Geometry, Fields and Cosmology

Techniques and Applications

Edited by

B.R. Iyer and C.V. Vishveshwara

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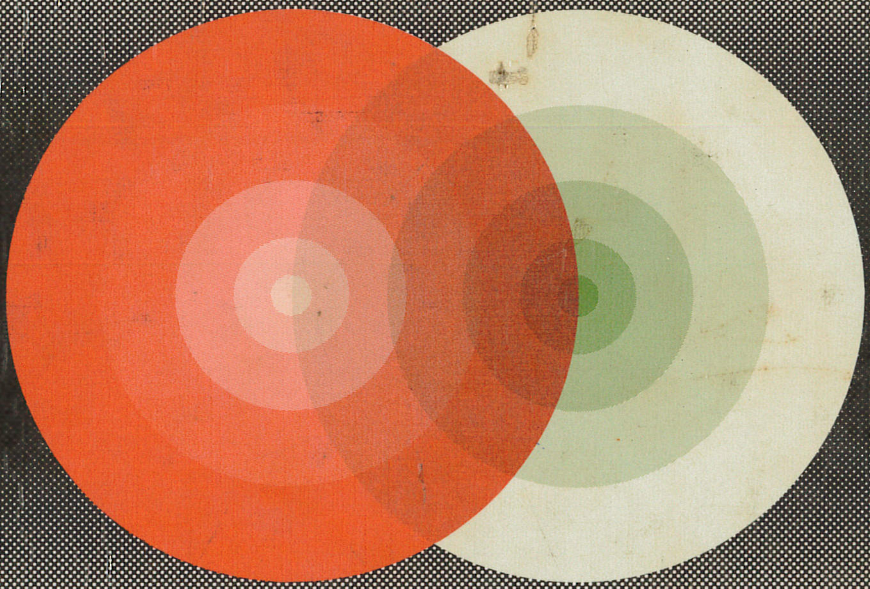

Fundamental Theories of Physics

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This volume is based on the lectures given at the First Inter-University Graduate School on Gravitation and Cosmology organized by IUCAA, Pune, India. The material offers a firm mathematical foundation for a number of subjects including geometrical methods for physics, quantum field theory methods and relativistic cosmology. It brings together the most basic and widely used techniques of theoretical physics today. A number of specially selected problems with hints and solutions have been added to assist the reader in achieving mastery of the topics.

Audience

The style of the book is pedagogical and should appeal to graduate students and research workers who are beginners in the study of gravitation and cosmology or related subjects such as differential geometry, quantum field theory and the mathematics of physics. This volume is also recommended as a textbook for courses or for self-study.

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