would notably entail controlling the quadratic interactions between one mass and one current quadrupole, $M_{ij} \times S_{kl}$, and between one mass quadrupole and one octupole, $M_{ij} \times M_{klm}$). Similarly for the radiative current-type quadrupole V_{ij} given by (2.4a). We have also to compute the 1.5PN terms in the corresponding expressions of U_{ijklm} and V_{ijkl}.

Concerning the inst(c) terms, which are the instantaneous terms coming from the difference between the canonical moments M_L, S_L and the general canonical moments I_L, J_L, [*c.f.* Eqs. (2.7)–(2.8)], one *cannot even guess* their structure. The crucial new input we would need at 3PN order concerns the relation between the canonical mass octupole M_{ijk} and current quadrupole S_{ij} to the corresponding source moments I_{ijk} and J_{ij} at 2.5PN order, using for instance an analysis similar to the one in [98].

[5] We are speaking here of the 3PN *waveform*. The computation of the 3PN *flux* is less demanding, because each multipolar order brings in a new factor $c^{-2} = \mathcal{O}(2)$ instead of $\mathcal{O}(1)$ in the case of the waveform, which explains why it is possible to control it up to the 3.5PN order in [143].