Chapter 8

CONCLUDING REMARKS

In this thesis we have studied geometric flows and the thermodynamic nature of black holes and have used the former to better understand black hole physics from a geometric point of view. We had started with a thermodynamic motivation triggered by the intriguing analogy between the approach of a system to its equilibrium state by maximizing its entropy and the approach of a system of geometric differential equation, namely Ricci flow to its fixed point by monotonically increasing some functional called the *entropy* functional. With this in mind we had tried to attack the Riemannian Penrose Inequality.

In the process we have studied the evolution of important geometric quantities, the area of a closed two surface, its compactness and its Hawking mass, which have clear physical significances in terms of entropy and energy. We have considered two geometric flows separately where in the first the metric is varied by a pure Ricci flow and in the second it is varied by a pure diffeomorphism.

The flow generated by a diffeomorphism does not affect the evolution of the area of an apparent horizon as it is a minimal surface and we have found that the area of an apparent horizon linearly decreases with the Ricci flow and ends in a singularity at a finite time.

The rate of change of area of a general round sphere under pure Ricci flow is found to be bounded from above by its compactness and hence also by its Hawking mass through an inequality which saturates for the Schwarzschild space. This inequality implies that for all surfaces in flat space, the Hawking mass is non positive. Also the converse statement, that, given positive scalar curvature, flat space is the only one for which Hawking mass is non

positive, is true.

The evolution of compactness is studied under both the flows and it is found that the compactness remains constant for the Schwarzschild space. In general both compactness and Hawking mass do not show any definite monotonic behaviour under Ricci flow. The non-monotonic behaviour is due to "shells" of matter which may cross the surface S of interest. However, the rate of change of Hawking mass is non positive for the exterior Schwarzschild space, since it is vacuum.

We have found a maximum principle for the compactness for spherically symmetric initial data set: If the maximum value of compactness is less than 16π, it always decreases under a Ricci flow. With this, we are led to the result that the flow exists for all time and thus we have a clear understanding of the existence of the Ricci flow in spherical symmetry. We conjecture that the maximum principle for compactness is true more generally (outside spherical symmetry). Such a principle would have implications for the long term existence of RF in general.

We have been partially successful while trying to generalize some results to a general (non spherically symmetric) asymptotically flat three manifold. While we derive some interesting new results, they do not lead to a new proof of the Riemannian Penrose Inequality. Looking into this problem we have found that this is due to the fact that Perelman's entropy which he defined in his gradient formulation of the Ricci flow is not quite the geometric entropy of black holes as introduced by Bekenstein and Hawking. We have shown that it is not possible to identify Perelman's entropy with black hole entropy and have proposed a new flow, which is very similar to Perelman flow, differing only in that the diffusion constant is space dependent. The new flow *does* seem to be connected to geometric entropy of black holes.

The general Penrose inequality (without restricting to time symmetric data) is an important unproved conjecture in the initial value formulation of GR. It would seem necessary to either prove it or produce a counterexample. Our hope is that the geometric flow techniques described here will lead to a solution of this important problem.

Appendix A

POSITIVE MASS THEOREM, PENROSE INEQUALITY AND INVERSE MEAN CURVATURE FLOW

INTRODUCTION

In this appendix we describe previous work on the Penrose inequality to bring out the connection with our efforts. This material is mostly review, but there are certain intriguing similarities, which suggest directions for future work. Using a pure diffeomorphism (instead of pure Ricci flow) we can see the connection with early work by Geroch, Jang and Wald [1, 2].

We recall the discussions about Positive Mass Theorem (PMT) and Penrose Inequality (PI) in chapter (1) where we consider an asymptotically flat, globally hyperbolic spacetime $(\mathcal{M}, g_{\mu\nu})$ (μ, $\nu = 0, 1, 2, 3$) (with signature $(-, +, +, +)$). Let (Σ, h_{ab}, K_{ab}) (a, $b = 1, 2, 3$) be an initial data set with mass M and A be the area of the outermost apparent horizon. Then the statements of PMT and PI are as follows:

- **Positive Mass Theorem:** The PMT states that for a nonsingular asymptotically flat initial data set, the ADM mass M is nonnegative and vanishes if and only if the initial data set is that of Minkowski space.

- **Penrose Inequality:** The Penrose inequality states that if the energy condition holds

then,

$$A \leq 16\pi M^2 \tag{A.1}$$

. The PI clearly implies PMT, which only states that $M \geq 0$.

GEROCH'S ARGUMENT TO PROVE POSITIVE MASS THEOREM

In 1973 Geroch gave a proof of the PMT [1] for the restricted case of maximal slicing ($K := h^{ab}K_{ab} = 0$) by defining a flow of 2-surfaces in Σ in which the surfaces flow in the outward normal direction at a rate equal to the inverse of their mean curvatures at each point. The flow is known as the Inverse Mean Curvature Flow (IMCF). Later in 1977 Jang and Wald extended the ideas of Geroch to give a proof of the Riemannian Penrose inequality [2]. Geroch's argument [1] establishes the validity of the PMT for an initial data set (Σ, h_{ab}, K_{ab}) having the topology $\mathbf{R}^3$ and with the trace of its extrinsic curvature, $K = 0$. The constraint equations in GR and the local energy condition (3.11-3.13) imply that the scalar curvature R of Σ cannot be negative i.e.,

$$R \geq 0. \tag{A.2}$$

We introduce a function τ on Σ such that the two dimensional surfaces $\tau = constant$ in Σ are nested topological 2-spheres with the innermost surface reducing to a point. For each value of τ let us assume that $S \subset \Sigma$ is one such surface and $\eta_a = D_a\tau$ defines the normal to S. The unit normal is then given by $n_a = (\eta.\eta)^{-1/2}\eta_a$. Let $\xi^a := un^a$ has the property that $\xi^a D_a\tau = 1$. Let $v^2 := \eta^a\eta_a$. So we have $\xi^a D_a\tau = uv = 1$. So $u = 1/v = (\eta^a\eta_a)^{-1/2}$. Note that $\eta_a = \xi_a/u^2$ is an exact differential form. Next we consider the function $C(\tau)$, which has been introduced earlier in this thesis by the name "compactness" and which, for each value of τ, is defined as

$$C(\tau) := \int_{S \in \Sigma} (2\mathcal{R} - k^2)dA \tag{A.3}$$

where the integration extends over the surface S and $\mathcal{R}$ and k denote the scalar curvature and the trace of the extrinsic curvature of the surface S as a submanifold of Σ, respectively.

Also the Hawking mass (as introduced earlier in the thesis) is defined as

$$M_H(\tau) = \frac{A^{1/2}}{64\pi^{3/2}} \int_{S \subset \Sigma} (2\mathcal{R} - k^2)dA = \frac{A^{1/2}}{64\pi^{3/2}} C(\tau). \tag{A.4}$$

We note that the Gauss-Bonnet theorem implies that

$$\int_{S \in \Sigma} \mathcal{R}dA = 8\pi. \tag{A.5}$$

The trace of the extrinsic curvature k of $S \subset \Sigma$ is defined as

$$k = D_a n^a. \tag{A.6}$$

The rate of change of any quantity with respect to τ is its Lie derivative by ξ^a. The rate of change of k with respect to τ is then given by

$$\frac{\partial k}{\partial \tau} = \xi^b D_b(D_a n^a) = \xi^b D_a D_b n^a - \xi^b R^a{}_{mab} n^b = \xi^b D_a D_b n^a - u R_{mb} n^b n^m. \tag{A.7}$$

Next after a calculation using the Gauss-Codazzi equation and standard projection techniques in GR [3] we arrive at the result that

$$\frac{d}{d\tau} C(\tau) = \int_{S \subset \Sigma} [2k\tilde{D}^a \tilde{D}_a u + ukk^{ab}k_{ab} - uk\mathcal{R} + ukR]dA \tag{A.8}$$

where $\tilde{D}_a$ is the covariant derivative operator on the 2-surface S with respect to the induced metric. Next we suppose that we can choose the 2-surfaces, S, such that

$$uk = 1. \tag{A.9}$$

This defines of the flow of the 2-surfaces considered. This is the IMCF. Under this condition the equation (A.8) becomes

$$\begin{aligned}
\frac{d}{d\tau} C(\tau) &= -\frac{1}{2}C(\tau) + \int_{S \subset \Sigma} [R + (k^{ab}k_{ab} - \frac{1}{2}k^2) + 2u^{-2}\tilde{D}_a u \tilde{D}^a u]dA \\
&\geq -\frac{1}{2}C(\tau). \tag{A.10}
\end{aligned}$$

But since $C(\tau) \to 0$ as the surface reduces to a point, equation (A.10) implies $C(\tau) \geq 0$ for all τ. As $\tau \to \infty$ the surface S expands to a round sphere at infinity and the Hawking mass M_H becomes the ADM mass M. By equation (A.4) we have $M \geq 0$. The equality holds (i.e., $M = 0$) only if $K^{ab} = 0$ ans h_{ab} is flat, i.e., initial data for flat space.

ARGUMENT BY JANG AND WALD TO PROVE PENROSE INEQUALITY

Jang and Wald extended the ideas of Geroch to give a proof of the Riemannian Penrose inequality [2]. Their argument runs as follows: We consider a time symmetric initial data set whose apparent horizon $\mathcal{H}$ has only one component. It is known that $\mathcal{H}$ must have the topology of a sphere. Consider a nested family of 2-sphere analogous to those used in the earlier section on Geroch's argument, but now this family has the additional property that the surface defined by $\tau = 0$ is $\mathcal{H}$. The family of surfaces for $\tau > 0$ is again defined by the property of the IMCF that

$$uk = 1. \tag{A.11}$$

Now by considering the function

$$C(\tau) := \int_{S \subset \Sigma} (2\mathcal{R} - k^2)dA \tag{A.12}$$

we again find that

$$\frac{d}{d\tau}C(\tau) \geq -\frac{1}{2}C(\tau) \tag{A.13}$$

i.e., we have

$$\frac{d}{d\tau}[exp(\tau/2)C] \geq 0 \tag{A.14}$$

. On the other hand, the rate of change of area $A(\tau)$ of the surface $\tau = const.$ is given by

$$\frac{d}{d\tau}A(\tau) = \int ku dA = \int dA = A(\tau). \tag{A.15}$$

Since, the surface $\tau = 0$ is just $\mathcal{H}$, we have

$$A(\tau) = A \, exp(\tau) \tag{A.16}$$

where A is the area of $\mathcal{H}$. Equation (A.14) then implies

$$\lim_{\tau \to \infty} \frac{A(\tau)^{1/2}}{A^{1/2}}C(\tau) \geq C(0) \tag{A.17}$$

. Using equation (A.4), the LHS of the equation (A.17) is

$$\lim_{\tau \to \infty} \frac{A(\tau)^{1/2}}{A^{1/2}}C(\tau) = \frac{64\pi^{3/2}}{A^{1/2}}M. \tag{A.18}$$

95

On the other hand, since $\mathcal{H}$ is extremal ($k = 0$), by the Gauss-Bonnet theorem we have

$$C(0) = 16\pi. \tag{A.19}$$

So from equations (A.17, A.18, and A.19) we have

$$A \leq 16\pi M^2 \tag{A.20}$$

which is PI (equation A.1).

CONCLUSION

We note that the effect of varying S in Σ can be achieved by using a diffeomorphism to change the metric and keeping S fixed. In earlier work (chapter (6)), we took the point of view that S is fixed and only the tensor field h_{ab} is changing. So the equation responsible for the change in the metric is

$$\frac{dh_{ab}}{d\tau} = D_a\xi_b + D_b\xi_a \tag{A.21}$$

which is the same equation for the rate of change of the metric under IMCF (where we vary S in Σ) given by the Lie derivative of the metric by ξ. In chapter (6) we have studied the evolution of geometric quantities of interest under this flow.

The proof of the PMT using Geroch's argument was only for the restricted case of time symmetric initial data set ($K = 0$). Although there is some evidence [4] for believing that all all nonsingular, asymptotically flat spacetimes must contain at least one asymptotically flat slice with $K = 0$, one would still want to prove the most general statement of PMT that an initially regular spacetime must have $M \geq 0$ even if it develops singularities later.

In both the proofs of PMT and PI described above, the existence of the flow of surfaces as required in the above arguments was assumed. It can happen that as the flow evolves there appear points where $k = 0$ and the flow $uk = 1$ becomes ill defined and difficulties with the surface evolution can occur.

The work of Huisken and Ilmanen in 1997 [5] reformulated the IMCF in such a way that the new generalized IMCF always exists. In the new formulation the surface sometimes

jumps outward. However when the flow is smooth it equals the original IMCF, and the Hawking mass still remains monotone.

Geometrically, the idea of Huisken and Ilmanen can be described as follows: Let $S(\tau)$ be the surface resulting from IMCF for "time" τ beginning with the minimal surface S_0. Define $\tilde{S}(\tau)$ to be the outermost minimal area enclosure of $S(\tau)$. Typically, $S(\tau) = \tilde{S}(\tau)$ in the flow, but in the case that the two surfaces do not coincide, we immediately replace $S(\tau)$ with $\tilde{S}(\tau)$ and then continue the IMCF. This is similar in essence to the geometric surgeries done in the Ricci flow when it hits a singularity in order to continue the flow further.

The proof of PI by the Inverse mean curvature flow applies only for a single black hole and also it does not work in higher dimensions. Bray considered a flow on the space of three metrics and presented a technique that proves the inequality for any number of black holes and which can likely be generalized to higher dimensions [6]. We hope that a combination of Ricci flow and diffeomorphism can be exploited to give a general approach to the Penrose inequality.

Bibliography

[1] Robert Geroch 1973 *Ann. NY Acad. of Science* **224**.

[2] P.S. Jang and R.M. Wald 1977 *J. Math. Phys.* **18** 41.

[3] Eric Poisson 2004 A Relativist's Toolkit *Cambridge University Press*.

[4] Choquet-Bruhat 1975 *C. R. Acad. Sci. Paris* **280** 169.

[5] Gerhard Huisken and Tom Ilmanen 2001 *J. Differential Geom.* 59 no. 3.

[6] Hubert L.Bray 2002 *Notices of the AMS* **49** 1372.

[7] R.M. Wald 1984 General Relativity *University of Chicago Press*.

[8] Roger Penrose 1973 *Ann. NY Acad. of Science* **224**.

[9] S.W. Hawking and G.F.R. Ellis 1973 The large scale structure of spacetime *Cambridge University Press*.

Appendix B

EVOLUTION OF THE SCALAR CURVATURE

Notation:

Our notation will be as follows:

1. <u>For Scalar and Tensor quantities we have the following rules:</u>

   - <u>In dimension 4:</u> the index of dimension is explicitly written *e.g* (^4R, $^4R_{\mu\nu}$) etc. except for the metric tensor while we use the symbol (g) for the 4 metric, *e.g* (g_{ab}) etc.

   - <u>In dimension 3:</u> the index of dimension is not written explicitly *e.g* (R, R_{ab}) etc. while we use the symbol (h) for the 3 metric, *e.g* (h_{ab}) etc.

2. <u>For operators:</u>

   - <u>In dimension 4:</u> '∇' is for 'Grad', '∇^2' is for 'Laplacian'

   - <u>In dimension 3:</u> 'D' is for 'Grad', 'D^2' is for 'Laplacian'

3. Our metric conventions are those of Poisson [Eric Poisson 2004 A Relativist's Toolkit *Cambridge Univ. Press*]

General expression for the evolution of the 3 scalar curvature

Here we calculate the general expression for the evolution of the 3-scalar curvature when h_{ab} is varied according to

$$\dot{h}_{ab} = H_{ab} \tag{B.1}$$

$$\dot{R} = \frac{d}{d\tau}(h^{bd}R_{bd}) = -R_{bd}H^{bd} + h^{bd}\dot{R}_{bd} \tag{B.2}$$

$$H^{bd} = h^{kb}h^{ld}H_{kl}. \tag{B.3}$$

Next we calculate the following term,

$$
\begin{aligned}
h^{bd}\dot{R}_{bd} &= h^{bd}(D_c\dot{\Gamma}^c_{bd} - D_d\dot{\Gamma}^c_{bc}) \\
&= h^{bd}\left[D_c\frac{1}{2}h^{ce}\left(D_bH_{ed} + D_dH_{eb} - D_eH_{bd}\right)\right. \\
&\quad \left. -D_d\frac{1}{2}h^{ce}\left(D_bH_{ce} + D_cH_{eb} - D_eH_{bc}\right)\right] \\
&= \delta^2 H - D^2(trH)
\end{aligned}
\tag{B.4}
$$

where,

$$\delta^2 H := D^a D^b H_{ab} = h^{ac}h^{bd}(D_cD_dH_{ab}) \tag{B.5}$$

$$trH := h^{ab}H_{ab} \tag{B.6}$$

and

$$D^2 := h^{ab}D_aD_b. \tag{B.7}$$

So, we have,

$$\dot{R} = \frac{d}{d\tau}(h^{bd}R_{bd}) = -R_{bd}H^{bd} + \delta^2 H - D^2(trH). \tag{B.8}$$

Appendix C

PERELMAN'S GRADIENT FORMULATION

Perelman flow using the variational technique

Here we give a derivation of the Perelman flow using a variational technique. This is based on Topping's exposition of Perelman's work [Peter Topping 2006 London Mathematical Society Lecture Notes Series *Cambridge University Press*]. Let us consider the following functional :

$$\mathcal{F}_{\mathcal{P}}(h, f) := \int_{\mathcal{M}} (R + |Df|^2)e^{-f}dV \tag{C.1}$$

where,

$$\tilde{k} := \frac{\partial f}{\partial \tau} \tag{C.2}$$

now,

$$\frac{\partial}{\partial \tau} |Df|^2 = -H^{ab}D_a f D_b f + 2h^{ab}D_a \tilde{k} D_b f \tag{C.3}$$

and

$$\frac{\partial}{\partial \tau}dV = \frac{1}{2}(trH)dV. \tag{C.4}$$

So,

$$\frac{d}{d\tau}\mathcal{F}_{\mathcal{P}}(h, f) = \int_{\mathcal{M}} \left[-H^{ab}D_a f D_b f + 2h^{ab}D_a \tilde{k} D_b f - R_{ab}H^{ab} + \delta^2 H - D^2(trH) \right]e^{-f}dV$$
$$+ \int_{\mathcal{M}} (R + |Df|^2)\left[-\tilde{k} + \frac{1}{2}(trH) \right]e^{-f}dV. \tag{C.5}$$

We notice the following things :

$$\int_M (2h^{ab} D_a \tilde{k} D_b f) e^{-f} dV = \int_M -2\tilde{k}(D^2 f - |Df|^2) e^{-f} dV \qquad (C.6)$$

$$\int_M (\delta^2 H) e^{-f} dV = \int_M (H^{ab} D_a f D_b f - H^{ab} D_a D_b f) e^{-f} dV \qquad (C.7)$$

and

$$\int_M -D^2(trH) e^{-f} dV = \int_M (D^2 f - |Df|^2)(trH) e^{-f} dV. \qquad (C.8)$$

So,

$$\frac{d}{d\tau} \mathcal{F}_{\mathcal{P}}(h, f) = \int_M \left[(-R^{ab} - D^a D^b f) H_{ab} + (2D^2 f - |Df|^2 + R)(-\tilde{k} + \frac{1}{2} trH) \right] e^{-f} dV. \qquad (C.9)$$

We demand that the scaled volume remains constant under the flow :

$$0 = \frac{\partial}{\partial \tau}(e^{-f} dV) = (-\tilde{k} + \frac{1}{2} trH) e^{-f} dV. \qquad (C.10)$$

So,

$$\tilde{k} = \frac{\partial f}{\partial \tau} = \frac{1}{2} trH. \qquad (C.11)$$

And we have,

$$\frac{d}{d\tau} \mathcal{F}_{\mathcal{P}}(h, f) = \int_M \left[(-R^{ab} - D^a D^b f) H_{ab} \right] e^{-f} dV \geq 0 \qquad (C.12)$$

i.e., $\mathcal{F}_{\mathcal{P}}$ is nondecreasing along the flow if we choose the metric h_{ab} to be varying according to the following equation

$$H_{ab} = \frac{\partial}{\partial \tau} h_{ab} = -2(R_{ab} + D_a D_b f) \qquad (C.13)$$

and hence

$$\tilde{k} = \frac{\partial f}{\partial \tau} = \frac{1}{2} trH = -R - D^2 f. \qquad (C.14)$$

102

Appendix D

EVOLUTION OF GEOMETRIC QUANTITIES UNDER PERELMAN'S RICCI FLOW

The rate of change of 3 scalar curvature under the Perelman fbw

Here we calculate the rate of change of 3 scalar curvature under the Perelman flow. (Here we follow Perelman's convention. The sign of f agrees with Perelman and is reversed between this and the bulk of the thesis).

Method:1

we have for the Perelman flow :

$$\frac{\partial}{\partial \tau} h_{ab} = -2(R_{ab} + D_a D_b f) \tag{D.1}$$

.

Using,

$$h^{ab} h_{bc} = \delta^a_c \tag{D.2}$$

we have,

$$\frac{dh^{ab}}{d\tau} := h^{am} h^{bn} \frac{dh_{mn}}{d\tau} = -\frac{d}{d\tau}(h^{ab}). \tag{D.3}$$

We move to a local flat coordinate system to calculate the following things :

$$\dot{\Gamma}^a_{bc} = \frac{1}{2} h^{ad}(D_b \dot{h}_{dc} + D_c \dot{h}_{db} - D_d \dot{h}_{bc}) \tag{D.4}$$

103

$$\dot{R}^a_{bcd} = D_c\dot{\Gamma}^a_{bd} - D_d\dot{\Gamma}^a_{bc} \tag{D.5}$$

$$\dot{R}_{bd} = D_c\dot{\Gamma}^c_{bd} - D_d\dot{\Gamma}^c_{bc}. \tag{D.6}$$

We introduce a standard notation for covariant derivative: $T^a_{b|c} := D_c T^a_b$ for any tensor T^a_b.

now,

$$\begin{aligned}
\dot{R} &= \dot{h}^{bd}R_{bd} + h^{bd}\dot{R}_{bd} \\
&= 2|R_{bd}|^2 + (2D^bD^df)R_{bd} + term
\end{aligned} \tag{D.7}$$

where,

$$\begin{aligned}
term &= h^{bd}\dot{R}_{bd} = h^{bd}(D_c\dot{\Gamma}^c_{bd} - D_d\dot{\Gamma}^c_{bc}) \\
&= h^{bd}\left[D_c\left\{\frac{1}{2}h^{ce}(\dot{h}_{ed|b} + \dot{h}_{eb|d} - \dot{h}_{bd|e})\right\} - D_d\left\{\frac{1}{2}h^{ce}(\dot{h}_{ec|b} + \dot{h}_{eb|c} - \dot{h}_{bc|e})\right\}\right] \\
&= h^{bd}h^{ce}\left[\left\{-R_{ed|bc} - (D_eD_df)_{|bc} - R_{eb|dc} - (D_eD_bf)_{|dc} + R_{bd|ec} + (D_bD_df)_{|ec}\right\}\right. \\
&\quad \left. -\left\{-R_{ec|bd} - (D_eD_cf)_{|bd} - R_{eb|cd} - (D_eD_bf)_{|cd} + R_{bc|ed} + (D_bD_cf)_{|ed}\right\}\right] \\
&= term1 + term2 + term3 + term4.
\end{aligned} \tag{D.8}$$

Now,

$$\begin{aligned}
term1 &= h^{bd}h^{ce}\left[R_{eb|cd} - R_{eb|dc}\right] \\
&= h^{bd}\left[R^c_{b|cd} - R^c_{b|dc}\right] \\
&= h^{bd}\left[-R^c_{mcd}R^m_b + R^m_{bcd}R^c_m\right] \\
&= h^{bd}\left[-R_{md}R^m_b + R^m_{bcd}R^c_m\right] = -R_{md}R^{md} + R_{mc}R^{mc} = 0
\end{aligned} \tag{D.9}$$

and

$$\begin{aligned}
term2 &= -2h^{bd}h^{ce}R_{ed|bc} \\
&= -h^{ce}D_c2D_bR^b_e \\
&= -h^{ce}D_cD_eR = -D^2R
\end{aligned} \tag{D.10}$$

104

where we have used the Bianchi Identity :

$$2D_a R^a_c = D_c R \tag{D.11}$$

also,

$$term3 = 2h^{bd}h^{ce}R_{bd|ec} = 2D^2 R \tag{D.12}$$

and

$$
\begin{aligned}
term4 \;=\; & -h^{bd}h^{ce}\Big[D_c D_b D_e D_d f + D_c D_d D_e D_b f - D_c D_e D_b D_d f \\
& \;-D_d D_b D_e D_c f - D_d D_c D_e D_b f + D_d D_e D_b D_c f \Big] \\
\;=\; & \; term(a) + term(b) + term(c)
\end{aligned}
\tag{D.13}
$$

where,

$$
\begin{aligned}
term(a) \;=\; & -h^{bd}h^{ce}\Big[(D_e D_b f)_{|(dc-cd)} \Big] \\
\;=\; & -h^{ce}\Big[(D_e \zeta^d)_{|(dc-cd)} \Big] \\
\;=\; & -h^{ce}\Big[-R^d_{\;ldc} D_e \zeta^l + R^l_{\;edc} D_l \zeta^d \Big] \\
\;=\; & -h^{ce}\Big[-R_{lc} D_e \zeta^l \Big] - R^{lc}_{\;\;dc} D_l \zeta^d \\
\;=\; & \; h^{ce} R_{lc} D_e \zeta^l - R^{cl}_{\;\;cd} D_l \zeta^d \\
\;=\; & \; h^{ce} R_{lc} D_e \zeta^l - h^{lm} R^c_{\;mcd} D_l \zeta^d = h^{ce} R_{lc} D_e \zeta^l - h^{lm} R_{md} D_l \zeta^d \\
\;=\; & \; 0
\end{aligned}
\tag{D.14}
$$

where,

$$\zeta_m := D_m f. \tag{D.15}$$

We notice that :

$$
\begin{aligned}
term(b) \;=\; & -h^{bd}h^{ce}\Big[(D_c D_b D_e D_d - D_c D_e D_b D_d)f \Big] \\
\;=\; & -h^{bd}h^{ce}\Big[(D_d D_e D_b D_c - D_d D_b D_e D_c)f \Big] \\
\;=\; & \; term(c).
\end{aligned}
\tag{D.16}
$$

So,

$$
\begin{aligned}
term(b) + term(c) &= -2h^{bd}h^{ce}\left[(D_cD_bD_eD_d - D_cD_eD_bD_d)f\right] \\
&= -2h^{bd}h^{ce}\left[(\zeta_{d|(eb-be)})_{|c}\right] \\
&= -2h^{ce}\left[(\zeta^b_{\ |(eb-be)})_{|c}\right] \\
&= -2h^{ce}\left[-R^b_{\ neb}\zeta^n\right]_{|c} \\
&= -2h^{ce}\left[R^b_{\ nbe}\zeta^n\right]_{|c} \\
&= -2h^{ce}\left[R_{ne}\zeta^n\right]_{|c} \\
&= -2\left[R^{nc}\zeta_n\right]_{|c} \\
&= -2(R_n^{\ c})_{|c}\zeta^n - 2R^{nc}\zeta_{n|c} \\
&= -\zeta^n D_n R - 2R_{bd}D^bD^d f.
\end{aligned}
\tag{D.17}
$$

Therefore,

$$
\begin{aligned}
\dot{R} &= \dot{h}^{bd}R_{bd} + h^{bd}\dot{R}_{bd} \\
&= 2|R_{bd}|^2 + D^2R - \zeta^n D_n R \\
&= 2|R_{bd}|^2 + D^2R - \mathcal{L}_\zeta R
\end{aligned}
\tag{D.18}
$$

where,

$$
\mathcal{L}_\zeta R = \zeta^m D_m R.
\tag{D.19}
$$

Method:2

This method is a shorter one where we will use the general evolution equation for the three
scalar curvature R:

We have here:

$$
H_{ab} = \dot{h}_{ab} = -2(R_{ab} + D_aD_b f)
\tag{D.20}
$$

and so,

$$
trH = -2(R + D^2 f)
\tag{D.21}
$$

and we also have the general equation for the evolution of R:

$$\dot{R} = -R_{ab}H^{ab} + D_a D_b H^{ab} - D^2 trH. \tag{D.22}$$

We calculate term by term the above expression of $\dot{R}$:

$$-R_{ab}H^{ab} = 2|R_{ab}|^2 + 2R_{ab}D^a D^b f. \tag{D.23}$$

Next we have:

$$D_a D_b H^{ab} = -2D_a D_b R^{ab} - 2D_a D_b D^a D^b f \tag{D.24}$$

and,

$$-D^2 trH = 2D^2 R + 2D^2 D^2 f. \tag{D.25}$$

We now note the following identities:

$$
\begin{aligned}
-2D_a D_b D^a D^b f + 2D^2 D^2 f &= 2D_a \left[(D^a D_b D^b - D_b D^a D^b) f \right] \\
&= 2D_a \left[R^b{}_{dab} D^d f \right] \\
&= 2D_a \left[R_{ad} D^d f \right] \\
&= -(D_a R)(D^a f) - 2R^{ad} D_a D_d f \tag{D.26}
\end{aligned}
$$

where to get the last step we had used the Bianchi identity. Collecting all the terms we have:

$$\dot{R} = 2|R_{ab}|^2 + D^2 R - \mathcal{L}_\zeta R \tag{D.27}$$

where,

$$\mathcal{L}_\zeta R = \zeta^a D_a R = (D^a f)(D_a R). \tag{D.28}$$