Large-scale dynamo action due to α fluctuations in a linear shear flow

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ABSTRACT

We present a model of large-scale dynamo action in a shear flow that has stochastic, zero-mean fluctuations of the α parameter. This is based on a minimal extension of the Kraichnan–Moffatt model, to include a background linear shear and Galilean-invariant α-statistics. Using the first-order smoothing approximation we derive a linear integro-differential equation for the large-scale magnetic field, which is non-perturbative in the shearing rate S, and the α-correlation time τ_α. The white-noise case, $\tau_\alpha = 0$, is solved exactly, and it is concluded that the necessary condition for dynamo action is identical to the Kraichnan–Moffatt model without shear; this is because white-noise does not allow for memory effects, whereas shear needs time to act. To explore memory effects we reduce the integro-differential equation to a partial differential equation, valid for slowly varying fields when τ_α is small but non-zero. Seeking exponential modal solutions, we solve the modal dispersion relation and obtain an explicit expression for the growth rate as a function of the six independent parameters of the problem. A non-zero τ_α gives rise to new physical scales, and dynamo action is completely different from the white-noise case; e.g. even weak α fluctuations can give rise to a dynamo. We argue that, at any wavenumber, both Moffatt drift and Shear always contribute to increasing the growth rate. Two examples are presented: (a) a Moffatt drift dynamo in the absence of shear and (b) a Shear dynamo in the absence of Moffatt drift.

Key words: dynamo – magnetic fields – MHD – turbulence – galaxies: magnetic fields.

1 INTRODUCTION

The magnetic fields observed in many astrophysical bodies – such as the Sun, stars, galaxies and clusters of galaxies – are thought to be generated by electric currents in the turbulent plasma flows in these objects (Moffatt 1978; Parker 1979; Krause & Rädler 1980; Ruzmaikin, Shukurov & Sokoloff 1998; Kulsrud 2004; Brandenburg & Subramanian 2005). The field shows structure on a wide range of scales, both smaller and larger than that of the underlying turbulence. Of particular interest to the present investigation is the large-scale ordered structure of the field. When the turbulent motions are helical (i.e. when mirror-symmetry is broken), the well-known α-effect can amplify seed magnetic fields and maintain them in the face of turbulent dissipation (Moffatt 1978; Parker 1979; Krause & Rädler 1980). However, it is not clear whether astrophysical turbulence has a mean helicity that is large enough to sustain such a large-scale turbulent dynamo. A natural question arises: is there dynamo action for turbulence whose helicity vanishes (not instantaneously, but) on average? Kraichnan (1976) considered dynamo action when α is a stochastic quantity with zero mean, and demonstrated that the α fluctuations would contribute to a decrease in the turbulent diffusivity. When the fluctuations are large enough, there is negative diffusion, and the magnetic field grows on all spatial scales, with the fastest rates of growth on the smallest scales. The Kraichnan model was generalized by Moffatt (1978), to include a statistical correlation between the fluctuations of α and its spatial gradient. In the simplest case this correlation contributes a constant drift velocity to the dynamo equation, and does not influence dynamo action. A feature common to astrophysical flows is velocity shear, and it is only relatively recently that the question of large-scale dynamo action in shear flows with α fluctuations began receiving attention (Vishniac & Brandenburg 1997; Sokolov 1997; Silant'ev 2000). The interest in this question has grown significantly, and a number of investigations have pursued this problem both analytically and numerically (Proctor 2007, 2012; Brandenburg et al. 2008; Yousef et al. 2008; Kleeorin & Rogachevskii 2008; Rogachevskii & Kleeorin 2008; Sur & Subramanian 2009; Heinemann,

McWilliams & Schekochihin 2011; McWilliams 2012; Mitra & Brandenburg 2012; Richardson & Proctor 2012; Tobias & Cattaneo 2013). The numerical simulations of Yousef et al. (2008) and Brandenburg et al. (2008) are demonstrations of large-scale dynamo action in a shear flow with turbulence that is, on average, non-helical. The evidence for the growth of a large-scale magnetic field seems compelling, as can be seen from the 'butterfly diagrams' of fig. 3 in Yousef et al. (2008) and fig. 8 in Brandenburg et al. (2008). It is also significant, as shown in fig. 10 of Brandenburg et al. (2008), that the α parameter fluctuates about a mean value close to zero.

The goal of this paper is to study large-scale dynamo action by extending the Kraichnan–Moffatt (KM) model in a minimal manner, in order to include a background linear shear flow. This is done in Section 2 where the α fluctuations are required to respect Galilean invariance (GC), a symmetry natural to linear shear flows. The integro-differential equation governing the evolution of the large-scale magnetic field is derived in Section 3, using the first-order smoothing approximation (FOSA), shearing coordinates and Fourier representation similar to Sridhar & Subramanian (2009a,b), Sridhar & Singh (2010), Singh & Sridhar (2011). This equation is non-perturbative in the shearing rate S, and the α-correlation time τ_α. With no further approximation, this is applied in Section 4 to the case of white-noise α fluctuations (for which $\tau_\alpha = 0$) and general conclusions are derived regarding the limited nature of dynamo action. In Section 5 the integro-differential equation is simplified to an ordinary differential equation, when $\tau_\alpha \neq 0$ is small and the large-scale magnetic field is slowly varying. This is used to explore one-dimensional propagating waves in Section 6: the dispersion relation is solved and an explicit expression for the growth rate function is obtained. In Section 7 we present an interpolation formula for the growth rate and study dynamo action due to Kraichnan diffusivity, Moffatt drift and shear. We conclude in Section 8.

2 DEFINITION OF THE MODEL

2.1 Outline of the Kraichnan–Moffatt model

Following Kraichnan (1976) we model small-scale turbulence in the absence of mean motions as random velocity fields, $\{v(X, \tau)\}$, where $X = (X_1, X_2, X_3)$ is the position vector with components given in a fixed orthonormal frame (e_1, e_2, e_3), and τ is the time variable. The ensemble is assumed to have *zero-mean isotropic velocity fluctuations, with uniform and constant kinetic energy density per unit mass, and slow helicity fluctuations*. Zero-mean isotropic velocity fluctuations imply that:

$$\langle v_i \rangle = 0 \, ; \qquad \langle v_i v_j \rangle = \delta_{ij} v_0^2 \, ; \qquad \left\langle v_i \frac{\partial v_j}{\partial X_n} \right\rangle = \epsilon_{inj} \, \mu(X, \tau) \, , \tag{1}$$

where $v_0^2 = \langle v^2/3 \rangle$ = two-thirds of the mean kinetic energy density per unit mass, and $\mu(X, \tau) = \langle v \cdot (\nabla \times v) \rangle / 6$ = one-sixth of the helicity density. Uniform and constant kinetic energy density per unit mass means that v_0^2 is a constant number across the ensemble. Let ℓ_0 be the eddy size and τ_c be the velocity correlation time. The correlation time need not equal the 'eddy turnover time' $\tau_0 = \ell_0/v_0$: for instance, $\tau_c \sim \tau_0$ for fully developed turbulence, whereas $\tau_c \ll \tau_0$ for a v which is close to white-noise. By slow helicity fluctuations we mean that the spatial and temporal scales of variation of $\mu(X, \tau)$ are assumed to be much larger than ℓ_0 and τ_c.

When electric currents in the fluid give rise to a magnetic field, the action of the stochastic velocity field makes the magnetic field stochastic as well. Let $B(X, \tau)$ be the *meso-scale magnetic field*, obtained by averaging over the above ensemble. Then the space-time evolution of $B(X, \tau)$ is given by the *dynamo equation* (Moffatt 1978; Krause & Rädler 1980; Brandenburg & Subramanian 2005):

$$\frac{\partial B}{\partial \tau} = \nabla \times [\alpha(X, \tau) B] + \eta_T \nabla^2 B \, ; \qquad \nabla \cdot B = 0 \, , \tag{2}$$

where

$$\alpha = -2\tau_c \mu(X, \tau) \, , \qquad\qquad \eta_T = \eta + \eta_t = \text{total diffusivity} \, ,$$

$$\eta = \text{microscopic diffusivity} \, , \qquad\qquad \eta_t = \tau_c v_0^2 = \text{turbulent diffusivity} \, . \tag{3}$$

The above expressions for the turbulent transport coefficients α and η_t are valid in the high-conductivity limit, under the FOSA, assuming isotropic background turbulence. The next step is to consider $\alpha(X, \tau)$ itself to be drawn from a *superensemble*, while η_T is kept constant. It is assumed that $\alpha(X, \tau)$ is a *statistically stationary, homogeneous, random function of X and τ with zero mean, $\overline{\alpha(X, \tau)} = 0$, and two-point space-time correlation function*:

$$\overline{\alpha(X, \tau)\alpha(X', \tau')} = 2\mathcal{A}(X - X')\,\mathcal{D}(\tau - \tau') \, , \qquad \text{with}$$

$$2\int_0^\infty \mathcal{D}(\tau)\,\mathrm{d}\tau = 1 \, , \qquad \mathcal{A}(0) = \eta_\alpha \, \geq \, 0 \, . \tag{4}$$

Here η_α is the α-diffusivity, introduced first by Kraichnan (1976). The meso-scale field is split as $B = \overline{B} + b$, where $\overline{B}$ is the *large-scale magnetic field* which is equal to the superensemble-average of the meso-scale field, and b is the *fluctuating magnetic field*. An equation for $\overline{B}$ is derived in section 7.11 of Moffatt (1978), using FOSA for the fluctuating field:

$$\frac{\partial \overline{B}}{\partial \tau} = \nabla \times [V_M \times \overline{B}] + \eta_K \nabla^2 \overline{B} \, ; \qquad \nabla \cdot \overline{B} = 0 \, , \tag{5}$$

where

$$\eta_K = \eta_T - \eta_\alpha = \text{Kraichnan diffusivity}\,,$$

$$\boldsymbol{V}_M = -\left(\frac{\partial \mathcal{A}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}}\right)_{\boldsymbol{\xi}=\boldsymbol{0}} = \int_0^\infty \overline{\alpha(\boldsymbol{X}, \tau)\nabla\alpha(\boldsymbol{X}, 0)}\,\mathrm{d}\tau = \text{Moffatt drift velocity}\,, \tag{6}$$

are the two new constants that determine the behaviour of the large-scale magnetic field. Note that the α diffusivity contributes a decrement to the diffusivity, and hence aids dynamo action. The general solution to (5) in an infinitely extended medium is:

$$\overline{\boldsymbol{B}}(\boldsymbol{X}, \tau) = \int \frac{\mathrm{d}^3 K}{(2\pi)^3} \exp[\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{X}]\,\widetilde{\overline{\boldsymbol{B}}}(\boldsymbol{K}, \tau)\,, \tag{7}$$

where

$$\widetilde{\overline{\boldsymbol{B}}}(\boldsymbol{K}, \tau) = \widetilde{\overline{\boldsymbol{B}}}_0(\boldsymbol{K})\exp\left[-\eta_K K^2 \tau - \mathrm{i}\boldsymbol{K}\cdot\boldsymbol{V}_M \tau\right]\,; \qquad \boldsymbol{K}\cdot\widetilde{\overline{\boldsymbol{B}}}_0 = 0\,. \tag{8}$$

Thus the large-scale magnetic field is a linear superposition of transverse waves that translate uniformly with velocity $\boldsymbol{V}_M$, while their amplitudes grow or decay according to whether η_K is negative or positive. The growth/decay rates are proportional to K^2, so the fastest growing/decaying modes are those with the smallest spatial scales $\sim$ a few ℓ_0.

2.2 The Kraichnan–Moffatt model with shear

In the presence of a background linear shear flow with velocity field $\boldsymbol{V}(\boldsymbol{X})$, the dynamo equation (2) acquires an additional term, $\nabla\times[\boldsymbol{V}\times\boldsymbol{B}]$, on the right hand side. For $\boldsymbol{V} = SX_1\boldsymbol{e}_2$, the dynamo equation is

$$\left(\frac{\partial}{\partial\tau} + SX_1\frac{\partial}{\partial X_2}\right)\boldsymbol{B} - SB_1\boldsymbol{e}_2 = \nabla\times[\alpha(\boldsymbol{X}, \tau)\boldsymbol{B}] + \eta_T\nabla^2\boldsymbol{B}\,, \qquad \nabla\cdot\boldsymbol{B} = 0\,. \tag{9}$$

Splitting $\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{b}$, and Reynolds averaging equation (9), we find that the evolution of the large-scale field is governed by

$$\left(\frac{\partial}{\partial\tau} + SX_1\frac{\partial}{\partial X_2}\right)\overline{\boldsymbol{B}} - S\overline{B}_1\boldsymbol{e}_2 = \nabla\times\overline{\boldsymbol{\mathcal{E}}} + \eta_T\nabla^2\overline{\boldsymbol{B}}\,, \qquad \nabla\cdot\overline{\boldsymbol{B}} = 0\,, \tag{10}$$

where

$$\overline{\boldsymbol{\mathcal{E}}} = \overline{\alpha(\boldsymbol{X}, \tau)\boldsymbol{b}(\boldsymbol{X}, \tau)}\,. \tag{11}$$

To calculate $\overline{\boldsymbol{\mathcal{E}}}$, the *electromotive force (EMF)*, we need to solve for the fluctuating field, $\boldsymbol{b}(\boldsymbol{X}, \tau)$, whose evolution is determined by

$$\left(\frac{\partial}{\partial\tau} + SX_1\frac{\partial}{\partial X_2}\right)\boldsymbol{b} - Sb_1\boldsymbol{e}_2 = \nabla\times\left[\alpha\overline{\boldsymbol{B}}\right] + \nabla\times\left[\alpha\boldsymbol{b} - \overline{\alpha\boldsymbol{b}}\right] + \eta_T\nabla^2\boldsymbol{b}\,,$$

$$\nabla\cdot\boldsymbol{b} = 0\,, \quad \text{with initial condition} \quad \boldsymbol{b}(\boldsymbol{X}, 0) = \boldsymbol{0}\,. \tag{12}$$

The physical assumption behind the choice of the initial condition $\boldsymbol{b}(\boldsymbol{X}, 0) = \boldsymbol{0}$ is that the homogeneous part of equation (12) has only decaying solutions. Then the origin of time can always be chosen such that the solution of the homogeneous part has already decayed. Then the $\boldsymbol{b}(\boldsymbol{X}, \tau)$ which contributes to the EMF in equation (11) is the one whose source is $\nabla\times[\alpha\overline{\boldsymbol{B}}]$.

We now need to specify the statistics of the α fluctuations. Shear flows possess a natural symmetry, called *Galilean invariance* in Sridhar & Subramanian (2009a,b), which is related to measurements made by observers, whose velocity with respect to the lab frame $(\boldsymbol{X}, \tau)$ is equal to that of the background shear flow. All these *comoving observers* may be labelled by $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$, the position of the origin of the comoving observer at the initial time zero. The position of the origin of the comoving observer at time τ is given by

$$\boldsymbol{X}_c(\boldsymbol{\xi}, \tau) = (\xi_1, \xi_2 + S\tau\xi_1, \xi_3)\,. \tag{13}$$

GI is the property of the form-invariance of a quantity that is transformed to the rest frame of any comoving observer, with $\boldsymbol{\xi} \in \mathbb{R}^3$. For GI α fluctuations, an n-point correlator given in the lab frame must equal a correspondingly constructed n-point correlator in the frame of any comoving observer. Therefore, for GI α fluctuations, we must have:

$$\overline{\alpha(\boldsymbol{X}^{(1)}, \tau_1)\ldots\alpha(\boldsymbol{X}^{(n)}, \tau_n)} = \overline{\alpha(\boldsymbol{X}^{(1)} + \boldsymbol{X}_c(\boldsymbol{\xi}, \tau_1), \tau_1) \ldots \alpha(\boldsymbol{X}^{(n)} + \boldsymbol{X}_c(\boldsymbol{\xi}, \tau_n), \tau_n)}\,. \tag{14}$$

It may be verified that, for GI α fluctuations, the dynamo equation (9), and equations (10) and (12) for the large-scale and fluctuating fields, is Galilean Invariant.[1]

Equations (9), (10) and (12) do not depend explicitly on time, so if we also require that the α fluctuations have time-stationary statistics, then we can expand the set of observers to include comoving observers whose origin of time may be shifted by an arbitrary constant amount.

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[1] We have argued that the GI α statistics of equation (14) is the one that respects the natural symmetry of a linear shear flow. However, this symmetry can be broken, if the sources of the α fluctuations picked out a special frame. In this case, equation (14) need not be true, and (9)–(12) would no longer respect the symmetry of GI.

All comoving observers can be labelled by the coordinates, $(\boldsymbol{\xi}, \tau_0)$, where τ_0 is the time read by the clock of a comoving observer at position $\boldsymbol{\xi}$ at time 0. Therefore α fluctuations that are GI and time-stationary must satisfy:

$$\overline{\alpha(\boldsymbol{X}^{(1)}, \tau_1)\ldots\alpha(\boldsymbol{X}^{(n)}, \tau_n)} = \overline{\alpha(\boldsymbol{X}^{(1)} + \boldsymbol{X}_c(\boldsymbol{\xi}, \tau_1 + \tau_0), \tau_1 + \tau_0)\ldots\alpha(\boldsymbol{X}^{(n)} + \boldsymbol{X}_c(\boldsymbol{\xi}, \tau_n + \tau_0), \tau_n + \tau_0)}. \tag{15}$$

It should be noted that the general requirements of Galilean–Invariance (and the additional specialization to time-stationarity), given in equations (14) and (15), will continue to hold, regardless of the details of an underlying more detailed theory based on velocity correlators.

Our model is now fully defined by equations (10) and (12) for the large-scale and fluctuating magnetic fields, $\overline{\boldsymbol{B}}(\boldsymbol{X}, \tau)$ and $\boldsymbol{b}(\boldsymbol{X}, \tau)$, together with either *general Galilean–Invariant α statistics* (equation 14) or *time-stationary Galilean–Invariant α statistics* (equation 15). To derive a closed equation for $\overline{\boldsymbol{B}}(\boldsymbol{X}, \tau)$, it is necessary to do the following: solve equation (12) to get $\boldsymbol{b}(\boldsymbol{X}, \tau)$ as a functional of $\alpha(\boldsymbol{X}, \tau)$ and $\overline{\boldsymbol{B}}(\boldsymbol{X}, \tau)$. Use this in equation (11) to get the EMF $\boldsymbol{\mathcal{E}}(\boldsymbol{X}, \tau)$ as a functional of $\overline{\boldsymbol{B}}(\boldsymbol{X}, \tau)$ and various n-point α correlators, which are required to be either general GI (equation 14) or time-stationary GI (equation 15). The α correlators are specified quantities, so $\overline{\boldsymbol{\mathcal{E}}}$ can be thought of as a functional only of $\overline{\boldsymbol{B}}$. Using this form for $\overline{\boldsymbol{\mathcal{E}}}$ in equation (10) will result in a closed integro-differential equation for $\overline{\boldsymbol{B}}$.

The above program is best realized in *sheared coordinates* $(\boldsymbol{x}, t)$, which are defined in terms of the lab coordinates $(\boldsymbol{X}, \tau)$ as:

$$x_1 = X_1; \qquad x_2 = X_2 - S\tau X_1; \qquad x_3 = X_3; \qquad t = \tau. \tag{16}$$

The inverse transformation is:

$$X_1 = x_1; \qquad X_2 = x_2 + Stx_1; \qquad X_3 = x_3; \qquad \tau = t. \tag{17}$$

The $\boldsymbol{x}$ are the Lagrangian coordinates of fluid elements in the background shear flow. Using equation (13) for the definition of the function $\boldsymbol{X}_c$, the shearing coordinate transformation and its inverse can also be written as $\boldsymbol{x} = \boldsymbol{X}_c(\boldsymbol{X}, -\tau)$ and $\boldsymbol{X} = \boldsymbol{X}_c(\boldsymbol{x}, t)$. Since all comoving observers are equivalent, *every comoving observer has corresponding sheared coordinates*. We want to recast all the equations (9)–(15) in terms of new fields that are functions of $(\boldsymbol{x}, t)$. These are the meso-scale field $\boldsymbol{H}(\boldsymbol{x}, t) = \boldsymbol{B}(\boldsymbol{X}, \tau)$, the large-scale field $\overline{\boldsymbol{H}}(\boldsymbol{x}, t) = \overline{\boldsymbol{B}}(\boldsymbol{X}, \tau)$, the fluctuating field $\boldsymbol{h}(\boldsymbol{x}, t) = \boldsymbol{b}(\boldsymbol{X}, \tau)$, the fluctuating alpha $a(\boldsymbol{x}, t) = \alpha(\boldsymbol{X}, \tau)$, and the large-scale EMF $\overline{\boldsymbol{E}}(\boldsymbol{x}, t) = \overline{\boldsymbol{\mathcal{E}}}(\boldsymbol{X}, \tau)$.[2] Then equations (9), (10) and (12) give:

$$\frac{\partial \boldsymbol{H}}{\partial t} - SH_1\boldsymbol{e}_2 = \nabla \times [a\boldsymbol{H}] + \eta_T\nabla^2\boldsymbol{H}, \qquad \nabla\cdot\boldsymbol{H} = 0; \tag{18}$$

$$\frac{\partial \overline{\boldsymbol{H}}}{\partial t} - S\overline{H_1}\boldsymbol{e}_2 = \nabla\times\overline{\boldsymbol{E}} + \eta_T\nabla^2\overline{\boldsymbol{H}}, \qquad \nabla\cdot\overline{\boldsymbol{H}} = 0, \qquad \overline{\boldsymbol{E}} = \overline{a\boldsymbol{h}}; \tag{19}$$

$$\frac{\partial \boldsymbol{h}}{\partial t} - Sh_1\boldsymbol{e}_2 = \nabla\times\left[a\overline{\boldsymbol{H}}\right] + \nabla\times\left[a\boldsymbol{h} - \overline{a\boldsymbol{h}}\right] + \eta_T\nabla^2\boldsymbol{h},$$

$$\nabla\cdot\boldsymbol{h} = 0, \quad \text{with initial condition} \quad \boldsymbol{h}(\boldsymbol{x}, 0) = \boldsymbol{0}; \tag{20}$$

$$\text{where} \qquad \nabla = \frac{\partial}{\partial\boldsymbol{x}} - \boldsymbol{e}_1St\frac{\partial}{\partial x_2} \qquad \text{is a time-dependent operator.} \tag{21}$$

Equations (18)–(20) are *homogeneous* in $\boldsymbol{x} = (x_1, x_2, x_3)$, although explicitly dependent on t through the operator ∇ of equation (21). We now rewrite the n-point correlators of equations (14) and (15) in the new variables. By definition, the left hand sides of both equations are $\overline{\alpha(\boldsymbol{X}^{(1)}, \tau_1)\ldots\alpha(\boldsymbol{X}^{(n)}, \tau_n)} = \overline{a(\boldsymbol{x}^{(1)}, t_1)\ldots a(\boldsymbol{x}^{(n)}, t_n)}$. The right hand sides can be worked out in a straightforward manner by applying the transformations of (16) and (17), and the results stated simply:

(i) A general Galilean–Invariant n-point correlator satisfies:

$$\overline{a(\boldsymbol{x}^{(1)}, t_1)\ldots a(\boldsymbol{x}^{(n)}, t_n)} = \overline{a(\boldsymbol{x}^{(1)} + \boldsymbol{\xi}, t_1)\ldots a(\boldsymbol{x}^{(n)} + \boldsymbol{\xi}, t_n)}, \tag{22}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^3$. Therefore, *general GI implies spatial homogeneity of the correlators in sheared coordinates*. This is entirely consistent with the homogeneity of equations (18)–(20) for the magnetic fields.

(ii) If a Galilean–Invariant n-point correlator has the additional symmetry of being time-stationary, then

$$\overline{a(\boldsymbol{x}^{(1)}, t_1)\ldots a(\boldsymbol{x}^{(n)}, t_n)} = \overline{a(\boldsymbol{x}^{(1)} + \boldsymbol{\xi} - S\tau_0x_1^{(1)}\boldsymbol{e}_2, t_1 + \tau_0)\ldots a(\boldsymbol{x}^{(n)} + \boldsymbol{\xi} - S\tau_0x_1^{(n)}\boldsymbol{e}_2, t_n + \tau_0)}, \tag{23}$$

for all $\boldsymbol{\xi} \in \mathbb{R}^3$ and $\tau_0 \in \mathbb{R}$. Equations (18)–(23) complete the definition of our model in sheared coordinates.

Henceforth we develop the theory for *time-stationary GI statistics*, for which the n-point correlators obey equation (23). We first note that, for $\tau_0 = 0$, this reduces to (22): therefore every time-stationary GI correlator is also spatially homogeneous, when the base temporal

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[2] The new vector fields are component-wise equal to the old vector fields, in that both are resolved along the fixed basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$. For instance, $H_1(\boldsymbol{x}, t) = B_1(\boldsymbol{X}, \tau)$ and so on.

points $(t_1, \ldots, t_n)$ are not shifted. But τ_0 need not be zero; it can take any real value. When the base temporal points are allowed to be shifted by $\tau_0 \neq 0$, the constraints on a time-stationary GI (instead of a general GI) n-point correlator are more severe. It must have the property of invariance under unequal displacements of the spatial base points $(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)})$. This is a richer, four-dimensional set of allowed variations, indexed by $(\boldsymbol{\xi}, \tau_0)$, and hence is satisfied by only a subset of all the GI functions obeying equation (22). This stronger constraint has entered the problem due to the additional requirement of time-stationarity. The restriction imposed is, indeed, strong enough to determine completely useful forms of the lowest order correlators. The one-point correlator $\overline{a(\boldsymbol{x}, t)} = 0$ by assumption. From equation (23), the time-stationary GI two-point correlator must satisfy:

$$\overline{a(\boldsymbol{x}, t)a(\boldsymbol{x}', t')} = \overline{a(\boldsymbol{x} + \boldsymbol{\xi} - St_0 x_1 \boldsymbol{e}_2, t + \tau_0)a(\boldsymbol{x}' + \boldsymbol{\xi} - St_0 x_1' \boldsymbol{e}_2, t' + \tau_0)},$$

for all $\boldsymbol{\xi} \in \mathbb{R}^3$ and $\tau_0 \in \mathbb{R}$, so it is also true for the particular values $\boldsymbol{\xi} = -\boldsymbol{x}' - St'x_1'\boldsymbol{e}_2$ and $\tau_0 = -t'$. Then $\overline{a(\boldsymbol{x}, t)a(\boldsymbol{x}', t')} = \overline{a(\boldsymbol{x} - \boldsymbol{x}' + St'(x_1 - x_1')\boldsymbol{e}_2, t - t')a(\boldsymbol{0}, 0)}$, so that the general form of the two-point correlator is:

$$\overline{a(\boldsymbol{x}, t)a(\boldsymbol{x}', t')} = \mathcal{F}(\boldsymbol{x} - \boldsymbol{x}' + St'(x_1 - x_1')\boldsymbol{e}_2, \, t - t'), \tag{24}$$

where the *two-point correlation function* $\mathcal{F}(\boldsymbol{R}, s)$ is a real function of one vector and one scalar argument. As expected from equation (22), the correlation function is homogeneous, and depends on $\boldsymbol{x}$ and $\boldsymbol{x}'$ only through their difference $(\boldsymbol{x} - \boldsymbol{x}')$. However, it is not time stationary in the sheared coordinates.[3] Henceforth we use the factored form, $\mathcal{F}(\boldsymbol{R}, t) = \mathcal{A}(\boldsymbol{R})\mathcal{D}(t)$, to make contact with equation (4) used in the earlier discussion of the KM model without shear. Then

$$\overline{a(\boldsymbol{x}, t)a(\boldsymbol{x}', t')} = 2\mathcal{A}\big(\boldsymbol{x} - \boldsymbol{x}' + St'(x_1 - x_1')\boldsymbol{e}_2\big)\,\mathcal{D}(t - t'),$$

$$2\int_0^\infty \mathcal{D}(t)\,\mathrm{d}t = 1, \qquad \mathcal{A}(\boldsymbol{0}) = \eta_\alpha > 0. \tag{25}$$

We also define the correlation time for the α fluctuations:

$$\tau_\alpha = 2\int_0^\infty \mathrm{d}t\, t\, \mathcal{D}(t). \tag{26}$$

The spatial correlation function is $\mathcal{A}(\boldsymbol{R})$, where $\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{x}' + St'(x_1 - x_1')\boldsymbol{e}_2$ is the relative separation vector of fluid elements due to shear. In the absence of shear, $S = 0$, the separation $\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{x}'$, and the right hand side of (25) is $2\mathcal{A}(\boldsymbol{x} - \boldsymbol{x}')\mathcal{D}(t - t') = 2\mathcal{A}(\boldsymbol{X} - \boldsymbol{X}')\mathcal{D}(\tau - \tau')$. Hence, as expected, for zero shear, equation (25) reduces to equation (4), which applies to the KM model described earlier. For non-zero shear the separation $\boldsymbol{R}$ is a time-dependent vector. Furthermore, for a general function $\mathcal{A}(\boldsymbol{R})$ and $St' \neq 0$, it is evident that the two-point correlator will be an anisotropic and time-dependent function in $(\boldsymbol{x} - \boldsymbol{x}')$.

Having presented our model in detail, it is useful to review the principal assumptions and limitations. The model is intended to be a minimal extension of the KM model of Section 2.1 to include a linear shear, and we have followed suit in making the simplifying assumptions of locality and isotropy. By this we mean: in the defining equation (9) for the meso-scale field, the relationship between the EMF and the meso-scale field is local and instantaneous, and the transport coefficients are isotropic. Specifically, EMF $\boldsymbol{\mathcal{E}}(\boldsymbol{X}, \tau) = \alpha(\boldsymbol{X}, \tau)\boldsymbol{B} - \eta_t \nabla \times \boldsymbol{B}$, where η_t is a scalar constant, and α is a pseudo-scalar stochastic function with GI statistics. The α fluctuations have a 2-point correlator which is taken to be of the factored form in equation (25). No restriction is placed on the forms of the functions $A(\boldsymbol{R})$ and $\mathcal{D}(t)$: they could be either dependent on shear or not; it is just that our model does not specify this.

3 EQUATION FOR THE LARGE-SCALE MAGNETIC FIELD

In order to derive a closed equation for the large-scale magnetic field we exploit the homogeneity of the problem in the sheared coordinates $\boldsymbol{x}$, and work with Fourier variables $\boldsymbol{k} \in \mathbb{R}^3$ which are conjugate to them. Given any $\mathcal{T}(\boldsymbol{x}, t)$ – which could be a (pseudo) scalar, vector or tensor function – let $\widetilde{\mathcal{T}}(\boldsymbol{k}, t) = \int \mathrm{d}^3x \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x})\mathcal{T}(\boldsymbol{x}, t)$ be its Fourier transform. Thus we use $\{\widetilde{\boldsymbol{H}}(\boldsymbol{k}, t), \widetilde{a}(\boldsymbol{k}, t), \overline{\widetilde{\boldsymbol{H}}}(\boldsymbol{k}, t), \widetilde{\boldsymbol{h}}(\boldsymbol{k}, t), \overline{\widetilde{\boldsymbol{E}}}(\boldsymbol{k}, t)\}$ to denote the Fourier transforms of $\{\boldsymbol{H}(\boldsymbol{x}, t), a(\boldsymbol{x}, t), \overline{\boldsymbol{H}}(\boldsymbol{x}, t), \boldsymbol{h}(\boldsymbol{x}, t), \overline{\boldsymbol{E}}(\boldsymbol{x}, t)\}$. Similarly, we can also define the Fourier variables $\boldsymbol{K} \in \mathbb{R}^3$, which are conjugate to the fixed coordinates $\boldsymbol{X}$. Then, given any $\mathcal{T}(\boldsymbol{X}, \tau)$ – which could be a (pseudo) scalar, vector or tensor function – let $\widetilde{\mathcal{T}}(\boldsymbol{K}, \tau) = \int \mathrm{d}^3X \exp(-\mathrm{i}\boldsymbol{K} \cdot \boldsymbol{X})\mathcal{T}(\boldsymbol{X}, \tau)$ be its Fourier transform.

The first step is to solve for $\widetilde{\boldsymbol{h}}(\boldsymbol{k}, t)$ as a functional of the stochastic field $\widetilde{a}(\boldsymbol{k}, t)$ and the large-scale magnetic field $\overline{\widetilde{\boldsymbol{H}}}(\boldsymbol{k}, t)$. This is carried out below in the FOSA, by dropping the terms $\nabla \times [a\boldsymbol{h} - \overline{a\boldsymbol{h}}]$ in equation (20). We also drop $\eta_\mathrm{T}\nabla^2\boldsymbol{h}$, although this is not a necessary feature of FOSA. Hence:

$$\frac{\partial\boldsymbol{h}}{\partial t} - Sh_1\boldsymbol{e}_2 = \nabla \times \boldsymbol{M}, \qquad \nabla \cdot \boldsymbol{h} = 0, \qquad \boldsymbol{h}(\boldsymbol{x}, 0) = \boldsymbol{0}, \tag{27}$$

---

[3] By construction the two-point correlator is, indeed, time stationary in the lab frame $(\boldsymbol{X}, \tau)$ and all the associated comoving observers. To see this, use (16) in (24): the left hand side is $\overline{\alpha(\boldsymbol{X}, \tau) \ldots \alpha(\boldsymbol{X}', \tau')}$ by definition. The right hand side is $\mathcal{F}(\boldsymbol{X} - \boldsymbol{X}' - S(\tau - \tau')X_1\boldsymbol{e}_2, \tau - \tau')$, so the correlation function is time-stationary in the lab coordinates but no longer homogeneous, as expected.

where $M(x, t) = a(x, t)\overline{H}(x, t)$ is a stochastic source field, and ∇ is the time-dependent operator defined in (21). Fourier transforming, we get

$$\frac{\partial \widetilde{h}}{\partial t} - S\widetilde{h}_1 e_2 = \mathrm{i} K \times \widetilde{M}, \qquad K \cdot \widetilde{h} = 0, \qquad \widetilde{h}(k, 0) = \mathbf{0},$$

$$K(k, t) = e_1(k_1 - St\, k_2) + e_2 k_2 + e_3 k_3,$$

$$\widetilde{M}(k, t) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3 k'\, \widetilde{a}^*(k', t)\, \widetilde{\overline{H}}(k + k', t). \tag{28}$$

Equation (28) can be integrated directly componentwise, and the solution which satisfies both constraints, $K \cdot \widetilde{h} = 0$ and $\widetilde{h}(k, 0) = \mathbf{0}$, written as:

$$\widetilde{h}(k, t) = \int_0^t \mathrm{d}t' \left[\mathrm{i} K(k, t') \times \widetilde{M}(k, t') \right] + e_2 S \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \left[\mathrm{i} K(k, t'') \times \widetilde{M}(k, t'') \right]_1. \tag{29}$$

The double time integral can be reduced to single time integral by noting that

$$\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \, f(t'') = \int_0^t \mathrm{d}t' (t - t') f(t').$$

Then the FOSA solution for the fluctuating magnetic field can be written as:

$$\widetilde{h}(k, t) = \int_0^t \mathrm{d}t' \left\{ \mathrm{i} K(k, t') \times \widetilde{M}(k, t') + e_2 S(t - t') \left[\mathrm{i} K(k, t') \times \widetilde{M}(k, t') \right]_1 \right\}. \tag{30}$$

Equation (30) should be used to calculate the Fourier transform of the EMF:

$$\widetilde{\overline{E}}(k, t) = \int \mathrm{d}^3 x \exp(-\mathrm{i} k \cdot x) \overline{E}(x, t) = \int \mathrm{d}^3 x \exp(-\mathrm{i} k \cdot x) \overline{a(x, t) h(x, t)}$$

$$= \frac{1}{(2\pi)^3} \int \mathrm{d}^3 k_1 \, \mathrm{d}^3 k_2 \, \delta(k_1 + k_2 - k) \, \overline{\widetilde{a}(k_1, t)\widetilde{h}(k_2, t)}$$

$$= \frac{1}{(2\pi)^3} \int \mathrm{d}^3 k_1 \, \mathrm{d}^3 k_2 \, \delta(k_1 + k_2 - k) \int_0^t \mathrm{d}t' \left\{ \left[\mathrm{i} K(k_2, t') \times \overline{\widetilde{a}(k_1, t)\widetilde{M}(k_2, t')} \right] + e_2 S(t - t') \left[\mathrm{i} K(k_2, t') \times \overline{\widetilde{a}(k_1, t)\widetilde{M}(k_2, t')} \right]_1 \right\} \tag{31}$$

is given in terms of the quantity $\overline{\widetilde{a}(k_1, t)\widetilde{M}(k_2, t')}$, which has to be calculated. Using equation (28) for $\widetilde{M}$,

$$\overline{\widetilde{a}(k_1, t)\widetilde{M}(k_2, t')} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3 k_3 \, \overline{\widetilde{a}(k_1, t)\widetilde{a}^*(k_3, t')} \, \widetilde{\overline{H}}(k_2 + k_3, t') \tag{32}$$

is a convolution of the large-scale magnetic field and the Fourier-space two-point correlator. We now compute the latter using equation (25) for time-stationary Galilean–Invariant α fluctuations:

$$\overline{\widetilde{a}(k_1, t)\widetilde{a}^*(k_3, t')} = \int \mathrm{d}^3 x_1 \mathrm{d}^3 x_3 \exp(-\mathrm{i} k_1 \cdot x_1 + \mathrm{i} k_3 \cdot x_3) \, \overline{a(x_1, t) a(x_3, t')}$$

$$= 2\mathcal{D}(t - t') \int \mathrm{d}^3 x_1 \mathrm{d}^3 x_3 \exp[-\mathrm{i}(k_1 \cdot x_1 - k_3 \cdot x_3)] \mathcal{A}(x_1 - x_3 + St'(x_{11} - x_{31})e_2).$$

Using new integration variables, $r = x_1 - x_3$ and $r' = \frac{1}{2}(x_1 + x_3)$, we get

$$\overline{\widetilde{a}(k_1, t)\widetilde{a}^*(k_3, t')} = 2\mathcal{D}(t - t') \int \mathrm{d}^3 r \, \mathrm{d}^3 r' \exp\left[-\mathrm{i}(k_1 - k_3) \cdot r' - \frac{\mathrm{i}}{2}(k_1 + k_3) \cdot r \right] \mathcal{A}(r + St' r_1 e_2)$$

$$= 2\mathcal{D}(t - t')(2\pi)^3 \, \delta(k_1 - k_3) \int \mathrm{d}^3 r \, \exp(-\mathrm{i} k_1 \cdot r) \, \mathcal{A}(r + St' r_1 e_2).$$

Another change of the integration variable to $R = r + St' r_1 e_2$ gives us a compact form for the two-point correlator:

$$\overline{\widetilde{a}(k_1, t)\widetilde{a}^*(k_3, t')} = 2\mathcal{D}(t - t')(2\pi)^3 \, \delta(k_1 - k_3) \widetilde{\mathcal{A}}\left(K(k_1, t')\right),$$

where $\quad \widetilde{\mathcal{A}}(K) = \int \mathrm{d}^3 R \exp(-\mathrm{i} K \cdot R) \, \mathcal{A}(R).$ \hfill (33)

$\widetilde{\mathcal{A}}(K)$ is the complex spatial power spectrum of α fluctuations, with $\widetilde{\mathcal{A}}(-K) = \widetilde{\mathcal{A}}^*(K)$ because $\mathcal{A}(R)$ is a real function. The δ-function in equation (33) implies that a shearing wave labelled by k_1 is uncorrelated with any other shearing wave with $k_1 \neq k_3$. Also each shearing wave has a time-dependent complex power because the argument of $\widetilde{\mathcal{A}}$ is a time-dependent wavevector, although the time correlation function

$\mathcal{D}$ is stationary. *Thus the α fluctuations may be thought of as a superposition of random and mutually uncorrelated shearing waves.* Using equation (33) in (32), we get a simple expression:

$$\overline{\widetilde{a}(\mathbf{k}_1, t)\widetilde{\mathbf{M}}(\mathbf{k}_2, t')} = 2\mathcal{D}(t - t')\widetilde{\mathcal{A}}\left(\mathbf{K}(\mathbf{k}_1, t')\right)\,\overline{\widetilde{\mathbf{H}}}(\mathbf{k}_1 + \mathbf{k}_2, t')\,. \tag{34}$$

When equation (34) is substituted in (31) we obtain a compact expression for the EMF:

$$\overline{\widetilde{\mathbf{E}}}(\mathbf{k}, t) = 2\int_0^t \mathrm{d}t'\,\mathcal{D}(t - t')\left\{ \widetilde{\mathbf{U}}(\mathbf{k}, t') \times \overline{\widetilde{\mathbf{H}}}(\mathbf{k}, t') \ + \ e_2 S(t - t')\left[\widetilde{\mathbf{U}}(\mathbf{k}, t') \times \overline{\widetilde{\mathbf{H}}}(\mathbf{k}, t')\right]_1 \right\}\,, \tag{35}$$

where

$$\widetilde{\mathbf{U}}(\mathbf{k}, t') = \int \frac{\mathrm{d}^3 k_1}{(2\pi)^3}\,\mathrm{i}\mathbf{K}(\mathbf{k} - \mathbf{k}_1, t')\widetilde{\mathcal{A}}\left(\mathbf{K}(\mathbf{k}_1, t')\right) \tag{36}$$

is a complex vector field that can be written in terms of the quantities, η_α and $\mathbf{V}_{\mathrm{M}}$, appearing in the KM theory without shear. Using the linearity of the function $\mathbf{K}$: i.e. $\mathbf{K}(\mathbf{k} - \mathbf{k}', t') = \mathbf{K}(\mathbf{k}, t') - \mathbf{K}(\mathbf{k}', t')$, we see that

$$\widetilde{\mathbf{U}}(\mathbf{k}, t') = \mathrm{i}\mathbf{K}(\mathbf{k}, t')\int \frac{\mathrm{d}^3 k'}{(2\pi)^3}\,\widetilde{\mathcal{A}}\left(\mathbf{K}(\mathbf{k}', t')\right) \ - \ \int \frac{\mathrm{d}^3 k'}{(2\pi)^3}\,\mathrm{i}\mathbf{K}(\mathbf{k}', t')\widetilde{\mathcal{A}}\left(\mathbf{K}(\mathbf{k}', t')\right)$$

$$= \mathrm{i}\mathbf{K}(\mathbf{k}, t')\int \frac{\mathrm{d}^3 K'}{(2\pi)^3}\,\widetilde{\mathcal{A}}\left(\mathbf{K}'\right) \ - \ \int \frac{\mathrm{d}^3 K'}{(2\pi)^3}\,\mathrm{i}\mathbf{K}'\widetilde{\mathcal{A}}\left(\mathbf{K}'\right)$$

$$= \mathrm{i}\mathbf{K}(\mathbf{k}, t')\mathcal{A}(\mathbf{0}) \ - \ \left.\frac{\partial \mathcal{A}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}}\right|_{\boldsymbol{\xi} = 0} \ = \ \mathrm{i}\mathbf{K}(\mathbf{k}, t')\eta_\alpha \ + \ \mathbf{V}_{\mathrm{M}}\,, \tag{37}$$

where we have transformed to a new integration variable $\mathbf{K}' = \mathbf{K}(\mathbf{k}', t') = (k_1' - St'k_2', k_2', k_3')$ – which has unit Jacobian giving $\mathrm{d}^3 k' = \mathrm{d}^3 K'$ – and used the definitions of η_α and $\mathbf{V}_{\mathrm{M}}$ given in equations (4) and (5).

Substituting (37) for $\widetilde{\mathbf{U}}(\mathbf{k}, t')$ in (35), we obtain an explicit expression for the EMF:

$$\overline{\widetilde{\mathbf{E}}}(\mathbf{k}, t) = 2\eta_\alpha \int_0^t \mathrm{d}t'\,\mathcal{D}(t - t')\left\{ \mathrm{i}\mathbf{K}(\mathbf{k}, t') \times \overline{\widetilde{\mathbf{H}}}(\mathbf{k}, t') \ + \ e_2 S(t - t')\left[\mathrm{i}\mathbf{K}(\mathbf{k}, t') \times \overline{\widetilde{\mathbf{H}}}(\mathbf{k}, t')\right]_1 \right\}\,,$$

$$+ 2\int_0^t \mathrm{d}t'\,\mathcal{D}(t - t')\left\{ \mathbf{V}_{\mathrm{M}} \times \overline{\widetilde{\mathbf{H}}}(\mathbf{k}, t') \ + \ e_2 S(t - t')\left[\mathbf{V}_{\mathrm{M}} \times \overline{\widetilde{\mathbf{H}}}(\mathbf{k}, t')\right]_1 \right\}\,. \tag{38}$$

Fourier transforming equation (19), the equation governing the large-scale field is

$$\frac{\partial \overline{\widetilde{\mathbf{H}}}}{\partial t} \ - \ S\overline{\widetilde{H}}_1 e_2 = \mathrm{i}\mathbf{K}(\mathbf{k}, t) \times \overline{\widetilde{\mathbf{E}}} \ - \ \eta_{\mathrm{T}} K^2(\mathbf{k}, t)\overline{\widetilde{\mathbf{H}}}\,, \qquad\qquad \mathbf{K}(\mathbf{k}, t) \cdot \overline{\widetilde{\mathbf{H}}} = 0\,. \tag{39}$$

Equation (39), together with equation (38), gives a closed, linear integro-differential equation for the large-scale magnetic field, $\overline{\widetilde{\mathbf{H}}}(\mathbf{k}, t)$, which is the principal general result of this paper. This mean-field equation depends on the three parameters (η_α, $\mathbf{V}_{\mathrm{M}}$, S), as well as the time-correlation function $\mathcal{D}(t)$. It should be noted that the theory is likely to overestimate the growth rates of high wavenumber modes because the dissipative term, $\eta_{\mathrm{T}}\nabla^2 \mathbf{h}$, was dropped in the FOSA equation (27).

4 WHITE-NOISE α FLUCTUATIONS

For white-noise α fluctuations the normalized correlation function is $\mathcal{D}_{\mathrm{WN}}(t) = \delta(t)$, the Dirac delta-function. From equation (26), we have $\tau_\alpha = 0$, which agrees with the notion of no memory [to explore memory effects it is necessary to consider $\mathcal{D}(t) \neq \delta(t)$ which, in general, will have $\tau_\alpha \neq 0$, and this is taken up later]. In the rest of this section we study dynamo action due to white-noise α fluctuations. From equation (38) the EMF for white-noise is

$$\overline{\widetilde{\mathbf{E}}}_{\mathrm{WN}}(\mathbf{k}, t) = [\,\mathrm{i}\mathbf{K}(\mathbf{k}, t)\eta_\alpha \ + \ \mathbf{V}_{\mathrm{M}}\,] \times \overline{\widetilde{\mathbf{H}}}(\mathbf{k}, t)\,, \tag{40}$$

which depends on the large-scale field at the present time only, because white-noise carries no memory of the past. Substituting (40) in (39), we get the following partial differential equation (PDE) for the large-scale magnetic field:

$$\frac{\partial \overline{\widetilde{\mathbf{H}}}}{\partial t} \ + \ \left[\,\eta_{\mathrm{K}} K^2 \ + \ \mathrm{i}\,\mathbf{K} \cdot \mathbf{V}_{\mathrm{M}}\,\right]\overline{\widetilde{\mathbf{H}}} = S\overline{\widetilde{H}}_1 e_2\,, \qquad\qquad \mathbf{K}(\mathbf{k}, t) \cdot \overline{\widetilde{\mathbf{H}}} = 0\,. \tag{41}$$

We can solve this by defining a new field $\overline{\widetilde{\mathbf{F}}}(\mathbf{k}, t)$ through

$$\overline{\widetilde{\mathbf{H}}}(\mathbf{k}, t) = \widetilde{\mathcal{G}}(\mathbf{k}, t)\,\overline{\widetilde{\mathbf{F}}}(\mathbf{k}, t)\,, \tag{42}$$

where

$$\widetilde{\mathcal{G}}(\boldsymbol{k}, t) = \exp\left\{ -\int_0^t \mathrm{d}t' \left[\eta_\mathrm{K} K^2(\boldsymbol{k}, t') + \mathrm{i}\,\boldsymbol{V}_\mathrm{M} \cdot \boldsymbol{K}(\boldsymbol{k}, t') \right] \right\}$$

$$= \exp\left\{ -\eta_\mathrm{K} \left[k^2 t - S k_1 k_2 t^2 + (S^2/3) k_2^2 t^3 \right] - \mathrm{i}\left[(\boldsymbol{V}_\mathrm{M} \cdot \boldsymbol{k})\,t - (S/2) V_{M1} k_2 t^2 \right] \right\} \tag{43}$$

is a shear-diffusive Green's function in which the Kraichnan diffusivity η_K contributes to the amplitude, and the Moffatt drift velocity $\boldsymbol{V}_\mathrm{M}$ contributes only to the phase. Using (42) and (43) in (41), we see that $\widetilde{\overline{\boldsymbol{F}}}(\boldsymbol{k}, t)$ must obey

$$\frac{\partial \widetilde{\overline{\boldsymbol{F}}}}{\partial t} = S \widetilde{\overline{F}}_1 \boldsymbol{e}_2\,, \qquad\qquad \boldsymbol{K}(\boldsymbol{k}, t) \cdot \widetilde{\overline{\boldsymbol{F}}}(\boldsymbol{k}, t) = 0\,. \tag{44}$$

The solution is given by $\widetilde{\overline{\boldsymbol{F}}}(\boldsymbol{k}, t) = \widetilde{\overline{\boldsymbol{F}}}(\boldsymbol{k}, 0) + \boldsymbol{e}_2 St\,\widetilde{\overline{F}}_1(\boldsymbol{k}, 0)$, where $\widetilde{\overline{\boldsymbol{F}}}(\boldsymbol{k}, 0)$ is any vector which satisfies $\boldsymbol{k} \cdot \widetilde{\overline{\boldsymbol{F}}}(\boldsymbol{k}, 0) = 0$. From (42) we have $\widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}, 0) = \widetilde{\overline{\boldsymbol{F}}}(\boldsymbol{k}, 0)$, so the solution to (41) may be written as:

$$\widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}, t) = \widetilde{\mathcal{G}}(\boldsymbol{k}, t) \left[\widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}, 0) + \boldsymbol{e}_2 St\,\widetilde{\overline{H}}_1(\boldsymbol{k}, 0) \right]\,, \qquad\qquad \boldsymbol{k} \cdot \widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}, 0) = 0\,. \tag{45}$$

Whereas (45) is a complete and compact form of the solution to the problem of white-noise α fluctuations, it is also useful to view the problem in real space, instead of Fourier space. We begin with the expression for the EMF in real space. Using (40), and the fact that $\mathrm{d}^3 k = \mathrm{d}^3 K$, $\boldsymbol{k} \cdot \boldsymbol{x} = \boldsymbol{K} \cdot \boldsymbol{X}$ and $\widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}, t) = \widetilde{\overline{\boldsymbol{B}}}(\boldsymbol{K}, \tau)$, we have

$$\overline{\boldsymbol{\mathcal{E}}}_\mathrm{WN}(\boldsymbol{X}, \tau) = \overline{\boldsymbol{E}}_\mathrm{WN}(\boldsymbol{x}, t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x})\,\widetilde{\overline{\boldsymbol{E}}}_\mathrm{WN}(\boldsymbol{k}, t)$$

$$= \int \frac{\mathrm{d}^3 K}{(2\pi)^3} \exp(\mathrm{i}\boldsymbol{K} \cdot \boldsymbol{X}) \left[\mathrm{i}\boldsymbol{K}\eta_\alpha + \boldsymbol{V}_\mathrm{M} \right] \times \widetilde{\overline{\boldsymbol{B}}}(\boldsymbol{K}, \tau)$$

$$= \eta_\alpha \nabla \times \overline{\boldsymbol{B}} + \boldsymbol{V}_\mathrm{M} \times \overline{\boldsymbol{B}}\,. \tag{46}$$

Thus the EMF for white-noise alpha fluctuations with shear is identical in form to the EMF for the KM case without shear. The physical reason why shear does not explicitly contribute to $\overline{\boldsymbol{\mathcal{E}}}_\mathrm{WN}$ is that white-noise has zero correlation time, giving shear no time to act. This is true for any finite value of the rate of shear parameter.

Substituting (46) in (10), we get the following PDE for the large-scale magnetic field:

$$\left(\frac{\partial}{\partial \tau} + S X_1 \frac{\partial}{\partial X_2} \right) \overline{\boldsymbol{B}} - S\overline{B}_1 \boldsymbol{e}_2 = \nabla \times \left(\boldsymbol{V}_\mathrm{M} \times \overline{\boldsymbol{B}} \right) + \eta_\mathrm{K} \nabla^2 \overline{\boldsymbol{B}}\,, \qquad\qquad \nabla \cdot \overline{\boldsymbol{B}} = 0\,. \tag{47}$$

The general solution of (47) can be written down, using the Fourier space solution (45) derived earlier:

$$\overline{\boldsymbol{B}}(\boldsymbol{X}, \tau) = \overline{\boldsymbol{H}}(\boldsymbol{x}, t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x})\,\widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}, t)$$

$$= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \exp\left[\mathrm{i}\boldsymbol{K}(\boldsymbol{k}, \tau) \cdot \boldsymbol{X} \right] \widetilde{\mathcal{G}}(\boldsymbol{k}, \tau) \left[\widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}, 0) + \boldsymbol{e}_2 S\tau\,\widetilde{\overline{H}}_1(\boldsymbol{k}, 0) \right]\,. \tag{48}$$

This is a superposition of shearing waves, whose growth or decay is controlled by Green's function $\widetilde{\mathcal{G}}(\boldsymbol{k}, \tau)$. We note some general properties:

(1) In the limit of no shear, $S = 0$, equation (47) reduces to (5). *Thus the KM problem corresponds to the case of white-noise ($\tau_\alpha = 0$) and no shear (S).*

(2) *Weak α fluctuations* have $\eta_\alpha < \eta_\mathrm{T}$ so that $\eta_\mathrm{K} > 0$. Waves of all wavenumbers $\boldsymbol{k}$ eventually decay. 'Axisymmetric' waves, whose shear-wise wavenumber $k_2 = 0$, decay as $\exp[-\eta_\mathrm{K} k^2 t]$.[4] The asymptotic decay of non-axisymmetric waves ($k_2 \neq 0$) is more rapid, like $\exp[-\eta_\mathrm{K}(S^2/3)k_2^2 t^3]$.

(3) *Strong α fluctuations* have $\eta_\alpha > \eta_\mathrm{T}$ so that $\eta_\mathrm{K} < 0$. Waves of all wavenumbers $\boldsymbol{k}$ eventually grow. Axisymmetric waves grow like $\exp[|\eta_\mathrm{K}| k^2 t]$, due to negative diffusion. Non-axisymmetric waves grow even more rapidly, like $\exp[|\eta_\mathrm{K}|(S^2/3)k_2^2 t^3]$, where shear acts in conjunction with the negative diffusion due to α fluctuations.

(4) The Moffatt drift velocity $\boldsymbol{V}_\mathrm{M}$ contributes only to the phase and does not determine the growth or decay of the large-scale magnetic field.

Therefore, the necessary condition for dynamo action for white-noise α fluctuations is that they must be strong, i.e. $\eta_\mathrm{K} < 0$. This condition is identical to the KM case of no shear. To explore the effects of the Moffatt drift $\boldsymbol{V}_\mathrm{M}$ and Shear S on the growth rate, it is necessary to consider $\tau_\alpha \neq 0$, thereby allowing for memory effects.

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[4] We use the term 'axisymmetric' in the sense in which it is often used in local treatments of disc-dynamos, where it is conventional to take $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ to be unit vectors along the radial, azimuthal and vertical directions.

5 AXISYMMETRIC LARGE-SCALE MAGNETIC FIELDS WHEN τ_α IS SMALL

We now ask: *is it possible that, when $\tau_\alpha \neq 0$, there is dynamo action even for weak α fluctuations, i.e. when $\eta_K > 0$?* To answer this question in generality, it is necessary to deal with the linear integro-differential equation defined by equations (38) and (39); such an analysis is beyond the scope of this paper. In this section we derive the PDE governing the evolution of large-scale magnetic field which evolves over times much larger than τ_α. Then we answer the above question by considering the dynamo action of the memory effect due to non-zero τ_α and η_α, but without either Moffatt drift or shear.

5.1 Derivation of the governing equation

We begin by noting that the normalized time correlation function, $\mathcal{D}(t)$, has a singular limit, i.e. $\lim_{\tau_\alpha \to 0} \mathcal{D}(t) = \mathcal{D}_{WN}(t) = \delta(t)$. The limit of small but non-zero τ_α corresponds to looking at functions $\mathcal{D}(t)$ that are, in some sense, close to the Dirac delta-function. However we are saved from the task of considering variations in function space, because the EMF depends on $\mathcal{D}(t)$ only through a time integral:

$$\widetilde{\overline{E}}(k, t) = 2 \int_0^t ds \, \mathcal{D}(s) \left\{ \widetilde{U}(k, t-s) \times \widetilde{\overline{H}}(k, t-s) + e_2 Ss \left[\widetilde{U}(k, t-s) \times \widetilde{\overline{H}}(k, t-s) \right]_1 \right\} \tag{49}$$

is the same as equation (35), rewritten by changing the integration variable from t' to $s = t - t'$. The complex velocity field $\widetilde{U} = iK(k, t')\eta_\alpha + V_M$ is as defined in (37). Since the limit $\lim_{\tau_\alpha \to 0} \widetilde{\overline{E}}(k, t) = \widetilde{\overline{E}}_{WN}(k, t) = \widetilde{U}(k, t) \times \widetilde{\overline{H}}(k, t)$ is evidently non-singular, we make the *ansatz* that, for small τ_α, the EMF can be expanded in a power series in τ_α:

$$\widetilde{\overline{E}}(k, t) = \widetilde{\overline{E}}_{WN}(k, t) + \widetilde{\overline{E}}^{(1)}(k, t) + \widetilde{\overline{E}}^{(2)}(k, t) + \cdots \tag{50}$$

where $\widetilde{\overline{E}}_{WN}(k, t) \sim \mathcal{O}(1)$ and $\widetilde{\overline{E}}^{(n)}(k, t) \sim \mathcal{O}(\tau_\alpha^n)$ for $n \geq 1$. Below we verify this ansatz up to $n = 1$, for slowly varying magnetic fields.

We want to work out $\widetilde{\overline{E}}(k, t)$ to first order in τ_α, for $t \gg \tau_\alpha$. Since $\mathcal{D}(s)$ is strongly peaked for times $s \lesssim \tau_\alpha$ and becomes very small for larger s, most of the contribution to the integral in (49) comes only from short times $0 \leq s < \tau_\alpha$. Hence in (49) we can (i) set the upper limit of the s-integral to $+\infty$; (ii) keep the terms inside the $\{\}$ in the integrand up to only first order in s. We also note that, for $k_2 \neq 0$, the magnitude of $K(k, t) = e_1(k_1 - St k_2) + e_2 k_2 + e_3 k_3$ increases without bound, leading to rapid time variations of $\widetilde{\overline{H}}(k, t)$. Henceforth we restrict attention to axisymmetric modes for which $k_2 = 0$. This implies that $K(k, t-s) = K(k, t) = k = (k_1, 0, k_3)$, and $\widetilde{U}(k, t-s) = \widetilde{U}(k) = \eta_\alpha i k + V_M$. We also write:

$$\widetilde{\overline{H}}(k, t-s) = \widetilde{\overline{H}}(k, t) - s \frac{\partial \widetilde{\overline{H}}(k, t)}{\partial t} + \cdots, \tag{51}$$

where it is assumed that $|\widetilde{\overline{H}}| \gg s|\partial \widetilde{\overline{H}}/\partial t| \gg s^2|\partial^2 \widetilde{\overline{H}}/\partial t^2|, s^3|\partial^3 \widetilde{\overline{H}}/\partial t^3|$, etc. Then

$$\{\} \text{ of equation (49)} = \widetilde{U}(k) \times \widetilde{\overline{H}}(k, t) + s \left\{ -\widetilde{U} \times \frac{\partial \widetilde{\overline{H}}}{\partial t} + e_2 S \left[\widetilde{U} \times \widetilde{\overline{H}} \right]_1 \right\} + \mathcal{O}(s^2). \tag{52}$$

The integral over s can now be performed. Using the properties of $\mathcal{D}(\tau)$, given in (4) and (26), we get:

$$\widetilde{\overline{E}}(k, t) = \widetilde{\overline{E}}_{WN}(k, t) + \tau_\alpha \left\{ -\widetilde{U} \times \frac{\partial \widetilde{\overline{H}}}{\partial t} + e_2 S[\widetilde{U} \times \widetilde{\overline{H}}]_1 \right\} + \mathcal{O}(\tau_\alpha^2). \tag{53}$$

In order to get an expression for the EMF accurate to $\mathcal{O}(\tau_\alpha)$, we need $\partial \widetilde{\overline{H}}/\partial t$ only to $\mathcal{O}(\tau_\alpha^0) = \mathcal{O}(1)$. Using (53) in (39) we have

$$\frac{\partial \widetilde{\overline{H}}}{\partial t}\bigg|_{\mathcal{O}(1)} = S\widetilde{\overline{H}}_1 e_2 + ik \times \widetilde{\overline{E}}_{WN}(k, t) - \eta_T k^2 \widetilde{\overline{H}} = S\widetilde{\overline{H}}_1 e_2 - [ik \cdot \widetilde{U} + \eta_T k^2]\widetilde{\overline{H}}. \tag{54}$$

When this is used in equation (53), we obtain an expression for the EMF,

$$\widetilde{\overline{E}}(k, t) = \left[1 + \eta_T k^2 \tau_\alpha + i(k \cdot \widetilde{U})\tau_\alpha \right] \widetilde{U} \times \widetilde{\overline{H}} + S\tau_\alpha \left[\widetilde{\overline{H}}_1 e_2 \times \widetilde{U} + \left(\widetilde{U} \times \widetilde{\overline{H}} \right)_1 e_2 \right], \tag{55}$$

accurate to $\mathcal{O}(\tau_\alpha)$, which verifies the ansatz of equation (50) up to $n = 1$, as claimed. Equation (55) for the EMF is valid only when the large-scale magnetic field is slowly varying; to lowest order this condition can be stated as $|\widetilde{\overline{H}}| \gg \tau_\alpha|\partial \widetilde{\overline{H}}/\partial t|$. Using equation (54) for $\partial \widetilde{\overline{H}}/\partial t$, we see that the sufficient condition for equation (55) to be valid is that the following three dimensionless quantities be small:

$$|S\tau_\alpha| \ll 1, \qquad |\eta_K k^2 \tau_\alpha| \ll 1, \qquad |k V_M \tau_\alpha| \ll 1. \tag{56}$$

Substituting (55) in (39) we obtain

$$\frac{\partial \widetilde{\overline{H}}}{\partial t} = \left[S\widetilde{\overline{H}}_1 \boldsymbol{e}_2 - \eta_K k^2 \widetilde{\overline{H}} - \mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{V}_M)\widetilde{\overline{H}} \right] \left[1 + \mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{V}_M)\tau_\alpha - \eta_\alpha k^2 \tau_\alpha \right]$$

$$+ S\tau_\alpha \left[V_{M2}\widetilde{\overline{H}}_3 - V_{M3}\widetilde{\overline{H}}_2 - \mathrm{i}\,\eta_\alpha k_3 \widetilde{\overline{H}}_2 \right] \left[-\mathrm{i}\,k_3 \boldsymbol{e}_1 + \mathrm{i}\,k_1 \boldsymbol{e}_3 \right],$$

$$\text{with} \quad \boldsymbol{k} = (k_1, 0, k_3), \qquad \text{and} \qquad k_1 \widetilde{\overline{H}}_1 + k_3 \widetilde{\overline{H}}_3 = 0. \tag{57}$$

Equation (57) is the linear PDE that determines the evolution of an axisymmetric, large-scale magnetic field which evolves over times that are much larger than τ_α. The three drivers of this evolution are (i) the diffusivities η_α and η_K; (ii) the Moffatt drift $\boldsymbol{V}_M$; (iii) shear S. These parameters must satisfy the three conditions given in (56), which also involves the wavenumber k. Only the condition on shear is independent of the wavenumber k, and it implies the following: whereas the full equations (39) and (38) are non-perturbative in both S and τ_α, equation (57) above is valid only when $|S\tau_\alpha| \ll 1$.

It is readily verified that the $\tau_\alpha = 0$ limit of equation (57) is identical to the $k_2 = 0$ case of equation (41), which describes axisymmetric white-noise α fluctuations. Hence the solution can be obtained by setting $k_2 = 0$ in equations (45) and (43). Here it is seen that shear contributes to the linear-in-time stretching of the field lines along the shear-wise ($\boldsymbol{e}_2$) direction, and the Moffatt drift causes the field to oscillate sinusoidally at angular frequency $\omega = \boldsymbol{k} \cdot \boldsymbol{V}_M$. It is only the Kraichnan diffusivity that contributes to the growth rate, $\gamma = -\eta_K k^2$. Hence the necessary condition for dynamo action is that $\eta_K = \eta_T - \eta_\alpha < 0$, just like in the more general non-axisymmetric white-noise case treated in Section 4.

5.2 The Kraichnan problem with non-zero τ_α

We want to understand the combined effect of the α fluctuations when $\boldsymbol{V}_M = \boldsymbol{0}$ and $S = 0$, but $\eta_\alpha > 0$ and $\tau_\alpha > 0$. Then there is a *new length scale* in the problem, whose corresponding wavenumber can be defined as

$$k_\alpha = (\eta_\alpha \tau_\alpha)^{-1/2} > 0. \tag{58}$$

When $S = 0$ and $\boldsymbol{V}_M = \boldsymbol{0}$, two of the three conditions in (56) are met trivially, and the third one implies that $|k|$ must be small enough such that $|\eta_K k^2 \tau_\alpha| \ll 1$. However, $|k|$ can be larger or smaller than k_α. We consider $|k| > k_\alpha$ to be *high wavenumbers*, and $|k| < k_\alpha$ to be *low wavenumbers*. Setting $S = 0$ and $\boldsymbol{V}_M = \boldsymbol{0}$ in equation (57), we get

$$\frac{\partial \widetilde{\overline{H}}}{\partial t} = \eta_K k^2 \left[\left(\frac{k}{k_\alpha} \right)^2 - 1 \right] \widetilde{\overline{H}}, \qquad \text{with} \quad \boldsymbol{k} = (k_1, 0, k_3) \quad \text{and} \quad k_1 \widetilde{\overline{H}}_1 + k_3 \widetilde{\overline{H}}_3 = 0. \tag{59}$$

This is the original Kraichnan problem for axisymmetric modes, modified by a new factor given in [ ]. The solutions are of the exponential form, $\widetilde{\overline{H}}(\boldsymbol{k}, t) = \widetilde{\overline{H}}_0(\boldsymbol{k}) \exp(\gamma t)$, where $\boldsymbol{k} \cdot \widetilde{\overline{H}}_0(\boldsymbol{k}) = 0$. Substituting this in equation (59), we get the growth rate as

$$\gamma = \left(\eta_K k_\alpha^2 \right) \left(\frac{k}{k_\alpha} \right)^2 \left[\left(\frac{k}{k_\alpha} \right)^2 - 1 \right], \qquad \text{when} \qquad |\eta_K k^2 \tau_\alpha| \ll 1. \tag{60}$$

Let us rewrite the growth rate in terms of a *new characteristic frequency*, defined by

$$\sigma = |\eta_K| k_\alpha^2 = \left(\frac{|\eta_T - \eta_\alpha|}{\eta_\alpha} \right) \frac{1}{\tau_\alpha} \geq 0. \tag{61}$$

Then (60) can be rewritten as

$$\gamma = \pm\, \sigma \left(\frac{k}{k_\alpha} \right)^2 \left[\left(\frac{k}{k_\alpha} \right)^2 - 1 \right], \qquad \text{when} \qquad |k| \ll \frac{k_\alpha}{\sqrt{\sigma \tau_\alpha}}, \tag{62}$$

where the $\pm$ signs correspond to the cases of *weak α fluctuations* ($\eta_K > 0$) and *strong α fluctuations* ($\eta_K < 0$), respectively. As Fig. 1 shows, for the weak case γ is negative for low wavenumbers and positive for high wavenumbers, and is exactly the opposite for the strong case. Hence there is a high wavenumber dynamo for weak α fluctuations, and a low wavenumber dynamo for strong α fluctuations.[5] As noted earlier, the high wavenumber behaviour is likely an overestimate because the dissipative term, $\eta_T \nabla^2 \boldsymbol{h}$, was dropped in the FOSA equation (27).

6 GROWTH RATES OF MODES WHEN τ_α IS NON-ZERO

We consider one-dimensional propagating modes for the general case when all the parameters (η_α, S, $\boldsymbol{V}_M$, τ_α) can be non-zero. Below we derive the dispersion relation and study the growth rate function. When the wavevector $\boldsymbol{k} = (0, 0, k)$ points along the 'vertical' ($\pm \boldsymbol{e}_3$)

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[5] It should be noted that this behaviour is *not* contradictory to the white-noise case: in the limit $\tau_\alpha \to 0$, both $k_\alpha \to \infty$ and $\sigma \to \infty$, and $\gamma \to \mp(\sigma/k_\alpha^2)k^2 = -\eta_K k^2$ which is in agreement with the results for the white-noise case.

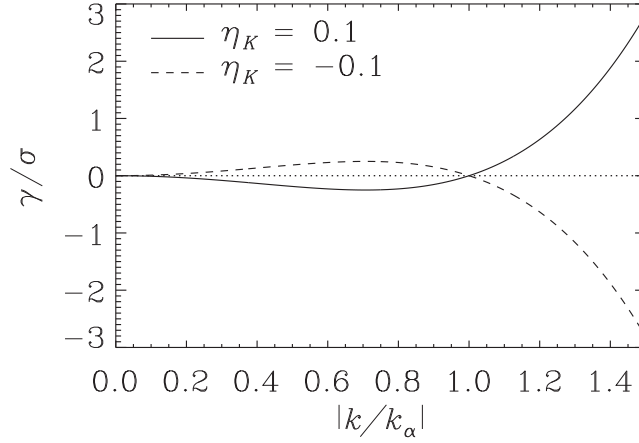**Figure 1.** Growth rate γ/σ as a function of $|k/k_\alpha|$, when $S = 0$ and $\boldsymbol{V}_M = \boldsymbol{0}$. Weak ($\eta_K > 0$) and strong ($\eta_K < 0$) α fluctuations are shown by bold and dashed curves, respectively.

directions, $\overline{\overline{H}}_3$ must be uniform and is of no interest for dynamo action. Hence we set $\overline{\overline{H}}_3 = 0$, and take $\overline{\overline{H}}(k, t) = \overline{\overline{H}}_1(k, t)\boldsymbol{e}_1 + \overline{\overline{H}}_2(k, t)\boldsymbol{e}_2$. The equation governing the time evolution of this large-scale magnetic field is obtained by setting $k_1 = 0$, $k_3 = k$ and $\overline{\overline{H}}_3 = 0$ in equation (57):

$$\frac{\partial \overline{\overline{H}}}{\partial t} = \left[S\overline{\overline{H}}_1 \boldsymbol{e}_2 - \eta_K k^2 \overline{\overline{H}} - \mathrm{i}kV_{\mathrm{M3}}\overline{\overline{H}} \right] \left[1 + \mathrm{i}kV_{\mathrm{M3}}\tau_\alpha - \eta_\alpha k^2\tau_\alpha \right] + S \left[\mathrm{i}kV_{\mathrm{M3}}\tau_\alpha - \eta_\alpha k^2\tau_\alpha \right] \overline{\overline{H}}_2 \boldsymbol{e}_1, \tag{63}$$

We seek modal solutions of the form

$$\overline{\overline{H}}(k, t) = \left[\overline{\overline{H}}_{01}(k)\boldsymbol{e}_1 + \overline{\overline{H}}_{02}(k)\boldsymbol{e}_2 \right] \exp(\lambda t). \tag{64}$$

Substituting this in equation (63) yields the following *dispersion relation for the two ($\pm$) modes*:

$$\lambda_\pm = - \left[\eta_K k^2 + \mathrm{i}kV_{\mathrm{M3}} \right] \left[1 + \mathrm{i}kV_{\mathrm{M3}}\tau_\alpha - \eta_\alpha k^2\tau_\alpha \right]$$

$$\pm |S| \sqrt{\left[\mathrm{i}kV_{\mathrm{M3}}\tau_\alpha - \eta_\alpha k^2\tau_\alpha \right] \left[1 + \mathrm{i}kV_{\mathrm{M3}}\tau_\alpha - \eta_\alpha k^2\tau_\alpha \right]}. \tag{65}$$

Of particular interest is the growth rate $\gamma = \mathrm{Re}\{\lambda\}$, because dynamo action corresponds to the case when $\gamma > 0$. From the dispersion relation (65) we have

$$\gamma_\pm = \mathrm{Re}\{\lambda_\pm\} = \eta_K k^2 \left(\eta_\alpha k^2\tau_\alpha - 1 \right) + \tau_\alpha(kV_{\mathrm{M3}})^2 \pm |S| \left[\chi_R^2 + \chi_I^2 \right]^{1/4} \cos(\psi/2),$$

where $\quad \tan(\psi) = \dfrac{\chi_I}{\chi_R}$,

$$\chi_R = \eta_\alpha k^2\tau_\alpha \left(\eta_\alpha k^2\tau_\alpha - 1 \right) - (kV_{\mathrm{M3}}\tau_\alpha)^2,$$

$$\chi_I = -kV_{\mathrm{M3}}\tau_\alpha \left(2\eta_\alpha k^2\tau_\alpha - 1 \right). \tag{66}$$

Note that both λ and γ are linear in the rate of shear S, when the parameters (η_K, η_α, V_{M3}, τ_α) are independent of S. Define the dimensionless quantities

$$\Gamma_\pm = \gamma_\pm\tau_\alpha, \qquad \beta = \eta_\alpha k^2\tau_\alpha,$$

$$\varepsilon_S = S\tau_\alpha, \qquad \varepsilon_K = \eta_K k^2\tau_\alpha, \qquad \varepsilon_M = kV_{\mathrm{M3}}\tau_\alpha. \tag{67}$$

Here $\Gamma_\pm$ are the dimensionless growth rates of the $\pm$ modes. $\beta = (k/k_\alpha)^2$ is a measure of the wavenumber k in units of the characteristic wavenumber k_α defined in equation (58), and can take any non-negative value. (ε_S, ε_K, ε_M) must all be small by the three conditions in (56) under which the basic mean field equation (57) is valid. There is just one constraint involving β and ε_K, coming from $\beta + \varepsilon_K = \eta_T k^2\tau_\alpha > 0$. Thus the parameter ranges are given by

$$0 \le \beta < \infty, \qquad \beta + \varepsilon_K > 0,$$

$$|\varepsilon_S| \ll 1, \qquad |\varepsilon_K| \ll 1, \qquad |\varepsilon_M| \ll 1. \tag{68}$$

Multiplying equation (66) by $\tau_\alpha > 0$, we obtain the dimensionless growth rates,

$$\Gamma_\pm = \varepsilon_K(\beta - 1) + \varepsilon_M^2 \pm |\varepsilon_S| \left[\beta^2(\beta - 1)^2 + \varepsilon_M^2 \left(2\beta^2 - 2\beta + 1 \right) + \varepsilon_M^4 \right]^{1/4} \cos(\psi/2),$$

where $\qquad \tan(\psi) = \dfrac{\varepsilon_M(1 - 2\beta)}{\beta(\beta - 1) - \varepsilon_M^2}$, $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (69)

as a function of the four dimensionless parameters $(\beta, \varepsilon_S, \varepsilon_K, \varepsilon_M)$. Let $\Gamma_>$ be the larger of Γ_+ and Γ_-, and $\Gamma_<$ be the smaller of Γ_+ and Γ_-. Then, from equation (69), we have

$$\Gamma_{\gtrless} = \varepsilon_K(\beta - 1) + \varepsilon_M^2 \pm |\varepsilon_S| \left[\beta^2(\beta - 1)^2 + \varepsilon_M^2 \left(2\beta^2 - 2\beta + 1\right) + \varepsilon_M^4 \right]^{1/4} |\cos(\psi/2)| . \qquad\qquad (70)$$

Note that $\Gamma_>$ increases, whereas $\Gamma_<$ decreases linearly with the shearing rate S, when the parameters $(\eta_K, \eta_\alpha, V_{M3}, \tau_\alpha)$ are independent of S.

6.1 The growth rate function $\Gamma_>$

We now study the behaviour of $\Gamma_>$, given in equation (70), as a function of the four parameters $(\beta, \varepsilon_S, \varepsilon_K, \varepsilon_M)$ for the range of parameter values given in (68). Of these, the three parameters $(\varepsilon_S, \varepsilon_K, \varepsilon_M)$ can be taken to be independently specified, taking positive and negative values, so long as their magnitudes are small. But $\beta \geq 0$ is subject to the constraint $\beta + \varepsilon_K > 0$. Therefore we can rewrite the conditions of (68) as

$$|\varepsilon_S| \ll 1, \qquad\qquad |\varepsilon_K| \ll 1, \qquad\qquad |\varepsilon_M| \ll 1,$$

For $\varepsilon_K \leq 0$, have $|\varepsilon_K| < \beta < \infty$; $\qquad$ For $\varepsilon_K > 0$, have $0 \leq \beta < \infty$. $\qquad\qquad\qquad\qquad\qquad\qquad$ (71)

Taking advantage of the smallness of $|\varepsilon_M|$, the expression for $\Gamma_>$ in equations (70) can be simplified by (i) noting that $(2\beta^2 - 2\beta + 1) \geq 1/2$ for any real β implies that $\varepsilon_M^2(2\beta^2 - 2\beta + 1) \gg \varepsilon_M^4$, so the ε_M^4 term inside the $[\ ]^{1/4}$ can be dropped; (ii) taking the limit $\varepsilon_M \to 0$ in $|\cos(\psi/2)|$. The two *special cases*, $\beta = 0$ and $\beta = 1$, are dealt with first. Then we take up the *general case*, and derive a simple expression for $\Gamma_>$ for arbitrary positive β, so long as it is not too close to the special values $(0, 1)$. *Taken together, the two special cases and the general case enable us to write a simple, good approximation to $\Gamma_>$ as a function of the four parameters $(\beta, \varepsilon_S, \varepsilon_K, \varepsilon_M)$, all independently varying: the εs being small in magnitude, and $0 < \beta < \infty$.*

1. Case $\beta = 0$:
$\qquad$ From (71) we must have $\varepsilon_K > 0$, whereas ε_S and ε_M can be of either sign. From (70) we have $\tan(\psi) = -(1/\varepsilon_M) \to \pm\infty$ for small and negative/positive ε_M, which implies that $|\cos(\psi/2)| = 1/\sqrt{2}$. Hence:

$$\Gamma_> \simeq -\varepsilon_K + \varepsilon_M^2 + \frac{1}{\sqrt{2}}|\varepsilon_S| |\varepsilon_M|^{1/2} . \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (72)$$

2. Case $\beta = 1$
$\qquad$ From (71) we can see that all the three small parameters, $(\varepsilon_K, \varepsilon_S, \varepsilon_M)$, can be of either sign. From (70) we have $\tan(\psi) = (1/\varepsilon_M) \to \pm\infty$ for small and positive/negative ε_M, which implies that $|\cos(\psi/2)| = 1/\sqrt{2}$. Hence

$$\Gamma_> \simeq \varepsilon_M^2 + \frac{1}{\sqrt{2}}|\varepsilon_S| |\varepsilon_M|^{1/2} . \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (73)$$

3. General Case: $\beta > 0$ but not too close to 0 or 1
$\qquad$ Excluding two narrow strips around 0 and 1, we consider the parameter ranges:

$$|\varepsilon_S| \ll 1, \qquad\qquad |\varepsilon_K| \ll 1, \qquad\qquad |\varepsilon_M| \ll 1,$$

and $\qquad \beta_{min} \leq \beta < (1 - \beta_{min}) \qquad$ and $\qquad (1 + \beta_{min}) < \beta < \infty$,

where $\qquad \text{Max}\{|\varepsilon_K|, |\varepsilon_M|\} \ll \beta_{min} \ll 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (74)

Note that this is not only compatible with conditions (71), but they are only slightly less general. The small parameters $(\varepsilon_K, \varepsilon_S, \varepsilon_M)$ can be of either sign. Since $\beta_{min} \gg |\varepsilon_M|$, we have: (i) In $\Gamma_>$ of (70), the term $[\ ]^{1/4} \simeq [\beta^2(\beta - 1)^2]^{1/4} = \sqrt{|\beta(\beta - 1)|}$; (ii) $\tan(\psi) \simeq \varepsilon_M(1 - 2\beta)/\beta(\beta - 1) \to 0$ for small and positive/negative ε_M, which implies that $|\cos(\psi/2)| = 1$. Hence

$$\Gamma_> \simeq \varepsilon_K(\beta - 1) + \varepsilon_M^2 + |\varepsilon_S| \sqrt{|\beta(\beta - 1)|} . \qquad\qquad\qquad\qquad\qquad\qquad\qquad (75)$$

Our analysis of dynamo action below is based on the above three cases.

7 DYNAMO ACTION DUE TO KRAICHNAN DIFFUSIVITY, MOFFATT DRIFT AND SHEAR

We are now ready to deal directly with the physical properties of dynamos driven by Kraichnan diffusivity, Moffatt drift and Shear. From the three cases considered above in (72)–(75), we write down:

$$\Gamma_> \simeq \varepsilon_K(\beta - 1) + \varepsilon_M^2 + |\varepsilon_S| \left[\beta^2(\beta - 1)^2 + \frac{\varepsilon_M^2}{4} \right]^{1/4} , \qquad\qquad\qquad\qquad (76)$$
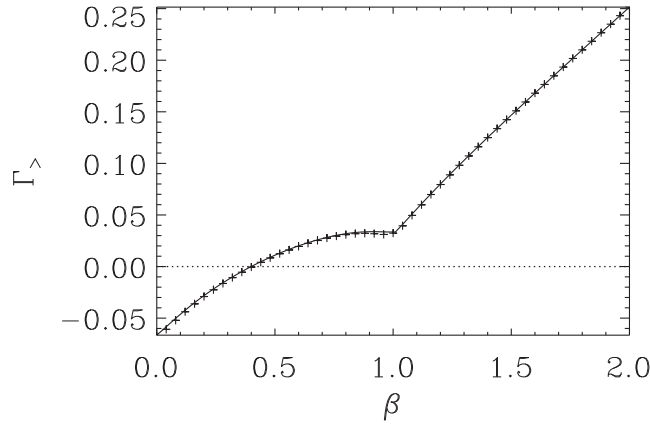
Figure 2. Growth rate function $\Gamma_>$ plotted as a function of β, for $\varepsilon_K = 0.1$, $|\varepsilon_M| = 0.1$ and $|\varepsilon_S| = 0.1$. The bold line corresponds to the exact expression of equation (70), and the '+' symbols correspond to the interpolation formula of equation (76).

as a simple interpolation formula, *valid for the independent variations of all four parameters*: as before the three εs are small, but can freely take positive and negative values; β is now free to take any positive value in an unconstrained manner:

$$|\varepsilon_S| \ll 1, \qquad |\varepsilon_K| \ll 1, \qquad |\varepsilon_M| \ll 1, \qquad 0 < \beta < \infty. \tag{77}$$

Fig. 2 compares the general formula (70) for the dimensionless growth rate $\Gamma_>$, with the interpolation formula given in equation (76) above; as may be seen it is an excellent fit to (70). Note that, even though $\varepsilon_K > 0$ (corresponding to weak α fluctuations), $\Gamma_>$ is positive for a range of values of β, allowing for dynamo action. A general property of equation (76) is that, for fixed β and ε_K, the dimensionless growth rate $\Gamma_>$ is a monotonically increasing function of both $|\varepsilon_M|$ and $|\varepsilon_S|$. Hence we arrive at an important conclusion:

> *Both Moffatt drift and shear always act to increase the growth rate of travelling waves of the form,* $\overline{\boldsymbol{B}}(X_3, \tau) = [\overline{B}_1 \boldsymbol{e}_1 + \overline{B}_2 \boldsymbol{e}_2] \exp(ikX_3 + \lambda\tau)$, *where* B_1 *and* B_2 *are arbitrary constants and* λ *is a complex frequency.*

We have already considered the combined effect of non-zero τ_α and Kraichnan diffusivity in Section 5.2. Below we present two examples of dynamo action due to non-zero τ_α in conjunction with Moffatt drift and Shear.

7.1 A Moffatt drift dynamo

We consider the case of zero shear, $\varepsilon_S = 0$, but general values of ε_K and ε_M. This can be thought of as the KM problem, discussed in Section 2.1, generalized to include a non-zero τ_α. From equation (63) we see that, for $S = 0$, the two components of $\widetilde{\overline{\boldsymbol{H}}}(k, t)$ evolve independently. The growth rate given by equation (76) is

$$\Gamma_> \simeq \varepsilon_K(\beta - 1) + \varepsilon_M^2. \tag{78}$$

The simplest cases are when one of ε_K and ε_M is zero: $\varepsilon_M = 0$ has already been discussed in Section 5.3, so let us consider the complementary case $\varepsilon_K = 0$. Then $\Gamma_> = \varepsilon_M^2$ is non-negative, so that the dimensional growth rate

$$\gamma_> = \left(V_{M3}^2 \tau_\alpha\right) k^2 \tag{79}$$

is negatively diffusive, with negative-diffusion coefficient equal to $(V_{M3}^2 \tau_\alpha)$. This growth is due to an extra term, $iV_{M3}^2 \tau_\alpha k(\boldsymbol{e}_3 \times \widetilde{\overline{\boldsymbol{H}}})$, in the EMF of equation (55). *Thus Moffat drift and a non-zero α-correlation time together can enable dynamo action.* This should be contrasted with the white-noise case where the Moffatt drift does not influence the growth rate. The remarkable thing is just that a constant drift velocity can enable a dynamo in the absence of any other generative effect. This is only possible if the relation between mean EMF and mean field has a memory effect. Rheinhardt et al. (2014) is another example where there is a drift velocity (a γ effect), analogous to the Moffat drift in the case of the Roberts flow III.

Now we go on to look at the more general problem. When ε_K and ε_M are both non-zero, then η_K and V_{M3} are also non-zero; this gives rise to a *new time-scale* in the problem:

$$\tau_* = (|\eta_K|/V_{M3}^2) > 0. \tag{80}$$

Below we consider the behaviour of the dimensional growth rate $\gamma_>$, as a function of the wavenumber k, for weak and strong α fluctuations. As we will see, the nature of dynamo action depends on whether τ_α is larger or smaller than τ_*.

Weak α fluctuations: This has $0 < \eta_\alpha < \eta_T$, so that η_K and ε_K are both positive. The dimensional growth rate can be written as

$$\gamma_> = \sigma\left\{\left(\frac{k}{k_\alpha}\right)^4 + \left[\frac{\tau_\alpha}{\tau_*} - 1\right]\left(\frac{k}{k_\alpha}\right)^2\right\}, \tag{81}$$
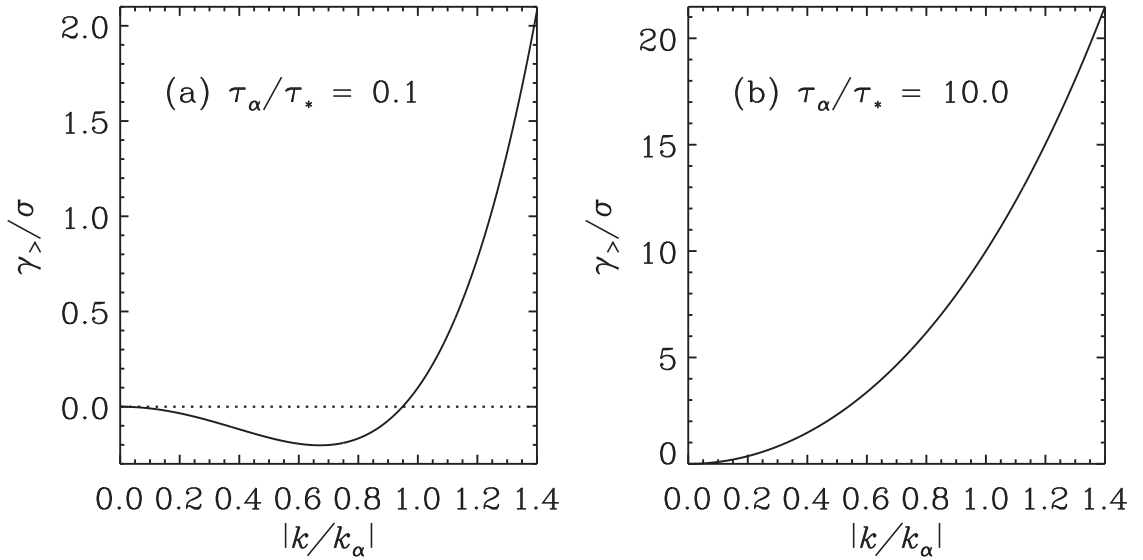
Figure 3. Growth rate $\gamma_>/\sigma$ plotted as a function of $|k/k_\alpha|$ for *weak α fluctuations*, when $S = 0$. Panels (a) and (b) correspond to the case when $\tau_\alpha/\tau_* = 0.1$ and 10.0, respectively. Note that the growth rate is always positive for high wavenumbers.

where $\sigma = |\eta_K|k_\alpha^2$ is the characteristic frequency defined in equation (61). High wavenumbers, $|k| > k_\alpha$, always grow, with $\gamma_>$ being a positive and monotonically increasing function of $|k|$. This behaviour is opposite, in a qualitative sense, to the white-noise case where weak α fluctuations always imply decaying large-scale fields. When $|k| \gg k_\alpha$, the growth rate tends to the asymptotic form $\gamma_> \to \sigma(k/k_\alpha)^4$ which is independent of the Moffatt drift. There are two cases to study: (a) *Case $\tau_\alpha < \tau_*$*: for small enough wavenumbers $|k| < k_\alpha\sqrt{1 - \tau_\alpha/\tau_*}$, the growth rate is negative; for very small wavenumbers, there is positive diffusion with $\gamma_> \propto -k^2$. For large enough wavenumbers $|k| > k_\alpha\sqrt{1 - \tau_\alpha/\tau_*}$, the growth rate becomes positive and tends to being negatively super-diffusive ($\gamma_> \propto k^4$) for very large wavenumbers. The graph is given in Fig. 3(a).

(b) *Case $\tau_\alpha > \tau_*$*: the growth rate $\gamma_>$ is always positive for all wavenumbers k. Therefore, $\tau_\alpha > \tau_*$ *is a sufficient condition for dynamo action at all wavenumbers*. This is equivalent to requiring that the Moffatt drift and the α-correlation time be large enough such that:

$$V_{M3}^2 \tau_\alpha > \eta_K > 0. \tag{82}$$

As can be seen from Fig. 3(b), the growth is negatively diffusive ($\gamma_> \propto k^2$) when $|k| \ll k_\alpha$, and negatively super-diffusive ($\gamma_> \propto k^4$) when $|k| \gg k_\alpha$.

Strong α fluctuations: this has $0 < \eta_T < \eta_\alpha$, so that η_K and ε_K are both negative. The dimensional growth rate is

$$\gamma_> = \sigma\left\{-\left(\frac{k}{k_\alpha}\right)^4 + \left[\frac{\tau_\alpha}{\tau_*} + 1\right]\left(\frac{k}{k_\alpha}\right)^2\right\}. \tag{83}$$

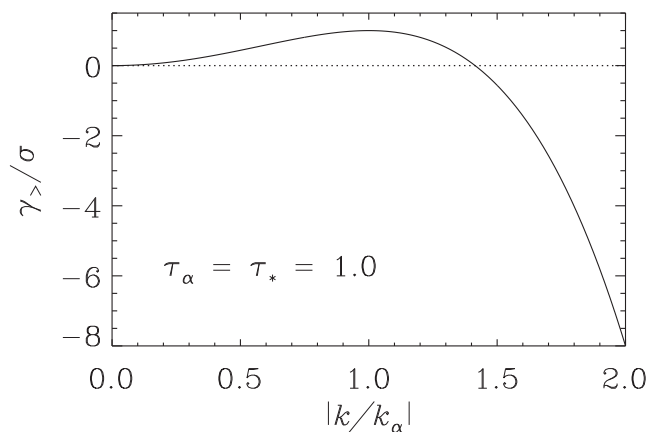As shown in Fig. 4, the growth rate $\gamma_>$ rises from zero at $|k| = 0$, to a maximum positive value,

Figure 4. Growth rate $\gamma_>/\sigma$ plotted as a function of $|k/k_\alpha|$ for *strong α fluctuations*, when $S = 0$. Note that the growth rate is always positive at low wavenumbers, and negative for high enough wavenumbers.

$$\gamma_{\mathrm{m}} = \frac{\sigma}{4} \left[1 + \frac{\tau_\alpha}{\tau_*} \right]^2 , \qquad \text{at} \quad |k| = k_{\mathrm{m}} = \frac{k_\alpha}{\sqrt{2}} \left[1 + \frac{\tau_\alpha}{\tau_*} \right]^{1/2} . \tag{84}$$

Then $\gamma_{>}$ decreases monotonically, turning negative for $|k| > k_\alpha \sqrt{1 + \tau_\alpha/\tau_*}$. For much larger $|k|$ the growth rate tends to the asymptotic form $\gamma_{>} \to -\sigma(k/k_\alpha)^4$ which is independent of the Moffatt drift. Note that this behaviour is quite opposite to the white-noise case where strong α fluctuations always imply growing large-scale fields, but it nevertheless approaches smoothly the white-noise case for $\tau_\alpha \to 0$.

7.2 A shear dynamo

We consider the case of zero Moffatt drift, $\varepsilon_{\mathrm{M}} = 0$, but general values of ε_{K} and ε_{S}. The growth rate is

$$\Gamma_{>} \simeq \varepsilon_{\mathrm{K}}(\beta - 1) + |\varepsilon_{\mathrm{S}}| \sqrt{|\beta(\beta - 1)|} , \tag{85}$$

As earlier we will look at the properties of the dimensional growth rate as a function of the wavenumber.

Weak α fluctuations: this has $0 < \eta_\alpha < \eta_{\mathrm{T}}$, so that η_{K} and ε_{K} are both positive. From (85) and (61) the dimensional growth rate is

$$\gamma_{>} = \sigma \left| \frac{k}{k_\alpha} \right| \sqrt{1 - \left| \frac{k}{k_\alpha} \right|^2} \left[\frac{|S|}{\sigma} - \left| \frac{k}{k_\alpha} \right| \sqrt{1 - \left| \frac{k}{k_\alpha} \right|^2} \right] , \quad \text{when} \quad |k| < k_\alpha ;$$

$$= \sigma \left| \frac{k}{k_\alpha} \right| \sqrt{\left| \frac{k}{k_\alpha} \right|^2 - 1} \left[\frac{|S|}{\sigma} + \left| \frac{k}{k_\alpha} \right| \sqrt{\left| \frac{k}{k_\alpha} \right|^2 - 1} \right] , \quad \text{when} \quad |k| > k_\alpha . \tag{86}$$

High wavenumbers, $|k| > k_\alpha$, always grow, with $\gamma_{>}$ being a positive and monotonically increasing function of $|k|$. This behaviour is opposite, in a qualitative sense, to the white-noise case where weak fluctuations decay. When $|k| \gg k_\alpha$, the growth rate tends to the asymptotic form $\gamma_{>} \to \sigma(k/k_\alpha)^4$. For low wavenumbers, $|k| < k_\alpha$, the growth rate is positive for all wavenumbers only when $(|S|/\sigma)$ is larger than the maximum value of $(|k|/k_\alpha)\sqrt{1 - (|k|/k_\alpha)^2}$, which is equal to $1/2$. *Therefore a sufficient condition for dynamo action at all wavenumbers is that $|S| > \sigma/2$.* This is equivalent to requiring that the shearing rate and α-correlation time be large enough such that:

$$|S|\tau_\alpha > \frac{1}{2} \frac{\eta_{\mathrm{K}}}{\eta_\alpha} > 0 . \tag{87}$$

When the shearing rate is not large enough, $|S| < \sigma/2$, then at low wavenumbers, the growth rate is negative in a range of wavenumbers centred around $|k| = (k_\alpha/\sqrt{2})$. Fig. 5 shows the growth rate as a function of wavenumber, for the two cases of weak and strong shear.
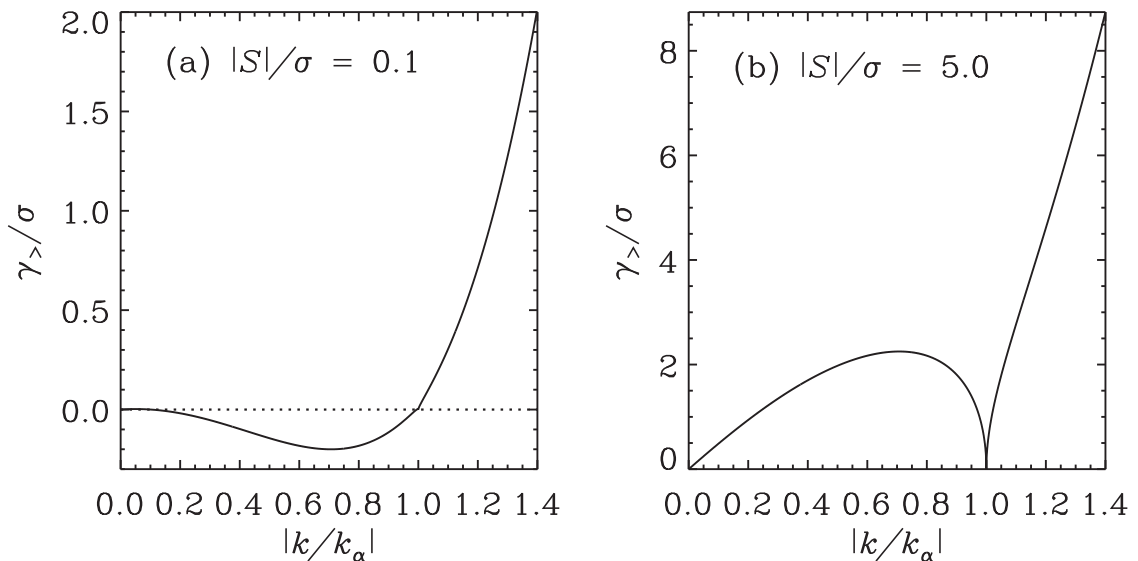

Figure 5. Growth rate $\gamma_{>}/\sigma$ plotted as a function of $|k/k_\alpha|$ for *weak α fluctuations*, when $V_{\mathrm{M3}} = 0$. Panels (a) and (b) correspond to the case when $|S|/\sigma = 0.1$ and 5.0, respectively. Note that the growth rate is always positive for high wavenumbers.
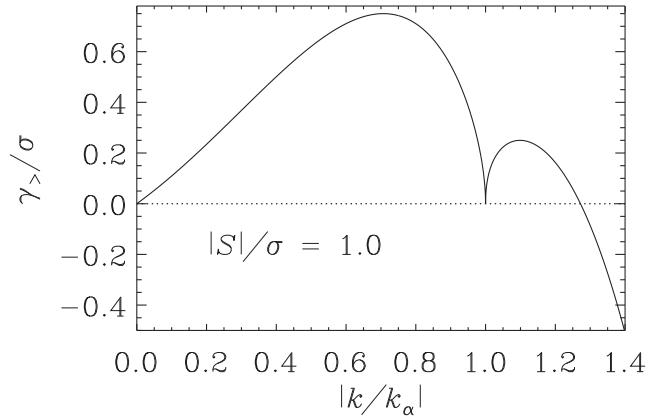
Figure 6. Growth rate $\gamma_>/\sigma$ plotted as a function of $|k/k_\alpha|$ for *strong α fluctuations*, when $V_{M3} = 0$. Note that the growth rate is always positive at low wavenumbers and negative for high enough wavenumbers.

Strong α fluctuations: this has $0 < \eta_T < \eta_\alpha$, so that η_K and ε_K are both negative. From (85) and (61) the dimensional growth rate is

$$\gamma_> = \sigma \left| \frac{k}{k_\alpha} \right| \sqrt{1 - \left| \frac{k}{k_\alpha} \right|^2} \left[\frac{|S|}{\sigma} + \left| \frac{k}{k_\alpha} \right| \sqrt{1 - \left| \frac{k}{k_\alpha} \right|^2} \right], \quad \text{when} \quad |k| < k_\alpha \, ;$$

$$= \sigma \left| \frac{k}{k_\alpha} \right| \sqrt{\left| \frac{k}{k_\alpha} \right|^2 - 1} \left[\frac{|S|}{\sigma} - \left| \frac{k}{k_\alpha} \right| \sqrt{\left| \frac{k}{k_\alpha} \right|^2 - 1} \right], \quad \text{when} \quad |k| > k_\alpha \, . \tag{88}$$

Fig. 6 shows the growth rate as a function of the wavenumber. $\gamma_>$ rises from zero at $|k| = 0$, to a maximum positive value,

$$\gamma_{m1} = \frac{|S|}{2} + \frac{\sigma}{4}, \quad \text{at} \quad |k| = k_{m1} = \frac{k_\alpha}{\sqrt{2}}, \tag{89}$$

and then decreases to zero at $|k| = k_\alpha$. For high wavenumbers, $\gamma_>$ again rises again from zero to a second maximum positive value,

$$\gamma_{m2} = \frac{S^2}{4\sigma}, \quad \text{at} \quad |k| = k_{m2} = \frac{k_\alpha}{\sqrt{2}} \left[1 + \sqrt{1 + \frac{S^2}{\sigma^2}} \right]^{1/2}, \tag{90}$$

and then decreases monotonically, turning negative for $|k| > (k_\alpha/\sqrt{2})[1 + \sqrt{1 + 4S^2/\sigma^2}]^{1/2}$. When $|k| \gg k_*$, the growth rate tends to the asymptotic form $\gamma_> \rightarrow -\sigma(k/k_\alpha)^4$. Note that this behaviour is quite opposite to the white-noise case where strong α fluctuations always imply growing large-scale fields, but it nevertheless approaches smoothly the white-noise case for $\tau_\alpha \rightarrow 0$.

8 CONCLUSIONS

We have presented a model of large-scale dynamo action in a linear shear flow that has stochastic, zero-mean fluctuations of the α parameter. This is based on a generalization of the KM model (Kraichnan 1976; Moffatt 1978), to include a background linear shear and a non-zero α-correlation time. Our principal result is a linear integro-differential equation for the large-scale magnetic field, which is non-perturbative in both the shear (S) and the α-correlation time (τ_α).

An immediate application is to the case of *white-noise α fluctuations* for which $\tau_\alpha = 0$. The EMF turned out to be identical in form to the KM model without shear; this is because white-noise carries no memory whereas shear needs time to act. With no further approximation, the integro-differential equation was reduced to a PDE, where shear entered it only through the background flow. We have presented a full solution of the initial value problem, and discussed the role of Kraichnan diffusivity (η_K), Moffatt drift (V_M) and shear (S). Some salient results are (a) V_M contributes only to the phase and does not influence the growth of the field, which is qualitatively similar to the KM model with an additional S-dependent term; (b) the growth of the field depends on both η_K and S; (c) however, the necessary condition for dynamo action is identical to the KM model (i.e. the α fluctuations must be strong, $\eta_K < 0$); then modes of all wavenumbers grow by shear-modified negative diffusion with the highest wavenumbers growing fastest.

To explore memory effects on dynamo action it is necessary to let τ_α be non-zero. The integro-differential equation is difficult to analyse, hence we restricted attention to a simpler yet physically meaningful case where it reduces to a PDE, i.e. for slowly varying 'axisymmetric' large-scale fields, small τ_α, and moderate values of (η_K, V_M, S) – the precise conditions were stated in terms of three dimensionless small parameters. Working perturbatively, we derived an expression for the EMF that is accurate to first order in τ_α, and obtained the PDE governing the evolution of the large-scale magnetic field. We solved for exponential propagating modes with wavenumber $\boldsymbol{k} = k\boldsymbol{e}_3$, and derived a dispersion relation. From this we obtained a formula for the growth rate, $\gamma_> = f + |S|g$, where f and g are explicitly given functions

of $(\eta_T, \eta_K, V_{M3}, \tau_\alpha, k)$. Therefore, if the parameters $(\eta_T, \eta_K, V_{M3}, \tau_\alpha)$ are independent of shear, then $\gamma_>$ is linear in $|S|$. The expression for $\gamma_>$ is somewhat complicated, so we simplified it to an interpolation formula that allowed us to explore dynamo action in some detail. Here we recall some of the salient results.

(1) There can be dynamo action for even weak α fluctuations (i.e. positive Kraichnan diffusivity), when $\tau_\alpha \neq 0$.

(2) Both Moffatt drift and Shear contribute to increasing the growth rate at any wavenumber.

(3) *Moffatt drift dynamo:* there can be dynamo action when both V_{M3} and τ_α are non-zero, but $S = 0$. This is different from the KM model in which the Moffatt drift contributes only to the phase of the large-scale field and does not influence dynamo action. For the simplest case when $\eta_K = 0$, the growth rate $\gamma_> = (V_{M3}^2 \tau_\alpha) k^2$ is negatively diffusive, with negative-diffusion coefficient equal to $(V_{M3}^2 \tau_\alpha)$; this growth is driven by an extra term, $i V_{M3}^2 \tau_\alpha k (\boldsymbol{e}_3 \times \widetilde{\overline{\boldsymbol{H}}})$, in the EMF of equation (55). For more general values of η_K, we recall two results: for weak α fluctuations, $V_{M3}^2 \tau_\alpha > \eta_K$ is a sufficient condition for dynamo action for all k. For strong α fluctuations, the growth rate as a function of the wavenumber has a single positive maximum value, and turns negative for large $|k|$.

(4) *Shear dynamo:* there can be dynamo action when both S and τ_α are non-zero, but $V_{M3} = 0$. For weak α fluctuations, $|S| \tau_\alpha > \eta_K/(2\eta_\alpha)$ is a sufficient condition for dynamo action for all k. For strong α fluctuations, the growth rate as a function of the wavenumber has two positive maxima, and turns negative for large $|k|$.

Cautionary note: in the two types of dynamos discussed above, it is likely that the growth of high $|k|$ modes has been overestimated, because the dissipative term was dropped in the FOSA equation (27) for the fluctuating field.

Our model is based on the dynamo equation (9) for the meso-scale magnetic field. It is a minimal extension of the KM dynamo equation (2) and inherits certain simplifying assumptions, such as the isotropy and locality of the transport coefficients η_t and α. Relaxation of these assumptions can, by itself, lead to dynamo action at the meso-scale level. Isotropy can be relaxed by making η_t and α tensorial [see e.g. Brandenburg et al. (2008), Devlen, Brandenburg & Mitra (2013), Singh & Jingade (2013), Rheinhardt et al. (2014) for the extraction of these transport coefficients from numerical simulations using the test-field method]. Relaxing locality in space implies a $\boldsymbol{k}$-dependence of the Fourier-transformed transport coefficients, which we denote here by $\tilde{\eta}_t$ and $\tilde{\alpha}$; Devlen et al. (2013) have demonstrated dynamo action due to $\tilde{\eta}_t$ turning negative at low k and overcoming molecular diffusivity. Relaxing locality in time (see e.g. Hubbard & Brandenburg 2009) makes $\tilde{\eta}_t$ and $\tilde{\alpha}$ dependent on ω, which then implies memory effects; Rheinhardt et al. (2014) demonstrate dynamo action due to delayed transport with a tensorial $\tilde{\alpha}$.

The other major assumption is the minimal role given to shear in our model; η_t has been assumed constant, and the α fluctuations have a Galilean-invariant two-point correlation function of a factored form; the time correlation part is $\mathcal{D}(t)$, and the spatial correlation part is $\mathcal{A}(\boldsymbol{R})$, where $\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{x}' + St'(x_1 - x_1')\boldsymbol{e}_2$. No restriction is placed on the value of η_t, or on the forms of the functions $\mathcal{D}(t)$ and $\boldsymbol{A}(\boldsymbol{R})$: they could be either dependent on shear or not; it is just that our model does not specify this. In a more general model shear is likely to play a deeper role, by influencing the very nature of the underlying turbulence. Then we can no longer expect $\tilde{\eta}_t$ and $\tilde{\alpha}$ to be so simply behaved; they can be general tensorial functions of the many variables $(\boldsymbol{k}, \omega, S)$. In this case, it is only natural to expect that the nature of dynamo action (criteria, wavenumber behaviour of the growth rate, etc.) may be different from items 1–4 above. Hence, particular results are less important than the formalism used to treat shear and α-correlation time non-perturbatively. Can this formalism be extended to obtain an integro-differential equation for the large-scale magnetic field, corresponding to (39) and (38), in the more general case? Even when the answer is in the affirmative, one is faced with the task of choosing functional forms and fluctuation statistics for tensorial, many-variable functions. It is here that the test-field method may serve a very useful purpose, by extracting transport coefficients from numerical simulations. Once we have plausible functional forms, there are still two more tasks at hand: (a) the study of the dynamo modes of the model and comparison with numerical simulations; (b) addressing the very nature of the underlying turbulence in a shear flow that gives rise to the transport coefficients.

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