Gravitational Waves – A New Astronomical Tool

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Gravitational waves have been detected in the centenary year of Einstein's formulation of general theory of relativity that predicts these waves. It has been heralded as a landmark discovery, since it not only confirms various predictions of the theory, but also opens up a new window of observing the universe. I will discuss a few basic characteristics of gravitational waves in this article, and a few astrophysical implications.

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1. Introduction

Gravitational waves are oscillations in the fabric of space that propagate outward from a source. They are travelling waves of space curvature. We know that according to Einstein's theory, gravity manifests itself as a distortion in the geometry of space around a massive object. This deformation can also travel through space, as gravitational waves. Such waves cannot exist in Newtonian theory, because there gravitation is regarded as a force that establishes instantaneously throughout space. The gravitational field of an object depends on the position, but not on time. On the contrary, in Einstein's relativity, space and time are treated on a similar footing, and therefore the gravitational field requires time to establish, and any change in it sends a ripple through space.

In electromagnetic waves, the changing electric field produces a magnetic field, and the change in (production of) the magnetic field produces on electric field. These two fields feed one another and a wave propagates. In the Newtonian theory, there is no difference between the gravitational field of a rotating and a non-rotating massive object. This is analogous to the electric field of a

Keywords
General theory of relativity, gravitational waves.

charged object. However, in Einstein's theory, movement of a massive object also has a gravitational effect, analogous to the magnetic field of a current. In other words, the gravitational field near a rotating and a non-rotating star would be different. These two aspects of Einstein's theory of gravity can be expressed in a set of equations like Maxwell's equations for electromagnetism. And these two 'types' of gravitational field play off one another and create ripples through space, just like in the case of the electromagnetic wave.

2. Sources of Gravitational Radiation

Let us consider a few general aspects of radiation. Firstly, radiation requires some kind of time variation of the source. Secondly, the amplitude of oscillation should not decrease rapidly with distance. Suppose the amplitude of the field at a distance r from the source is given by $A(r)$ (in the spherically symmetric case). Therefore, the flux through a spherical surface at a distance r will be $A^2(r)$. Recall that, in the electrostatic or magnetostatic case, the field $A(r) \propto 1/r^2$, and the luminosity falls rapidly like $1/r^2$, and no energy is carried to infinity. In order for radiation to carry energy far away, one needs $A(r) \propto 1/r$.

What kind of sources produce radiation? In the electromagnetic case, if we expand the radiation field in moments, the first term is the monopole radiation, due to the total charge of the source. But this cannot vary. We could have radiation from a source that created or destroyed charge periodically, but that is not allowed by charge conservation. So there is no monopole radiation. The same holds for gravitational radiation, because of mass-energy conservation.

The next moment is the dipole moment, $P = \Sigma Q_i r_i$. Now, this is not a conserved quantity – we can change it by freely moving a charge around, or separating pairs of positive or negative charges. So a dipole radiation is possible. However, in gravity, the dipole moment of mass-energy density $P = \Sigma m_i r_i$. This is a conserved quantity, because of Newton's third law or the conservation of momentum for an isolated system.

Therefore, gravitational waves can be radiated by the quadrupole moments[1], which has the dimensions of mass $\times$ length^2. We can draw a few general conclusions based on this discussions. Firstly, a spherically symmetric variation cannot produce gravitational radiation, however large an explosion or collapse it may be – there has to be some asymmetry in it. Also, a rotation that is axisymmetric (like a totally smooth neutron star rotating) also cannot radiate.

[1] For a mass density distribution $\rho(\mathbf{r})$, it is defined as $I_{ij} = \int r(\mathbf{r}) r_i r_j d^3 r$.

3. Nature and Effects of Gravitational Waves

Let us now try to understand the nature of gravitational radiation. Recall that in special relativity, the space–time interval between two events is an invariant quantity in all frames of reference. For two infinitesimally separated events, the interval is written in the differential form as,

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \,. \qquad (1)$$

In shorthand, we can write this as (what is called the Minkowski metric) $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, where the summation over the indices μ and ν are implied and both indices range over $0, 1, 2, 3$, corresponding to t, x, y, z. Here $\eta_{\mu\nu}$ is given by

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

This implies an Euclidean geometry. In general relativity, we allow the space to have a non-Euclidean geometry, and write the metric as $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, where $g_{\mu\nu}$ would depend upon the distribution of mass. Einstein's equation connects the distribution of mass (energy) with $g_{\mu\nu}$. The idea is to find the values of $g_{\mu\nu}$ that describe a particular physical situation. This metric carries all the information about the geometry of space, and therefore, embodies gravitational effects.

We will only consider the case of a weak gravitational field whose metric is almost flat, where

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad h_{\mu\nu} \ll 1\,. \tag{2}$$

In this approximation, one can linearize Einstein's equations. The perturbed gravitational field then obeys the wave equation in free space,

$$\Big(-\frac{\partial^2}{\partial t^2} + \nabla^2\Big) h_{\mu\nu} = 0\,. \tag{3}$$

We can easily recognize the wave equation here, and can assume solutions of the type of a plane wave,

$$h_{\mu\nu} = A_{\mu\nu} \exp(i k_\alpha x^\alpha)\,. \tag{4}$$

In writing the wave equation in this manner, a particular gauge has been chosen. Recall that we choose the Lorentz gauge, $\partial^\mu A_\mu = 0$, in simplifying the electromagnetic wave equation. Similarly, there is a particular coordinate system in which the equation for gravitational waves looks simple as shown above in (3). This is called the transverse-traceless gauge. In this system, one can show that $A_{\mu\nu} k^\nu = 0$, which means that the wave propagates perpendicular to the direction of the oscillations, and so it is a transverse wave like electromagnetic waves. Also, one can choose the tensor $A_{\mu\nu}$ to be traceless, which would mean that the oscillations preserve the surface area. This coordinate system can be thought of as being made by free-falling test particles. Why do we need to choose a gauge like this? By doing this, we basically do away with any *waving of the coordinates*, which is not a physical effect, since coordinates are human constructs. So, once we get onto this

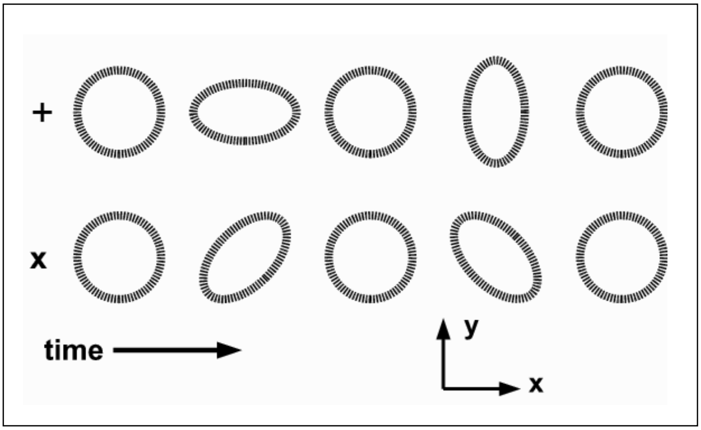

Figure 1. The two polarisations of gravitational waves.

coordinate system of free-falling particles, what is left is the waving of the curvature of space itself.

The final result of this exercise is that $A_{\mu\nu}$ (or equivalently $h_{\mu\nu}$) depends on two parameters:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

This means that there are two polarisations, or two modes in which gravitational waves propagate. Consider a ring of particles in the xy plane, while the wave propagates in the z direction. If we put $A_{xy} = 0$, then the result of the travelling wave will be to alternately stretch and compress in the x and y directions. This is called + (plus) polarisation state. If instead $A_{xx} = 0$, then the ring is stretched and compressed along the two 45° lines. This is called the × (cross) polarisation state (see *Figure* 1).

4. Detection of Gravitational Waves

How do we detect gravitational waves? In the case of electromagnetic waves, we could use an antenna, in which the passive wave would move the electrons periodically and produce a current that we can measure. But in the case of a wave in the curvature of space itself, accelerating a mass particle will not help. In general relativity, any acceleration can be made to vanish if one gets into the local free-falling frame. This is the equivalence principle. Therefore, local acceleration cannot be measured. However, since the gravitational wave produces a field that is a function of space and time, two neighbouring particles would be accelerated in a different manner, and this would allow the detection of a wave. In other words, one detects a tidal field. J Wheeler gave an analogy in his book *Spacetime Physics*: "A cork floating all alone on the Pacific Ocean may not reveal the passage of a wave. But when a second cork is floating near it, then the passing of the wave is revealed by the fluctuating separation between the two corks." (See *Box* 1).

Consider the plus polarisation state (see *Figure* 2). The distortion is maximised when one considers the distances in x and y directions. In one direction, the distance will decrease, while it will increase in the other direction,

Box 1.

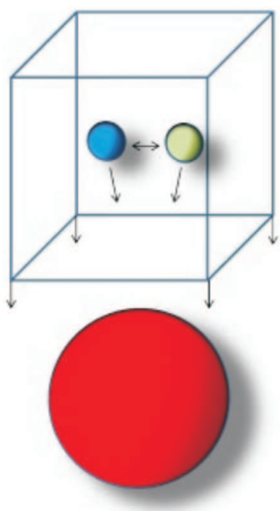

Figure A.

A thought experiment with Einstein's elevator can illustrate the importance of tidal field in gravity. An observer in an elevator cannot tell if he is floating in space, far from all sources of gravity, or if the elevator is falling in a gravitational field. This is because all objects fall in a gravitational field with the same acceleration, and so inside a closed elevator, there is no way of telling if the elevator is falling in a gravitational field or is floating far away from one. However, if the gravitational field is *not uniform*, then there will be a way to tell the difference. If the elevator is very large, then two balls inside an elevator that is freely falling in Earth's gravitational field, will appear to move closer to one another (*Figure* A). This is due to the *difference* in gravitational field at two places, or, what is called the tidal field. (Tides in the oceans occur due to the difference in gravitational field of the Moon at different points on Earth.) Therefore, tidal force is the true measure of gravitational field.

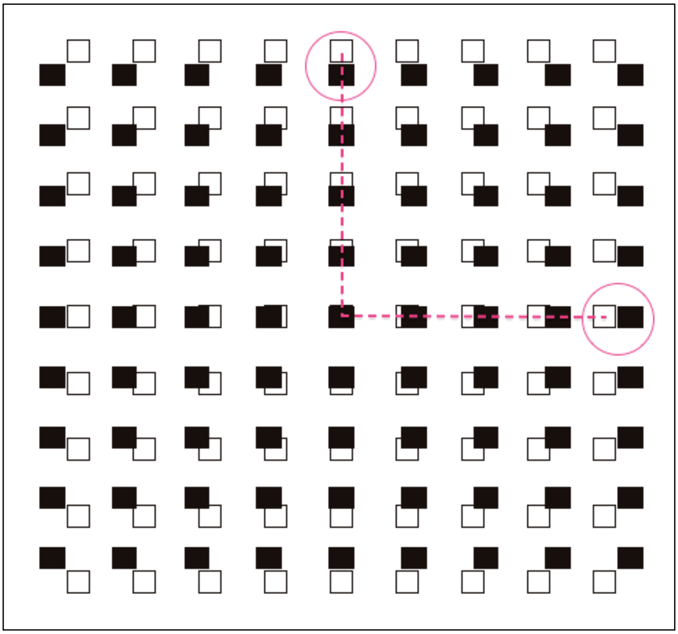

Figure 2. The black (and white) squares represent test particles before (and after) a gravitational wave with plus polarisation passes. The particles which can detect the maximum deviation are marked by red circles. The dashed lines show the positions of the interferometer arms.

and then half a cycle later, the directions of stretching and compression will reverse. Now, as the wave passes, the coordinate system also changes! In other words, the coordinate distance between the test particles also changes. That was the whole idea of choosing the transverse-traceless gauge, of a coordinate system defined by free-falling masses. How then can we ever hope to detect the change?

However, even if the space curvature changes, what remains constant is the speed of light. If one bounces off light between two mirrors in the x-arm, we can estimate the change in the *proper distance* between the mirrors as the wave passes. The space-time interval between the two events of light travelling from one mirror to the other, is (zero, since we are considering the path taken by light)

$$ds^2 = -c^2dt^2 + (1 + h_{xx})dx^2 = 0\,. \qquad (5)$$

Assume that h_{xx} remains constant during the light's travel through the arm. Rearranging, and replacing the square root with the first two terms of the binomial expansion, and integrating from $x = 0$, to $x = L$, we get

$$\Delta t = \frac{h_{xx}L}{2c}\,. \tag{6}$$

The round-trip along x and y arms (i.e, doing this calculation four times), takes a proper time interval,

$$\Delta \tau = \frac{2h_{xx}L}{c}\,. \tag{7}$$

This introduces a phase shift in light, which can be detected by looking at the fringe pattern of the interferometer.

How large can this change be in the curvature of space? Recall that the deviation h has to fall as $1/r$ and is proportional to the quadrupole moment, which has the dimensions of MR^2. But h is dimensionless.

In general relativity, one often uses the geometric or natural unit system, in which the fundamental constants that appear in it, such as G and c, are equated to unity (and made dimensionless), and all physical parameters such as mass, length and time are identified with geometric quantities such as length, or area. If we put $G = c = 1$, we can express mass as some sort of length by multiplying it by G/c^2. In this unit system, mass, length and time all have the same effective 'unit'. This system is quite useful in separating the physical parameters from the rest, and so, in order to make h dimensionless, we can proceed in two steps: first make it independent of all physical parameters, and then make it totally dimensionless.

The quantity $h \propto MR^2/r$ has the dimensions of mass (or length, or time) squared. So, we need to divide it by mass (or length, or time) squared. We know that there should be a time derivative somewhere, because the source has to change with time, in order to produce waves. Therefore, two time derivatives can make h dimensionless in the GR units, and we can write,

$$h \sim \frac{\partial^2 (MR^2)}{\partial t^2} \frac{1}{r} \,. \tag{8}$$

Now, we can put in the factors of G and c carefully so that we can write the expression in traditional units. We know that GM/c^3 has the dimensions of time, so we have, approximately,

$$h \sim \frac{G}{c^4 r} \frac{\partial^2 (MR^2)}{\partial t^2} \sim \frac{GMv^2}{c^4 r} \,. \tag{9}$$

We can estimate the change in the case of a binary black hole system. If the total mass is $M \approx 60\ \mathrm{M}_\odot$, situated at a distance of $r \sim 400$ Mpc, and spiralling around one another with a speed of $v \sim 0.5c$, then we have,

$$h \sim 10^{-21} \left(\frac{M}{60\, M_\odot}\right) \left(\frac{v}{0.5c}\right)^2 \left(\frac{400\,\mathrm{Mpc}}{r}\right) . \tag{10}$$

This is the magnitude of the amplitude that has been recorded by LIGO. Clearly, it helps to have massive black holes that are moving with a large velocity.

The power radiated in these waves is, however, not small. Recall the Poynting vector in the case of electromagnetic waves, which is proportional to $E^2 \propto (\partial A/\partial t)^2$ (where A is the vector potential). One then expects the energy flux in gravitational waves to also scale as $(\partial h/\partial t)^2$. One can think of $(\partial/\partial t) \propto f$, the frequency of radiation. Note that because of the quadrupolar nature of gravitational waves, the frequency of the waves is *twice* the orbital frequency of such a binary system. (In the electromagnetic case, the frequency of the emitted radiation from a rotating dipole is the same as the frequency of rotation.) Therefore the flux is $\propto h^2 f^2$. This should be multiplied by combinations of G and c in order to give us the traditional units of energy flux, and one gets,

$$F \approx \frac{c^3}{G} h^2 f^2 \,. \tag{11}$$

Consider a binary system of compact stars. As the binary system rotates, it emits gravitational waves and loses energy and angular momentum, consequently shrinking in size. As the semi-major axis decreases, the orbital frequency increases. This phenomenon was confirmed in the case of a binary system of neutron stars, one of which is a pulsar. Joseph Taylor, Jr. and Russell Hulse noticed the shrinking size and increasing orbital frequency of the binary pulsar system 1913+16. The shrinking of the orbit and the change in the period can be predicted from the knowledge of how much energy is lost in the form of gravitational waves. And the observed value was in excellent agreement with the predictions, and was an indirect proof of the existence of gravitational waves. For this, they were awarded the Nobel Prize in physics in 1993.

As the binary system shrinks, it will coalesce after a certain time. The loss of energy through gravitational waves will keep increasing as the binary system collapses. The detection of gravitational waves of 14 September 2015 in LIGO came from a system of binary black holes in its final throes of coalescence. For a binary black hole system of total mass M, when the (event horizons of the) black holes are separated by roughly ten times their Schwarzschild radii[2], the time scale of rotation is $T \sim \sqrt{R^3/GM}$, where we can use $R \sim 10R_{\rm sch}(= 2GM/c^2)$. The corresponding frequency is $\sim 2\pi/T \sim 10^2$ Hz, for a total mass of 60 $M_\odot$. With $h \sim 10^{-21}$ and $f \sim 10^2$ Hz, the energy flux is a few erg cm^{-2} s^{-1}. Compare this to the luminosity of Sirius, the brightest star in the sky, which is about 10^{-4} erg cm^{-2} s^{-1}. In other words, if we could 'see' the gravitational waves, the event would have blinded us with its brightness!

[2] Schwarzschild radius of a black hole is the radius of the event horizon, where the escape speed equals that of light.

The total power radiated by gravitational waves for a source with quadrupole moment of MR^2, and which is changing with a time scale T is roughly,

$$L_{\rm GW} = \frac{G}{5c^5}\Big(\frac{\partial^3 MR^2}{\partial t^3}\Big)^2 \approx \frac{G}{5c^5}\Big(MR^2/T^3\Big)^2 . \qquad (12)$$

The time scale is roughly $1/\sqrt{G\rho}$, as is the case for any system whose dynamics is determined by gravity. (Recall the textbook problem of a ball dropped into a tunnel through the centre of the earth, which executes harmonic oscillation with a time period of roughly $1/\sqrt{G\rho}$.) We can write this as $\sqrt{R^3/GM}$. Remembering the Schwarzschild radius $R_{\rm sch} = 2GM/c^2$, and writing $v = R/T$, we can write,

$$L_{\rm GW} = \frac{G}{5c^5}\Big(\frac{M}{R}\Big)^2 v^6 \sim \frac{c^5}{G}\Big(\frac{R_{\rm sch}}{R}\Big)^2\Big(\frac{v}{c}\Big)^6 . \qquad (13)$$

Clearly, the maximum luminosity in gravitational waves can be achieved if the size of the source is close to its Schwarzschild radius, and the typical speed with which parts of the source are moving is close to the speed of light. Therefore, the best sources of gravitational waves are highly relativistic compact objects. The maximum limit to the power radiated is given by $G/5c^5 \sim 3.6\times10^{59}$ erg s^{-1}, which is a very large quantity indeed. The event of 14 September 2015 corresponded to $R \sim 10R_{\rm sch}$ and $v \sim 0.5c$, and the luminosity was $\sim 4 \times 10^{56}$ erg s^{-1}, almost 50 times the luminosity of all the stars in the universe.

5. New Astronomy

The detection of gravitational waves has now opened up a new vista of the universe. Binary black holes of masses as large as 30 $M_\odot$ opens up the questions of their origin. Black holes are supposed to form as a remnant of massive stars, after they explode as supernova explosions when they run out of fuel for nuclear reactions. A black hole of mass $\sim$ 30 $M_\odot$ would have had a progenitor which was more massive. However, there is a problem in such a scenario. Massive stars are also very luminous, and lose a lot of mass by stellar wind, when the gas in the outer surface is pushed out by radiation pressure of the starlight. This is facilitated by the presence of ions of heavy elements, which absorb starlight of certain frequencies (corresponding to the energy level differences in these ions) and get pushed by the process of absorption. Therefore, a very massive star is likely to lose a lot of its mass and is unlikely to leave a remnant behind as massive as $\sim 30\ M_\odot$. This may happen if the abundance of heavy elements happens to be small for some reason. This then opens up the questions of where and in which cases can it happen.

The advantage of gravitational waves as an astronomical tool is that unlike electromagnetic waves, they are not absorbed by intervening medium. Light waves are attenuated by their interactions with matter, and astronomers are handicapped by the presence of gas and dust particles in space. The disadvantage is that the deviation caused by gravitational waves is so tiny. Even when a large amount of energy is radiated away in the form of gravitational waves, the value of h is minuscule (one way to think of it is to say that space is 'stiff' and acts like a tough rubber.) The recent detection has shown that scientists have been able to overcome this barrier to a large extent.

Whenever a new mode of observation has become possible in astronomy, it has revealed new aspects of the universe. From cosmic rays, or neutrinos, or observations in X-ray or infrared or radio wavelengths, new and often serendipitous discoveries have challenged the existing ideas of the universe. Gravitational wave astronomy will not be an exception.

Suggested Reading

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