Chapter 5

NUMERICAL STUDIES

5.1   INTRODUCTION

We are interested in applying the Ricci flow (RF) in general relativity (GR) to better understand the quantities of physical interest like mass (energy) and area of the apparent horizon (entropy) in GR as these quantities can be treated as purely geometric ones in the domain of the RF techniques and one can study the evolution of these quantities under the RF to study the thermodynamics of GR.

While analytic techniques are general and powerful, we would also like to get a physical feel for the evolution of the metric under RF. While it may be possible with present day computers to evolve a general metric under the RF, in this thesis we make a modest beginning by studying spherically symmetric flows numerically. Our motivation is to have a few examples of evolution under RF and to get a feel for the phenomena that we encounter. This serves as a check on the analytic techniques of the last chapter. As we will see, even in this simple situation there are several practical problems to be overcome, having to do with the use of coordinates.

5.2   THE RICCI FLOW IN SPHERICAL SYMMETRY

In this section we set up the formalism for evolving the metric according to the Perelman's RF in spherical symmetry. Perelman's RF is

$$\frac{dh_{ab}}{d\tau} = -2R_{ab} + 2D_a D_b f \qquad (5.1)$$

where f is a scalar function generating a diffeomorphism. Since we work in spherical symmetry we choose the gauge so that the metric is

$$ds^2 = a(r)dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2). \qquad (5.2)$$

We will need another diffeomorphism to maintain the gauge in (5.2) and as this diffeomorphism is purely radial and hence can be written as a gradient, we can combine this with f without any further loss of generality.

The independent equations of the (5.1) can be written as

$$h^{ab}\frac{dh_{ab}}{d\tau} = -2R + 2D^2 f \qquad (5.3)$$

$$\frac{dh_{\theta\theta}}{d\tau} = -2R_{\theta\theta} + 2D_\theta D_\theta f. \qquad (5.4)$$

We will use (5.4) to solve for f and will plug this into (5.3) to get an autonomous equation for a(r)

$$\frac{\partial a}{\partial \tau} = \frac{-2a}{r^2} + \frac{2}{r^2} - \frac{a'}{ra} + \frac{a'}{r} - \frac{3a'^2}{2a^2} + \frac{a''}{a}. \qquad (5.5)$$

We will rewrite $a(r)$ in the form

$$a(r) = \left(1 - \frac{2M(r)}{r}\right)^{-1}, \qquad (5.6)$$

which can be done in the absence of an apparent horizon. $M(r)$ physically means the total mass contained within a shell of radius r. This form is useful because we can easily implement the constraint that the local energy density is positive.

The scalar curvature is given by

$$R = \frac{4M'}{r^2} \qquad (5.7)$$

where the mass density ρ is

$$\rho = \frac{R}{16\pi}. \tag{5.8}$$

We therefore can start with an initial positive mass density profile $\rho(r)$ and can derive the mass profile $M(r)$ from (5.7) accordingly.

We choose $\rho(r)$ to be positive from $r = 0$ to a finite radius and is zero after wards. This will mean that

$$M(r) = \frac{1}{2} \int_0^r r^2 \rho(r) dr \tag{5.9}$$

will increase from $M(0) = 0$ to $M(r) = M_\infty$.

After we have got M from an assumed mass density ρ maintaining the energy condition $(R > 0)$ we form the initial data set by working out $a(r)$ from $M(r)$

$$a(r) = \left(1 - \frac{2M(r)}{r}\right)^{-1}. \tag{5.10}$$

Next we look for a numerical solution of the evolution equation (5.5) of $a(r)$. (5.5) is a forward type nonlinear heat equation which can be evolved numerically using *Mathematica*.

We need to be careful about two regions : near $r = 0$ and near $r \to \infty$. Near $r = 0$ there is an effect due to the coordinate singularity and we need to handle this analytically. Similarly the asymptotic region, where r is infinite, needs special handling in order to put it on a computer.

5.3 ANALYTICAL STUDIES AT THE BOUNDARIES

- **Analytical study near the origin** $(r = 0)$**:**

  Here we present some analytical calculations near the origin $(r = 0)$.

  From the boundary conditions for $a(r)$ we have $a(0) = 1$ and $a'(0) = 0$. So the Taylor expansion for $a(r)$ around $r = 0$ looks like,

$$a(r) = 1 + \alpha r^2 + \beta r^3 + ... \tag{5.11}$$

.

First we note that $a = 1$ is a fixed point as the RHS of (5.5) is zero $(-2/r^2 + 2/r^2 = 0)$.

Taking $a(r) = 1 + \alpha r^2$, we have, at $r = 0$

$$\frac{\partial a}{\partial \tau} = \frac{-2}{r^2}(1 + \alpha r^2) + \frac{2}{r^2} + \frac{2\alpha r}{r} - \frac{2\alpha r}{r(1 + \alpha r^2)} - \frac{3(2\alpha r^2)}{2} + 2\alpha = 0. \tag{5.12}$$

So $a(0) = 1$ is maintained.

Next we will assume that $a(r)$ is an even function of r. This follows if $a(r)$ is analytic about the origin $(r = 0)$ and is spherically symmetric. We write $a(r)$ up to the tenth order in r

$$a(r) = 1 + \alpha r^2 + \beta r^4 + \gamma r^6 + \delta r^8 + \mu r^{10} \tag{5.13}$$

and then calculate the RHS of (5.5) to same order.

Next we collect the coefficients of each order in r from the RHS of (5.5) and get the new coefficients of each order in r from these old coefficients by solving the evolution equation (5.5) numerically. We find that after evolving τ by an amount ϵ

$$\alpha_{new} = \alpha_{old} + \epsilon(-6\alpha^2 + 10\beta) \tag{5.14}$$

$$\beta_{new} = \beta_{old} + 4\epsilon(3\alpha^3 - 8\alpha\beta + 7\gamma) \tag{5.15}$$

$$\gamma_{new} = \gamma_{old} + \epsilon(-18\alpha^4 + 68\alpha^2\beta - 32\beta^2 - 60\alpha\gamma + 54\delta) \tag{5.16}$$

$$\delta_{new} = \delta_{old} + 4\epsilon[6\alpha^5 - 29\alpha^3\beta + 27\alpha^2\gamma - 26\beta\gamma + 4\alpha(7\beta^2 - 6\delta) + 22\mu] \tag{5.17}$$

$$\mu_{new} = \mu_{old} - 2\epsilon[15\alpha^6 - 88\alpha^4\beta - 28\beta^3 + 84\alpha^3\gamma + 39\gamma^2 + 3\alpha^2(43\beta^2 - 26\delta) + 76\beta\delta + \alpha(-164\beta\gamma + 70\mu)].$$

(5.18)

We keep updating the coefficients in the above iterative way to evolve the RF near the origin.

- **Analytical study near the $(r \to \infty)$ limit:**

  Here we do a very similar kind of analysis as in the earlier section. This time we assume that $a(r \to \infty) = 1$ and Taylor expand $a(r)$ around $a(\infty)$ in powers of $\frac{1}{r}$. Unlike the analysis done in the earlier section, here we do not have analyticity about the point $(r = \infty)$ and we therefore keep both even and odd powers of $\frac{1}{r}$ up to order $\frac{1}{r^5}$ in the expansion as below (These coefficients, α, β, γ, δ, μ, are not the same as those earlier in the region around the origin.)

$$a(r) = 1 + \frac{\alpha}{r} + \frac{\beta}{r^2} + \frac{\gamma}{r^3} + \frac{\delta}{r^4} + \frac{\mu}{r^5}$$

(5.19)

and then, as before, we calculate the RHS of (5.5) to same order.

Next we collect the coefficients of each order in $\frac{1}{r}$ from the RHS of (5.5) and get the new coefficients of each order in $\frac{1}{r}$ from these old coefficients by solving the evolution equation (5.5) numerically. We find that

$$\alpha_{new} = \alpha_{old}$$

(5.20)

$$\beta_{new} = \beta_{old}$$

(5.21)

$$\gamma_{new} = \gamma_{old}$$

(5.22)

$$\delta_{new} = \delta_{old} + \epsilon(-\frac{9\alpha^2}{2} + 4\beta) \tag{5.23}$$

$$\mu_{new} = \mu_{old} + \epsilon(6\alpha^3 - 17\alpha\beta + 10\gamma). \tag{5.24}$$

We keep updating the coefficients in the above iterative way to evolve the RF near $r \to \infty$ limit.

5.4 DISCUSSION

We first create the initial data set $a(r)$ by starting with an initial positive mass density $\rho(r)$ profile and then obtaining the mass profile (fig. 5.1) by integrating this density over a spherical shell. Next we numerically evolve $a(r)$ and note that if we put a small lump of mass then the flow makes this extra lump smoother (fig. 5.2). This smoothing feature of the flow reminds us of the smoothing property of the heat equation and indeed, when written in geodesic coordinate system RF looks like heat equation. We also verify that an initially increasing $M(r)$ remains increasing. This verifies the analytical result that the RF preserves positivity of the scalar curvature. Our treatment is limited by the fact that we use the "a-form" of the metric. This form is not possible if there are apparent horizons. However, in the absence of apparent horizons, it appears that an initially non-singular flow remains non-singular. For long times the program develops cusps at the origin which may be due to the problems of matching the analytical treatment (at $r = 0$) with the numerical scheme. Next we notice that the maximum value of the compactness decreases as the flow parameter τ increases (fig. 5.3). This result was derived earlier in chapter (4) where we discussed that this property of compactness will ascertain the existence of Ricci flow and ensures that it will not hit a singularity.
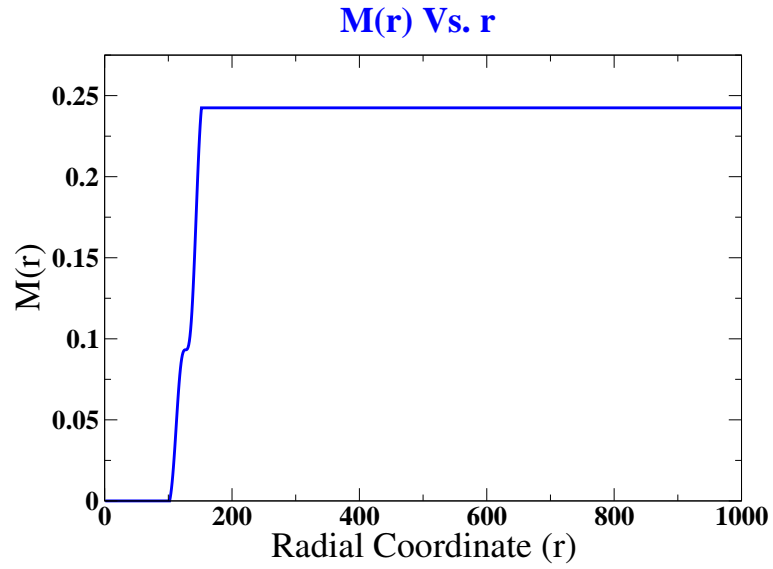
M(r) Vs. r

Figure 5.1: The initial mass profile: $M(r)$ increases from $M(0) = 0$ to $M(r) = M_\infty$
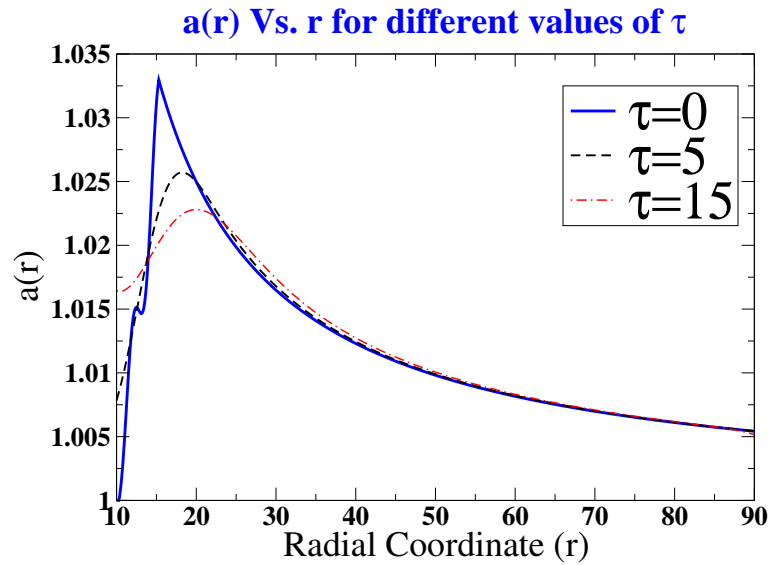
a(r) Vs. r for different values of τ

Figure 5.2: $a(r)$ vs r for different values of the flow parameter τ: Starting from an initial distribution with a peak it gets smoother as τ increases.
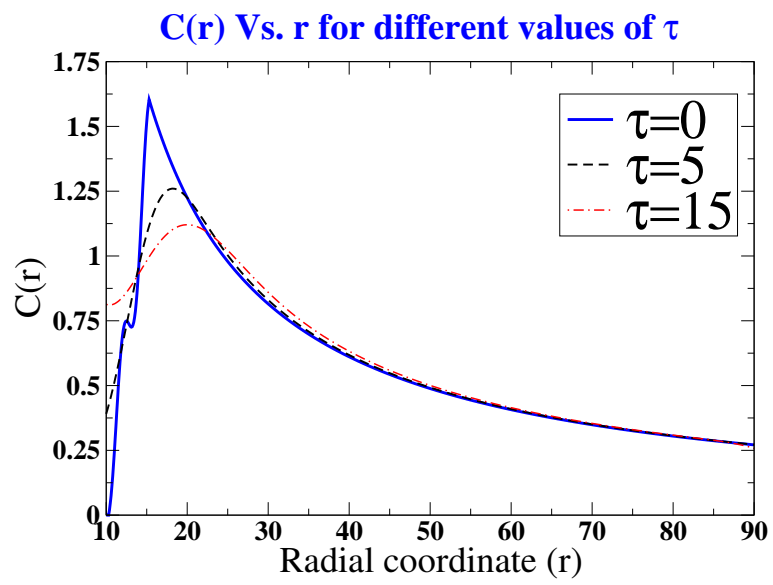
Figure 5.3: The maximum value of the compactness decreases as τ increases.

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