Unsharp measurements and joint measurability

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We give an overview of joint unsharp measurements of non-commuting observables using positive operator valued measures (POVMs). We exemplify the role played by joint measurability of POVMs in entropic uncertainty relation for Alice's pair of non-commuting observables in the presence of Bob's entangled quantum memory. We show that Bob should record the outcomes of incompatible (non-jointly measurable) POVMs in his quantum memory so as to beat the entropic uncertainty bound. In other words, in addition to the presence of entangled Alice–Bob state, implementing incompatible POVMs at Bob's end is necessary to beat the uncertainty bound and hence predict the outcomes of non-commuting observables with improved precision. We also explore the implications of joint measurability to *validate* a moment matrix constructed from average pairwise correlations of three dichotomic non-commuting qubit observables. We prove that a classically acceptable moment matrix – which ascertains the existence of a legitimate joint probability distribution for the outcomes of all the three dichotomic observables – could be realized if and only if compatible POVMs are employed.

Keywords: Incompatibility, joint measurability, positive operator valued measures, unsharp measurements.

Introduction

IN the classical perspective, all physical observables can be measured jointly. In contrast, quantum theory places restrictions on the precision with which non-commuting observables can be measured. In the conventional quantum framework, *sharp* measurement of an observable is done through the corresponding spectral projection valued (PV) operators. Joint *sharp* values can only be assigned to a commuting set of observables, when one restricts to PV measurements. More specifically, joint measurability or compatibility of observables is entirely linked with their commutativity, if *sharp* PV measurements are employed. However, with the introduction of generalized measurements – formulated in terms of positive operator valued measures (POVMs) – it has been possible to refine the notion of joint measurability of non-commuting observables[1–4]. While *sharp* values for non-

commuting observables cannot be assigned jointly via their PV measurements, their *unsharp* joint values could be perceived when *compatible* POVMs are employed. Active research efforts are dedicated to investigate operationally significant criteria of joint measurability (compatibility) of two or more POVMs and also to develop a resource theory of measurement incompatibility[5–17]. In this article, we review the notion of compatible POVMs in the generalized measurement setting. We then illustrate two different physical situations, where incompatible unsharp measurements are crucial to bring forth non-classical features. First, we identify the significance of incompatible POVMs to predict the outcomes of a pair of non-commuting observables with enhanced precision – by focusing on the entropic uncertainty relation in the presence of quantum entangled memory[18]. Beating the entropic uncertainty bound relies on both entanglement and measurement incompatibility as necessary quantum resources[19]. Next, we explore when a moment matrix, constructed from the pairwise correlation outcomes of joint unsharp measurements of three dichotomic non-commuting qubit observables, admits a classical joint probability distribution. We find that the optimal value of the unsharpness parameter, below which the moment matrix is positive (and hence, admits a joint probability distribution for the fuzzy measurement outcomes of all the three qubit observables), matches identically with the compatibility of the observables.

The article is organized as follows. First, we begin with an overview of generalized measurements comprised of POVMs. We discuss the notion of joint measurability of two or more POVMs and show that this is much broader than commutativity. Next we explore the implications of joint measurability on entropic uncertainty relation for Alice's pair of non-commuting observables, in the presence of Bob's quantum memory. We show that when Bob is restricted to employ only jointly measurable POVMs, it is not possible to achieve enhanced precision for predicting Alice's measurement outcomes, even if entangled state is shared between them. Further, we explore the role of joint measurability of three dichotomic Pauli qubit observables $\vec{\sigma} \cdot \hat{n}_k$; $k = 1, 2, 3$ with $\hat{n}_1 \cdot \hat{n}_2 = \hat{n}_2 \cdot \hat{n}_3 = \hat{n}_1 \cdot \hat{n}_3 = -1/2$, on the positivity of the moment matrix – which is constructed from the average pairwise correlations of the outcomes of the dichotomic observables arising in their sequential unsharp–sharp measurements. We

find that the corresponding moment matrix is positive if and only if the unsharpness parameter lies in the joint measurability range of the observables. The last section is devoted to a summary of our results.

Compatible POVMS

In the conventional framework, quantum measurements are described in terms of projection operators of the corresponding self-adjoint observables. And joint measurability of two commuting observables means that one can produce the results for both the observables by performing only *one* PV measurement. A necessary requirement for joint measurability is that there exists a joint probability distribution for the measurement outcomes of a set of compatible observables, such that it yields correct marginal probability distributions for the outcomes of all the subsets of observables. Introduction of POVMs in 1960s by Ludwig[1] and subsequent investigations on their applicability[2], led to a conceptually sound and mathematically rigorous generalization of measurement theory. The usual PV measurements constitute a special case of generalized measurements. Generalized observables are represented by POVMs, which are termed as *unsharp* in contrast to their *sharp* PV counterparts. Commutativity of POVMs has a restricted meaning than their compatibility in the framework of generalized measurements.

Mathematically, POVM is a set $\mathbb{E} = \{E(x)\}$ comprising positive self-adjoint operators $0 \leq E(x) \leq 1$, called *effects*, satisfying $\sum_x E(x) = 1$; x denotes the outcomes of measurement and 1 is the identity operator. The notion of a POVM $\mathbb{E}$ to be a generalized observable provides a physical representation for any possible events (effects $E(x)$) to occur as outcomes x in a measurement process.

When a quantum system is prepared in the state ρ, measurement of the observable $\mathbb{E}$ gives rise to generalized Lüder's transformation of the state, i.e.

$$\rho \mapsto \sum_x \sqrt{E(x)} \rho \sqrt{E(x)}, \tag{1}$$

and an outcome x occurs with probability $p(x) = \text{Tr}[\rho E(x)]$. The expectation value of the observable is given by

$$\langle \mathbb{E} \rangle = \sum_x x \, \text{Tr}[\rho E(x)] = \sum_x x p(x). \tag{2}$$

The usual scenario of PV measurements is recovered as a special case when $\{E(x)\}$ forms a set of complete, orthogonal projectors.

A finite collection of POVMs $\mathbb{E}_1, \mathbb{E}_2, ..., \mathbb{E}_n$ is said to be jointly measurable (or compatible), if there exists a *grand* POVM $\mathbb{G} = \{G(\lambda); 0 \leq G(\lambda) \leq 1, \sum_\lambda G(\lambda) = 1\}$ from

which the observables $\mathbb{E}_i$ can be obtained by post-processing as follows. Suppose a measurement of the global POVM $\mathbb{G}$ is carried out in a state ρ and the probability of obtaining the outcome λ is denoted by $p(\lambda) = \text{Tr}[\rho G(\lambda)]$. If the effects $E_i(x_i)$ constituting the POVM $\mathbb{E}_i$ can be obtained as *marginals* of the *grand* POVM $\mathbb{G} = \{G(\lambda), \lambda \equiv \{x_1, x_2, ...\}\}$, (where λ corresponds to a collective index $\{x_1, x_2, ...\}$), i.e. if there exists a grand POVM $\mathbb{G}$ such that[7]

$$E_1(x_1) = \sum_{x_2, x_3, ...} G(x_1, x_2, ..., x_n),$$

$$E_2(x_2) = \sum_{x_1, x_3, ...} G(x_1, x_2, ..., x_n),$$

$$\vdots$$

$$E_n(x_n) = \sum_{x_1, x_3, ...} G(x_1, x_2, ..., x_n), \tag{3}$$

the POVMs $\mathbb{E}_1, \mathbb{E}_2, ..., \mathbb{E}_n$ are said to be jointly measurable. Thus, a collection of compatible POVMs $\mathbb{E}_1, \mathbb{E}_2, ..., \mathbb{E}_n$ is obtained from a global POVM $\mathbb{G}$ via post processing of the form given in eq. (3). We emphasize once again that compatibility of POVMs does not require their commutativity, but it demands the existence of a global POVM.

As an example, consider Pauli spin observables σ_x, σ_z of a qubit. Sharp measurements of the observables $\sigma_x = \sum_{x=\pm 1} x \Pi_{\sigma_x}(x)$ and $\sigma_z = \sum_{z=\pm 1} z \Pi_{\sigma_z}(z)$ are performed using the two outcome projection operators

$$\Pi_{\sigma_x}(x) = \frac{1}{2}(1 + x\sigma_x),$$

$$\Pi_{\sigma_z}(z) = \frac{1}{2}(1 + z\sigma_z). \tag{4}$$

The observables σ_x and σ_z are non-commuting and hence cannot be measured jointly using PV measurements. However, it is possible to consider joint fuzzy measurements of σ_x, σ_z in terms of their POVM counterparts, which are constructed by adding uniform white noise to the PV operators of eq. (4). One then obtains binary POVMs $\mathbb{E}_{\sigma_x} = \{E_{\sigma_x}(x); \ x = \pm 1\}$, $\mathbb{E}_{\sigma_z} = \{E_{\sigma_z}(z); \ z = \pm 1\}$, where

$$E_{\sigma_x}(x) = \eta \Pi_{\sigma_x}(x) + (1-\eta)\frac{1}{2}$$

$$= \frac{1}{2}(1 + \eta x \sigma_x),$$

$$E_{\sigma_z}(z) = \eta \Pi_{\sigma_z}(z) + (1-\eta)\frac{1}{2}$$

$$= \frac{1}{2}(1 + \eta z \sigma_z), \tag{5}$$

where $0 \leq \eta \leq 1$ denotes the unsharpness parameter. It may be noted that when $\eta = 1$, the fuzzy POVMs $\mathbb{E}_{\sigma_x} = \{E_{\sigma_x}(x)\}$; $\mathbb{E}_{\sigma_z} = \{E_{\sigma_z}(z)\}$ reduce to their corresponding *sharp* PV versions $\{\Pi_{\sigma_x}(x)\}$, $\{\Pi_{\sigma_x}(z)\}$.

The binary POVMs $\mathbb{E}_{\sigma_x}$, $\mathbb{E}_{\sigma_z}$ are compatible if there exists a four-element grand POVM $\mathbb{G} = \{G(x, z); x, z = \pm 1\}$ satisfying

$$\sum_{z=\pm 1} G(x,z) = G(x,1) + G(x,-1) = E_{\sigma_x}(x),$$

$$\sum_{x=\pm 1} G(x,z) = G(1,z) + G(-1,z) = E_{\sigma_z}(z),$$

$$\sum_{x,z=\pm 1} G(x,z) = 1, \quad G(x,z) \geq 0. \tag{6}$$

It has been shown[3,7] that the POVMs $\mathbb{E}_{\sigma_x}$, $\mathbb{E}_{\sigma_z}$ are compatible in the range $0 \leq \eta \leq 1/\sqrt{2}$ of the unsharpness parameter, as it is possible to construct a global POVM $\mathbb{G}$ comprising the effects

$$G(x,z) = \frac{1}{4}(1 + \eta x \sigma_x + \eta z \sigma_z), \quad 0 \leq \eta \leq 1/\sqrt{2} \tag{7}$$

satisfying the required conditions in eq. (6). Measurement of a *single* generalized observable (POVM) $\mathbb{G}$ enables one to produce the results of measurement of both the POVMs $\mathbb{E}_{\sigma_x}$ and $\mathbb{E}_{\sigma_z}$, when they are compatible. And, as a consequence, joint measurability of POVMs $\mathbb{E}_{\sigma_x}$, $\mathbb{E}_{\sigma_y}$ ensures the existence of a joint probability distribution $p(x, z) = \text{Tr}[\rho G(x, z)]$ obeying $p(x) = \sum_z p(x, z) = \text{Tr}[\rho \sum_z G(x, z)] = \text{Tr}[\rho E_{\sigma_x}(x)]$, $p(z) = \sum_x p(x, z) = \text{Tr}[\rho \sum_x G(x, z)] = \text{Tr}[\rho E_{\sigma_z}(z)]$, over the measurement outcomes x, z of the unsharp POVMs $\mathbb{E}_{\sigma_x}$, $\mathbb{E}_{\sigma_z}$ in any arbitrary quantum state ρ.

Triple-wise joint measurements of all the three Pauli observables σ_x, σ_y and σ_z can be envisaged by considering the fuzzy binary outcome POVMs $\mathbb{E}_{\sigma_x} = \{E_{\sigma_x}(x) = (1/2)(1 + \eta x \sigma_x); x = \pm 1\}$, $\mathbb{E}_{\sigma_y} = \{E_{\sigma_y}(y) = (1/2)(1 + \eta y \sigma_y); y = \pm 1\}$, $\mathbb{E}_{\sigma_z} = \{E_{\sigma_z}(z) = 1/2(1 + \eta z \sigma_z); z = \pm 1\}$ in the range $0 \leq \eta \leq 1/\sqrt{3}$ of the unsharpness parameter[7,10]. Further, it has also been shown[10] that the noisy versions $\mathbb{E}_{\vec{\sigma} \cdot \hat{n}_k} = \{E_{\vec{\sigma} \cdot \hat{n}_k}(x_k = \pm 1) = (1/2)(1 + \eta x_k \vec{\sigma} \cdot \hat{n}_k)\}$ of the qubit spin, oriented along the unit vectors $\hat{n}_k$, $k = 1, 2, 3$, which are equally separated in a plane (i.e. separated by an angle 120°), are pairwise jointly measurable if the unsharpness $\eta \leq \sqrt{3} - 1$, but are triple-wise jointly measurable when $\eta \leq 2/3$.

These examples bring forth the possibility of quantum measurements of three observables that can be implemented jointly pairwise – but not triplewise – in a two-dimensional Hilbert space, which could not be realized within the PV measurement framework. This identification led towards an extension of the notion of Kochen–

Specker contextuality[20] recently and a generalized non-contextuality inequality[10,21] is shown to be violated in a two-dimensional Hilbert space[22], if a set of three dichotomic POVMs, which has pairwise joint measurability – but no triple-wise joint measurability – is employed. Moreover, it has been recognized[3,5–8,13,14] that if Bob is restricted to employ only local compatible POVMs on his system, irrespective of Alice's measurements and of the bipartite state shared between them, a local classical probabilistic model could be realized. And hence it is not possible to witness non-local quantum features like steering[23,24] (the ability to non-locally alter the states of one part of a composite system by performing measurements on another spatially separated part) and the violation of Bell inequality[25]. An intrinsic connection between non-local steering and incompatible measurements has been independently established by Quintino *et al.*[13] and Uola *et al.*[14], who proved that a set of POVMs is not jointly measurable if and only if it is useful to demonstrate quantum steering. In addition to bringing out the fact that measurement compatibility is not synonymous with commutativity of the observables, these research efforts highlight the quantum resource nature of incompatible measurements. There are ongoing investigations recently, which focus towards developing a resource theory of measurement incompatibility[16,17].

Beating entropic uncertainty bound using incompatible measurements

In this section, we investigate entropic uncertainty relation for Alice's pair of non-commuting observables, in the presence of an entangled quantum memory at Bob's end, when Bob is restricted to measure only compatible POVMs[19].

The Shannon entropies $H(\mathbb{E}_X) = -\sum_x p(x) \log_2 p(x)$, $H(\mathbb{E}_Z) = -\sum_z p(z) \log_2 p(z)$, associated with the probabilities $p(x) = \text{Tr}[\rho E_X(x)]$, $p(z) = \text{Tr}[\rho E_{\mathbb{E}_Z}(z)]$ of measurement outcomes x, z of a pair of POVMs $\mathbb{E}_X \equiv \{E_X(x) | 0 \leq E_X(x) \leq 1; \sum_x E_X(x) = 1\}$, $\mathbb{E}_Z \equiv \{E_Z(z) | 0 \leq E_Z(z) \leq 1; \sum_z E_Z(z) = 1\}$, offer a more general framework to quantify uncertainties in predicting the measurement outcomes of two observables $\mathbb{E}_X$, $\mathbb{E}_Z$ in a given quantum state ρ.

The uncertainties of the measurement outcomes of $\mathbb{E}_X$ and $\mathbb{E}_Z$ in a quantum state of finite dimension d reveal a trade-off, which is expressed in terms of the entropic uncertainty relation[26,27]

$$H(\mathbb{E}_X) + H(\mathbb{E}_Z) \geq -2 \log_2 \mathcal{C}(\mathbb{E}_X, \mathbb{E}_Z), \tag{8}$$

where $\mathcal{C}(\mathbb{E}_X, \mathbb{E}_Z) = \max_{x,z} \| \sqrt{E_X(x)} \sqrt{E_Z(z)} \|$. (Here $\|A\| = \text{Tr}[\sqrt{A^\dagger A}]$).

A generalized version of the entropic uncertainty relation, for Alice's pair of observables $\mathbb{E}_X$, $\mathbb{E}_Z$, when assisted by Bob's quantum memory, led to a refinement of the

uncertainty bound as in eq. (8) and brought out that the outcomes of non-commuting observables could be predicted more precisely with the help of an entangled state shared between Alice and Bob[18].

The entropic uncertainty relation in the presence of quantum memory is better introduced in terms of a game[18]: to begin with, two players Alice and Bob decide to measure a pair of observables $\mathbb{E}_X$ and $\mathbb{E}_Z$. Bob prepares a quantum state of his choice and sends it to Alice. Alice measures $\mathbb{E}_X$ or $\mathbb{E}_Z$ randomly and communicates only her choice of measurements (not the outcomes of her measurement) to Bob. To win the game, Bob should prepare a suitable quantum state such that he is able to predict Alice's measurement outcomes in every experimental run with as much precision as possible. In other words, Bob's task is to minimize the uncertainties in the measurements of a pair of chosen observables $\mathbb{E}_X$, $\mathbb{E}_Z$, by appropriate state preparation and measurements at his end. The uncertainties of $\mathbb{E}_X$, $\mathbb{E}_Z$ are bounded as in eq. (8), if Bob can access only classical information. On the other hand, when Bob prepares an entangled state and sends one part of it to Alice, retaining the other part, he can beat the uncertainty bound of eq. (8).

The entropic uncertainty relation for the observables $\mathbb{E}_X$, $\mathbb{E}_Z$, measured on Alice's subsystem of the entangled state ρ_{AB}, is given by[18]

$$S(\mathbb{E}_X \mid B) + S(\mathbb{E}_Z \mid B) \geq -2\log_2 \mathcal{C}(\mathbb{E}_X, \mathbb{E}_Z) + S(A \mid B), \quad (9)$$

where

$$S(\mathbb{E}_X \mid B) = S(\rho_{AB}^{(\mathbb{E}_X)}) - S(\rho_B),$$
$$S(\mathbb{E}_Z \mid B) = S(\rho_{AB}^{(\mathbb{E}_Z)}) - S(\rho_B) \quad (10)$$

are the conditional von Neumann entropies of the post-measured states $\rho_{AB}^{(\mathbb{E}_X)}$, $\rho_{AB}^{(\mathbb{E}_Z)}$, which are obtained after Alice measures $\mathbb{E}_X$, $\mathbb{E}_Z$ on her system and stores the outcomes x, z in an orthonormal basis $\{|x\rangle\}$ ($\{|z\rangle\}$) respectively

$$\rho_{AB}^{(\mathbb{E}_X)} = \sum_x |x\rangle\langle x| \otimes \rho_B^{(x)},$$

$$\rho_{AB}^{(\mathbb{E}_Z)} = \sum_z |z\rangle\langle z| \otimes \rho_B^{(z)}. \quad (11)$$

Here, $\rho_B^{(x)} = \text{Tr}_A[\rho_{AB}(E_X(x) \otimes \mathbb{1}_B)]$ and $\rho_B^{(z)} = \text{Tr}_A[\rho_{AB} \times (E_\mathbb{Z}(z) \otimes \mathbb{1}_B)]$. The probabilities of measurement outcomes x, z are given by $p(x) = \text{Tr}[\rho_B^{(x)}] = \text{Tr}[\rho_{AB}(E_X(x) \otimes \mathbb{1}_B)]$, $p(z) = \text{Tr}[\rho_B^{(z)}] = \text{Tr}[\rho_{AB}(E_Z(z) \otimes \mathbb{1}_B)]$; $S(A|B) = S(\rho_{AB}) - S(\rho_B)$ denotes the conditional von Neumann entropy of the state ρ_{AB}; the von Neumann entropy $S(\rho)$ of the quantum state ρ is given by $S(\rho) = -\text{Tr}[\rho \log_2 \rho]$.

It may be noted that when the state ρ_{AB} shared between Alice and Bob is maximally entangled, the second term on the right-hand side of eq. (9) takes the value $S(A|B) = -\log_2 d$. And, as the first term $-2\log_2 \mathcal{C}(\mathbb{E}_X, \mathbb{E}_Z)$ cannot be larger than $\log_2 d$ (the maximum value of $\log_2 d$ for $-2\log_2 \mathcal{C}(\mathbb{E}_X, \mathbb{E}_Z)$ is realized when Alice employs pairs of unbiased projective measurements[28]), a trivial lower bound of zero can be achieved in the entropic uncertainty relation, showing that Bob can predict Alice's outcomes with certainty. In general, by sharing an appropriate entangled state with Alice, Bob can in fact beat the uncertainty bound of eq. (8) with the help of suitable measurements on his part of the state.

Let us denote $\mathbb{E}_{X'}$ or $\mathbb{E}_{Z'}$ as the POVMs which Bob choses to measure at his end, after Alice announces her choice $\mathbb{E}_X$ or $\mathbb{E}_Z$ of observables in each experimental run. Probabilities of Alice obtaining an outcome x for $\mathbb{E}_X$, and Bob getting an outcome x' in his measurement of $\mathbb{E}_{X'}$ are given by

$$p(x, x') = \text{Tr}[\rho_{AB} E_X(x) \otimes E_{X'}(x')]. \quad (12)$$

Shannon conditional entropies of Alice's measurement outcomes $x \in X$, $z \in Z$ – conditioned by Bob's measurement outcomes $x' \in X'$, $z' \in Z'$ – are given by

$$H(X \mid X') = -\sum_{x,x'} p(x, x') \log_2 p(x \mid x'),$$

$$H(Z \mid Z') = -\sum_{z,z'} p(z, z') \log_2 p(z \mid z'), \quad (13)$$

Here, $p(x|x') = p(x, x')/p(x')$ denotes the conditional probability that Alice registers an outcome x, when Bob finds the outcome to be x'; $p(x') = \text{Tr}[\rho_B E_{X'}(x')] = \sum_x p(x, x')$ is the probability of Bob getting an outcome x' in the measurement of $\mathbb{E}_{X'}$.

Recalling that measurements can never decrease entropy, i.e. $H(X|X') \geq S(\mathbb{E}_X \mid B)$, $H(Z|Z') \geq S(\mathbb{E}_Z|B)$, the entropic uncertainty relation in the presence of quantum memory in eq. (9) can also be expressed in terms of Shannon conditional entropies as,

$$H(X \mid X') + H(Z \mid Z') \geq -2\log_2 \mathcal{C}(\mathbb{E}_X, \mathbb{E}_Z) + S(A \mid B). \quad (14)$$

On the other hand, it has been shown[29,30] that the sum of conditional Shannon entropies $H(X|X')$, $H(Z|Z')$ is constrained to obey the following *entropic steering inequality*

$$H(X \mid X') + H(Z \mid Z') \geq -2\log_2 \mathcal{C}(\mathbb{E}_X, \mathbb{E}_Z), \quad (15)$$

when Bob is unable to remotely steer Alice's state by his local measurements[24]. As it has been shown recently[13,14]

that measurement incompatibility is necessary and sufficient to demonstrate the violation of any steering inequality, it turns out that Bob should perform incompatible measurements at his end, so as to be able to *beat the uncertainty bound* of eq. (8) below the value '$-2\log_2 \mathcal{C}(\mathbb{E}_X, \mathbb{E}_Z)$' (or equivalently, to witness the violation of entropic steering inequality (eq. (15)).

We now proceed to illustrate, with a particular example of qubits, how Bob can beat the upper bound on entropic uncertainties with the help of an entangled state and appropriate measurements. Suppose that Alice and Bob decide to measure a pair of qubit observables σ_x and σ_z initially. Bob then prepares a pure maximally entangled two-qubit state

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0_A, 1_B\rangle - |1_A, 0_B\rangle), \qquad (16)$$

and sends one of the subsystems to Alice. Alice performs any one of the *sharp* PV measurements

$$\Pi_{\sigma_X}(x) = \frac{1}{2}(1 + x\sigma_x), \quad x = \pm 1,$$

$$\Pi_{\sigma_Z}(z) = \frac{1}{2}(1 + z\sigma_z), \quad z = \pm 1, \qquad (17)$$

randomly on her qubit and announces her choice (σ_x or σ_z) to Bob. Now Bob's task is to predict Alice's measurement outcomes x or z by performing suitable measurements at his end. Suppose that he performs *unsharp* measurements

$$E_{\sigma_x}(x') = \frac{1}{2}(1 + \eta x'\sigma_x),$$

$$E_{\sigma_z}(z') = \frac{1}{2}(1 + \eta z'\sigma_z), \qquad (18)$$

and announces his outcomes x' or z' in every experimental run.

The joint probabilities $p(x, x')$ (or $p(z, z')$) of Alice's *sharp* outcome x (or z) and Bob's *unsharp* outcome x' (or z'), are obtained (see eq. (12)) to be

$$p(x, x') = \langle \psi_{AB} | \Pi_{\sigma_x}(x) \otimes E_{\sigma_x}(x') | \psi_{AB}\rangle$$

$$= \frac{1}{4}(1 - \eta x x'),$$

$$p(z, z') = \langle \psi_{AB} | \Pi_{\sigma_z}(z) \otimes E_{\sigma_z}(z') | \psi_{AB}\rangle$$

$$= \frac{1}{4}(1 - \eta z z'). \qquad (19)$$

While the right-hand side of the entropic uncertainty relation in eq. (14) reduces to zero in this case, the left-hand side can be simplified (by substituting eq. (19) in eq. (13) and simplifying) to obtain

$$H(X \mid X') + H(Z \mid Z') = -\sum_{x, x' = \pm 1} p(x, x') \log_2 p(x \mid x')$$

$$- \sum_{z, z' = \pm 1} p(z, z') \log_2 p(z \mid z')$$

$$= 2H[(1 + \eta)/2], \qquad (20)$$

where $H(p) = -p \log_2 p - (1-p) \log_2(1-p)$ is the Shannon binary entropy $0 \le H(p) \le 1$. Noting that the binary entropy function $H[(1 + \eta)/2] = 0$ only if $\eta = 1$, the trivial bound zero of the uncertainty relation in eq. (14) can be achieved only when Bob too performs *sharp* PV measurements of the observables σ_x and σ_z at his end, in which case Bob can predict Alice's outcomes *precisely*. Reduction in the uncertainty bound, below $-2\log_2 \mathcal{C}(\mathbb{E}_{\sigma_x}, \mathbb{E}_{\sigma_z}) = 1$, can only be realized if $H[(1 + \eta)/2] \le 0.5$, i.e. for $\eta > 0.78$.

It may be recalled that unsharp joint measurements of the observables σ_x, σ_z (i.e. compatibility of the POVMs $\mathbb{E}_{\sigma_x}, \mathbb{E}_{\sigma_z}$) places the restriction $\eta \le 1/\sqrt{2} \approx 0.707$ on the *unsharpness* parameter[3,7]. If Bob confines only to the joint measurability range $0 \le \eta \le 1/\sqrt{2}$, the entropic steering inequality (see eq. (15))

$$H(X \mid X') + H(Z \mid Z') \ge 1 \qquad (21)$$

is always satisfied. Bob's inability to steer Alice's state remotely (to be able to violate the steering inequality (eq. (21)) translates itself into his inability to predict Alice's outcomes with enhanced precision (i.e. to beat the entropic uncertainty bound below 1), when he is restricted to employ compatible measurements – irrespective of the fact that he shares an entangled state with Alice.

Moment matrix positivity and measurement incompatibility

Foundational conflicts about the quantum–classical worldviews of nature arise due to strikingly different statistical features in the two domains. Pioneering investigations by Bell[25], Kochen and Specker[20] and Leggett and Garg[31] brought out the puzzling features of probabilities of measurement outcomes, arising within the quantum framework in terms of various no-go theorems. A common underlying feature that gets highlighted in these no-go theorems is *the non-existence of a joint probability distribution for the measurement outcomes of all the observables in the quantum framework*[32–35].

From an entirely different perspective, *the classical moment problem*[36,37] addressed the issue of determining the probability distribution, given a sequence of valid statistical moments. The classical moment problem identifies that a given set of real numbers qualifies to be a legitimate moment sequence of a probability distribution,

if and only if the associated moment matrix is positive. In other words, *existence of a valid joint probability distribution*, consistent with the set of moments, may be put to test in terms of the moment matrix positivity[34,35]. In the present context, we focus our attention on the role of incompatibile measurements on the positivity of the moment matrix, constructed based on the statistics of fuzzy measurements of a set of three non-commuting dichotomic observables.

Let X_k, $k = 1, 2, 3$ denote the dichotomic random variables with outcomes $x_k = \pm 1$. Consider a row with four elements $\xi^T = (1, x_1 x_2, x_2 x_3, x_1 x_3)$. The average pairwise correlations $\langle X_k X_l \rangle = \sum_{x_k, x_l = \pm 1} p(x_k, x_l) x_k x_l$, $k, l = 1, 2, 3$ of the random variables X_k can be used to construct a 4×4 moment matrix $M = \langle \xi \xi^T \rangle$

$$M = \begin{pmatrix} 1 & \langle X_1 X_2 \rangle & \langle X_2 X_3 \rangle & \langle X_1 X_3 \rangle \\ \langle X_1 X_2 \rangle & 1 & \langle X_1 X_3 \rangle & \langle X_2 X_3 \rangle \\ \langle X_2 X_3 \rangle & \langle X_1 X_3 \rangle & 1 & \langle X_1 X_2 \rangle \\ \langle X_1 X_3 \rangle & \langle X_2 X_3 \rangle & \langle X_1 X_2 \rangle & 1 \end{pmatrix}. \tag{22}$$

Note that in order to obtain the diagonal elements of M, we find that

$$\langle X_k^2 X_l^2 \rangle = \sum_{x_k, x_l = \pm 1} p(x_k, x_l) x_k^2 x_l^2 = \sum_{x_k, x_l = \pm 1} p(x_k, x_l) = 1.$$

Further, the off-diagonal elements are identified as follows: we have

$$M_{23} = \langle X_1 X_2^2 X_3 \rangle = \sum_{x_1, x_2, x_3 = \pm 1} p(x_1, x_2, x_3) x_1 x_2^2 x_3$$

$$= \sum_{x_1, x_3 = \pm 1} p(x_1, x_3) x_1 x_3 = \langle X_1 X_3 \rangle$$

and so on, which leads to the above structure in eq. (22) for the moment matrix M involving only average pairwise correlations of the variables.

In the construction of M, it is implicit that a joint probability distribution $p(x_1, x_2, x_3)$ for the statistical outcomes of three random variables X_1, X_2, X_3 exists, and pairwise probabilities $p(x_1, x_2)$, $p(x_2, x_3)$, $p(x_1, x_3)$ are obtained as marginal distributions, i.e.

$$\sum_{x_3} p(x_1, x_2, x_3) = p(x_1 x_2), \quad \sum_{x_1} p(x_1, x_2, x_3) = p(x_2 x_3),$$

$$\sum_{x_2} p(x_1, x_2, x_3) = p(x_1 x_3).$$

By construction, the moment matrix M is symmetric and positive definite. Thus, all the four eigenvalues of M are positive, i.e.

$$1 + \langle X_1 X_2 \rangle - \langle X_2 X_3 \rangle - \langle X_1 X_3 \rangle \geq 0, \tag{23}$$

$$1 - \langle X_1 X_2 \rangle + \langle X_2 X_3 \rangle - \langle X_1 X_3 \rangle \geq 0, \tag{24}$$

$$1 - \langle X_1 X_2 \rangle - \langle X_2 X_3 \rangle + \langle X_1 X_3 \rangle \geq 0, \tag{25}$$

$$1 - \langle X_1 X_2 \rangle + \langle X_2 X_3 \rangle + \langle X_1 X_3 \rangle \geq 0. \tag{26}$$

Note that the positivity condition in eq. (25) on one of the eigenvalues of the moment matrix directly corresponds to the three-term Leggett–Garg inequality[31,38]

$$\langle X_1 X_2 \rangle + \langle X_2 X_3 \rangle - \langle X_1 X_3 \rangle \leq 1. \tag{27}$$

It has been shown[31,38] that sequential measurements of a set of three dichotomic quantum observables, with possible outcomes ± 1, violate the inequality in eq. (27) – as the average quantum pairwise correlations on the left-hand side of eq. (27) can sum up to a maximum value 3/2. The maximal violation of the three-term Leggett–Garg inequality can be realized from the statistics of outcomes in the sequential PV measurements of three dichotomic qubit observables $\vec{\sigma} \cdot \hat{n}_k$, with the unit vectors $\hat{n}_k$, $k = 1, 2, 3$ equally separated in a plane by an angle 120°, in the completely mixed initial state $\rho = 1/2$ of a qubit. In other words, the moment matrix constructed based on the results of sharp PV sequential measurements of three dichotomic qubit observables $\vec{\sigma} \cdot \hat{n}_k$ could turn out to be non-positive[35] – and hence, points towards the non-existence of a joint probability distribution $p(x_1, x_2, x_3)$ for the outcomes.

In the following, we consider unsharp measurements of the trine axes observables $\vec{\sigma} \cdot \hat{n}_k$ using the POVMs $\mathbb{E}_{\vec{\sigma} \cdot \hat{n}_k} = \{E_{\vec{\sigma} \cdot \hat{n}_k}(x_k = \pm 1)\}$, where the effects $E_{\vec{\sigma} \cdot \hat{n}_k}(x_k = \pm 1)$ are given by,

$$E_{\vec{\sigma} \cdot \hat{n}_k}(x_k = \pm 1) = \frac{1}{2}(1 + \eta x_k \vec{\sigma} \cdot \hat{n}_k). \tag{28}$$

Our intention is to obtain the range of the unsharpness parameter η for which the 4×4 moment matrix – constructed from the average pairwise correlations $\langle X_k X_l \rangle$ of the three dichotomic Pauli observables – is positive.

We consider a maximally mixed qubit state $\rho = 1/2$ and perform sequential unsharp–sharp pairwise measurements of the observables $\vec{\sigma} \cdot \hat{n}_k$, $k = 1, 2, 3$. Suppose that the first unsharp measurement of $\mathbb{E}_{\vec{\sigma} \cdot \hat{n}_k}$ gives an outcome x_k. The initial quantum state transforms to

$$\rho \rightarrow \rho_k = \sqrt{E_{\vec{\sigma} \cdot \hat{n}_k}(x_k)} \rho \sqrt{E_{\vec{\sigma} \cdot \hat{n}_k}(x_k)} / p(x_k)$$

$$= E_{\vec{\sigma} \cdot \hat{n}_k}(x_k) / p(x_k), \tag{29}$$

with probability $p(x_k) = \mathrm{Tr}[E_{\vec{\sigma} \cdot \hat{n}_k}(x_k)]/2 = 1/2$. A subsequent sharp measurement of $\vec{\sigma} \cdot \hat{n}_l$, yielding an outcome x_l, results in the post-measured state $\Pi_{\vec{\sigma} \cdot \hat{n}_k(x_l)} E_{\vec{\sigma} \cdot \hat{n}_k}(x_k)$ $\Pi_{\vec{\sigma} \cdot \hat{n}_k(x_l)}/p(x_l \mid x_k)$ with the probability $p(x_l|x_k)$ of obtaining the sharp outcome x_l for $\vec{\sigma} \cdot \hat{n}_l$, given that the first unsharp measurement of $E_{\vec{\sigma} \cdot \hat{n}_k}$ has resulted in the outcome x_k given by

$$
\begin{aligned}
p(x_l \mid x_k) &= \mathrm{Tr}[E_{\vec{\sigma} \cdot \hat{n}_k}(x_k)\Pi_{\vec{\sigma} \cdot \hat{n}_k(x_l)}] \\
&= \frac{1}{2}(1 + \eta x_k x_l \hat{n}_k \cdot \hat{n}_l) \\
&= \frac{1}{2}\left(1 - \frac{\eta}{2} x_k x_l\right).
\end{aligned} \tag{30}
$$

The pairwise joint probabilities $p(x_k, x_l)$ of the sequential measurement are then evaluated as

$$
\begin{aligned}
p(x_l, x_k) &= p(x_k)p(x_l \mid x_k) \\
&= \frac{1}{4}\left(1 - \frac{\eta}{2} x_k x_l\right).
\end{aligned} \tag{31}
$$

Using the above joint probabilities, the average pairwise correlations $\langle X_k^{(\mathrm{u})} X_l^{(\mathrm{s})} \rangle$, $k < l = 1, 2, 3$ of the unsharp–sharp sequential measurements are evaluated to obtain

$$
\langle X_1^{(\mathrm{u})} X_2^{(\mathrm{s})} \rangle = \sum_{x_1, x_2} p(x_1, x_2)x_1 x_2 = -\eta/2,
$$

$$
\langle X_2^{(\mathrm{u})} X_3^{(\mathrm{s})} \rangle = \sum_{x_2, x_3} p(x_2, x_3)x_2 x_3 = -\eta/2,
$$

$$
\langle X_1^{(\mathrm{u})} X_3^{(\mathrm{s})} \rangle = \sum_{x_1, x_3} p(x_1, x_3)x_1 x_3 = -\eta/2. \tag{32}
$$

The corresponding 4×4 moment matrix (see eq. (22)) is then given by

$$
M = \begin{pmatrix} 1 & -\eta/2 & -\eta/2 & -\eta/2 \\ -\eta/2 & 1 & -\eta/2 & -\eta/2 \\ -\eta/2 & -\eta/2 & 1 & -\eta/2 \\ -\eta/2 & -\eta/2 & -\eta/2 & 1 \end{pmatrix}. \tag{33}
$$

The eigenvalues of the moment matrix are readily found to be $\lambda_1 = (2 + \eta)/2 = \lambda_2 = \lambda_3$ and $\lambda_4 = (2 - 3\eta)/2$. Positivity of the moment matrix implies that $\eta \leq 2/3$, which matches exactly with the range of the unsharpness parameter over which the POVMs $\mathbb{E}_{\vec{\sigma} \cdot \hat{n}_k}$, $k = 1, 2, 3$ for the trine axes $\hat{n}_k$ are compatible[10]. We thus obtain the result that moment matrix positivity and joint measurability of the observables are synonymous.

Conclusion

In the classical framework, physical observables are all compatible and they can be measured jointly. In contrast, measurements of observables, which do not commute, are declared to be incompatible in the quantum scenario. The notion of compatibility of measurements is synonymous with commutativity of the observables only when one restricts to PV measurements. A broader notion of compatibility emerged in a generalized framework of unsharp measurements using POVMs. In the generalized measurement theory, joint measurability of two or more POVMs does not, in general, require their commutativity, but it necessarily requires the existence of a grand POVM, measurement of which suffices to construct the results of measurements of the set of compatible (jointly measurable) POVMs. In this article we have reviewed the notion joint measurability of POVMs. We have also given a detailed discussion on the importance of incompatible measurements, to be employed by Bob at his end, so as to beat the entropic uncertainty bound for a pair of non-commuting observables of Alice's spatially separated quantum system, entangled with Bob's state. Further, we have brought out the connection between measurement compatibility and positivity of a moment matrix, by considering a specific example of sequential unsharp–sharp measurements of pairs of qubit observables $\vec{\sigma} \cdot \hat{n}_k$; $k = 1, 2, 3$, where the three unit vectors $\hat{n}_1, \hat{n}_2, \hat{n}_3$ are equally separated in a plane, making an angle 120°. Specifically, we have shown that the moment matrix is positive if and only if the unsharpness parameter $\eta \leq 2/3$, which coincides exactly with that for the joint measurability of all the three qubit observables $\vec{\sigma} \cdot \hat{n}_k$; $k = 1, 2, 3$. Our example indicates that positivity of the moment matrix, existence of joint probabilities and compatibility of POVMs are all equivalent notions.

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