Chapter 7

RICCI SOLITONS AND BLACK HOLES

7.1   INTRODUCTION

Perelman has given a gradient formulation for the Ricci flow (RF), introducing an "entropy function" which increases monotonically along the flow. In this chapter we pursue our thermodynamic analogy and apply Ricci flow ideas to general relativity. We investigate whether Perelman's entropy is related to (Bekenstein-Hawking) geometric entropy as familiar from black hole thermodynamics. From a study of the fixed points of the flow we conclude that Perelman entropy is not connected to geometric entropy. However, we notice that there is a very similar flow which *does* appear to be connected to geometric entropy. The new flow may find applications in black hole physics suggesting for instance, new approaches to the Penrose inequality.

This chapter is organized as follows. In section II, we briefly describe Perelman's gradient formulation of the Ricci flow. In section III, we motivate the application of these ideas to general relativity. In section IV, we study fixed points of the Perelman flow and note that the Schwarzschild space is not a fixed point. In section V, we describe a slight modification of the Perelman flow. The fixed points of the new flow turn out to be initial data for the Schwarzschild black hole. Section VI generalizes the discussion to include AdS. Section VII is a concluding discussion.

7.2   THE RICCI FLOW AS A GRADIENT FLOW

Let us recall the gradient formulation for the Ricci flow discussed in section (3.3) where, given a Riemannian manifold (Σ, h_{ab}) (a, b run over 1,2,3 as we restrict our discussion to three dimensional manifolds.) with h_{ab} as the initial metric, the Ricci flow describes an evolution equation, which evolves the metric according to its Ricci tensor. The evolution parameter is τ and the family of metrics on Σ, $h_{ab}(\tau)$ satisfies the Ricci flow equation

$$\frac{\partial h_{ab}}{\partial \tau} = -2R_{ab} + 2D_a D_b f \tag{7.1}$$

supplemented by a diffeomorphism $\mathcal{L}_\xi h_{ab} = D_a \xi_b + D_b \xi_a$ generated by a vector field ξ^a, which is itself a gradient $\xi_a = D_a f$, where f is a scalar function on Σ.

Recent mathematical applications use the Ricci flow to address questions of three dimensional topology. While Perelman's motivation was topological, the techniques he introduced are tools of geometry and analysis. It seems reasonable to hope that these tools will also find physical applications in general relativity. The work in the present chapter is a preliminary step in this direction. We suggest that the flow may be useful in understanding elusive concepts like energy and entropy in general relativity. Both of these quantities are well understood in special relativistic physics. In GR[1], they take on a new global geometric interpretation, respectively the mass at infinity and the area of a horizon.

From a physical point of view, the most striking property of the Ricci flow (which it shares with the heat equation) is its tendency to lose memory of initial conditions. This is very similar to the approach of an isolated physical system to thermal equilibrium. The question we address here is: can we in some sense regard the Ricci flow as representing the increase of entropy and the approach of a gravitating system to its equilibrium maximum entropy state. The question can be split into two parts. 1) Is there a functional of the metric which increases monotonically along the Ricci flow? 2) Does this functional physically represent geometric entropy?

One answer to the first question is provided by Zamolodchikov's C theorem [2], proved in the context of two dimensional σ models. This theorem formalizes the intuition that renor-

malization is an irreversible process. The flow of coupling constants under renormalization is such that a function $C(g_1, g_2...)$ of the coupling constants of the theory is monotonically increasing along the flow. In σ models the role of the coupling constants is played by the metric on the target space and the β function is the Ricci tensor[3]. The C theorem guarantees that the RG flow is irrotational and does not have limit cycles. However, the Zamolodchikov C theorem appears to be a very specific result: it applies only to two dimensional field theories and fails if the target space is noncompact. This does not appear to be a general answer to the question we raised.

Another answer, which is more general and relevant to the present context is provided by Perelman's gradient formulation of the Ricci flow. We have briefly described this formulation in section (3.3). The reader is referred to Topping [4] for a detailed exposition of Perelman's gradient formulation.

Recall the remark of Perelman quoted earlier. In his paper on "The entropy formula for the Ricci flow and its geometric application", Perelman remarks (section 5.3): "The interplay between statistical physics and (pseudo) Riemannian geometry occurs in the subject of black hole thermodynamics developed by Hawking et al. Unfortunately, this subject is beyond my understanding at the moment". Our objective here is to pursue this remark and see if there is any connection between Perelman entropy and the geometric entropy familiar from black hole thermodynamics. This motivates our application of Ricci flow techniques to GR.

7.3   APPLICATION TO GENERAL RELATIVITY

How does one apply Perelman's ideas to general relativity? The Perelman flow is defined in the space of three dimensional Riemannian metrics, while GR deals with four dimensional Lorentzian ones. A natural approach is to use spatial slices of four dimensional spacetimes in order to define the flow. The induced metric on Σ is a three dimensional Riemannian metric. In the initial value formulation of GR, one uses the induced metric on Σ and the extrinsic curvature of Σ as initial data for the spacetime. In this thesis we restrict attention to time symmetric initial data. The extrinsic curvature then vanishes and we need to deal only with

(Σ, h_{ab}) . Our idea is to apply the Ricci flow to the initial data h_{ab}. For time symmetric initial data, the dominant energy condition implies that the scalar curvature R is positive. The Ricci flow has the happy feature that it preserves the positivity of scalar curvature [4]. Thus, it may be used to understand general statements about initial data that rely on such assumptions as positivity of curvature. Examples of such statements are the positive energy theorem [5, 6, 7] and the Penrose inequality [8, 9, 10, 11, 12, 13]. In all these examples Einstein's equations do not play any direct role: they are replaced by energy conditions, which are geometric inequalities (for example, the positivity of scalar curvature).

In general relativity, energy and entropy are only well defined in situations where the space has an asymptotic region. The total mass of an isolated system (the ADM mass) is defined using a fixed metric at infinity (either flat or AdS). The Bekenstein-Hawking entropy of black holes uses the notion of an horizon and "escape to infinity". It is therefore clear that our (Σ, h_{ab}) must be a non-compact space, unlike the compact situations considered by Perelman in the study of three dimensional topology. Let us assume that (Σ, h_{ab}) has just one end at infinity and that h_{ab} tends to a fixed flat metric δ_{ab} at infinity [1]. Even in ordinary thermodynamics, entropy is only well defined in stationary (or quasistationary) situations. Since we assumed for simplicity that we deal only with time symmetric situations, we can replace the word 'stationary' in the last sentence by static. Correspondingly, in pursuing the thermodynamic analogy we will restrict our attention to static, four dimensional, Lorentzian, asymptotically flat space times, subject to an Energy condition.

We now address the second question: is the Perelman entropy related to geometric entropy, as familiar from black hole physics? Perelman's entropy is non-decreasing along the flow. However, the property of being monotonic along a flow is a fairly generic one and there may be many functionals which have this property. For instance, in black hole physics, the area theorem asserts that the area of a horizon can only increase with time. But then the same is true for the square of the horizon area. In the motivating example of the copper wire in chapter (3), replacing $a(T) = -T \log T$ by any other convex function ($a'' < 0$) would also lead to a functional which monotonically non-decreasing along the flow. Only the choice

$a(T) = \log T$ results in $S[T(x)]$ being proportional to the actual thermodynamic entropy of the wire. While all these "entropy functionals" differ, they share one property in common: they are all maximized at the fixed point of the flow: the final state of uniform temperature.

Rather than look at the entropy function, it is more effective to look at the fixed points of the Perelman flow and ask if these correspond to maximum entropy in the physical sense. This motivates a study of the fixed points of the Perelman flow. These fixed points are referred to as 'Ricci solitons'.

The stationary spacetime which maximizes its geometric entropy for a fixed energy[14],is the static, spherically symmetric Schwarzschild black hole[15]. Black holes have enormously more entropy than a star of the same mass (some 10^{20} times more for a solar mass). If Perelman's entropy were related to geometric entropy, we would expect the Schwarzschild space to be a fixed point of the Perelman flow. This expectation can be tested by a direct calculation which is presented in the next section.

7.4 RICCI SOLITONS: FIXED POINTS OF THE PERELMAN FLOW

We have already given a brief introduction to Ricci solitons in general in section (2.2). Let us now focus specifically on the Perelman flow. The end points of the Perelman flow are called Ricci solitons and are characterised by a vanishing of the RHS of (7.1). A Ricci soliton consists of a pair (h_{ab}, f) subject to the equations:

$$R_{ab} = D_a D_b f. \tag{7.2}$$

We wish to check if the Schwarzschild space solves these equations for *some* choice of f. We can assume the spherically symmetric form

$$ds^2 = a(r)dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2) \tag{7.3}$$

for h_{ab} and suppose that f depends only on r. In order to preserve this gauge (7.3) we will need to supplement the RHS of (7.2) with an additional diffeomorphism generated by a radial

vector field $U(r)\frac{\partial}{\partial r}$. Since this vector field is obviously a gradient of a scalar function, we can absorb it in a redefinition of f.

The independent fixed point (7.2) equations are $D^2 f = R$ and $R_{\theta\theta} = (DDf)_{\theta\theta}$. These are two equations for the pair $(a(r), f(r))$. We can use the second of these equations to eliminate $f(r)$ in favour of $a(r)$ and arrive at the autonomous differential equation for $a(r)$.

$$\frac{d^2 a}{dr^2} = \frac{3(a')^2}{2a} + \frac{2a^2 - 2a + ra'(1-a)}{r^2}. \tag{7.4}$$

It is easily seen that (apart from the trivial case $M = 0$) the Schwarzschild space, which has $a(r) = (1 - 2M/r)^{-1}$ does not satisfy (7.4). Another way to see that the Schwarzschild space is not a solution of the fixed point equation of the Perelman flow is to go through the following calculation of the evolution of the total curvature under the Ricci flow:

- **The total curvature under Ricci flow:**

The total curvature is given as

$$I = \int d^3 x \sqrt{h} R. \tag{7.5}$$

So,

$$\frac{dI}{d\tau} = \int d^3 x \left[\frac{d\sqrt{h}}{d\tau} R + \sqrt{h} \frac{dR}{d\tau} \right]. \tag{7.6}$$

We then use the Ricci flow equation $\frac{\partial h_{ab}}{\partial \tau} = -2R_{ab}$ without adding a possible diffeomorphism term in the RF here as the total curvature is diffeomorphism invariant.

We then have

$$\frac{dR}{d\tau} = 2R_{ab}R^{ab} + D^2 R \tag{7.7}$$

where D^2 is the Laplacian operator with respect to the metric h_{ab} of Σ. Also,

$$\frac{d\sqrt{h}}{d\tau} = -\sqrt{h}R. \tag{7.8}$$

Neglecting the total divergence terms inside the integration, we then have

$$\frac{dI}{d\tau} = 2 \int d^3 x \sqrt{h}(R_{ab}R^{ab} - (1/2)R^2) \tag{7.9}$$

83

for a general spacetime with integrable curvature where we have used the Ricci flow equation $\frac{\partial h_{ab}}{\partial \tau} = -2R_{ab}$.

The evolution equation for the total curvature gives a quick way to prove that Schwarzschild metric is not a fixed point of the following flow

$$\frac{dh_{ab}}{d\tau} = -2R_{ab} + D_a\xi_b + D_b\xi_a \tag{7.10}$$

for any choice of ξ_b, (not necessarily a gradient) as $R = 0$ identically for Schwarzschild metric and all the additional terms coming from the additional diffeomorphism terms will be zero and we find from equation (7.9) that

$$\frac{dI}{d\tau} \geq 0. \tag{7.11}$$

This is a contradiction if Schwarzschild metric is a fixed point. Only way out is $R_{ab} = 0$ which means that the metric is that of flat space.

Thus Schwarzschild is *not* a fixed point of the flow *no matter what choice is made for* f. This shows that fixed points of the Perelman flow do not maximise geometric entropy as familiar from black hole physics. Flowing the Schwarzschild space using the Perelman flow would lead to a *further* increase of Perelman's entropy. We conclude that Perelman's entropy function is not connected to Bekenstein-Hawking entropy. However, as we argued earlier, many aspects of the Ricci flow are strongly reminiscent of thermodynamics. Are there other similar flows in the space of metrics which do describe the thermodynamics of geometry? This question is addressed in the next section.

7.5 THE MODIFIED RICCI FLOW AND ITS FIXED POINTS

Let us consider how one can modify the Ricci flow to make connections with black hole entropy. We would like to have a geometric differential equation, (*i.e* one constructed from tensor fields) of a parabolic nature similar to the forward heat equation. The right hand side must contain no more than second derivatives of the basic fields (h_{ab}, f). Choosing the

simplest possibility consistent with these requirements, we define the modified Ricci flow (MRF) by the equation

$$\frac{\partial h_{ab}}{\partial \tau} = -2fR_{ab} + 2D_a D_b f. \tag{7.12}$$

This differs from Perelman's flow (7.1) only in the presence of f in the first term. f can be interpreted as an effective space dependent diffusion constant. It describes a forward type heat equation only in regions where f is positive and bounded away from zero. In these regions we would expect the flow to have the same behaviour as Perelman's flow, smoothing out inhomogeneities and losing memory of initial conditions. Let us supplement the Ricci flow with a forward type heat equation for the evolution of f:

$$\frac{\partial f}{\partial \tau} = D^2 f. \tag{7.13}$$

The pair (h_{ab}, f) evolved according to (7.12,7.13) constitute the modified Ricci flow [16]. Does the MRF have any connection with geometric entropy?

As we mentioned before, the most efficient way to investigate this question is to study the fixed points of the flow, characterised by

$$fR_{ab} = D_a D_b f \tag{7.14}$$

where f satisfies

$$D^2 f = 0. \tag{7.15}$$

It is easily seen that with

$$f = (1 - 2M/r)^{1/2} \tag{7.16}$$

the Schwarzschild exterior space

$$ds^2 = (1 - 2M/r)^{-1} dr^2 + r^2 (d\theta^2 + \sin \theta^2 d\phi^2) \tag{7.17}$$

$(r > 2M)$ is a fixed point of the MRF. This is the second main observation we make in this chapter. The modified Ricci flow does seem to have some connection with geometric entropy: its fixed points are extrema of the Bekenstein-Hawking entropy. Note that the function f is positive in the Schwarzschild exterior and that it degenerates to zero on the

inner boundary $r = 2M$. The form of f motivates the identification of the pair (h_{ab}, f) with the static spacetime

$$ds^2 = -f(x)^2 dt^2 + h_{ab} dx^a dx^b. \tag{7.18}$$

This gives a Lorentzian spacetime interpretation of f as the lapse function or the redshift factor.

7.6   GENERALIZATION TO ADS

For the compact spatial topologies studied by Perelman, there was room to modify the RHS of the Ricci flow by the addition of a "cosmological constant" λh_{ab}. This led to the possibility of additional Ricci solitons, shrinking and expanding breathers. In the last two sections, we required that our metric h_{ab} tend to a *fixed flat* metric at infinity and so λ had to vanish to preserve the asymptotic conditions. It is interesting to ask what happens if we replace asymptotic flatness by asymptotically AdS. The arguments of the last two sections go through with appropriate slight modifications. In the equations (7.1,7.2), one simply replaces the Ricci tensor R_{ab} by $R_{ab} - \lambda h_{ab}$, where λ is a negative cosmological constant. Correspondingly, (7.4) acquires an extra term $\lambda raa'$ on the right hand side. As before we see that the AdS Schwarzschild space, which has $a(r) = (1 - 2M/r - \lambda r^2/3)^{-1}$ does not satisfy the soliton equation.

However, we modify the flow to read

$$\frac{\partial h_{ab}}{\partial \tau} = -2f(R_{ab} - \lambda h_{ab}) + 2D_a D_b f \tag{7.19}$$

supplemented by an evolution equation for f:

$$\frac{\partial f}{\partial \tau} = D^2 f + \lambda f. \tag{7.20}$$

Then we find that the AdS Schwarzschild is indeed a Ricci Soliton of the modified flow with f given by $f = (1 - 2M/r - \lambda r^2/3)^{1/2}$. Thus our main conclusions generalise easily to include AdS space.

7.7 CONCLUSION

In his paper[21], Perelman remarks that the interplay between statistical physics and (Lorentzian) geometry occurs in the subject of black hole thermodynamics. He appears to suggest possible connections between his entropy functional and black hole thermodynamics. Perelman was working with compact Riemannian manifolds. It seems clear that for a discussion of geometric entropy we need to have non-compact manifolds, so that we can define a region of no escape. The main result of this chapter is the observation that the Perelman entropy is *not* the geometric entropy, at least if one uses the natural identifications we have made between initial data and spacetimes.

Another observation we make is that a small modification of Perelman's flow does seem to relate to geometric entropy. It would seem important to understand the properties of the new flow and possibly relate it to the Perelman flow. For instance, What can one say about the long term behaviour of both flows starting with an arbitrary initial positive curvature metric in the asymptotically flat context. How does one choose the initial f? Does the MRF admit a gradient formulation? Such questions would be interesting to pursue in the future.

Several papers have appeared [20, 18, 19, 17] in the physics literature about physical applications of the Ricci flow. Many of them are motivated by the renormalization group, string theory or conformal field theory. Our motivation is from the viewpoint of classical general relativity. In GR, there are a number of results of a purely differential geometric character which also have a strong thermodynamic flavour. Examples abound in black hole physics, but let us mention two: the Penrose inequality and the Hawking area theorem. These results do not appeal to Einstein's equations but only to energy conditions. The Ricci flow techniques of Perelman do seem to be a natural tool to attack such problems.

Bibliography

[1] R.M. Wald 1984 General Relativity *University of Chicago Press*.

[2] A.B. Zamolodchikov 1986 *JETP Lett.* **43** 731; 1987 *Sov. J. Nucl. Phys.* **46** 1090.

[3] D. Friedan 1980 *Phys. Rev. Lett.* **45**, 1057.

[4] Peter Topping 2006 London Mathematical Society Lecture Notes Series *Cambridge University Press*.

[5] Richard Schoen and Shing-Tung Yau 1979 Positivity of the Total Mass of a General Space-Time *Phys. Rev. Lett.* Volume 43 Number 20.

[6] Edward Witten 1981 A New Proof of the Positive Energy Theorem *Commun. Math. Phys.* 80 381-402.

[7] R.Penrose, R.D Sorkin and E.Woolgar A Positive Mass Theorem Based on the Focusing and Retardation of Null Geodesics 1993 *Preprint* gr-qc/9301015.

[8] Roger Penrose 1973 *Ann. NY Acad. of Science* 224.

[9] Robert Geroch 1973 *Ann. NY Acad. of Science* 224.

[10] P.S. Jang and R.M. Wald 1977 *J. Math. Phys.* **18** 41.

[11] Gerhard Huisken and Tom Ilmanen 2001 *J. Differential Geom.* 59 no. 3.

[12] Hubert L.Bray 2003 *Preprint* math.DG/0304261.

[13] Hubert L.Bray 2001 J. Diff. Geom. **59** 177-267.

[14] R.D. Sorkin and M. Varadarajan 1996 Class. Quant. Grav. **13** 1949.

[15] We are suppressing a subtlety here. Quantum effects lead to black hole evaporation and a further increase of entropy. This chapter is concerned only with classical GR.

[16] The term "modified Ricci flow" is sometimes used in the mathematical literature to describe Ricci flow with surgery. This is not to be confused with the flow described here.

[17] Ioannis Bakas 2004 *Preprint* hep-th/0410093, 2004 *Preprint* hep-th/0702034.

[18] T. Oliynyk, V. Suneeta and E. Woolgar 2006 *Nucl. Phys.* **B739** 441.

[19] Matthew Headrick and Toby Wiseman, Class. Quantum Grav. **23** (2006), 6683.

[20] T. Oliynyk and E. Woolgar 2006 *Preprint* math-DG/0607438.

[21] Grisha Perelman 2002 The entropy formula for the Ricci flow and its geometric applications *Preprint* math.DG/0211159.

[22] R. S. Hamilton 1982 Three-manifolds with positive Ricci curvature *J. Differential Geom.* 17.

[23] Huai-Dong Cao and Bennett Chow 1999 Recent developments on the Ricci flow *Bull. Amer. Math. Soc.* 36 59-74.