Chapter 2

REVIEW OF RICCI FLOW

2.1 THE RICCI FLOW

One of the fundamental problems in differential geometry is to find metrics of constant curvature on Riemannian manifolds. The existence of such a metric is important to topologists due to Thurston's programme of geometrizing 3-manifolds. In 1982 Richard Hamilton introduced the Ricci flow (RF) for producing metrics of constant sectional curvature. Intuitively, the idea is to set up a partial differential equation that evolves a metric according to its Ricci curvature tensor.

The Ricci flow has much in common with the heat equation. Indeed in a suitable co-ordinate system (Riemann normal co-ordinates) the Ricci flow becomes the heat equation for the metric functions h_{ij}. The heat equation tends to uniformize a given temperature distribution. Analogously the Ricci flow evolves an initial metric into a homogeneous one. The mathematical motivation for studying Ricci flows is that it leads to homogeneous metrics on manifolds.

Consider a compact 3-manifold Σ and a smooth 1-parameter family of Riemannian metrics $h_{ij}(\tau), \tau \in [0, \Gamma), \Gamma \leq \infty$ on Σ. We use the variable τ to describe the family to emphasize that τ is not the physical time variable t. The family $h_{ij}(\tau)$ is said to be a *Ricci flow* if it satisfies the equation:

$$\frac{\partial h_{ij}}{\partial \tau} = -2R_{ij} + \mathcal{L}_{\xi} h_{ij} + \lambda h_{ij}. \tag{2.1}$$

The first term is the Ricci tensor of h_{ij}, the second term is a diffeomorphism by the vector field ξ on Σ and the last term is an overall scaling of the metric. A few special cases are described below.

- **The unnormalized Ricci flow:** This was first introduced by Hamilton in 1982 where only the first term in the RHS of the equation (2.1) is present, i.e.,

$$\frac{\partial h_{ij}}{\partial \tau} = -2R_{ij}. \tag{2.2}$$

  Since this version of Ricci flow does not preserve the volume in general it is called the unnormalized Ricci flow.

- **The normalized Ricci flow:** This flow is defined as

$$\frac{\partial h_{ij}}{\partial \tau} = -2R_{ij} + \frac{2}{n} r h_{ij} \tag{2.3}$$

  where $r = \int RdV / \int dV$ is the average scalar curvature and n is the dimension of the Riemannian manifold. Under this flow the volume is preserved and this flow is obtained from (2.1) by keeping only the first and the last term in the RHS and choosing λ so as to preserve volume. In our work, we do not add a term like the last one, responsible for an overall scaling of the metric. We are interested in asymptotically flat 3-manifolds (not compact ones) and such a term will violate our asymptotic conditions.

- **Perelman's Ricci flow:** Perelman wrote the Ricci flow in a new way by supposing that the diffeomorphism generator ξ_i is a gradient

$$\xi_i = D_i f \tag{2.4}$$

  for some function f on Σ. Following Perelman we write the coupled equations:

$$\frac{\partial h_{ij}}{\partial \tau} = -2(R_{ij} + D_i D_j f) \tag{2.5}$$

$$\frac{\partial f}{\partial \tau} = -R - D^2 f \tag{2.6}$$

  to generate a flow $(h_{ij}(\tau), f(\tau))$ for a pair (h, f) consisting of a metric and a scalar function.

- **Examples and special solutions:** A simple example of Ricci flow is one starting from a round sphere. This will evolve by shrinking homothetically to a point in finite time. More generally if we take a metric $h_{ij}(0)$ such that

$$R_{ij}(h(0)) = \lambda h_{ij}(0) \tag{2.7}$$

for some constant $\lambda \in R$ (these metrics are known as Einstein metrics) then a solution $h_{ij}(\tau)$ of the Ricci flow equation

$$\frac{\partial h_{ij}}{\partial \tau} = -2R_{ij} \tag{2.8}$$

with $h_{ij}(\tau = 0) = h_{ij}(0)$ is given by

$$h_{ij}(\tau) = (1 - 2\lambda\tau)h_{ij}(0). \tag{2.9}$$

It is worth pointing out here that the Ricci tensor is invariant under uniform scaling of the metric. In particular, for the round 'unit' sphere $(S^n, h(0))$, we have $R_{ij}(h(0)) = (n-1)h_{ij}(0)$, so the evolution is $h(t) = (1 - 2(n-1)\tau h(0))$ and the sphere collapses to a point at 'time' $\tau = 1/2(n-1)$.

2.2 RICCI SOLITONS

A soliton for the normalized Ricci flow (2.3) is a metric that changes only by pull back by a one-parameter family of diffeomorphisms as it evolves under (2.3). This is equivalent to the initial metric satisfying

$$\mathcal{L}_X h_{ij} = -2R_{ij}(h) + (2r/n)h_{ij} \tag{2.10}$$

for some vector field X ($\mathcal{L}_X$ denotes the Lie derivative with respect to the vector field X which generates the diffeomorphism).

- **Gradient soliton:** A Ricci soliton whose vector field can be written as the gradient of some function $f : \Sigma \to R$ is known as a 'gradient Ricci soliton'. An example is the fixed point of the Perelman's Ricci flow (2.5,2.6) where the soliton equation is obtained by setting the RHS of these equations to be zero that is

$$-2(R_{ij} + D_i D_j f) = 0 \tag{2.11}$$

and

$$-R - D^2 f = 0 \tag{2.12}$$

examples of gradient solitons are Hamilton's cigar soliton and Bryant's soliton.

- **Solitons and Breathers:** In general, a "Breather" is a solution to a evolution equation that is periodic over time which, in our case of the normalized Ricci flow (2.3), means a solution to the flow that is periodic, up to diffeomorphism; that is

$$h_T = \phi^* h_0 \tag{2.13}$$

for some fixed period $\tau = T$ and diffeomorphism ϕ.

2.3 THE EVOLUTION OF SCALAR CURVATURE UNDER RICCI FLOW

The derivation of the general evolution equation for the scalar curvature under a geometric flow is given in the appendix- A and is as follows

$$\dot{R} = \frac{d}{d\tau}(h^{bd} R_{bd}) = -R_{bd} H^{bd} + \delta^2 H - D^2(trH) \tag{2.14}$$

where

$$H_{bd} := \dot{h}_{bd} \tag{2.15}$$

and

$$H^{bd} = h^{kb} h^{ld} H_{kl} \tag{2.16}$$

also

$$\delta^2 H := D^a D^b H_{ab} = h^{ac} h^{bd} (D_c D_d H_{ab}) \tag{2.17}$$

and

$$trH := h^{ab}H_{ab} \tag{2.18}$$

and the Laplacian operator

$$D^2 := h^{ab}D_aD_b. \tag{2.19}$$

Substituting $H_{ab} = -2R_{ab}$ in (2.14) gives us the evolution of R under the Ricci flow.

- **Ricci flow preserves positivity of scalar curvature:** *If the scalar curvature $R > 0$ at $\tau = 0$, then it remains so under the evolution.* This result shows that the evolution equation "prefers" positive curvature. The proof goes as follows

  Proof: The evolution of the scalar curvature under the unnormalized Ricci flow (2.2) is given by

  $$\dot{R} = \frac{\partial R}{\partial \tau} = D^2R + 2|R_{bd}|^2 \tag{2.20}$$

  where $|R_{bd}|^2 := R_{bd}R^{bd} > 0$ always and when R attains its minimum value at some "time" τ, we have $D^2R \geq 0$ and hence $\dot{R} \geq 0$ which proves that the minimum of R is non-decreasing along the flow. So if $R \geq 0$ to start with then it will remain positive throughout the evolution.

  Also for the Perelman Ricci flow the evolution of the scalar curvature is given by

  $$\frac{\partial R}{\partial \tau} = D^2R + 2R_{ab}R^{ab} + \mathcal{L}_X R, \tag{2.21}$$

  where X is the vector field generating the diffeomorphism as mentioned earlier. We argue by contradiction. Suppose that the RF evolves a positive scalar curvature metric to a negative scalar curvature one. R has to cross zero at some τ. Let τ_1 be the *first* τ for which R vanishes. This point $p \in \Sigma$ where this happens is a minimum of R since R is positive elsewhere on Σ (where Σ is the manifold considered). Since $D^2R \geq 0$ and $\mathcal{L}_X R = 0$ at the minimum, it follows from (2.21) that $\frac{\partial R}{\partial \tau} \geq 0$. Thus R increases with τ and remains positive. This contradiction proves that the positivity of R is preserved by the RF.

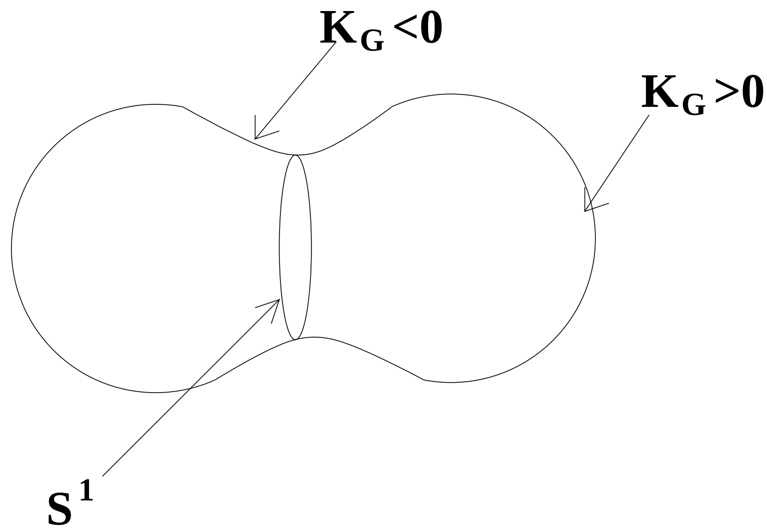

Figure 2.1: A (topological) 2-sphere.

2.4 RICCI FLOW IN TWO DIMENSIONS

In two dimensions, we know that the Ricci curvature in 2D can be written in terms of the Gaussian curvature K_G as $R_{ij}(h) = K_G h_{ij}$ where h_{ij} denotes the metric in 2D. Working directly from the equation (2.8), we then see that regions, in which $K_G < 0$, tend to expand, and regions, where $K_G > 0$, tend to shrink. From the inspection of fig.(2.1), one might guess that the Ricci flow tends to make a 2-sphere "rounder". This is indeed the case, and it is shown that the Ricci flow on any closed surface tends to make the Gaussian curvature constant, after renormalization of the flow. This gives a qualitative feel for the Ricci flow and its tendency to uniformize.

2.5 RICCI FLOW AND RENORMALIZATION GROUP FLOW

The gradient formulation of the Ricci flow (RF) by Perelman was motivated by the connection between the RF and renormalization group (RG) flow. In his paper on "The entropy formula for the Ricci flow and its geometric application" (section 5.1) [3], Perelman showed that, after fixing a closed manifold M with an appropriate probability measure m and a metric $h_{ij}(\tau)$ which depends on the 'temperature' τ, his entropy functional $\mathcal{F}_{\mathcal{P}}$ is in a sense analogous to minus the actual thermodynamic entropy of the system considered.

The Ricci flow first appeared in physics in the context of statistical physics and the use of the RG flow. To start with, we consider viewing the world through a microscope called the Ricci microscope as described below.

- **The Ricci microscope:** We imagine a microscope that allows us to look at objects with a variable magnification M, which can range from 1 to a very large (theoretically infinite) number. Suppose that we are able to view a 2 manifold under such a microscope. At a higher magnification $M = e^{-\tau}$ we will be able to see the *bumps* and *wiggles* on the manifold. These features will disappear as we lower the magnification.

  Let us define the Ricci microscope by the property that as we change the magnification from $M = e^{\tau}$ to $M = e^{\tau+d\tau}$ the metric on the 2 manifold changes according to the RF

  $$\frac{\partial h_{ij}}{\partial \tau} = -2R_{ij} + \mathcal{L}_{\xi} h_{ij}. \tag{2.22}$$

  If one looks at the manifold with a high M (small τ), each frame of the picture will cover a small part of the manifold and since the frame has a fixed resolution (number of pixels), storing the data would take a large number of files whereas a lower value of M (i.e., a higher τ) would lower the number of frames and hence the number of files resulting in a decrease in the total information about the manifold. From a physical point of view, the decrease in the information content can be viewed as the increase of "entropy". In fact the notion of entropy in useful in understanding the RF. The RF is interesting precisely because it decreases the amount of information that we have to look at. We are not interested in the information contained in the initial geometry of the manifold. We use the RF to decrease the amount of information and bring it down to a manageable size. In the programme developed by Thurston and Hamilton, one is interested in the topology of the manifold and this is considerably less information than the geometry of the initial manifold. With this motivation we would like to construct the Ricci microscope. The way to do this can be borrowed from physics where the Ricci microscope is realized as the RG. RG is a technique which is used in statistical mechanics and quantum field theory for understanding physics at different scales. The first point to note about RG is that it is not a group in the mathematical sense. It is a flow or a vector field. This vector field "lives" on the space of parameters $(p^1,\ p^2,\ ...,\ p^n)$ where these parameters are used for building up a theoretical model in statistical mechanics to make predictions about physical phenomena and for each value of the parameter the model makes certain predictions about the outcome of experiments.

The parameter space may be n-dimensional. One can then perform n experiments (assuming that the outcome depends on all the p's) to fix all the n parameters. The outcome of subsequent experiments can then be predicted from the theory. Note that if the parameter space is infinite dimensional, the theory has no predictive power as one has to perform infinitely many experiments to fix all the parameters, which is impossible. Such theories are called *non-renormalizable*.

The RG is a flow or a vector field on the parameter space $(p^1,\ p^2,\ ...,\ p^n)$. The integral curves of this vector field are obtained by solving the differential equation

$$\frac{dp^i}{d\tau} = \beta^i(p), \quad i = 1,\ 2,\ ...,\ n. \tag{2.23}$$

The RHS is called the beta function. The zeros of beta are the fixed point of the RG flow. τ describes the scale at which one probes the system under consideration. Over the years one of the ideas that has emerged from statistical mechanics is that physics is scale dependent i.e., the parameters of a model may depend on scale.

- **Models in statistical mechanics (SM):** A model consists of a configuration space C and an energy functional $E : C \rightarrow R$. We require E to be bounded below. The problem of SM is to compute the partition function

$$Z(T, V; p^1,\ p^2,\ ...,\ p^n) = \sum_C exp(-E(C, p^1,\ p^2,\ ...,\ p^n)/k_B T) \tag{2.24}$$

where T is the temperature and V is the volume of the system, and k_B is Boltzmann's constant. The partition function is like a generating function: by differentiating this with respect to T, V one can derive all experimental consequences of the model. e.g. the average energy is given by $< E >= -(\frac{\partial}{\partial \tilde{\beta}}) log Z$ where $\tilde{\beta} := (k_B T)^{-1}$, also the entropy is $S = \tilde{\beta} < E > + log Z$ and the energy fluctuation $\sigma =< (E- < E >)^2 >= \frac{\partial^2}{(\partial \tilde{\beta})^2} log Z$.

Example: Let us take an example where $C = R^+$, the positive z axis $E(z) = mgz$ (where mg is a parameter). and $Z(T, mg) = \int_0^\infty dz exp(-mgz/k_B T)$. This model describes a particle restricted to the half line in a uniform and infinite constant gravitational field.

This was a simple example where the partition function is easily calculable and finite. More usually the partition function cannot be calculated exactly and has infinities. This happens when the configuration space C is itself a map f from one space to the other.

$$f : R^n \to M. \tag{2.25}$$

Let us first take the case where M is the linear space R, with a norm. $x \in R^n$, $f(x) \in R$. A typical energy functional is

$$E[C] = \frac{1}{2} \int [a \nabla f \nabla f + b f^2 + c f^4] d^n x. \tag{2.26}$$

The sum over configurations $\sum_C$ means that we have to sum over all functions $f(x)$. If we work in Fourier space, the sum is over all Fourier coefficients $\tilde{f}(k)$. We do the sum in stages. Let us fix an absolute k space cutoff Λ_0, which corresponds to the smallest spatial scale of the theory (large k is small spatial scale). We write $\Lambda_\tau = \Lambda_0 exp(-\tau)$ and use Λ_τ as a sliding scale. Starting from $\tau = 0$, we integrate $exp(-E[C]/k_B T)$ successively over all $\tilde{f}(k)$ for which $\Lambda_{\tau+d\tau} < k < \Lambda_\tau$. After integrating $exp(-E[C]/k_B T)$ over this shell of k vectors, we rewrite the answer again as $exp(-E[C]/k_B T)$, where $E[C]$ is the same functional (2.26), possibly with changed (renormalized) parameters $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ which are functions of a, b, c and T and $d\tau$. Taking $d\tau$ to be infinitesimal, the flow can be written as a vector field

$$\frac{d}{d\tau} A^i = \beta^i(A^i) \tag{2.27}$$

where $i = 1, 2, 3$ and $A_1 = a$, $A_2 = b$ and $A_3 = c$.

This is an example of the RG flow. An important class of models, called the σ- models, is the case where (M, h) is a Riemannian manifold. In this case there is a natural energy functional

$$E[C] = \frac{1}{2} \int d^n x h_{ij} \nabla f^i \nabla f^j \tag{2.28}$$

where $h_{ij}(f)$ are the metric functions on M and f^i are the local coordinates on M. In this case, a calculation using perturbation theory gives

$$\frac{\partial h_{ij}}{\partial \tau} = -2R_{ij} + \lambda h_{ij} + \mathcal{L}_X h_{ij} \tag{2.29}$$

where λ is a scaling on M, R_{ij} is the Ricci tensor of M and X is an arbitrary vector field on M.

- **Remarks:**

  1. The parameter τ is just $-log\Lambda$ and represents the scale of observation.

  2. A fixed point of RG flow is of great importance to physicists. Many different choices of parameters p "flow" to the fixed point $p*$. The large scale description of the system is dictated by the region near $p*$. The set of p values that flow to $p*$ is called the "basin of attraction" of $p*$. All theories in the same basin are said to be in the same universality class. Fixed points of the Ricci flow are called Ricci solitons. They are also important for physicists, since they represent a universality class of theories .

  3. One illustration of RG at work is superconductivity theory. The RG flow is such that even a tiny attraction between electrons at a small scale grows with the flow and finally dominates the physics at large scales. The transition between normal and superconducting behaviour is abrupt because of this reason. A small change in the microscope parameters leads to a large change in macroscopic behaviour.

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