Chapter 6

ENERGY, ENTROPY AND THE RICCI FLOW

6.1 INTRODUCTION

The Ricci flow [1, 2, 3, 4] has been used by mathematicians to understand the topology of three manifolds. It appears likely that these mathematical developments will also be useful in physics in the study of geometric theories like general relativity. The Ricci flow is a (degenerate) parabolic differential equation, and is very similar to the heat equation. We hope that this will help us understand thermodynamic features of GR.

In this chapter we look at the Ricci flow to see how some physically interesting quantities evolve with the flow. Energy and entropy are quantities of physical interest from the thermodynamic point of view. In general relativity these quantities take on a purely differential geometric meaning: the entropy is related to the area of horizons and the energy to the ADM mass at infinity. In the present chapter we write down the equations governing the general evolution of these quantities (without assuming any particular symmetry such as spherical symmetry) and derive some inequalities relating them. In this context let us recall our initial motivation. In the seventies, Roger Penrose, in an attempt to pick holes in the "establishment view" of gravitational collapse, which includes the idea of cosmic censorship, wrote down an inequality relating the ADM mass of an initial data set for GR and the area of apparent horizons it contained: $M \geq \sqrt{A/16\pi}$. A counterexample to this inequality would imply a flaw in the establishment view. No counter example has so far been found.

6.2 GEOMETRIC QUANTITIES OF INTEREST

We have already introduced the following geometric quantities in chapter (4). We go through them briefly again. Let S be a closed surface in Σ, γ_{ij} the pull back or induced metric on S, $\mathcal{R}$ the scalar curvature of (S,γ) and k the trace of its extrinsic curvature. We will be interested in the evolution of some geometric properties of S under the Ricci flow. Our interest in these quantities stems from their physical significance. These are :

the area of S,

$$A(S) = \int_S dA = \int_S d^2x\,\sqrt{\gamma} \tag{6.1}$$

the "compactness" of S,

$$C(S) = \int_S dA(2\mathcal{R} - k^2) \tag{6.2}$$

and its Hawking mass

$$M_H(S) = \frac{\sqrt{A(S)}}{64\pi^{3/2}}C(S). \tag{6.3}$$

Using the *Gauss Bonnet theorem*, if S is of spherical topology

$$2\int_S \mathcal{R}dA = 2\int 2kdA = 16\pi. \tag{6.4}$$

So the compactness is

$$C = \left[16\pi - \int_S k^2 dA\right]. \tag{6.5}$$

We will be interested in the evolution of $A(S)$, $C(S)$, and $M_H(S)$ as the metric changes according to

$$\frac{dh_{ab}}{d\tau} = -2R_{ab} + D_a\xi_b + D_b\xi_a. \tag{6.6}$$

The first term $-2R_{ab}$ in equation (6.6) is the RF and the rest is a diffeomorphism. The effect of varying S in Σ can be achieved by using a diffeomorphism. In our work, we take the point of view that S is fixed and only the tensor field h_{ab} is changing. We will consider the evolution under RF and diffeomorphism separately considering one at a time. So the equations responsible for the change in the metric in these two cases are

$$\frac{dh_{ab}}{d\tau} = -2R_{ab} \tag{6.7}$$

and

$$\frac{dh_{ab}}{d\tau} = D_a\xi_b + D_b\xi_a \tag{6.8}$$

respectively.

6.3 THE EVOLUTION OF AREA

The area is

$$A(S) = \int d^2x\,\sqrt{\gamma}. \tag{6.9}$$

Let η_a be the normal to S. η_a is defined only up to a nonzero multiple. The unit normal

$$n_a = \frac{\eta_a}{(\eta.\eta)^{1/2}} \tag{6.10}$$

depends on the metric. As the metric changes, the unit normal can only change by a multiple of itself

$$\frac{dn_a}{d\tau} = \alpha n_a. \tag{6.11}$$

Also,

$$\frac{d}{d\tau}(h^{ab}n_an_b) = -\frac{dh^{ab}}{d\tau}n_an_b + 2h^{ab}n_a\frac{dn_b}{d\tau} = 0 \tag{6.12}$$

$\Big($in our notation, $\frac{dh^{ab}}{d\tau} := h^{am}h^{bn}\frac{dh_{mn}}{d\tau}$ and we note that $\frac{d}{d\tau}(h^{ab}) = \frac{d}{d\tau}(h^{am}h^{bn}h_{mn}) = -\frac{dh^{ab}}{d\tau}\Big)$,

which implies that

$$\alpha = \frac{1}{2}\frac{dh^{ab}}{d\tau}n_an_b \tag{6.13}$$

$\frac{dn^b}{d\tau}$ need not point along n^b

$$\frac{d}{d\tau}n^b = \frac{d}{d\tau}(h^{bc}n_c) = -\frac{dh^{bc}}{d\tau}n_c + h^{bc}\frac{dn_c}{d\tau} = -\frac{dh^{bc}}{d\tau}n_c + \alpha n^b. \tag{6.14}$$

While the second term does point along n^b, the first need not. For convenience we pick $\eta_a = D_a\eta$ where η is a function on Σ which is constant over S. Then η_a is exact and therefore closed. This will be useful later.

Now we compute $\frac{dA(S)}{d\tau}$ for a general evolution of the metric h_{ab} with τ.

$$A(S) = \int dA = \int d^2x \sqrt{\gamma} \tag{6.15}$$

$$\frac{dA(S)}{d\tau} = \int d^2x \frac{d}{d\tau} \sqrt{\gamma} \tag{6.16}$$

$$\frac{1}{\sqrt{\gamma}} \frac{d}{d\tau} \sqrt{\gamma} = \frac{1}{2} \gamma^{ij} \frac{d\gamma^{ij}}{d\tau} = \frac{1}{2}(h^{ab} - n^a n^b) \frac{d}{d\tau}(h_{ab} - n_a n_b). \tag{6.17}$$

Note that

$$\frac{d}{d\tau}(n_a n_b) = 2\alpha n_a n_b \tag{6.18}$$

which is projected out by $\gamma^{ab} = h^{ab} - n^a n^b$

$$\frac{dA}{d\tau} = \frac{1}{2} \int (h^{ab} - n^a n^b) \frac{dh_{ab}}{d\tau} \sqrt{\gamma} d^2x. \tag{6.19}$$

- **Evolution under RF:**

  We first calculate the evolution of area under RF which is equation (6.7) where the metric changes as $\frac{dh_{ab}}{d\tau} = -2R_{ab}$

$$\frac{dA}{d\tau} = \frac{1}{2} \int \sqrt{\gamma} d^2x \Big[n^a n^b R_{ab} - R \Big]. \tag{6.20}$$

  We use the following relations (contracted form of the Gauss-Codazzi equation [5])

$$-2(R_{ab} - \frac{1}{2} R h_{ab}) n^a n^b = \mathcal{R} + (k^{ij} k_{ij} - k^2) \tag{6.21}$$

  or,

$$-2R_{ab} n^a n^b = -R + \mathcal{R} + (k^{ij} k_{ij} - k^2) \tag{6.22}$$

$$R_{ab} n^a n^b - R = -\frac{1}{2} \Big[R + \mathcal{R} + (k^{ij} k_{ij} - k^2) \Big] \tag{6.23}$$

$$
\begin{aligned}
\frac{dA}{d\tau} &= -\frac{1}{2}\int \sqrt{\gamma}d^2x\Big[R + (k^{ij} - \frac{1}{2}k\gamma^{ij})(k_{ij} - \frac{1}{2}k\gamma_{ij}) + \mathcal{R} - \frac{k^2}{2}\Big] \\
&= -\frac{1}{2}\int \sqrt{\gamma}d^2x\Big[R + (k^{ij} - \frac{1}{2}k\gamma^{ij})(k_{ij} - \frac{1}{2}k\gamma_{ij})\Big] \\
&\quad -\frac{1}{4}\int \sqrt{\gamma}d^2x(2\mathcal{R} - k^2).
\end{aligned}
\tag{6.24}
$$

The second integral in (6.24) is identified as the $C(S)/4$, one fourth the compactness integral of S and the first integral, which is of definite sign can be dropped to arrive at the inequality

$$
\frac{dA}{d\tau} \leq -\frac{1}{4}C(S). \tag{6.25}
$$

This inequality is one of the main results of this chapter. This result can be re expressed in terms of the Hawking Mass:

$$
\frac{dA}{dt} \leq -\frac{16\pi^{3/2}}{\sqrt{A}}M_H(S). \tag{6.26}
$$

Thus the rate of decrease of area under Ricci flow is bounded by the Hawking mass.

The inequality (6.25) is saturated in the case of the spheres of Schwarzschild space $\Big($which is given by $ds^2 = (1 - 2M(r)/r)^{-1}dr^2 + (d\theta^2 + \sin^2\theta d\phi^2)$ with $M(r) = M\Big)$. In this case $R = 0$ and the spheres are shear free ($k_{ij} = \frac{1}{2}k\gamma_{ij}$), so the first integral in (6.24) vanishes.

So for the case of the spheres of Schwarzschild space, we have

$$
\frac{dA}{d\tau} = -\frac{1}{4}C(S) \tag{6.27}
$$

which was obtained by a direct calculation in spherical symmetry in chapter (4).

As a simple application of this inequality, let us consider flat space. Since the Ricci tensor vanishes we have that $dA/d\tau = 0$ and so the LHS of 6.26) vanishes. We arrive at the conclusion that for all surfaces in flat space, the Hawking mass is non positive! This fact has also been noticed in [6], where a direct proof is given. In fact, the converse of this statement is also true: Given positive scalar curvature, flat space is the only one for which the Hawking mass is non-positive. To see this, note that the supremum of the Hawking mass is the ADM mass and if this supremum vanishes, it follows from the positive mass theorem that the space must be flat.

- **Evolution under diffeomorphism:**

Now we calculate the evolution of area under the flow given in equation (6.8) where the metric changes due to a diffeomorphism as $\frac{dh_{ab}}{d\tau} = D_a\xi_b + D_b\xi_a$

so we have

$$\frac{dA}{d\tau} = \frac{1}{2}\int \sqrt{\gamma}d^2x(h^{ab} - n^a n^b)2D_a\xi_b = \int \sqrt{\gamma}d^2x[D_a\xi^a - n^a n^b D_a\xi_b]. \tag{6.28}$$

If we suppose that ξ^a is tangent to S, then

$$\tilde{D}_a\xi_b = \gamma_a{}^{a'}\gamma_b{}^{b'}D_{a'}\xi_{b'} \tag{6.29}$$

$$\tilde{D}_a\xi^a = \gamma^{ab}\tilde{D}_a\xi_b = (h^{ab} - n^a n^b)D_a\xi_b. \tag{6.30}$$

So

$$\frac{dA}{d\tau} = \int \sqrt{\gamma}d^2x[\tilde{D}_a\xi^a] = 0. \tag{6.31}$$

Since this is a divergence over a boundary less surface S. It is therefore enough to consider the component of ξ normal to S. Let

$$\xi^a = un^a \tag{6.32}$$

then

$$\frac{dA}{d\tau} = \int \sqrt{\gamma}d^2x(h^{ab} - n^a n^b)D_a(un_b). \tag{6.33}$$

When we differentiate u, n^b comes out and is killed by γ^{ab} so

$$\frac{dA}{d\tau} = \int \sqrt{\gamma}d^2x\, u\gamma^{ab}D_a n_b = \int \sqrt{\gamma}d^2x\, uk \tag{6.34}$$

- **Application to apparent horizon:**

If S is a minimal surface, we find that since $k = 0$, the area changes according to

$$\frac{dA}{d\tau} \leq \frac{1}{4}C = -16\pi/4 = -4\pi. \tag{6.35}$$

So for apparent horizon we have

$$\frac{dA}{d\tau} \leq -4\pi \tag{6.36}$$

which has already been checked for the special case of spherical symmetry in chapter (4).

This is the behaviour of the area of an apparent horizon under the Ricci flow. The presence of a diffeomorphism does not matter for apparent horizon since $k = 0$. Under a Ricci flow we expect S to shrink to a point and disappear. We take S to be the outermost horizon, i.e.,the boundary of the region having trapped surfaces. Under the Ricci flow, this region will not disappear suddenly, but shrinks. Near S there will be a new apparent horizon with the same area. If the initial area is A_0, within a time $A_0/4\pi$ the apparent horizon shrinks to a point of zero area. This implies a finite time singularity.

6.4 THE EVOLUTION OF COMPACTNESS

The compactness C is

$$C(\tau) = \int_S \sqrt{\gamma} d^2x (2\mathcal{R} - k^2). \tag{6.37}$$

We have

$$\frac{dC}{d\tau} = -\int_S 2k \frac{dk}{d\tau} \sqrt{\gamma} d^2x - \int_S k^2 \frac{d\sqrt{\gamma}}{d\tau} d^2x. \tag{6.38}$$

- **Evolution under RF:**

We first calculate the evolution of compactness under RF which is equation (6.7) where the metric changes as $\frac{dh_{ab}}{d\tau} = -2R_{ab}$.

The compactness C is

$$C(\tau) = \int_S \sqrt{\gamma} d^2x (2\mathcal{R} - k^2). \tag{6.39}$$

We have

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We next calculate $\frac{d\sqrt{\gamma}}{d\tau}$. We express the induced metric γ_{ab} of $S \subset \Sigma$ in terms of the metric h_{ab} of Σ, and the normal n to S as

$$\gamma_{ab} = h_{ab} - n_a n_b \tag{6.41}$$

and note (using the fact that $\frac{d}{d\tau}(n_a n_b)$ is projected out by $\gamma_{ab} = (h_{ab} - n_a n_b)$) that

$$\frac{d\sqrt{\gamma}}{d\tau} = (1/2)\sqrt{\gamma}(h^{ab} - n^a n^b)\frac{\partial hab}{\partial\tau} = -\sqrt{\gamma}[R - R_{ab}n^a n^b] \tag{6.42}$$

where we have used the Ricci flow (6.7).

We use the Gauss-Codazzi equation (contracted form) [5]

$$-2(R_{ab} - (1/2)Rh_{ab})n^a n^b = \mathcal{R} + (k^{ij}k_{ij} - k^2) \tag{6.43}$$

to have

$$\frac{d\sqrt{\gamma}}{d\tau} = -(1/2)\sqrt{\gamma}[R + (k^{ij}k_{ij} - k^2/2) + \mathcal{R} - k^2/2]. \tag{6.44}$$

Next we calculate the rate of change of the trace of the extrinsic curvature $k = D_a n^a$. From an earlier discussion given in the section for calculating the rate of change of area under RF, it is clear that the rate of change of the normal n to the surface S is given as

$$\frac{dn_a}{d\tau} = \alpha n_a \tag{6.45}$$

where

$$\alpha = (1/2)\frac{dh^{ab}}{d\tau}n_a n_b \tag{6.46}$$

also

$$\frac{d}{d\tau}(h^{ab}) = -\frac{dh^{ab}}{d\tau}. \tag{6.47}$$

With these facts we see that

$$\frac{dn^a}{d\tau} = \frac{d}{d\tau}(h^{ab}n_b) = -\frac{dh^{ab}}{d\tau}n_b + h^{ab}n_b\Big((1/2)\frac{dh^{cd}}{d\tau}n_c n_d\Big). \tag{6.48}$$

So we have

$$\frac{dk}{d\tau} = \frac{d}{d\tau}(D_a n^a) = \Big(\frac{d}{d\tau}\Gamma^a{}_{am}\Big)n^m + D_a\frac{dn^a}{d\tau}. \tag{6.49}$$

A calculation then shows that

$$\frac{dk}{d\tau} = 2R^{ab}D_a n_b - k(R^{cd}n_c n_d) - n^a D_a(R^{cd}n_c n_d). \tag{6.50}$$

So we have the expression for the rate of change of compactness, C under Ricci flow

$$\frac{dC}{d\tau} = \int dA\Big[k^2(R + R_{ab}n^a n^b) - 2k[2R^{ab}D_a n_b - n^a D_a(R^{cd}n_c n_d)]\Big]. \tag{6.51}$$

We note that for Schwarzschild space, $R = 0$ and the metric h_{ab} of Σ is given as

$$ds^2 = (1 - 2M/r)^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{6.52}$$

With these, a calculation shows that for Schwarzschild metric

$$\frac{dC}{d\tau} = 0 \tag{6.53}$$

which was obtained by a direct calculation in spherical symmetry in chapter (4).

- **Evolution under diffeomorphism:**

Now we calculate the evolution of compactness under the flow given in equation (6.8) where the metric changes due to a diffeomorphism as $\frac{dh_{ab}}{d\tau} = D_a\xi_b + D_b\xi_a$. A general diffeomorphism ξ^a can be decomposed as $\tilde{\xi}^a$ tangent to S and $\xi^a_{normal} = un^a$ normal to S. The tangential components do not move S because of diffeomorphism invariance and therefore do not affect geometric quantities. We take $\xi^a = \xi^a_{normal} = un^a$ where n_a is the unit normal to the surface S. Then the rate of change of a quantity will be its Lie derivative by ξ^a. We see that

$$\frac{1}{\sqrt{\gamma}}\frac{d\sqrt{\gamma}}{d\tau} = uk. \tag{6.54}$$

And using the Gauss-Codazzi equation [5]

$$-2n^a n^b R_{ab} + R = \mathcal{R} + (k^{ij}k_{ij} - k^2) \tag{6.55}$$

we have

$$\frac{dk}{d\tau} = -\tilde{D}^a\tilde{D}_a u + \frac{u}{2}[\mathcal{R} - R - k^{ij}k_{ij} - k^2] \tag{6.56}$$

where $\tilde{D}_a$ denotes the intrinsic covariant derivative operator within the surface. So we have

$$\frac{dC}{d\tau} = \int_S [2k\tilde{D}^a\tilde{D}_a u + ukk^{ij}k_{ij} - uk\mathcal{R} + ukR]\sqrt{\gamma}d^2x. \tag{6.57}$$

From the results described above we can get an expression for the evolution of Hawking mass. The Hawking mass is

$$M_H(S) = \frac{\sqrt{A(S)}}{64\pi^{3/2}}C(S). \tag{6.58}$$

We then have

$$\frac{d}{d\tau}M_H = \left(\frac{1}{64\pi^{3/2}}\right)\left(\frac{1}{2\sqrt{A}}\frac{dA}{d\tau}C + \sqrt{A}\frac{dC}{d\tau}\right). \tag{6.59}$$

Knowing $\frac{dA}{d\tau}$ and $\frac{dC}{d\tau}$ for both RF and diffeomorphism from equations (6.24, 6.34, 6.40, and 6.51) we can calculate $\frac{d}{d\tau}M_H$ for both cases.

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