Condensation and Intermittency in an Open-Boundary Aggregation-Fragmentation Model

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We study real space condensation in aggregation-fragmentation models where the total mass is not conserved, as in phenomena such as cloud formation and intracellular trafficking. We study the scaling properties of the system with influx and outflux of mass at the boundaries using numerical simulations, supplemented by analytical results in the absence of fragmentation. The system is found to undergo a phase transition to an unusual condensate phase, characterized by strong intermittency and giant fluctuations of the total mass. A related phase transition also occurs for biased movement of large masses, but with some crucial differences which we highlight.

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Condensation transitions constitute an important class of nonequilibrium phase transitions and occur generically in many mass transport models [1] such as the zero range process and its variants [2] and the aggregation-chipping model [3]. These systems are characterized by a fixed total mass (number of particles) and stochastic rules for exchange of mass between sites. When the total mass of the system exceeds a critical value, condensation sets in, with a finite fraction of the total mass forming a macroscopic cluster that occupies a single site. The phenomenon is akin to Bose condensation, but in real space.

Does the condensation transition survive in a system when the total mass is not conserved but can undergo large fluctuations due to the exit of clusters of all sizes? This question is important in a number of physical situations, ranging from formation of clouds and aerosols to intracellular trafficking and organelle formation in living cells [4–6]. We address this within a simple but generic 1D model with aggregation and fragmentation (chipping) of masses in the bulk and influx and outflux of masses at the boundaries. Our main finding is that the open system *does* undergo a condensation transition upon increasing the influx or decreasing the chipping rate. However, the nature of the condensate is very different from that in the closed model [3], in that the mass in the condensate shows giant number fluctuations and has a broad distribution, in contrast to the sharply peaked distribution in the closed system [3,7]. The condensate, however, has a well-defined, finite mean mass for a fixed system size and is thus quite different from the indefinitely growing aggregates in open models which allow only single particles to exit at the boundaries [6,8].

The intermittent and fluctuating nature of the condensate gives rise to novel signatures: The total mass M itself shows giant fluctuations and has a distribution characterized by a prominent non-Gaussian condensate tail whose width scales with system size. Furthermore, the exit of the condensate from the boundaries and the accompanying sharp drops in M give rise to interesting "charge and fire" behavior of M: The time series $M(t)$ departs strongly from self-similarity and shows quantitative features of intermittency, which we characterize in terms of appropriately defined structure functions, as in turbulence phenomena. Turbulence, in the sense of multiscaling of n-point mass-mass correlation functions, has been studied earlier in aggregation models [9], but our characterization of turbulencelike behavior is quite different, being associated with *temporal* fluctuations of *total* mass. Our results are based on both analytical and numerical work. In the limit of zero chipping, we analytically calculate the moments of total mass in the steady state and also the dynamical structure functions, whereas for nonzero chipping, we perform numerical Monte Carlo simulations.

Recently, it has been demonstrated that giant number fluctuations are related to anomalous, non-Porod behavior of spatial correlation functions in a wide class of systems [10]. Our work points to a connection between giant number fluctuations and anomalous dynamical behavior, namely, temporal intermittency, which is related to higher order correlation functions *in time* [11]. It also raises the interesting general question of whether temporal intermittency is present in other systems with giant number fluctuations and suggests that dynamical structure functions, as used in this paper, provide a useful probe of intermittency in these systems also.

We work with a general lattice model incorporating diffusion, aggregation, fragmentation, influx, and outflux, which goes beyond earlier studies of aggregation with input [12–14] and aggregation and fragmentation in a closed system [3,15]. Starting with an empty lattice of L sites at $t = 0$, a site i is chosen at random, and one of the following moves occurs (see Fig. 1): (i). *Influx.*—A single particle of unit mass is injected at rate a at the first site ($i = 1$). (ii). *Diffusion and aggregation.*—With rate D (or D'), the full stack on site i (i.e., all particles on the site collectively) hops to site $i+1$ (or $i-1$) and adds to the mass already there. (iii). *Chipping of unit mass.*—With rate $2w$, a unit mass breaks off from the mass at i and hops to site $i-1$ or $i+1$ with equal probability, adding to the mass already there. (iv). *Outflux of mass from boundaries.*—With rate D (or D'), the entire mass at site L (or site 1) exits the system; with rate w, a unit mass breaks off from site L (or site 1) and exits.

We find that the results depend strongly on two factors: one, whether motion of particles is biased or not, and two, whether or not exit of masses is allowed from the boundary where influx occurs. In this paper, we consider only the effect of bias [16]. We find that the occurrence of the phase transition is robust with respect to bias in the movement of stacks, but not chipping. As in the closed periodic case, if the forward and backward chipping rates are unequal, an aggregate is not expected to form [17]. Thus, chipping is taken to be unbiased in both the cases we study in this paper: (A) *unbiased stack hopping.*—$D = D'$; exit allowed from sites 1 and L; (B) *biased stack hopping.*—$D' = 0$; exit allowed from site L. Influx and chipping occur at rates a and $2w$, respectively, in both the above.

(A) Unbiased stack hopping ($D' = D$). —We discuss both the phases and the critical point below.

Normal (*large* w) *phase.*—In this phase, a typical configuration does not show very large fluctuations about the average mass profile. The total mass M too has normal fluctuations, i.e., the rms fluctuations $\Delta M \equiv \sqrt{\langle M^2\rangle - \langle M\rangle^2} \propto \sqrt{L}$, with the distribution for the rescaled mass variable $(M - \langle M\rangle)/\Delta M$, approaching a Gaussian at large L. The mass distribution $P(m, j, L)$ at a given site j is found to depend on j and L only through the rescaled position variable $x = j/L$ (see Supplemental Material [18]), implying that, for a given x, all moments of mass are independent of L to leading order.

Condensate (*small* w) *phase.*—A typical configuration deviates strongly from the average profile, with the largest (local) fluctuations scaling as system size L. On monitoring the largest mass m_1 in the system, we find that its average value $\langle m_1\rangle$ is proportional to L [18], implying that the system contains a macroscopic condensate. The presence of the condensate has a strong effect on all steady state properties of the system such as the total mass M, mass at a site, etc. The probability distributions of all

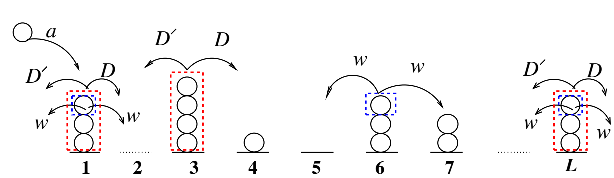

FIG. 1 (color online). Model: Influx of unit masses at site 1. Forward and backward stack movement at rates D and D', respectively. Forward and backward chipping at equal rates w. Outflux of full stack or unit mass (via chipping) from site L (and 1).

these quantities have an exponential tail with a characteristic mass M_0, where $M_0 \propto L$ for a given w and a. We refer to this exponential tail as the "condensate" tail and describe, below, how it appears in various steady state distributions: (i). The steady state distribution $P(M)$ of total mass of M in the system behaves as $P(M) \sim 1/M_0 \exp(-M/M_0)$ at large M [Fig. 2]. Consequently, the rms fluctuation ΔM of total mass shows non-Gaussian behavior, scaling as L rather than $\sqrt{L}$. We have analytically calculated various moments of the total mass in the limit $w = 0$ [18]. We find that $\Delta M/L \simeq 0.46(a/D)$, in the limit of large L, which agrees well with numerics. (ii). The distribution $P(m_1)$ of the largest mass m_1 also follows $P(m_1) \sim 1/M_0 \exp(-m_1/M_0)$ for large m_1 [18]. (iii). The distribution of masses exiting from the left or right boundary [18] is found to follow $P_{\text{exit}}(m) \sim 1/L^2[1/M_0 \exp(-m/M_0)]$, for large m [19]. (iv). The single site mass distribution $P(m, j, L)$ [18] follows $1/Lf(j/L)[1/M_0 \exp(-m/M_0)]$ at large m [19]. The factor $1/L$ arises as the aggregate can be at any one of the L sites, and $f(j/L)$ reflects that the aggregate does not visit all sites with the same probability. The rms fluctuation $\Delta m(x, L)$ of mass at a given $x = j/L$ is thus anomalously large as well: It increases as $\sqrt{L}$ with L rather than being $O(1)$, as in the normal phase.

That there is no constraint on the total number of particles per site in our model is crucial for L-dependent fluctuations to arise. Systems such as vibrated needles [20] and passive particles in fluctuating fields [21,22] also display giant number fluctuations, but, in these systems, fluctuations in a region of linear size Δl depend primarily on Δl rather than L [10]. This is traceable to hard core interactions between particles in these models. Once this constraint is removed, macroscopic stacks can form and mass fluctuations depend on L [23].

Critical point w_c.—The transition from the normal to the condensate phase takes place at a critical chipping rate

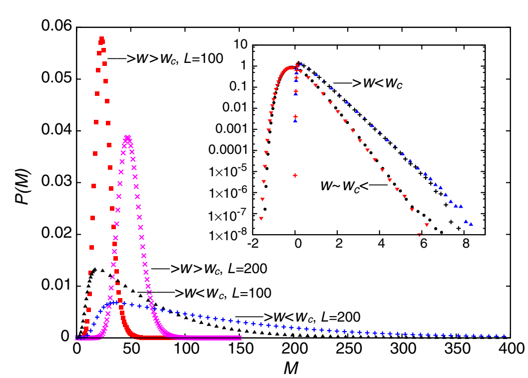

FIG. 2 (color online). $P(M)$ vs M for $L = 100$ and $L = 200$ in the normal phase ($a = 1$, $D = 0.75$, $w = 2$) and condensate phase ($a = 1$, $D = 0.75$, $w = 0.25$). Inset: Scaling collapse of tails on plotting $LP(M)$ vs M/L in the condensate phase and $L^{2/3}P(M)$ vs $(M - \langle M\rangle)/L^{2/3}$ near the critical point ($a = 1$, $D = 0.75$, $w = 1.5$).

w_c, which increases with injection rate a if D is held constant [24]. At criticality also, large fluctuations of the total mass are found, consistent with $\Delta M \propto L^{\theta_c}$ where $\theta_c \simeq 2/3$. The mass distribution has a tail of the form $P(M) \sim 1/M_2 \exp[-(M - \langle M \rangle)/M_2]$, where $\langle M \rangle \sim L$ and $M_2 \sim L^{\theta_c}$ [inset, Fig. 2]. Interestingly, we find that there is a similar L-dependent tail in the distribution of masses exiting from the left but not the right. This is presumably because, although an L-dependent aggregate forms close to the left boundary, it dissipates due to chipping on diffusing through the bulk of the system and does not survive up to the right boundary.

Contrasting signatures of the phases also appear in dynamical properties: $M(t)$ is self-similar in time in the normal phase but exhibits breakdown of self-similarity in the condensate phase. The breakdown of self-similarity is captured in the behavior of the structure functions $S_n(t) = \langle [M(t) - M(0)]^n \rangle$ [25], where $\langle ... \rangle$ denotes average over histories. Self-similar signals typically show $S_n(t) \propto t^{\gamma n}$ as $t/\tau \to 0$, where γ is a constant and τ is a time scale which characterizes the lifetime of the largest structures in the system. A deviation from $S_n(t) \propto t^{\gamma n}$ reflects the breakdown of self-similarity and may occur, for example, if the signal $M(t)$ alternates between periods of quiescence (small or no activity) and bursts (sudden large changes) [11]. Such an alternation is characteristic of intermittency. The most well-studied measures of intermittency are the flatness, defined as $\kappa(t) = S_4(t)/S_2^2(t)$, and the hyperflatness $h(t) = S_6(t)/S_2^3(t)$. For intermittent signals, both $\kappa(t)$ and $h(t)$ diverge as $t/\tau \to 0$ [11]. Below, we present evidence for intermittency in our model.

Normal phase.—In this phase, the structure functions are independent of L at small t and scale as $S_{2n} \sim t^{\beta_n}$ where the dependence of β_n on n is close to linear [18], indicating self-similarity of the time series $M(t)$ [Fig. 3(a)]. The flatness $\kappa(t)$ and hyperflatness $h(t)$ approach a finite, L-independent value as $t \to 0$ [Fig. 3(c) and Fig. 4 in Ref. [18]].

Condensate phase.—In the condensate phase, $M(t)$ builds up as mass is injected and drops as masses exit, with occasional large crashes [Fig. 3(b)] corresponding to the exit of condensates with $O(L)$ mass. The structure functions are found to scale as $S_n \sim L^n f_n(t/L^2)$, where f_n is consistent with the form $f_n(y) \sim (-1)^n y g_n[\log(y)]$ for small y (with g_n chosen to be a polynomial) [26] and approaches an n-dependent constant value at large y [18]. Thus, the system shows strong intermittency: At small t, all structure functions S_n behave as $\sim t$ with the n dependence entering only through the multiplicative $\log t$ terms. It follows that $\kappa(t)$ and $h(t)$ diverge at small times in a strongly L-dependent way [Fig. 3(c) and 4 in Ref. [18]]. In fact, they are functions of t/L^2 and diverge as $t/L^2 \to 0$ [inset, Fig. 3(c) and inset, Fig. 4 in Ref. [18]]. We have also analytically calculated $S_2(t)$ in the zero chipping limit $w = 0$ [18] and find that it agrees well with numerical results.

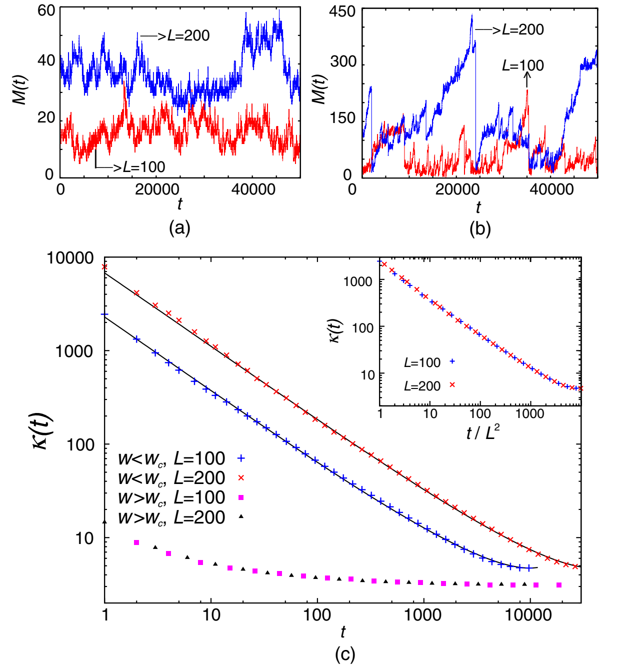

FIG. 3 (color online). (a),(b) Realizations of $M(t)$ vs t for different L in the (a) normal phase ($a = 1$, $D = 0.75$, $w = 3.0$) and (b) condensate phase ($a = 1$, $D = 0.75$, $w = 0.25$). Note that the y axis in (a) and (b) has a different scale. (c) $\kappa(t)$ vs t for $L = 100$ and $L = 200$ in the two phases. Solid lines are fits to the form described in the text for $t \ll L^2$ for $w < w_c$. Inset: Scaling collapse of $\kappa(t)$ vs t for different L on scaling time as t/L^2 in the condensate phase.

Critical point w_c.—$M(t)$ continues to show intermittency at the critical point with flatness and hyperflatness diverging as $t \to 0$ in an L-dependent manner. However, there seems to be no simple scaling which collapses the curves for different L.

(B) Biased stack hopping ($D' = 0$).—The steady state can be obtained exactly in the limiting cases of only chipping $D = 0$ [8] and only aggregation $w = 0$ [27].

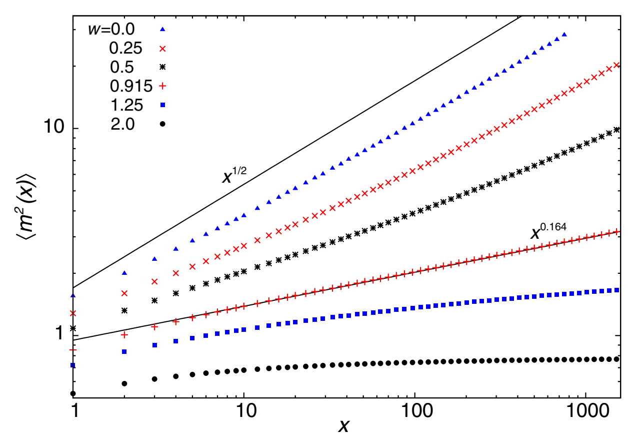

FIG. 4 (color online). $\langle m^2(x) \rangle$ vs x for $a = 1$, $D = 1.5$, and different w. Note the upward (downward) bending of curves on the log-log plot in the aggregation (chipping) dominated phase. There is no bending at w_c.

The probability distribution of the rescaled mass, $(M - \langle M \rangle)/\Delta M$, is Gaussian with $\Delta M \propto \sqrt{L}$, in both limits but for different reasons. In the pure chipping limit $D = 0$, this follows from the independence of masses at different sites, implied by the product measure of the mass distribution [8]; in the pure stack hopping limit $w = 0$, on the other hand, it is associated with the formation and exit of aggregates of typical size $\sqrt{L}$ [27,28]. This essential difference is well captured by the time series data. For $w = 0$, the total mass M shows intermittency on time scales of the order of $\sqrt{L}$, corresponding to a typical time interval of $O(\sqrt{L})$ between exit events. Flatness and hyperflatness are functions of $t/\sqrt{L}$ and diverge as power laws as $t/\sqrt{L} \to 0$. By contrast, for $D = 0$, the time series $M(t)$ is not intermittent. Thus, intermittency, rather than anomalous steady state fluctuations of M, is a key signature of aggregate formation when stack hopping is driven.

As w is decreased, there is a phase transition from a normal phase to an aggregation-dominated phase characterized by intermittency. Unlike the unbiased case, however, the typical size of aggregates that exit the system is now expected to scale as $\sqrt{L}$ rather than L. This is consistent with the behavior of $\langle m^2(x) \rangle$ vs x [Fig. 4]. For large w, the plots of $\langle m^2(x) \rangle$ approach a constant value at large x, thus indicating that there are no x-dependent aggregates at large x and no intermittency. For small w, the plots of $\langle m^2(x) \rangle$ vs x bend upwards, consistent with an approach to $\sqrt{x}$ at large x. Exit of $\sqrt{L}$ sized aggregates leads to intermittency [18], as for $w = 0$. The transition takes place at w_c (corresponding to the curve with no bending on the log-log plot), at which $\langle m^2(x) \rangle$ behaves as $\sim x^{\alpha_c}$ with $\alpha_c \simeq 0.16$. $M(t)$ shows intermittency at the critical point also.

In conclusion, the principal result of this work is to establish the existence of a condensate phase in unbiased aggregation-chipping models where total mass is not conserved due to influx and outflux at the boundaries. This phase is characterized by anomalous steady state fluctuations of the total mass and by intermittency in the dynamics, as quantified by the divergence of the flatness. It is likely that flatness would be a useful measure in other mass exchange models also. The phase transition also occurs when the movement of stacks is biased, but the intermittent, aggregation-dominated phase in this case is different.

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[19] This holds for most $w < w_c$. For w very close to w_c, no condensate tail is visible in $P_{\mathrm{exit}}(m)$ for right-exiting masses and in $P(m, j, L)$ for j in a small region near the right boundary.
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