4 Particle pump with symmetric exclusion process.

4.1 Introduction

The symmetric exclusion process (SEP) is one of the simplest and well studied models of a stochastic interacting particle system. In this model which can be defined on a d-dimensional hypercubic lattice, particles move diffusively while satisfying the hardcore constraint that two particles cannot be on the same site. A number of exact results have been obtained for this model, particularly in one dimension [95 – 97]. If the model is defined on a ring and conserves the total density, the system obeys the equilibrium condition of detailed balance in the steady state and thus does not support any net current. A lot of attention has also been given to non-equilibrium steady states of driven SEP in which the particles can enter or leave the bulk at the boundaries. For this model, the time-dependent correlation functions [100] and dynamical exponents have been obtained using the equivalence of the transition matrix (W-matrix) to the Heisenberg model [101]. Recently, large deviation functional and current fluctuations have also been calculated for the driven SEP [100 – 102]. Experimentally it has been shown that SEP can be used to model the diffusion of colloidal particles in narrow pores [103 – 108].

Here we study the SEP for the case where hopping rates are time-dependent. This is one of the few studies of a many-particle interacting stochastic model with time-dependent transition rates and as we demonstrate shows a lot of interesting behaviour. The initial motivation for this study comes from quantum pump models discussed in the previous chapter [85, 88, 90 – 92, 109 – 117]. We saw there that classical heat pumps could be built on simi-

lar principles. Here we investigate the question whether, by using similar driving protocols, particle pumping can be achieved in a classical stochastic model.

Classical pumping of particles in time-dependent stochastic models of non-interacting particles has earlier been studied [118 – 120] and seen in experiments [123]. Systems exhibiting pumping effect have often been modeled as Brownian ratchets in which non-interacting particles move in an external periodic potential and we have discussed various such models in chapter (1). Our model differs from such models in that here we are dealing with a many body particle system with interactions, and particle interactions seem necessary for the pumping effect.

We have studied the time-dependent SEP by simulations and also analytically by using perturbation theory. The first perturbation uses the driving amplitude as the small parameter. The other uses the inverse of driving frequency as a small parameter. Within this perturbative approach, we are able to obtain exact expressions for various physical quantities, and find very good agreement with simulation results. The most interesting result is that in the model with time-dependent rates at all sites, a $\mathcal{DC}$ current of order unity can be obtained. We note that the hopping rates though time-dependent, are still symmetric and hence our result is surprising.

4.2 Definition of Model

The model is defined on a ring with L sites (see Fig. (4.1)). A site $l = 1, 2, 3, ...L$ can be occupied by $n_l = 0$ or 1 particle and the system contains a total of $N = \rho L$ particles where ρ is the total density. A particle at site l hops to an empty site either on the left or right with equal rates given by:

$$\begin{aligned} u_l &= f_0 + f_1 \nu_l \\ \text{where } \nu_l &= \alpha_l \sin(\omega t + \phi_l) = \nu_l e^{i\omega t} + \nu_l^* e^{-i\omega t} \,. \end{aligned} \tag{4.1}$$

Here the site-dependent complex amplitudes are defined by $\nu_l = \alpha_l \, e^{i\phi_l}/2i$ with α_l as a real amplitude and f_1 is chosen such that all hopping rates are positive.

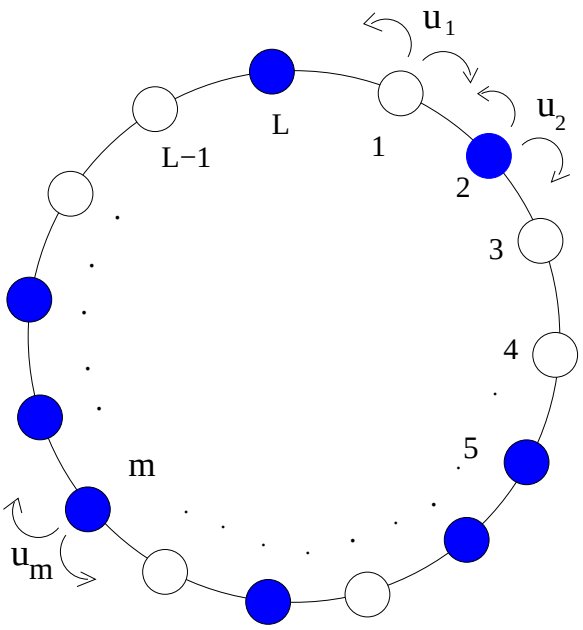

Figure 4.1: Schematic representation of the SEP model with periodic boundary conditions where a particle hops to next or previous unoccupied site with equal rates. Blue and white colors denote occupied and unoccupied sites respectively. For example particle at site 2 can hop to site 1 or 3 with equal probability where as particle at site 5 can hop to the previous site but not to the next site in this particular configuration.

A configuration of the system can be specified by the set $\{n_l\}$, $l = 1, 2, ...L$. Let us define $\mathbf{P}(t)$ as the probability vector in the configuration space, with elements $P(C, t)$ giving the probability of the system being in the configuration $C = \{n_l\}$ at time t. Then the stochastic dynamics of the many particle system is described by the master equation:

$$\frac{d\mathbf{P}(t)}{dt} = \mathbf{W}(t)\,\mathbf{P}(t) = \mathbf{W_0}\,\mathbf{P}(t) + \mathbf{W_1}(t)\,\mathbf{P}(t) \tag{4.2}$$

where $\mathbf{W}$ is the transition matrix, which we have split into a time-independent and a time-dependent part. One can also consider the time-evolution equations for m-point equal-time correlation functions $C_{l_1,l_2,l_3,....,l_m}(t) = \langle n_{l_1}...n_{l_m}\rangle = \sum_{\{n_l\}} n_{l_1}...n_{l_m} P(\{n_l\}, t)$. Thus, for example, the density $\rho_l(t) = \langle n_l\rangle$ and the two-point correlation function $C_{l,m}(t)$ satisfy the following

equations:

$$\frac{\partial \rho_l}{\partial t} + 2u_l\rho_l - u_{l-1}\rho_{l-1} - u_{l+1}\rho_{l+1} = u_l(C_{l-1,l} + C_{l,l+1}) - u_{l+1}C_{l,l+1} - u_{l-1}C_{l-1,l} \quad (4.3)$$

$$\begin{aligned} &\frac{\partial C_{l,m}}{\partial t} + 2(\,u_l + u_m\,)C_{l,m} - u_{l-1}\,C_{l-1,m} - u_{l+1}\,C_{l+1,m} - u_{m-1}\,C_{l,m-1} - u_{m+1}\,C_{l,m+1} \\ = \;& u_l\,(\,C_{l-1,l,m} + C_{l,l+1,m}\,) + u_m\,(\,C_{l,m-1,m} + C_{l,m,m+1}\,) - u_{l-1}\,C_{l-1,l,m} - u_{l+1}\,C_{l,l+1,m} \\ & -u_{m-1}\,C_{l,m-1,m} - u_{m+1}\,C_{l,m,m+1}, \quad \text{for} \quad |l-m| \neq 1 \\ &\frac{\partial C_{l,l+1}}{\partial t} + (\,u_l + u_{l+1}\,)C_{l,l+1} - u_{l-1}\,C_{l-1,l+1} - u_{l+2}\,C_{l,l+2} \\ = \;& u_l\;C_{l-1,l,m} + u_{l+1}\,C_{l,l+1,l+2} - u_{l-1}\,C_{l-1,l,l+1} - u_{l+2}\,C_{l,l+1,l+2}\,. \end{aligned} \quad (4.4)$$

From Floquet's theorem [124], it is expected that the long time state of the system (assumed to be unique) will be periodic in time with period $T = 2\pi/\omega$. Here we will be mainly interested in the *DC* current $\bar{J}$ defined as

$$\bar{J_l} = \frac{1}{T}\int_0^T J_{l,l+1}(t)\,dt, \quad (4.5)$$

where the current $J_{l,l+1}$ in a bond connecting sites l and $l+1$ is given by

$$J_{l,l+1} = u_l(\rho_l - C_{l,l+1}) - u_{l+1}(\rho_{l+1} - C_{l,l+1}) \quad (4.6)$$

and the local density $\rho_l = \langle n_l \rangle$. From the periodicity of the state and particle conservation, it follows that the *DC* current is uniform in space and therefore, using Eq. (4.6), we can write for the *DC* current:

$$\bar{J} = \frac{1}{LT}\int_0^T \sum_{l=1}^{L} J_{l,l+1}(t)\,dt \quad (4.7)$$

$$= \frac{f_1}{LT}\int_0^T \sum_{l=1}^{L} (v_{l+1} - v_l)C_{l,l+1}\,dt \quad (4.8)$$

Thus, to find the *DC* current, we need to compute 2-point correlation function $C_{l,l+1}(t)$. In this chapter, we will first develop a perturbation theory, for general v_l, and then apply it to some special cases.

Note that for $f_1 = 0$, the above model reduces to the homogeneous SEP with periodic

boundary conditions whose properties are known exactly. In this case the steady state is an equilibrium state which obeys detailed balance and hence the average current is zero (This result holds even when the u_l's are site dependent, but time independent). In the steady state, all configurations are equally probable i.e. $P(C) = 1/\binom{L}{N}$ when $f_1 = 0$. Then one can show that the density and correlation functions for the homogeneous SEP are given by:

$$\begin{aligned} \rho_l^{(0)} &= \rho = \frac{N}{L} \\ C_{l_1,l_2}^{(0)} &= \rho\frac{(N-1)}{(L-1)} \\ C_{l_1,l_2,l_3,....,l_m}^{(0)} &= \binom{L-m}{N-m}/\binom{L}{N}. \end{aligned} \tag{4.9}$$

4.3 Perturbation theory in f_1

For $f_1 \neq 0$, the knowledge of the exact steady state of homogeneous SEP enables us to set up a perturbation expansion in f_1 of various observables. We now describe this perturbation theory within which we calculate an expression for *DC* current $\bar{J}$ in the bulk of the system. A similar perturbation technique was developed for a two-state system in [125]. We expand various quantities of interest with f_1 as the perturbation parameter about the homogeneous steady state corresponding to $f_1 = 0$. Thus we write

$$\rho_l(t) = \langle n_l(t)\rangle = \rho + \sum_{r=1}^{\infty} f_1^r \rho_l^{(r)}(t) \tag{4.10}$$

$$C_{l,m}(t) = \langle n_l(t) n_m(t)\rangle = C_{l,m}^{(0)} + \sum_{r=1}^{\infty} f_1^r C_{l,m}^{(r)}(t)\,, \tag{4.11}$$

and similar expressions for higher correlations. Plugging in Eq. (4.11) into Eq. (4.8), we find that the lowest order contribution to $\bar{J}$ is at $O(f_1^2)$ and given by:

$$\bar{J}^{(2)} = \frac{f_1^2}{T\,L}\int_0^T \sum_{l=1}^{L} (\,v_l - v_{l+1}\,)\, C_{l,l+1}^{(1)}\, dt\,. \tag{4.12}$$

To develop our perturbation theory and finding $C_{l,m}^{(1)}$'s, we start with the time evolution equation for density $\rho_l(t)$ which is given by Eq. (4.3). Plugging in the expansions in Eqs. (4.10)

and (4.11), we get the following equation for the density $\rho_l^{(r)}$ at r^{th} order:

$$\begin{aligned}\frac{\partial \rho_l^{(r)}}{\partial t} &- f_0 \Delta_l \rho_l^{(r)} + 2\nu_l \rho_l^{(r-1)} - \nu_{l-1}\rho_{l-1}^{(r-1)} - \nu_{l+1}\rho_{l+1}^{(r-1)} \\ &= \nu_l (C_{l-1,l}^{(r-1)} + C_{l,l+1}^{(r-1)}) - \nu_{l-1} C_{l-1,l}^{(r-1)} - \nu_{l+1} C_{l,l+1}^{(r-1)}, \end{aligned} \tag{4.13}$$

where $\Delta_l g_l = g_{l+1} - 2g_l + g_{l-1}$ defines the discrete Laplacian operator. Thus the density at r^{th} order is obtainable in terms of density and two point correlation function at $(r-1)^{th}$ order. We check that at the zeroth order, we obtain the homogeneous SEP for which the density and all equal time correlations are given by Eq. (4.9). At first order, the above equation then gives:

$$\frac{\partial \rho_l^{(1)}}{\partial t} - f_0 \Delta_l \rho_l^{(1)} = r_0 \Delta_l \nu_l, \tag{4.14}$$

where $r_0 = \rho - C_{l,m}^{(0)}$. The solution for this equation is the sum of a homogeneous part which depends on initial conditions and a particular integral. At long times the homogeneous part vanishes while the particular integral has the following asymptotic form:

$$\rho_l^{(1)}(t) = A_l^{(1)} e^{i\omega t} + A_l^{*(1)} e^{-i\omega t}. \tag{4.15}$$

Substituting Eq. (4.15) in Eq. (4.14) we obtain the following equation for $\{A_l^{(1)}\}$:

$$(i\omega + 2f_0) A_l^{(1)} - f_0 A_{l-1}^{(1)} - f_0 A_{l+1}^{(1)} = r_0 (\nu_{l+1} - 2\nu_l + \nu_{l-1}) \,. \tag{4.16}$$

This can be written in matrix form as:

$$\hat{Z}(\omega) \;\; \mathbf{A} = -r_0 \; \hat{B} \, \mathbf{\Phi}, \tag{4.17}$$

where

$$\begin{aligned} Z_{lm} &= -f_0 \, \delta_{l,m+1} + (\, i\omega + 2f_0 \,) \, \delta_{l,m} - f_0 \, \delta_{l,m-1} \\ B_{lm} &= -\delta_{l,m+1} + 2 \, \delta_{l,m} - \delta_{l,m-1} \\ \mathbf{A} &= \{A_1^{(1)}, A_2^{(1)}, ..., A_L^{(1)}\}^T, \mathbf{\Phi} = \{\nu_1, \nu_2, ..., \nu_L\}^T, \end{aligned} \tag{4.18}$$

and periodic boundary conditions are implicitly taken. The above equation can be solved for **A** and we get:

$$\mathbf{A} = -r_0\, \hat{G}(\omega)\, \hat{B}\, \mathbf{\Phi}, \tag{4.19}$$

where $\hat{G}(\omega) = \hat{Z}^{-1}(\omega)$. Both $\hat{G}(\omega)$ and $\hat{B}$ are cyclic matrices and so can be diagonalized simultaneously. The eigenvalues of $\hat{Z}(\omega)$ are $i\omega + 4f_0 \sin^2(p\pi/L)$, while that of $\hat{B}$ are $4\sin^2(p\pi/L)$ with $p = 1, 2, ..., L$, and eigenvector elements are $e^{i2\pi pl/L}/L^{1/2}$. Hence $A_l^{(1)}$ can be written as:

$$A_l^{(1)} = -\frac{4r_0}{L} \sum_{m=1}^{L} \sum_{p=1}^{L} \frac{e^{-i\frac{2\pi p(l-m)}{L}} \sin^2(p\pi/L)}{i\omega + 4f_0 \sin^2(p\pi/L)}\, \nu_m, \tag{4.20}$$

which in the large L limit gives:

$$A_l^{(1)} = -\frac{r_0}{f_0}\, \nu_l + \frac{ir_0\omega}{f_0^2} \frac{1}{z_+ - z_-} \sum_{m=1}^{L} [\, z_-^{|m-l|} + z_-^{L-|m-l|}\,]\, \nu_m, \tag{4.21}$$

where, $z_- = y/2 - [(y/2)^2 - 1]^{1/2}$, $z_+ = 1/z_-$ and $y = 2 + (i\omega/f_0)$.

To compute the $O(f_1^2)$ contribution to $\bar{J}$, we need to evaluate $C_{l,m}^{(1)}$, which we now proceed to obtain. Inserting the perturbation series in Eqs. (4.10) and (4.11) into Eq. (4.4) we get the following equation for the correlation $C_{l,m}^{(r)}$ at r^{th} order for $|m - l| \neq 1$:

$$\begin{aligned}
\frac{\partial C_{l,m}^{(r)}}{\partial t} &- f_0\, (\, \Delta_l + \Delta_m\,)\, C_{l,m}^{(r)} + 2\nu_l\, C_{l,m}^{(r-1)} - \nu_{l-1}\, C_{l-1,m}^{(r-1)} - \nu_{l+1}\, C_{l+1,m}^{(r-1)} \\
&+ 2\nu_m\, C_{l,m}^{(r-1)} - \nu_{m-1}\, C_{l,m-1}^{(r-1)} - \nu_{m+1}\, C_{l,m+1}^{(r-1)} \\
&= \nu_l\, (\, C_{l-1,l,m}^{(r-1)} + C_{l,l+1,m}^{(r-1)}\,) + \nu_m\, (\, C_{l,m-1,m}^{(r-1)} + C_{l,m,m+1}^{(r-1)}\,) \\
&- \nu_{l-1}\, C_{l-1,l,m}^{(r-1)} - \nu_{l+1}\, C_{l,l+1,m}^{(r-1)} - \nu_{m-1}\, C_{l,m-1,m}^{(r-1)} - \nu_{m+1}\, C_{l,m,m+1}^{(r-1)},
\end{aligned}$$

while for $m = l + 1$:

$$\begin{aligned}
\frac{\partial C_{l,l+1}^{(r)}}{\partial t} &+ f_0\, (\, 2C_{l,l+1}^{(r)} - C_{l-1,l+1}^{(r)} - C_{l,l+2}^{(r)}\,) \\
&= \nu_{l+2}\, (\, C_{l,l+2}^{(r-1)} - C_{l,l+1,l+2}^{(r-1)}\,) + \nu_{l-1}\, (\, C_{l-1,l+1}^{(r-1)} - C_{l-1,l,l+1}^{(r-1)}\,) \\
&- \nu_l\, (\, C_{l,l+1}^{(r-1)} - C_{l-1,l,l+1}^{(r-1)}\,) - \nu_{l+1}\, (\, C_{l,l+1}^{(r-1)} - C_{l,l+1,l+2}^{(r-1)}\,).
\end{aligned} \tag{4.22}$$

At first order we get:

$$\begin{aligned}
\frac{\partial C^{(1)}_{l,m}}{\partial t} &- f_0(\Delta_l + \Delta_m) C^{(1)}_{l,m} = k_0(\Delta_l \nu_l + \Delta_m \nu_m)\,, \\
\frac{\partial C^{(1)}_{l,l+1}}{\partial t} &+ f_0\left(2C^{(1)}_{l,l+1} - C^{(1)}_{l-1,l+1} - C^{(1)}_{l,l+2}\right) = k_0(\nu_{l-1} + \nu_{l+2} - \nu_l - \nu_{l+1}),
\end{aligned} \tag{4.23}$$

where $k_0 = C^{(0)}_{l_1,l_2} - C^{(0)}_{l_1,l_2,l_3}$ and these are known from Eq. (4.9). The computation of even the homogeneous solution of the above set of equations is in general a nontrivial task because of the form of the equations involving nearest neighbor indices and requires a Bethe ansatz or dynamic product ansatz [99, 100]. However it turns out that the long time solution can still be found exactly and is given by:

$$C^{(1)}_{l,m}(t) = \frac{k_0}{r_0}[\rho^{(1)}_l(t) + \rho^{(1)}_m(t)] = A^{(1)}_{l,m} e^{i\omega t} + A^{*(1)}_{l,m} e^{-i\omega t}\,, \tag{4.24}$$

where $A^{(1)}_{l,m} = (k_0/r_0)(A^{(1)}_l + A^{(1)}_m)$. It is easily verified that this satisfies Eq. (4.23) for all l, m. To determine whether the system indeed has a product measure requires a more detailed analysis of the higher order terms in the perturbation series and higher correlations. We have verified that, at first order in perturbation theory, all correlation functions in fact have the same structure as the two-point correlation function in Eq. (4.24).

We now plug the solution in Eq. (4.24) into Eq. (4.12) for the average current in the system and after some simplifications obtain:

$$\bar{J}^{(2)} = -\frac{f_1^2}{L}\,\frac{k_0}{r_0}\,\sum_{l=1}^{L}\,(\,A^{*(1)}_{l+1}\nu_l \,+\, A^{(1)}_{l+1}\nu^*_l \,-\, A^{*(1)}_l \nu_{l+1} \,-\, A^{(1)}_l \nu^*_{l+1}\,)\,, \tag{4.25}$$

with $A^{(1)}_l$ given by Eq. (4.21). For any given choice of the rates ν_l, this general expression can be used to explicitly evaluate the net *DC* current in the system.

We now consider two special choices of the rates $\{\nu_l\}$.

(i) The choice $\alpha_1 = \alpha_L = 1$, all other $\alpha_l = 0$, and $\phi_1 = 0, \phi_L = \phi$ corresponds to the two-site pumping problem. In the limit of large L, this gives:

$$\bar{J}^{(2)} = \left(\frac{f_1}{f_0}\right)^2 \frac{k_0 \omega \sin\phi}{L}\,\mathrm{Re}[z_-]. \tag{4.26}$$

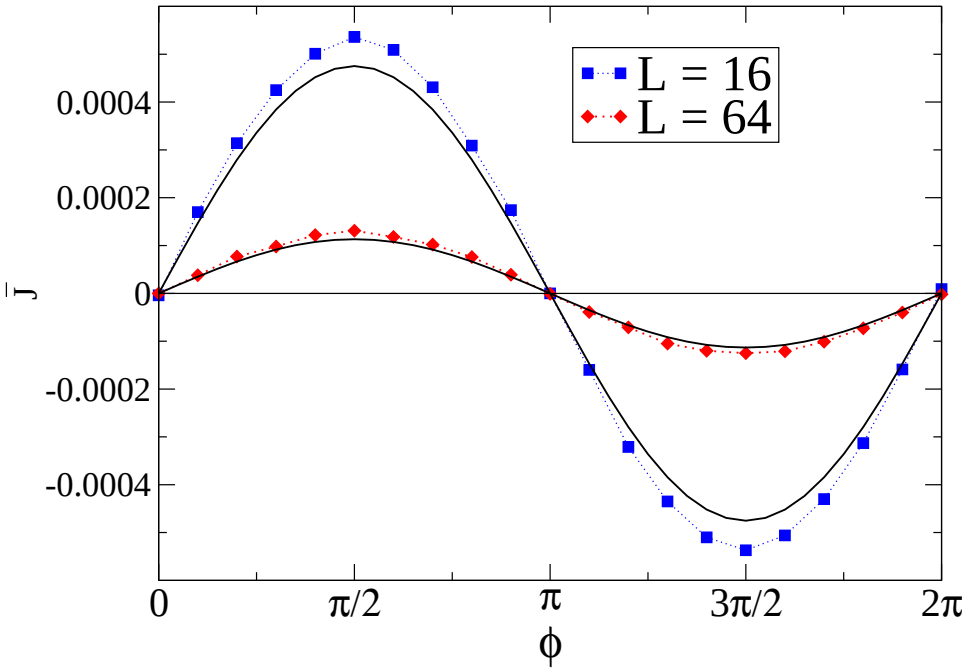

Figure 4.2: Plot of current $\bar{J}$ versus the phase difference ϕ. For parameters as in Fig. (4.4). The solid lines are from the perturbation theory.

Writing $z_+ = re^{i\eta}$, we find that for $\omega \ll \omega^* = 2f_0$, the magnitude $r \approx 1 + \sqrt{\omega/\omega^*}$ and the angle $\eta \approx \sqrt{\omega/\omega^*}$. In the opposite limit, $r \approx 2\omega/\omega^*$ and $\eta \approx \pi/2 - \omega^*/\omega$. Using $z_+ = 1/z_-$, we find that the current has the scaling form:

$$\bar{J}^{(2)} = \frac{f_1^2 k_0 \sin\phi}{f_0 L} G\left(\frac{\omega}{2f_0}\right), \tag{4.27}$$

where the scaling function $G(x) = 2x$ for $x \ll 1$ and $1/x$ for $x \gg 1$. We summarize the most interesting features of the above result. These are: (1) A *DC* current $\bar{J}$ is obtained, which decays with system size L as $\bar{J} \sim 1/L$. (2) The *DC* current $\bar{J}$ depends sinusoidally on the phase difference between rates at two sites. (3) The dependence of $\bar{J}$ on driving frequency ω shows a peak at a frequency ω^* with $\bar{J} \to 1/\omega$ as $\omega \to \infty$ and $\bar{J} \to \omega$ as $\omega \to 0$. The latter result means that a finite number of particles are circulated even in the adiabatic limit. We discuss this point in detail in Sec. (4.5). We have performed direct numerical simulations of the time-dependent SEP and compared them with our analytic results. We plot $\bar{J}$ versus phase difference ϕ and driving frequency ω in Figs. (4.2) and (4.3) respectively.

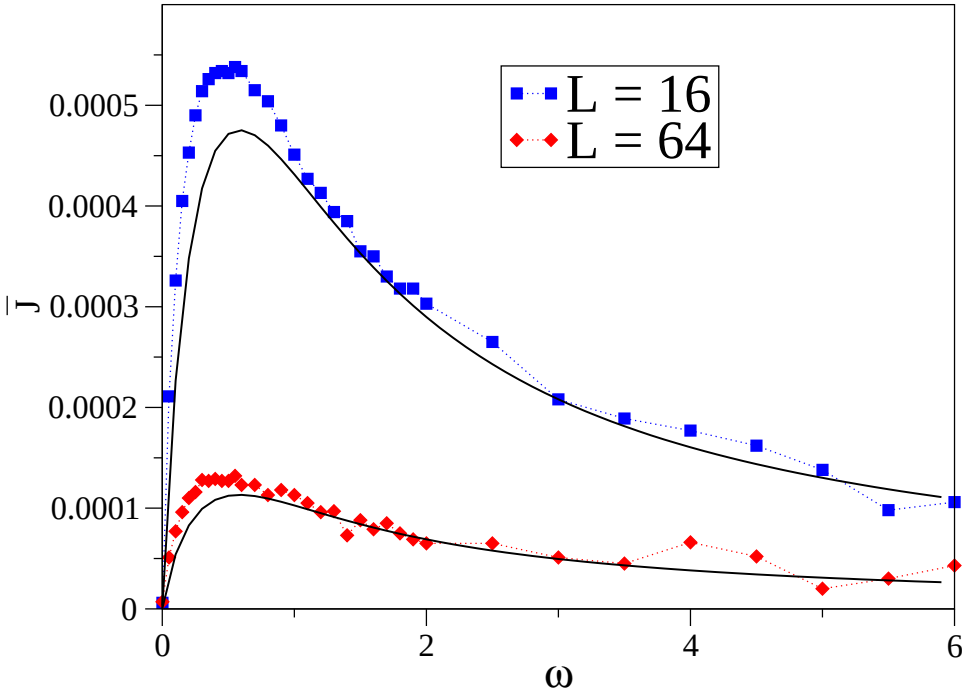

Figure 4.3: Plot of current $\bar{J}$ versus driving frequency ω for the same parameters as in Fig. (4.4). Solid lines are from perturbation theory.

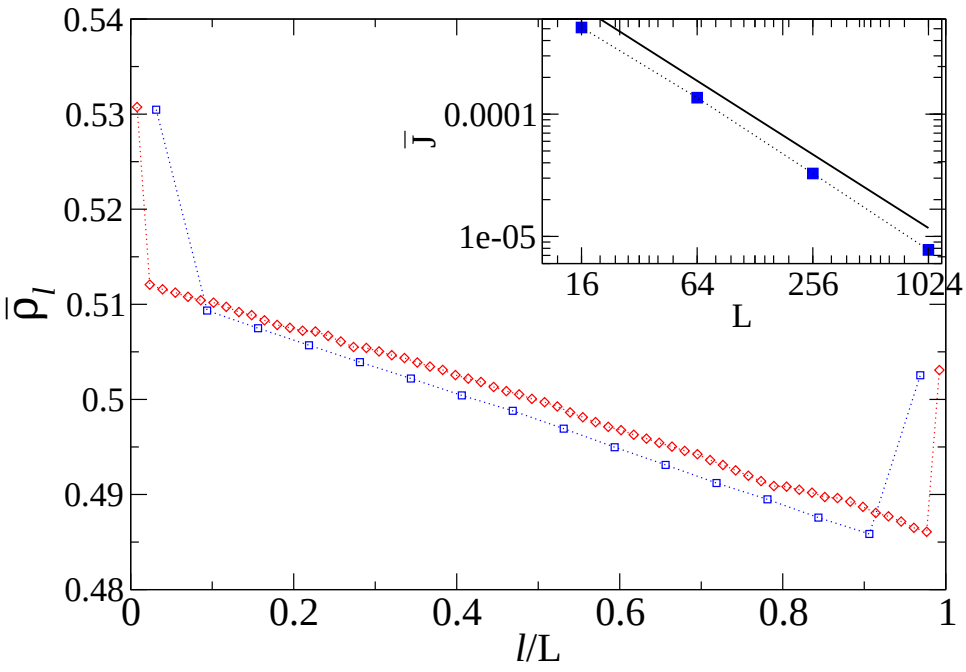

Figure 4.4: Plot of $\mathcal{DC}$ density $\bar{\rho}_l$ across the ring for $f_0 = 0.3$, $f_1 = 0.2$, $\omega = 0.2\pi$ and $\phi = \pi/2$ at half filling for two system sizes obtained from simulations. Inset: $\mathcal{DC}$ current (from simulations) $\bar{J} \sim 1/L$ as shown by solid line of slope -1.

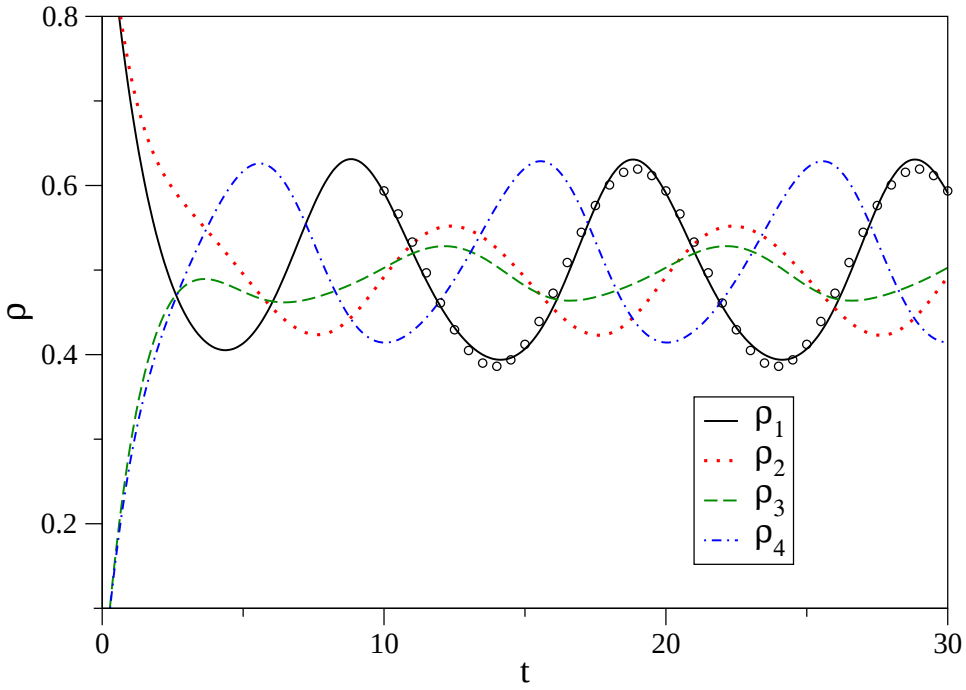

Figure 4.5: Plot of time-dependent densities at the four sites of a $L = 4$ lattice. In the initial configuration, sites 1 and 2 have one particle each and other sites are empty. The averages over one time period give: $\bar{\rho}_1 = 0.503493$, $\bar{\rho}_2 = 0.498702$, $\bar{\rho}_3 = 0.497417$, $\bar{\rho}_4 = 0.500388$ and $\bar{J} = 0.000514$. The points show the curve $\rho + f_1\rho_1^{(1)} + f_1^2\rho_1^{(2)}$. [Parameters: $f_0 = 0.4$, $f_1 = 0.1$, $\phi = \pi/2$ and $\omega = 0.2\pi$].

In the simulations we have also looked at the steady state density profiles. The results from simulation are shown in Fig. (4.4). The linear profile is expected since in the bulk of the system we have $J = -\nabla\rho$. From Eq. (4.15) it is clear that at first order correction, $\mathcal{DC}$ part $\bar{\rho}_l^{(1)}$ vanishes. Hence, we need to look at the higher order contribution, namely $\rho_l^{(2)}(t)$. This can be found exactly and has the form:

$$\rho_l^{(2)}(t) = \bar{\rho}_l^{(2)} + A_l^{(2)} e^{i2\omega\, t} \;+\; A_l^{*(2)} e^{-i2\omega\, t}. \tag{4.28}$$

The general expression for the $\mathcal{DC}$ part is given by:

$$\begin{aligned}
\bar{\rho}_l^{(2)} &= bl + h\,,\ l = 2, ..., L-1 \\
\bar{\rho}_1^{(2)} &= b + h + \frac{2}{f_0}\,\mathrm{Re}[\nu_1^*(A_{1,2}^{(1)} - A_1^{(1)})] \\
\bar{\rho}_L^{(2)} &= bL + h + \frac{2}{f_0}\,\mathrm{Re}[\nu_L^*(A_{L-1,L}^{(1)} - A_L^{(1)})]\,,
\end{aligned} \tag{4.29}$$

where the slope b of the linear density profile is given by

$$b = \frac{2}{L f_0} \, \mathrm{Re}[\nu_1^*(A_{1,2}^{(1)} - A_{1,L}^{(1)}) + \nu_L^*(A_{1,L}^{(1)} - A_{L-1,L}^{(1)})] \, , \tag{4.30}$$

and the intercept h can be found using the particle conservation condition $\sum_l \rho_l^{(2)} = 0$. This agrees with the form seen in results in Fig. (4.4). Finally in Fig. (4.5) we plot the density $\rho_l(t)$ as a function of time for $L = 4$ and $N = 2$ problem, which can be exactly solved numerically. As can be seen, the results from the perturbation theory match very well with the exact ones.

We also note that $\bar{J}$ is independent of f_0 for large x. This can be seen by writing the master equation as:

$$\frac{d\mathbf{P}}{d(\omega t)} = \frac{f_0}{\omega}\mathbf{W_0}\mathbf{P(t)} + \frac{f_1}{\omega}\mathbf{W_1}\mathbf{P(t)} \, . \tag{4.31}$$

For $\omega \gg f_0$, the first term on the right hand side can be neglected thus giving the probability distribution to be a function of f_1/ω.

(ii) The second case we consider is one where $\alpha_l = 1$ at all sites and $\phi_l = ql$, where $q = 2\pi s/L$ with $s = 1, 2...L/2$, so that there is a constant phase difference q between successive sites. In this case, $A_l^{(1)}$'s given by Eq. (4.20), evaluated at large L gives:

$$\begin{aligned} A_l^{(1)} &= \frac{ir_0}{2f_0} e^{iql} a \\ \text{where } a &= \frac{1 - \cos q}{y/2 - \cos q} \end{aligned} \tag{4.32}$$

and from Eq. (4.25) we get for the average current:

$$\begin{aligned} \bar{J}^{(2)} &= -\frac{f_1^2 k_0}{f_0} \sin q \, \mathrm{Im}[a] \\ &= \frac{2 \, f_1^2 \, k_0 \, \omega \, \sin q \, (1 - \cos q)}{[\, \omega^2 \, + \, 4 f_0^2 \, (1 - \cos q)^2 \,]} \, . \end{aligned} \tag{4.33}$$

Thus we see that for most values of q we get a finite current, even in the limit $L \to \infty$. For $q \sim 1/L$ and $q \sim \pi - 1/L$, the current goes to zero for large system size as $\bar{J} \sim L^{-3}$. From the current expression in Eq. (4.33), we can find out the value $q = q^*$, at which the current is a

maximum. By differentiating Eq. (4.33) with respect to q we get:

$$\cos(q^*) = (1 + \Omega^2) - \sqrt{(1 + \Omega^2)^2 - (1 - \Omega^2)}, \tag{4.34}$$

where $\Omega = \omega/2f_0$. It turns out that for large ω the maximum is at $q^* = 2\pi/3$, while for small frequencies we get $q^* \sim \sqrt{\omega}$. Also we find from Eq. (4.33) that in the adiabatic and fast drive limits, the currents are respectively given by:

$$\bar{J}^{(2)} \begin{cases} = \frac{f_1^2 k_0}{2 f_0^2} \cot(q/2)\, \omega & \omega/f_0 << (1 - \cos q) \\ = 2 f_1^2 k_0 \sin q (1 - \cos q)\, \frac{1}{\omega} & \omega/f_0 >> 1\ . \end{cases} \tag{4.35}$$

The perturbation theory results turn out to be quite accurate, as can be seen from the comparisons with simulation results, shown in Figs. (4.6) and (4.7), for different choices of q namely $q = \pi/2$ and $q = 2\pi/L$, for case (ii) discussed above. In these figures we have plotted the current for different system sizes and verify the $\bar{J} \sim L^0$ dependence and $\bar{J} \sim L^{-3}$ dependence for these two q's. Using the expression for k_0 in Eqs. (4.26, 4.33), we find that $\bar{J}^{(2)} \sim \rho^2(1 - \rho)$ which has a maximum at $\rho^* = 2/3$ and breaks particle-hole symmetry. This particle-hole asymmetry can be understood easily. From the definition of the model we see that, unlike the particles, the hopping rates of a hole are not symmetric: a hole at site l hops towards right with rate u_{l+1} and left with u_{l-1}. In Fig. (4.8) we have plotted simulation results for the average current as a function of particle density, for different system sizes, and find good agreement with our perturbative result, even at a relatively large value of f_1/f_0.

In simulations we have looked at the density profiles and find that the site wise density profile $\bar{\rho}_l$ in case (ii) is flat. This is unlike in case (i), where we found high densities at the two special sites and then a linear density profile in the bulk (see Fig. (4.4)). The flat density profile, for case (ii), is understood because here there are no special *pumping* sites. It is interesting that we can get current in the system even in the absence of Fick's law. We also note that even if the hop-out rates are made biased in one direction, like in the asymmetric exclusion process (ASEP), we can still get a current opposing this bias (for small biases).

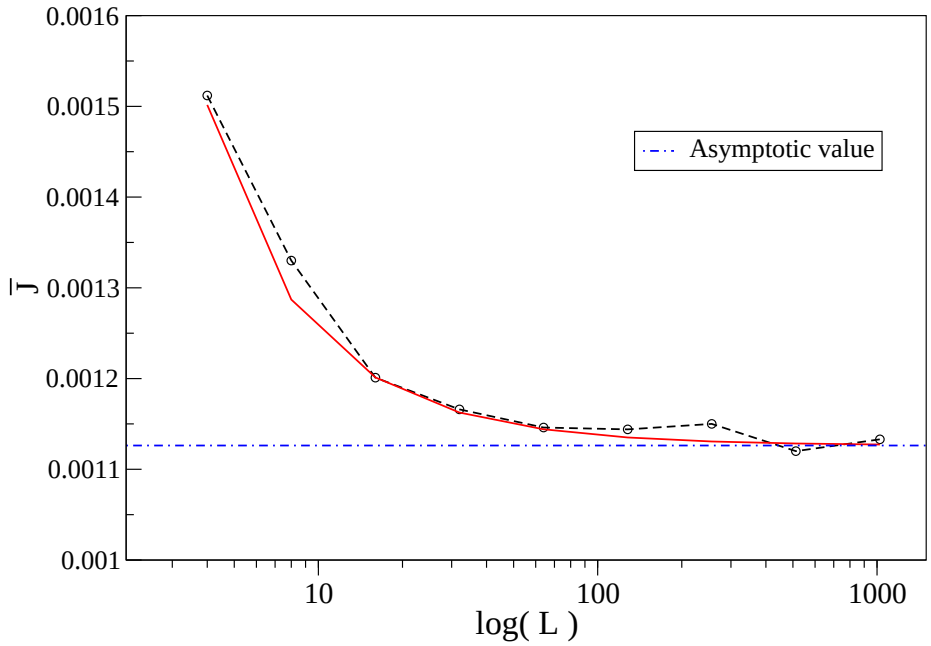

Figure 4.6: Plot of $\mathcal{DC}$ current $\bar{J}$ versus system size L for parameters $f_0 = 0.5$, $f_1 = 0.1$, $\omega = 0.2\pi$ and for $q = \pi/2$. Continuous line from perturbation theory and dotted line from simulations. $\bar{J}$ goes to a constant value can also be seen from Eq. (4.33) for this phase difference.

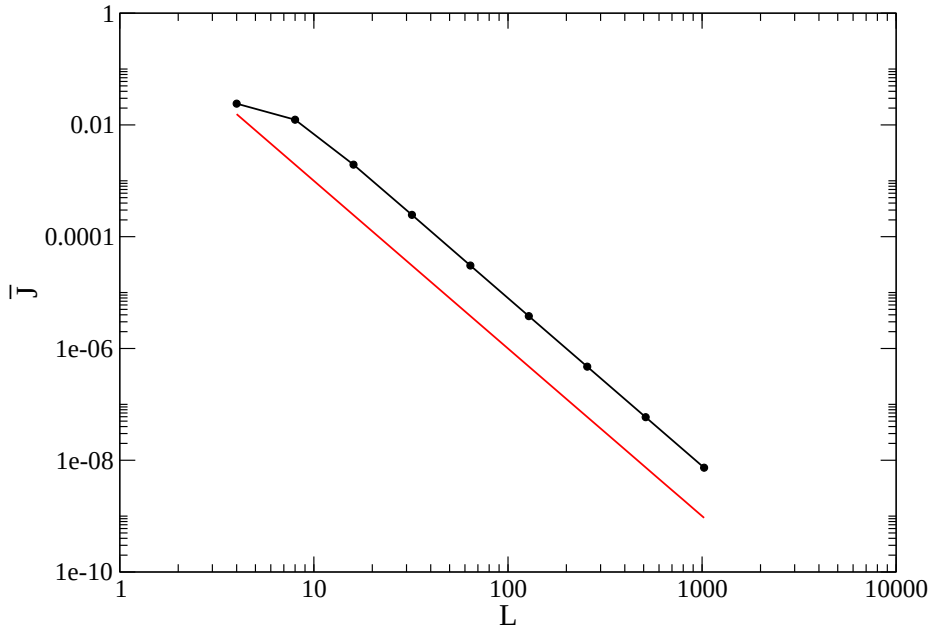

Figure 4.7: Log-log plot of $\mathcal{DC}$ current $\bar{J}$ (dotted line from Eq. (4.33), numerical values) versus system size L for $q = 2\pi/L$. The current decays as $1/L^3$ (continuous line) as predicted by Eq. (4.33). Parameter values are $f_0 = 0.5$, $f_1 = 0.4$, $\omega = 0.2\pi$.

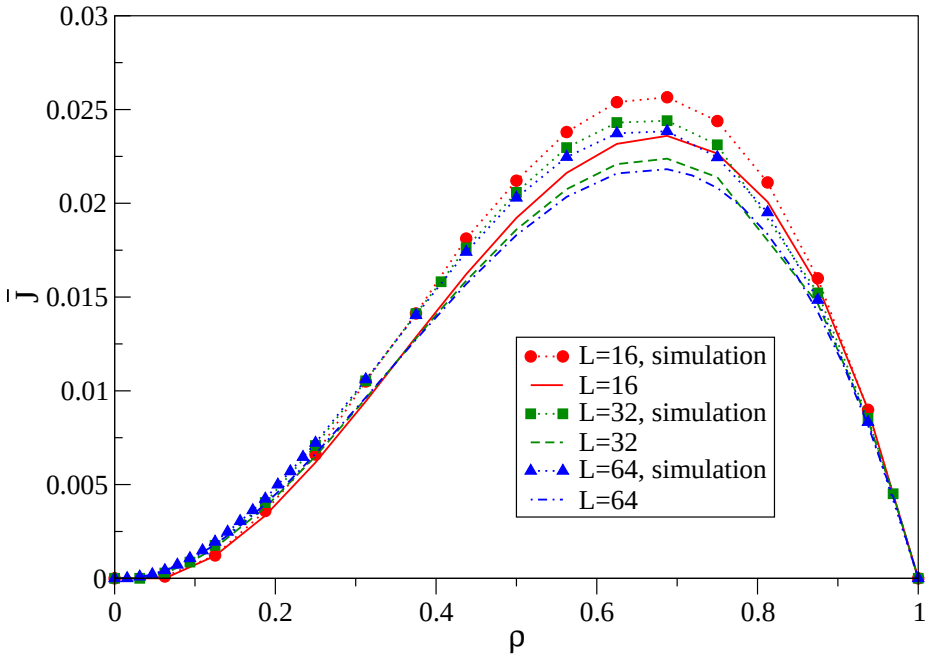

Figure 4.8: Plot of *DC* current $\bar{J}$ versus density $\rho = N/L$ for parameters $f_0 = 0.5$, $f_1 = 0.4$, $\omega = 0.2\pi$ and $\phi_l = \pi l/2$ for system sizes $L = 16, 32$ and 64. Both the results from simulations (symbols connected by dotted lines) and from the perturbation theory (lines) are plotted.

4.4 Perturbation theory in $1/\omega$

In this section, we find the *DC* current within sudden approximation following the procedure of [126]. Calling $\theta = \omega t$, the master equation Eq. (4.2) can be rewritten as

$$\frac{d\mathbf{P}(\theta)}{d\theta} = \frac{1}{\omega}\left[\mathbf{W_0} + \mathbf{W_1}(\theta)\right]\mathbf{P}(\theta) \tag{4.36}$$

which can be expanded in powers of $1/\omega$ by using $\mathbf{P}(\theta) = \sum_{n=0}^{\infty} \omega^{-n}\mathbf{P}^{(n)}(\theta)$ to give

$$\frac{d\mathbf{P}^{(0)}}{d\theta} = 0 \tag{4.37}$$

$$\frac{d\mathbf{P}^{(1)}(\theta)}{d\theta} - \mathbf{W_1}(\theta)\mathbf{P}^{(0)} = \mathbf{W_0}\mathbf{P}^{(0)} \tag{4.38}$$

and so on. From the zeroth order equation, we see that $\mathbf{P}^{(0)}$ is independent of θ. In fact, for $\omega \to \infty$, we expect the system to behave as the unperturbed homogeneous SEP for which $\mathbf{W_0}\mathbf{P}^{(0)} = 0$ is satisfied and as discussed in Section 4.2, all the elements of the vector $\mathbf{P}^{(0)}$ are known. Using this fact, the first order correction $\mathbf{P}^{(1)}$ can be found by integrating Eq. (4.38)

over θ. Following steps as those leading to Eq. (4.12), we now get an average current, $\bar{J_s}$, at order $O(1/\omega)$. This is given by:

$$\bar{J}_s^{(1)} = \frac{f_1}{2\pi\omega L}\int_0^{2\pi} d\theta \sum_{l=1}^{L}(v_{l+1}-v_l)C_{l,l+1}^{[1]} \tag{4.39}$$

where we have expanded the nearest neighbor correlation function $C_{l,l+1} = \sum_{n=0}^{\infty}\omega^{-n}C_{l,l+1}^{[n]}$ in powers of $1/\omega$ and again use the expression for $C_{l,l+1}^{[0]} = C_{l,l+1}^{(0)}$ given by Eq. (4.9). The first order correction to correlation function can be obtained by perturbatively expanding Eq. (4.4) and obeys the following simple equation:

$$\frac{dC_{l,l+1}^{[1]}}{d\theta} = f_1 k_0\,(v_{l+2}+v_{l-1}-v_l-v_{l+1})\,. \tag{4.40}$$

We now again discuss the two special choices of rates v_l, discussed in the previous section.

(i) In this case, only two sites have time-dependent hopping rates. Solving the equations above for the correlation function, we get:

$$C_{1,2}^{[1]} = f_1 k_0(\cos(\theta)-\cos(\theta+\phi)) + c_{1,2} \tag{4.41}$$

$$C_{L-1,L}^{[1]} = -f_1 k_0(\cos(\theta)-\cos(\theta+\phi)) + c_{L-1,L} \tag{4.42}$$

$$C_{L,1}^{[1]} = f_1 k_0(\cos(\theta)+\cos(\theta+\phi)) + c_{L,1} \tag{4.43}$$

where c's are constant of integration (which do not contribute to current). Using the above equations in the expression for $\bar{J}_s^{(1)}$, we finally obtain

$$\bar{J}_s^{(1)} = \frac{2f_1^2 k_0 \sin\phi}{\omega L}\,. \tag{4.44}$$

Thus, we find that to leading order in $1/\omega$ (and arbitrary f_1), the *DC* current is the same as the one obtained by taking large ω limit in the current expression Eq. (4.27) obtained from the f_1 expansion.

(ii) In this case with $\alpha_l = 1$ at all sites, the equations for the first order correlation functions can be solved for arbitrary phases ϕ_l, and we get:

$$C_{l,l+1}^{[1]} = k_0 f_1\left[\cos(\theta+\phi_l)+\cos(\theta+\phi_{l+1})-\cos(\theta+\phi_{l-1})\cos(\theta+\phi_{l+2})\right]\,. \tag{4.45}$$

Using these in the current expression and after some simplifications, we get:

$$\bar{J}_s^{(1)} = \frac{k_0 f_1^2}{\omega L} \sum_{l=1}^{L} \left[2\sin(\phi_{l+1} - \phi_l) - \sin(\phi_{l+1} - \phi_{l-1}) \right] . \tag{4.46}$$

Note that the above expression depends on the phase difference between nearest and next nearest neighbor sites. For $\phi_l = ql$, we recover the result stated in the second line of Eq. (4.35).

4.5 Adiabatic calculation

We now discuss an adiabatic calculation similar to that of Astumian for a two state model [123]. The model considered by Astumian consists of a single site connected to two reservoirs with input rates $\alpha_1(t)$, $\alpha_2(t)$ and output rates $\beta_1(t), \beta_2(t)$. The rate equation of the particle density at the site is given by:

$$\begin{aligned} \frac{dQ}{dt} &= I_1 + I_2 \qquad (4.47) \\ \text{where} \quad I_1 &= \alpha_1(1-Q) - \beta_1 Q, \quad I_2 = \alpha_2(1-Q) - \beta_2 Q \, . \end{aligned}$$

The instantaneous rates satisfy the conditions, $\alpha_1(t)/\beta_1(t) = \alpha_2(t)/\beta_2(t) = e^{\epsilon(t)}$ and $\alpha_2(t)/\alpha_1(t) = \beta_2(t)/\beta_1(t) = e^{u(t)}$. For low driving frequencies $Q(t)$ can be expanded about the instantaneous equilibrium solution $Q^{(0)}(t)$ as $Q(t) = Q^{(0)}(t) + \omega Q^{(1)}(t)$, where $Q^{(0)}$, $Q^{(1)}$ satisfy the following equations:

$$\alpha_1(1 - Q^{(0)}) - \beta_1 Q^{(0)} = \alpha_2(1 - Q^{(0)}) - \beta_2 Q^{(0)} = 0 \tag{4.48}$$

$$\frac{dQ^{(0)}}{dt} = -\omega(\alpha_1 + \beta_1 + \alpha_2 + \beta_2) Q^{(1)} \tag{4.49}$$

The instantaneous equilibrium solution, from Eq. (4.48) is:

$$Q^{(0)} = \frac{\alpha_1}{\alpha_1 + \beta_1} = \frac{1}{1 + e^{-\epsilon}} \, . \tag{4.50}$$

The net particle transport $\mathcal{N}$ (from reservoir 1 into system) over one period $T = 2\pi/\omega$ can be written as:

$$\begin{aligned} \mathcal{N} &= \int_0^T I_1 dt = -\int_0^T (\alpha_1 + \beta_1)\omega Q^{(1)} dt \\ &= \int_0^T \frac{\alpha_1 + \beta_1}{\alpha_1 + \beta_1 + \alpha_2 + \beta_2} \frac{dQ^{(0)}}{dt}\, dt = \int_C F\, dQ^{(0)}, \end{aligned}$$

$$\text{with} \quad F = \frac{1}{(1+e^{u})}\,, \tag{4.51}$$

and where $\int_C$ denotes the integral over a cycle.

In our case formally one can obtain an exact expression for the net particle transport. For this we start with the master equation $\partial\mathbf{P}/\partial t = \mathbf{W}(t)\mathbf{P}$. Let $\mathbf{P}^{(0)}(t)$ be the instantaneous equilibrium solution satisfying $\mathbf{W}(t)\mathbf{P}^{(0)} = 0$. Then, for slow rates ω, $\mathbf{P}(t)$ will have the form $\mathbf{P}^{(0)}(t) + \omega\mathbf{P}^{(1)}(t)$ where the correction is given by: $\omega\mathbf{P}^{(1)} = \mathbf{W}^{-1}\,\partial\mathbf{P}^{(0)}/\partial t$. The net particle transported across any bond in one time cycle, $\mathcal{N}$, can then be expressed as:

$$\mathcal{N} = \int_0^T dt \sum_C J(C)P(C,t) = -\int_0^{2\pi} dx \sum_{C,C'} J(C)\frac{\partial W^{-1}_{C,C'}(x)}{\partial x} P^{(0)}(C',x)\,, \tag{4.52}$$

where J refers to the current on any given bond. Thus we have a formal expression, for the net particle transported, in terms of an integral over an *equilibrium average* of some quantity. However this expression does not appear to have any simple physical interpretation and neither is it easy to obtain any explicit results, unlike the fast case treated in section (4.4). The above equation has to be interpreted carefully, since $\mathbf{W}$ has a zero eigenvalue and $\mathbf{W}^{-1}$ is not strictly defined.

4.6 Conclusions

Here we have considered a lattice model of diffusing particles with hard core interactions and shown that if the hopping rates at various sites are made time-dependent, but still symmetric, then a *DC* current can be generated in the system. Thus, a ratchet effect is obtained in the sense that a directed current occurs even though there is no applied external biasing

force. Unlike many other examples of models of classical ratchets, there is no asymmetric potential in our model. However asymmetry is incorporated in the modulation of the hopping rates, and this is best seen when we consider the case where the modulation is given by $v_l(t) = \sin(\omega t - ql)$. This of-course corresponds to a wave travelling in a *g*iven direction. A non-trivial aspect of the problem studied is the fact that the effect goes away as soon as we switch off the hard-core interactions. For non-interacting the *DC* current is given by $\bar{J} = (1/LT)\int_0^T dt \ \sum_{l=1}^{L} u_l\rho_l - u_{l+1}\rho_{l+1}$, and is seen to be exactly zero, for arbitrary choice of the time-dependent rates. On the other hand, having interactions in the system is not a sufficient condition to generate a *DC* current. For the models considered in this chapter, the hopping rate is site-wise symmetric. But if the hopping rates are symmetric bond-wise, *i.e.*, the hop rate $u_{l,l+1}$ from site l to $l+1$ is the same as that from $l+1$ to l, then the *DC* current is zero for any choice of phases ϕ_l. To see this, consider the density evolution equation obeyed by bond-wise symmetric SEP:

$$\frac{\partial \rho_l}{\partial t} = u_{l-1,l}(\rho_{l-1} - \rho_l) + u_{l,l+1}(\rho_{l+1} - \rho_l) \tag{4.53}$$

Unlike Eq. (4.3) for site-wise symmetric SEP, $\rho_l = \rho$ is a solution of the above equation for any choice of rates u_l. In fact, an inspection of the master equation shows that, even with a time-dependent **W**-matrix, all configurations are equally likely, thus leading to the zero current. Thus the exclusion process with bond-wise symmetric rates does not give the ratchet effect. It is not completely clear as to what are the necessary and sufficient conditions to get a directed current.

For the model considered here, since the equations for any *n*-point correlation function do not close, it does not seem simple to solve the model exactly. We have therefore studied the system analytically using a perturbation theory in the amplitude f_1 and the inverse frequency $1/\omega$. In this study, we have been able to obtain the *DC* current at order f_1^2 by solving the evolution equations for density and two point correlation function to order f_1. Also, we have been able to obtain results for large driving frequency by solving the correlation function alone by such perturbative approaches. Comparing with simulations we find that

the perturbative results turn out to be quite accurate.

Finally, we point out that an experimental realization of the effect observed in our model should be possible in colloidal systems. For instance, consider a colloidal suspension in an externally applied laser field. This constitutes a system of diffusive interacting particles in an external potential (generated by the laser field) of the form $V(x,t) = V_0 \sin(\omega t - qx)$. This system is similar to the model that we have studied. There are some differences, namely, in this case because the external field is space dependent, hence the effective hopping rates are not symmetric in the forward and backward directions. It would be interesting to study this model to see if a current can be generated here, and perhaps one can make detailed predictions for experimental observation.

Bibliography

[1] C. Bustamante, J. Liphardt and F. Ritort, Physics Today 43 (July 2005).

[2] M. J. Perrin, *Brownian Movement and Molecular Reality*, page 6 (Taylor and Francis, London, 1910).

[3] C. Jarzynski, Phys. Rev. Lett. **78**, 2690 (1997).

[4] C. Jarzynski, Phys. Rev. E **56**, 5018 (1997).

[5] E. G. D. Cohen, D. Mauzerall, J. Stat. Mech. P07006 (2004).

[6] C. Jarzynski, J. Stat. Mech: Theor. Exp. P09005 (2004).

[7] D. J. Evans, E G. D. Cohen, and G. P. Morriss, Phys. Rev. Lett. **71**, 2401 (1993).

[8] D. J. Evans and D. J. Searles, Phys. Rev. E **50**, 1645 (1994).

[9] G. Gallavotti and E. G. D. Cohen, Phys. Rev. Lett. **74**, 2694(1995).

[10] G. E. Crooks, Phys. Rev. E **60**, 2721 (1999).

[11] Udo Seifert, Phys. Rev. Lett. **95**, 040602 (2005)

[12] J. Kurchan, J. Phys. A: Math. Gen. **31**, 3719 (1998).

[13] J. L. Lebowitz and H. Spohn, J. Stat. Phys. **95**, 333 (1999).

[14] C. Maes, J. Stat. Phys. **95**, 367 1999.

[15] O. Narayan and A. Dhar, J. Phys. A: Math. Gen. **37** 63 (2004).

[16] R. van Zon and E. G. D. Cohen, Phys. Rev. Lett. **91**, 110601 (2003).

[17] R. van Zon, S. Ciliberto and E. G. D. Cohen, Phys. Rev. Lett. **92**, 130601 (2004).

[18] R. Marathe and A. Dhar, Phys. Rev. E **72**, 066112 (2005).

[19] J. Liphardt *et al.*, Science **296**, 1832 (2002).

[20] G. M. Wang *et al.*, Phys. Rev. Lett. **89**, 050601 (2002).

[21] K. Feitosa and N. Menon, Phys. Rev. Lett. **92**, 164301 (2004).

[22] S. Aumatre *et al.*, Eur. Phys. J. B **19**, 449 (2001).

[23] W. I. Goldburg, Y. Y. Goldschmidt and H. Kellay, Phys. Rev. Lett. **87**, 245502 (2001).

[24] N. Garnier and S. Ciliberto, Phys. Rev E **71**, 060101(R) (2005).

[25] F. Douarche *et al.*, Europhys. Lett. **70**, 593 (2005).

[26] G. Hummer and A. Szabo, Proc. Natl. Acad. Sci **98**, 3658 (2001).

[27] J. Gore, F. Ritort and C. Bustamante, Proc. Natl. Acad. Sci. **100**, 12564 (2003).

[28] R. F. Fox, Proc. Natl. Acad. Sci. **100**, 12537 (2003).

[29] S. Park *et al.*, J. Chem. Phys. **119**, 3559 (2003).

[30] T. Tome and M. J. de Oliveira, Phys. Rev. A **41**, 4251(1990).

[31] M. Rao, H. R. Krishnamurthy and R. Pandit, Phys. Rev. B **42**, 856 (1990).

[32] B. Chakrabarti and M. Acharyya, Rev. Mod. Phys. **71**, 847 (1999).

[33] S. W. Sides, P. A. Rikvold, and M. A. Novotny, Phys. Rev. E **57**, 6512 (1998).

[34] F. Ritort, C. Bustamante and I. Tinoco Jr., Proc. Natl. Acad. Sci. **99**, 13544 (2002).

[35] A. Imparato and L. Peliti, Europhys. Lett., **69**, 643 (2005).

[36] T. Speck and U. Seifert, Phys. Rev. E **70**, 066112 (2004).

[37] A. Dhar, Phys. Rev. E **71**, 36126 (2005).

[38] Petr Chvosta *et al.*, Phys. Rev. E **75**, 041124 (2007).

[39] Evzen Subrt and Petr Chvosta, J. Stat. Mech. P09019 (2007).

[40] A. M. Jayannavar and M. Sahoo, Phys. Rev. E **75**, 032102 (2007).

[41] S. Saikia *et al.*, Phys. Lett. A **369**, 367 (2007).

[42] M. Sahoo *et al.*, arXiv:0708.0496v2 (2007).

[43] P. Jop, A. Petrosyan, S. Ciliberto, EPL **81**, 5, 50005 (2008).

[44] S. Joubaud, N. B. Garnier, S. Ciliberto, EPL **82** 3, 30007 (2008).

[45] I. Dornic and C. Godrche, J. Phys. A **31**, 5413 (1998).

[46] T.J. Newman and Z. Toroczkai, Phys. Rev. E **58**, R2685 (1998).

[47] A. Dhar and S. N. Majumdar, Phys. Rev. E **59**, 6413 (1999).

[48] C. Chatelain, J. Stat. Mech: Theor. Exp. **18** P06005 (2006).

[49] R. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Adison-Wesley, Reading Massachussets, vol.1, chap. 46, 1966).

[50] J. M. R. Parrondo and P. Español, Am. J. Phys. **64**, 1125 (1996).

[51] M.O.Magnasco and G.Stolovitzky, J. Stat. Phys. **93**, 615 (1998).

[52] C. Jarzynski and O. Mazonka, Phys. Rev. E **59**, 6448 (1999).

[53] C. van den Broeck, R. Kawai, and P. Meurs, Phys. Rev. Lett. **93**, 090601 (2004).

[54] C. Van den Broeck and R. Kawai, Phys. Rev. Lett. **96**, 210601 (2006).

[55] A. L. R. Bug, B. J. Berne, Phys. Rev. Lett. **59** 948 (1987).

[56] R. Bartussek, P. Hanggi, J. G. Kissner, Europhys. Lett. **28** 459 (1994).

[57] N. G. van Kampen, Z. Phys. B **68**,135 (1987).

[58] A. M. Jayannavar, M. C. Mahato, Pramana-J. Phys. **45**, 369 (1996).

[59] N. G. van Kampen, IBM J. Res. Develop. **32** 107 (1988).

[60] K. Sekimoto, J. Phys. Soc. Jpn. **68**, 1448 (1999).

[61] M. Matsuo, S. Sasa, Physica A **276**, 188 (2000).

[62] J. D. Bao, Y. Abe, Y. Z. Zhuo, Physica A **265**, 111 (1999).

[63] R. H. Luchsinger, Phys. Rev. E **62**, 272 (2000).

[64] K. Sekimoto, Jr. Phy. Soc. Jpn. **66**, 1234 (1997).

[65] T. Hondou and F. Takagi, J. Phy. Soc. Jpn. **67**, 2974 (1998).

[66] H. Kamegawa, T. Hondou, and F. Takagi, Phys. Rev. Lett. **80**, 5251 (1998).

[67] F. Takagi and T. Hondou, Phys. Rev. E **60**, 4954 (1999).

[68] J. M. R. Parrondo and B. J. de Cisneros, Applied Physics A: Materials Science & Processing **75**, 179 (2002).

[69] I. M. Sokolov, Phys. Rev. E **60**, 4946 (1999).

[70] H. Sakaguchi, J. Phy. Soc. Jpn. **67**, 709 (1998).

[71] Y. Lee *et al.*, Phys. Rev. Lett. **91**, 220601 (2003).

[72] M. O. Magnasco, Phys. Rev. Lett. **71**, 1477 (1993).

[73] I. Derényi and T. Vicsek, Phys. Rev. Lett. **75**, 374 (1995).

[74] F. Jülicher, A. Ajdari, and J. Prost, Rev. Mod. Phys. **69**, 1269 (1997).

[75] P. Reimann, Phys. Rep. **361**, 57 (2002).

[76] R. D. Astumian, Science **276**, 917 (1997).

[77] R. D. Astumian and P. Hanggi, Phys. Today **55**, No. 11, 33 (2002).

[78] J. M. R. Parrondo, Phys. Rev. E **57**, 7297 (1998).

[79] O. Usmani, E. Lutz and M. Buttiker, Phys. Rev. E **66**, 021111 (2002).

[80] T. Feldmann, E. Geva, R. Kosloff, and P. Salamon, Am. J. Phys. **64**, 485 (1996).

[81] I. Derényi and R. D. Astumian, Phys. Rev. E **59**, 6219 (1999).

[82] A. Gomez-Marin and J. M. Sancho, Phys. Rev. E **71**, 021101 (2005).

[83] A. Gomez-Marin and J. M. Sancho, Phys. Rev. E **73**, 045101 (2006).

[84] M. C. Mahato, T. P. Pareek, and A. M. Jayannavar, International Journal of Modern Physics B **10**, 3857 (1996).

[85] R. D. Astumian and I. Dernyi, European Biophysics Journal **27**, 474 (1997).

[86] M. Buttiker, H. Thomas, and A. Pretre, Z. Phys. B. **94**, 133 (1994).

[87] P.W. Brouwer, Phys. Rev. B **58**, R10135 (1998).

[88] L. Arrachea, Phys. Rev. B **72**, 125349 (2005a).

[89] L. Arrachea, Phys. Rev. B **72**, 121306 (2005b).

[90] M. Strass, P. Hänggi, and S. Kohler, Phys. Rev. Lett. **95**, 130601 (2005).

[91] S. Kohler, J. Lehmann, and P. Hänggi, Phy. Rep. **406**, 379 (2005).

[92] A. Agarwal and D. Sen, J. Phys. Condens. Matter **19**, 046205 (2007).

[93] M. Switkes *et al.*, Science **283**, 1905 (1999).

[94] P. Leek *et al.*, Phys. Rev. Lett. **95**, 256802 (2005).

[95] D. Segal and A. Nitzan, Phys. Rev. E **73**, 026109 (2006).

[96] R. Glauber, Jr. Math. Phys. **4**, 294 (1963).

[97] T. Liggett, *Interacting particle systems* (Springer-Verlag, New York, 1985).

[98] B. Schmittmann and R. K. P. Zia, in *Phase transitions and Critical Phenomena, vol. 17,* edited by C. Domb and J. Lebowitz (Academic Press, London, 1995).

[99] G.M. Schütz, in *Phase transitions and Critical Phenomena*, edited by C. Domb and J. Lebowitz (Academic Press, London, 2000), pp. 3–242.

[100] J.E. Santos and G.M. Schütz, Phys. Rev. E **64**, 036107 (2001).

[101] R. B. Stinchcombe and G. M. Schütz, Phys. Rev. Lett. **75**, 140 (1995).

[102] H. Spohn, J. Phys. A **16**, 4275 (1983).

[103] B. Derrida, J. L. Lebowitz, and E. R. Speer, J. Stat. Phys. **107**, 599 (2002).

[104] B. Derrida, B. Douçot, and P.-E. Roche, J. Stat. Phys. **115**, 717 (2004).

[105] V. Gupta *et al.*, Chem. Phys. Lett. **247**, 596 (1995).

[106] K. Hahn, J. Kärger, and V. Kukla, Phys. Rev. Lett. **76**, 2762 (1996).

[107] V. Kukla *et al.*, Science **272**, 702 (1996).

[108] Q.-H. Wei, C. Bechinger, and P. Leiderer, Science **287**, 625 (2000).

[109] T. Chou, Phys. Rev. Lett. **80**, 85 (1998).

[110] T. Chou and D. Lohse, Phys. Rev. Lett. **82**, 3552 (1999).

[111] D. J. Thouless, Phys. Rev. B **27**, 6083 (1983).

[112] M. Büttiker, H. Thomas, and A. Prêtre, Z. Phys. B **94**, 133 (1994).

[113] I. Aleiner and A. Andreev, Phys. Rev. Lett. **81**, 1286 (1998).

[114] B. L. Altshuler and L. I. Glazman, Science **283**, 1864 (1999).

[115] P. Sharma and C. Chamon, Phys. Rev. Lett. **87**, 096401 (2001).

[116] O. Entin-Wohlman, A. Aharony, and Y. Levinson, Phys. Rev. B **65**, 195411 (2002).

[117] S. Watson, Phys. Rev. Lett. **91**, 258301 (2003).

[118] R. Citro, N. Andrei, and Q. Niu, Phys. Rev. B **68**, 165312 (2003).

[119] E. Sela and Y. Oreg, Phys. Rev. Lett. **96**, 166802 (2006).

[120] R.D. Astumian and I. Derenyi, Phys. Rev. Lett. **86**, 3859 (2001).

[121] N.A. Sinitsyn, I. Nemenman, Europhys. Lett. **77**, 58001 (2007).

[122] J. Ohkubo, J. Stat. Mech., P02011 (2008).

[123] R. D. Astumian, Phys. Rev. Lett. **91**, 118102 (2003).

[124] P. Jung, Phys. Rep. **234**, 175 (1993).

[125] R.D. Astumian and B. Robertson, J. Chem. Phys. **91**, 4891 (1989).

[126] P. Reimann *et al.*, Phys. Lett. A **215**, 26 (1996).