Third post-Newtonian gravitational waveforms for compact binary systems in general orbits: Instantaneous terms

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We compute the *instantaneous* contributions to the spherical harmonic modes of gravitational waveforms from compact binary systems in general orbits up to the third post-Newtonian (PN) order. We further extend these results for compact binaries in quasielliptical orbits using the 3PN quasi-Keplerian representation of the conserved dynamics of compact binaries in eccentric orbits. Using the multipolar post-Minkowskian formalism, starting from the different mass and current-type multipole moments, we compute the spin-weighted spherical harmonic decomposition of the instantaneous part of the gravitational waveform. These are terms which are functions of the retarded time and do not depend on the history of the binary evolution. Together with the *hereditary* part, which depends on the binary's dynamical history, these waveforms form the basis for construction of accurate templates for the detection of gravitational wave signals from binaries moving in quasielliptical orbits.

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I. INTRODUCTION

Compact binary systems composed of neutron stars and/or black holes (BHs) are one of the most promising sources for the second generation of earth-bound gravitational-wave (GW) detectors such as Advanced LIGO [1] and Advanced Virgo [2] as well as for the proposed space-based detector eLISA [3]. Detection of such systems in GW detectors relies on a data analysis technique known as *matched filtering* which in turn requires very accurate modeling of GW signals from these sources [4]. The compact binaries are known to have significant eccentricities when they are formed. However, since the GW radiation reaction effects tend to circularize the binary's orbit [5,6], for most long-lived binary systems one can expect that their orbits would have circularized by the time they enter the sensitivity band of ground-based detectors. This has motivated the GW community to perform searches of GW signals from coalescing compact binary systems (CCBs) using circular orbit templates.

Many astrophysical scenarios have been proposed which suggest the possible existence of close eccentric binary systems.[1] One such scenario may exist in the cores of dense globular clusters due to a mechanism known as the "Kozai mechanism" [7]. This mechanism can also come into effect in scenarios involving formation of hierarchical triples of supermassive black holes due to subsequent mergers of galaxies [8]. Another scenario might involve formation of *close* eccentric compact binary systems in dense stellar systems like globular clusters [9]. Compact binaries involving intermediate mass black holes in globular clusters might be seen in the eLISA band with residual eccentricities of $0.1 \lesssim e \lesssim 0.2$ [10]. Other scenarios involve formation of close eccentric compact binary systems at centers of galaxies [11] and NS-BH binary systems which can become eccentric as a consequence of multistage mass transfer from the NS to the BH [12]. In light of these possibilities, it becomes necessary to compute accurate waveforms accounting for the eccentricity of the binary's orbit.

A number of investigations concerning the sensitivity of searches using circular orbit templates to detect eccentric binary systems have been performed in the past. The first such investigation was presented in Ref. [13] where the authors studied the loss of signal-to-noise ratio in detecting signals from binaries with residual eccentricities using template banks constructed with quasicircular waveform models with leading-order effects (both *conservative* and *secular*). They argued that even if the system has a residual eccentricity ($e_0 \lesssim 0.13$ for binary system with two $1.4M_\odot$ neutron stars or $e_0 \lesssim 0.3$ for a binary with two $6M_\odot$ black holes), use of circular orbit templates will be sufficient to detect signals from such systems.[2] However, this result has

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[1] Eccentric binary systems, with small orbital separations, formed through the capture process. Unlike the long-lived CCBs such systems are expected to enter the detector with nonzero eccentricities.

[2] They chose a lower cutoff for the fitting factor (FF| min = 90%) corresponding to a loss in event rates of about less than 27%.

been subsequently weakened due to two independent investigations [14,15]. Both of the investigations suggest that if the eccentricity of the binary when it enters the sensitivity band of detector is greater than 0.1, then it will not be possible to detect such systems using circular orbit templates. These investigations only dealt with sources that will be seen in ground-based detectors. However, the capabilities of circular waveforms to detect signals from the coalescence of supermassive black holes (visible in the eLISA frequency band) have been investigated in [16]. The results presented in Ref. [16] suggest that even to search signals from sources with initial eccentricities of the order 10^{-4} one would need waveforms which accurately account for the effects of eccentricities. In addition, in a recent work, Huerta and Brown [17] showed that searches for CCBs with eccentricity ≥ 0.05 would require eccentric template banks to avoid significant loss in the sensitivity of the search. Lastly, systematic errors due to the orbital eccentricity in measuring the source parameters of double NS systems was investigated recently by Favata [18] which again indicated the necessity to incorporate the effects of eccentricity to measure the parameters of a double NS system if it has non-negligible eccentricity when detected.

Evolution of a compact binary system can be divided into three stages: the early inspiral, late inspiral and merger and the final ringdown. The early inspiral phase can be very well modeled using the approximation schemes such as multipolar post-Minkowskian (MPM) approximation matched to post-Newtonian(PN) [19] whereas the late inspiral, merger and ringdown phases can be modeled using numerical relativity (NR) [20] or effective one body approach [21]. In fact, it is now possible to perform numerical simulations to track the evolution of the BH binary systems over many inspiral orbits and the subsequent merger and ringdown phases. However, *high* computational cost of generating numerical waveforms covering the entire parameter space of coalescing binary black holes (BBHs) has led to the construction of hybrid waveforms (by combining PN and NR waveforms), which further are used to phenomenologically construct a waveform model which has sufficient overlap with the hybrid waveform [22–27]. In addition to this, one needs to check the consistency between these two waveforms (PN\NR) in a regime where both of them are valid. This would not only tell us about the compatibility of the two waveforms but also would indicate the limits up to which PN waveforms are reliable. There have been many such investigations involving nonspinning or nonprecessing BBH in quasicircular orbits [28–38] and quasieccentric orbits [39]. The need for such comparisons and matching of the two waveforms (PN and NR) has led to the high accuracy computations of spherical harmonic modes of the PN waveforms in case of CCBs moving in quasicircular orbits [40–42]. Evidently, in order to perform similar comparisons for eccentric binaries, one would need high accuracy eccentric PN waveforms for such systems.

The leading order (or Newtonian) expressions for the GW polarizations (h_+ and $h_\times$) were obtained in the context of spacecraft Doppler detection of GWs from an isolated compact binary in an eccentric orbit [43]. This work was then extended to 1PN and the next 1.5PN order in [44–48]. At the 2PN order, the transverse-traceless radiation field (h^{TT}_{ij}) due to an isolated binary composed of two compact stars moving in eccentric orbits was computed in [49,50]. Although the two works, [49] and [50], followed two different approaches, their final findings were in perfect agreement with each other. Under the *adiabatic* approximation, associated 2PN GW polarizations ($h_+, h_\times$) were obtained in [51] for the inspiral phase of binaries in quasieccentric orbits. Later in Refs. [52,53], the method of variation of constants was used to compute post-adiabatic corrections (varying on the orbital time scale and $1/c^5$ times smaller) to the secular variation due to radiation reaction. Using the 3PN generalized quasi-Keplerian representation of the conservative dynamics of compact binary systems with arbitrary mass ratios moving in eccentric orbits presented in [54], Ref. [53] provides the evolution of the orbital phase with relative 1PN accuracy (absolute 3.5PN). The energy and angular momentum fluxes as well as evolution of orbital elements up to 3PN order was calculated in Refs. [55–57]. Recently, computations of the frequency domain waveforms and the orbital dynamics (both at the 2PN order) were presented for eccentric binaries in harmonic coordinates [58]. On the NR front, the first simulations involving nonspinning equal mass BBHs in bound eccentric orbits were performed in [59,60] and the effects of eccentricity on the final mass and spin were studied. Another recent work [39] presents numerical simulations for a nonspinning equal mass binary system with an initial eccentricity of $e \sim 0.1$ and compares the NR waveforms with those of the PN models.

In this paper we present the computation of instantaneous part[3] of various modes of the waveform ($h^{\ell m}$) for general orbits using the basis of spherical harmonics of spin weight -2. In addition we also specialize to the case of compact binaries in quasielliptical orbits and provide 3PN instantaneous expressions for various modes using 3PN quasi-Keplerian representation of the conserved dynamics of compact binaries in eccentric orbits [54,56]. Note again that investigations presented here involve only the contributions from the instantaneous terms which must be complemented by computations accounting for the hereditary effects.[4] Computations of hereditary parts to various modes of the waveform will form the basis for a companion paper [61].

[3]The part of the gravitational radiation which depends on the state of its source at a given retarded time.

[4]The part of the gravitational radiation which depends on the entire dynamical history of the source and is complementary to the instantaneous part of the radiation.

This paper is organized in the following manner. In Sec. II we first introduce general formulas for spherical harmonic modes of the gravitational waveform, $h^{\ell m}$, in terms of the radiative mass and current multipole moments, $U^{\ell m}$ and $V^{\ell m}$. Section III A recalls some of the important aspects of the MPM-PN formalism and lists various inputs that are needed for computing 3PN expressions for various modes. These inputs involve relations connecting radiative moments to source moments, expressions for various source multipole moments for an isolated compact binary system and equations of motion. In Sec. IV we provide our results related to the instantaneous part of the spherical harmonic modes of the waveform for a nonspinning compact binary system in terms of variables that describe the radiation from a generic compact binary. We find that these expressions are quite large and run over several pages. Keeping this in mind we choose to list only the dominant mode (h_{22}) in the main text of the paper and provide all the relevant modes contributing to the 3PN waveform in a separate file readable in MATHEMATICA (Hlm-GenOrb.m) that will be made available on the journal web page as Supplemental Material [62] along with the paper. In Sec. V we specialize to the case of CCBs moving in quasielliptical orbits and provide the corresponding expression for the dominant mode, h^{22}, in terms of the time-eccentricity e_t, a PN parameter related to the orbital frequency x and the eccentric anomaly u. Similar to the general orbit case, in the case of CCBs in quasielliptical orbits, expressions for all the relevant modes contributing to the 3PN waveform will be listed in a separate file (Hlm-EllOrb.m). Finally, in Sec. VI we conclude the paper by providing a brief summary of our results and the future plans.

II. SPHERICAL HARMONIC MODES OF THE GRAVITATIONAL WAVEFORM

For an isolated source of GWs, the spherical harmonic modes of the waveform ($h^{\ell m}$), in terms of the radiative mass-type ($U^{\ell m}$) and current-type multipole moments ($V^{\ell m}$) [40–42,63], are given as

$$h^{\ell m} = -\frac{G}{\sqrt{2}Rc^{\ell+2}}\left[U^{\ell m} - \frac{i}{c}V^{\ell m}\right], \tag{2.1}$$

where R is the distance of the source in radiative coordinates, G is Newton's gravitational constant and c is the speed of the light. The radiative multipole moments, $U^{\ell m}$ and $V^{\ell m}$, appearing above are related to the symmetric trace-free (STF) radiative moments U_L and V_L as

$$U^{\ell m} = \frac{4}{\ell!}\sqrt{\frac{(\ell+1)(\ell+2)}{2\ell(\ell-1)}}\alpha_L^{\ell m}U_L, \tag{2.2a}$$

$$V^{\ell m} = -\frac{8}{\ell!}\sqrt{\frac{\ell(\ell+2)}{2(\ell+1)(\ell-1)}}\alpha_L^{\ell m}V_L. \tag{2.2b}$$

Here $\alpha_L^{\ell m}$ denote STF tensors which connect the usual basis of spherical harmonics $Y^{\ell m}(\Theta,\Phi)$ to the set of STF tensors $N_{\langle L\rangle} = N_{\langle i_1}\cdots N_{i_\ell\rangle}$ as[5]

$$N_{\langle L\rangle}(\Theta,\Phi) = \sum_{m=-\ell}^{\ell}\alpha_L^{\ell m}Y^{\ell m}(\Theta,\Phi), \tag{2.3a}$$

$$Y^{\ell m}(\Theta,\Phi) = \frac{(2\ell+1)!!}{4\pi\ell!}\bar{\alpha}_L^{\ell m}N_{\langle L\rangle}(\Theta,\Phi). \tag{2.3b}$$

In the above, $\mathbf{N} = \mathbf{X}/R$ is a unit vector pointing towards the detector along the line joining the source to the detector. For instance, if the binary's plane is the x-y plane, then $\mathbf{N}$, in terms of angles (Θ,Φ) giving the location of the binary, can be given as

$$\mathbf{N} = \sin\Theta\cos\Phi\hat{\mathbf{e}}_x + \sin\Theta\sin\Phi\hat{\mathbf{e}}_y + \cos\Theta\hat{\mathbf{e}}_z. \tag{2.4}$$

The STF tensorial coefficients $\alpha_L^{\ell m}$ in terms of $N_{\langle i_1}\cdots N_{i_\ell\rangle}$ and $Y^{\ell m}(\Theta,\Phi)$ can be written as[6]

$$\alpha_L^{\ell m} = \int d\Omega N_{\langle L\rangle}\bar{Y}^{\ell m}, \tag{2.5}$$

where the usual basis of spherical harmonics is given as

$$Y^{\ell m}(\Theta,\Phi) = (-)^m\frac{1}{2^\ell\ell!}\left[\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} \times \mathrm{e}^{im\Phi}(\sin\Theta)^m\frac{\mathrm{d}^{\ell+m}}{\mathrm{d}(\cos\Theta)^{\ell+m}}(\cos^2\Theta-1)^\ell. \tag{2.6}$$

It is important to note that for nonspinning binaries, there exists a mode separation as pointed out in Ref. [41] and explicitly shown in Ref. [42]. The mode $h^{\ell m}$ is completely determined by mass-type radiative multipole moment ($U^{\ell m}$) when $\ell+m$ is even, and by current-type radiative multipole moment ($V^{\ell m}$) when $\ell+m$ is odd. This allows us to write for various modes

$$h^{\ell m} = -\frac{G}{\sqrt{2}Rc^{\ell+2}}U^{\ell m} \quad \text{if } \ell+m \text{ is even}, \tag{2.7a}$$

$$h^{\ell m} = \frac{iG}{\sqrt{2}Rc^{\ell+3}}V^{\ell m} \quad \text{if } \ell+m \text{ is odd}. \tag{2.7b}$$

[5] Here $L = i_1 i_2\cdots i_l$ represents a multi-index composed of l spatial indices and the angular brackets ($\langle\rangle$) surrounding indices denote symmetric trace-free projections.

[6] The notation used in [40,42] (which we follow here) to the one in [41,63] is related by $\mathcal{Y}_L^{\ell m} = \frac{(2\ell+1)!!}{4\pi\ell!}\bar{\alpha}_L^{\ell m}$.

III. INPUTS FOR COMPUTING THE 3PN WAVEFORM

A. Relations connecting the radiative moments to the source moments

In the MPM-PN formalism [19,64–68], the radiative multipole moments (U_L, V_L) are first written in terms of two sets of canonical moments (M_L, S_L), which in turn are expressed in terms of six sets of source moments (I_L, J_L, W_L, Y_L, X_L, Z_L). Relations connecting radiative moments to the canonical moments and those connecting the canonical moments to the source moments, with the PN accuracy desired for the waveform computations at the 3PN order, have been established and have been listed in Ref. [40] (see Eqs. (5.4)–(5.8) and Eqs. (5.9)–(5.11) there). Using these inputs, we can parametrize the set of radiative moments (and, hence, the modes) in terms of source multipole moments. Below we list all the relevant radiative multipole moments in terms of the source multipole moments with PN accuracy desired for the present work. Further, these expressions can be decomposed into two parts namely the *instantaneous* contribution and the *hereditary* contribution.

The only radiative moment required at the 3PN order is the one related to the mass quadrupole (U_{ij}) and is given by

$$U_{ij} = U^{\rm inst}_{ij} + U^{\rm hered}_{ij}, \tag{3.1}$$

where the instantaneous and hereditary parts in terms of the source multipole moments read

$$U^{\rm inst}_{ij}(U) = I^{(2)}_{ij}(U) + \frac{G}{c^5}\Big\{\frac{1}{7}I^{(5)}_{a\langle i}I_{j\rangle a} - \frac{5}{7}I^{(4)}_{a\langle i}I^{(1)}_{j\rangle a} - \frac{2}{7}I^{(3)}_{a\langle i}I^{(2)}_{j\rangle a} + \frac{1}{3}\varepsilon_{ab\langle i}I^{(4)}_{j\rangle a}J_b + 4[W^{(4)}I_{ij} + W^{(3)}I^{(1)}_{ij} - W^{(2)}I^{(2)}_{ij} - W^{(1)}I^{(3)}_{ij}]\Big\} + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{3.2a}$$

$$U^{\rm hered}_{ij}(U) = \frac{2GM}{c^3}\int_{-\infty}^{U} d\tau\left[\ln\left(\frac{U-\tau}{2\tau_0}\right) + \frac{11}{12}\right]I^{(4)}_{ij}(\tau) + \frac{G}{c^5}\left\{-\frac{2}{7}\int_{-\infty}^{U} d\tau I^{(3)}_{a\langle i}(\tau)I^{(3)}_{j\rangle a}(\tau)\right\} + 2\left(\frac{GM}{c^3}\right)^2\int_{-\infty}^{U} d\tau\left[\ln^2\left(\frac{U-\tau}{2\tau_0}\right) + \frac{57}{70}\ln\left(\frac{U-\tau}{2\tau_0}\right) + \frac{124627}{44100}\right]I^{(5)}_{ij}(\tau) + \mathcal{O}\left(\frac{1}{c^7}\right). \tag{3.2b}$$

In the above, the quantity M represents the mass monopole moment or the Arnowitt, Deser, and Misner (ADM) mass of the source. The constant τ_0 appearing in the above integrals is related to an arbitrary length scale r_0 by $\tau_0 = r_0/c$ and was originally introduced in the MPM formalism. Note that, numbers in the parenthesis (appearing as superscripts of the source moments) denote the $p^{\rm th}$ time derivatives. The Levi-Civita tensor is denoted by ε_{ijk}, such that $\varepsilon_{123} = +1$ and $\mathcal{O}(1/c^7)$ indicates that we ignore contributions of order 3.5PN and higher.

As may be seen from the above, computing the instantaneous part requires source multipole moments given at a retarded time U. On the other hand, the hereditary part involves integrals over time and would require the knowledge of the source multipole moments at any instant of time before U in the past dynamical history of the source. Further, the hereditary terms are of two kinds: those with and without the logarithmic factors (see Eq. (3.2b) above). The first integral appearing in Eq. (3.2b) (the one with the logarithmic kernel inside) is called the "tail integral" and the one in the last line is called the "tail-of-tail integral," whereas the integral without the logarithmic factor (in the first line) is known as the "memory integral." This paper only focuses on computing the *instantaneous* contribution to various modes of gravitational waveforms and the computation of hereditary contributions shall be discussed elsewhere [61].

Moments required with 2.5PN accuracy are the mass octopole moment U_{ijk} and the current quadrupole moment V_{ij}. The mass octopole moment U_{ijk} is given as

$$U_{ijk} = U^{\rm inst}_{ijk} + U^{\rm hered}_{ijk}, \tag{3.3}$$

where $U^{\rm inst}_{ij}$ and $U^{\rm hered}_{ijk}$ in terms of source multipole moments read

$$U^{\rm inst}_{ijk}(U) = I^{(3)}_{ijk}(U) + \frac{G}{c^5}\Big\{-\frac{4}{3}I^{(3)}_{a\langle i}I^{(3)}_{jk\rangle a} - \frac{9}{4}I^{(4)}_{a\langle i}I^{(2)}_{jk\rangle a} + \frac{1}{4}I^{(2)}_{a\langle i}I^{(4)}_{jk\rangle a} - \frac{3}{4}I^{(5)}_{a\langle i}I^{(1)}_{jk\rangle a} + \frac{1}{4}I^{(1)}_{a\langle i}I^{(5)}_{jk\rangle a} + \frac{1}{12}I^{(6)}_{a\langle i}I_{jk\rangle a} + \frac{1}{4}I_{a\langle i}I^{(6)}_{jk\rangle a} + \frac{1}{5}\varepsilon_{ab\langle i}\Big[-12J^{(2)}_{ja}I^{(3)}_{k\rangle b} - 8I^{(2)}_{ja}J^{(3)}_{k\rangle b} - 3J^{(1)}_{ja}I^{(4)}_{k\rangle b} - 27I^{(1)}_{ja}J^{(4)}_{k\rangle b} - J_{ja}I^{(5)}_{k\rangle b} - 9I_{ja}J^{(5)}_{k\rangle b} - \frac{9}{4}J_a I^{(5)}_{jk\rangle b}\Big] + \frac{12}{5}J_{\langle i}J^{(4)}_{jk\rangle} + 4[W^{(2)}I_{ijk} - W^{(1)}I^{(1)}_{ijk} + 3I_{\langle ij}Y^{(1)}_{k\rangle}]^{(3)}\Big\} + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{3.4a}$$

$$U^{\rm hered}_{ijk}(U) = \frac{2GM}{c^3}\int_{-\infty}^{U} d\tau\left[\ln\left(\frac{U-\tau}{2\tau_0}\right)+\frac{97}{60}\right]I^{(5)}_{ijk}(\tau) + \frac{G}{c^5}\left\{\int_{-\infty}^{U} d\tau\left[-\frac{1}{3}I^{(3)}_{a\langle i}(\tau)I^{(4)}_{jk\rangle a}(\tau) - \frac{4}{5}\varepsilon_{ab\langle i}I^{(3)}_{ja}(\tau)J^{(3)}_{k\rangle b}(\tau)\right]\right\} + \mathcal{O}\left(\frac{1}{c^6}\right). \tag{3.4b}$$

The current quadrupole moment V_{ij} is given as

$$V_{ij} = V^{\rm inst}_{ij} + V^{\rm hered}_{ij}, \tag{3.5}$$

where $V^{\rm inst}_{ij}$ and $V^{\rm hered}_{ij}$ in terms of source multipole moments read

$$\begin{aligned} V^{\rm inst}_{ij}(U) = {}& J^{(2)}_{ij}(U) + \frac{G}{7c^5}\Bigg\{4J^{(2)}_{a\langle i}I^{(3)}_{j\rangle a} + 8I^{(2)}_{a\langle i}J^{(3)}_{j\rangle a} + 17J^{(1)}_{a\langle i}I^{(4)}_{j\rangle a} - 3I^{(1)}_{a\langle i}J^{(4)}_{j\rangle a} + 9I_{a\langle i}I^{(5)}_{j\rangle a} - 3I_{a\langle i}J^{(5)}_{j\rangle a} \\ & - \frac{1}{4}J_a I^{(5)}_{ija} - 7\varepsilon_{ab\langle i}J_a J^{(4)}_{j\rangle b} + \frac{1}{2}\varepsilon_{ac\langle i}\Bigg[3I^{(3)}_{ab}I^{(3)}_{j\rangle bc} + \frac{353}{24}I^{(2)}_{j\rangle bc}I^{(4)}_{ab} - \frac{5}{12}I^{(2)}_{ab}I^{(4)}_{j\rangle bc} + \frac{113}{8}I^{(1)}_{j\rangle bc}I^{(5)}_{ab} \\ & - \frac{3}{8}I^{(1)}_{ab}I^{(5)}_{j\rangle bc} + \frac{15}{4}I_{j\rangle bc}I^{(6)}_{ab} + \frac{3}{8}I_{ab}I^{(6)}_{j\rangle bc}\Bigg] + 14[\varepsilon_{ab\langle i}(-I^{(3)}_{j\rangle b}W_a - 2I_{j\rangle b}Y^{(2)}_a + I^{(1)}_{j\rangle b}Y^{(1)}_a) \\ & + 3J_{\langle i}Y^{(1)}_{j\rangle} - 2J^{(1)}_{ij}W^{(1)}]^{(2)}\Bigg\} + \mathcal{O}\left(\frac{1}{c^6}\right). \end{aligned} \tag{3.6a}$$

$$V^{\rm hered}_{ij}(U) = \frac{2GM}{c^3}\int_{-\infty}^{U} d\tau\left[\ln\left(\frac{U-\tau}{2\tau_0}\right)+\frac{7}{6}\right]J^{(4)}_{ij}(\tau) + \mathcal{O}\left(\frac{1}{c^6}\right). \tag{3.6b}$$

At the 2PN order, the required moments are U_{ijkl} and V_{ijk}. The moment U_{ijkl} is given by

$$U_{ijkl} = U^{\rm inst}_{ijkl} + U^{\rm hered}_{ijkl}, \tag{3.7}$$

where $U^{\rm inst}_{ijkl}$ and $U^{\rm hered}_{ijkl}$ are related to the source multipole moments by

$$U^{\rm inst}_{ijkl}(U) = I^{(4)}_{ijkl}(U) + \frac{G}{c^3}\left\{-\frac{21}{5}I^{(5)}_{\langle ij}I_{kl\rangle} - \frac{63}{5}I^{(4)}_{\langle ij}I^{(1)}_{kl\rangle} - \frac{102}{5}I^{(3)}_{\langle ij}I^{(2)}_{kl\rangle}\right\} + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{3.8a}$$

$$U^{\rm hered}_{ijkl}(U) = \frac{G}{c^3}\left\{2M\int_{-\infty}^{U} d\tau\left[\ln\left(\frac{U-\tau}{2\tau_0}\right)+\frac{59}{30}\right]I^{(6)}_{ijkl}(\tau) + \frac{2}{5}\int_{-\infty}^{U} d\tau I^{(3)}_{\langle ij}(\tau)I^{(3)}_{kl\rangle}(\tau)\right\} + \mathcal{O}\left(\frac{1}{c^5}\right). \tag{3.8b}$$

The moment V_{ijk} is given by

$$V_{ijk} = V^{\rm inst}_{ijk} + V^{\rm hered}_{ijk}, \tag{3.9}$$

where $V^{\rm inst}_{ijk}$ and $V^{\rm hered}_{ijk}$ are given in terms of the source multipole moments as

$$V^{\rm inst}_{ijk}(U) = J^{(3)}_{ijk}(U) + \frac{G}{c^3}\left\{\frac{1}{10}\varepsilon_{ab\langle i}I^{(5)}_{ja}I_{k\rangle b} - \frac{1}{2}\varepsilon_{ab\langle i}I^{(4)}_{ja}I^{(1)}_{k\rangle b} - 2J_{\langle i}I^{(4)}_{jk\rangle}\right\} + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{3.10a}$$

$$V^{\rm hered}_{ijk}(U) = \frac{2GM}{c^3}\int_{-\infty}^{U} d\tau\left[\ln\left(\frac{U-\tau}{2\tau_0}\right)+\frac{5}{3}\right]J^{(5)}_{ijk}(\tau) + \mathcal{O}\left(\frac{1}{c^5}\right). \tag{3.10b}$$

The moments required at the 1.5PN order are U_{ijklm} and V_{ijkl}. The mass-type moment U_{ijklm} is given as

$$U_{ijklm} = U^{\rm inst}_{ijklm} + U^{\rm hered}_{ijklm}, \tag{3.11}$$

where in terms of the source multipole moments, $U^{\rm inst}_{ijklm}$ and $U^{\rm hered}_{ijklm}$ read

$$U^{\rm inst}_{ijklm}(U) = I^{(5)}_{ijklm}(U) + \frac{G}{c^3}\left\{-\frac{710}{21}I^{(3)}_{\langle ij}I^{(3)}_{klm\rangle} - \frac{265}{7}I^{(2)}_{\langle ijk}I^{(4)}_{lm\rangle} - \frac{120}{7}I^{(2)}_{\langle ij}I^{(4)}_{klm\rangle} - \frac{155}{7}I^{(1)}_{\langle ijk}I^{(5)}_{lm\rangle} - \frac{41}{7}I^{(1)}_{\langle ij}I^{(5)}_{klm\rangle} - \frac{34}{7}I_{\langle ijk}I^{(6)}_{lm\rangle} - \frac{15}{7}I_{\langle ij}I^{(6)}_{klm\rangle}\right\} + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{3.12a}$$

$$U^{\rm hered}_{ijklm}(U) = \frac{G}{c^3}\left\{2M\int_{-\infty}^{U} d\tau\left[\ln\left(\frac{U-\tau}{2\tau_0}\right) + \frac{232}{105}\right]I^{(7)}_{ijklm}(\tau) + \frac{20}{21}\int_{-\infty}^{U} d\tau I^{(3)}_{\langle ij}(\tau)I^{(4)}_{klm\rangle}(\tau)\right\} + \mathcal{O}\left(\frac{1}{c^4}\right). \tag{3.12b}$$

The current-type moment V_{ijkl} is given by

$$V_{ijkl} = V^{\rm inst}_{ijkl} + V^{\rm hered}_{ijkl}, \tag{3.13}$$

where $V^{\rm inst}_{ijkl}$ and $V^{\rm hered}_{ijkl}$ in terms of the source multipole moments read

$$V^{\rm inst}_{ijkl}(U) = J^{(4)}_{ijkl}(U) + \frac{G}{c^3}\left\{-\frac{35}{3}S^{(2)}_{\langle ij}I^{(3)}_{kl\rangle} - \frac{25}{3}I^{(2)}_{\langle ij}J^{(3)}_{kl\rangle} - \frac{65}{6}J^{(1)}_{\langle ij}I^{(4)}_{kl\rangle} - \frac{25}{6}I^{(1)}_{\langle ij}J^{(4)}_{kl\rangle} - \frac{19}{6}J_{\langle ij}I^{(5)}_{kl\rangle} - \frac{11}{6}I_{\langle ij}J^{(5)}_{kl\rangle} - \frac{11}{12}J_{\langle i}I^{(5)}_{jkl\rangle} + \frac{1}{6}\varepsilon_{ab\langle i}\left[-5I^{(3)}_{ja}I^{(3)}_{kl\rangle b} - \frac{11}{2}I^{(4)}_{ja}I^{(2)}_{kl\rangle b} - \frac{5}{2}I^{(2)}_{ja}I^{(4)}_{kl\rangle b} - \frac{1}{2}I^{(5)}_{ja}I^{(1)}_{kl\rangle b} + \frac{37}{10}I^{(1)}_{ja}I^{(5)}_{kl\rangle b} + \frac{3}{10}I^{(6)}_{ja}I_{kl\rangle b} + \frac{1}{2}I_{ja}I^{(6)}_{kl\rangle b}\right]\right\} + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{3.14a}$$

$$V^{\rm hered}_{ijkl}(U) = \frac{2GM}{c^3}\int_{-\infty}^{U} d\tau\left[\ln\left(\frac{U-\tau}{2\tau_0}\right) + \frac{119}{60}\right]J^{(6)}_{ijkl}(\tau) + \mathcal{O}\left(\frac{1}{c^4}\right). \tag{3.14b}$$

Other mass-type moments U_L contributing to 3PN waveform are given as

$$U_L = U^{\rm inst}_L + U^{\rm hered}_L, \tag{3.15}$$

where $U^{\rm inst}_L$ and $U^{\rm hered}_L$ are related to the source multipole moments as

$$U^{\rm inst}_L(U) = I^{(\ell)}_L(U) + \mathcal{O}\left(\frac{1}{c^3}\right), \tag{3.16a}$$

$$U^{\rm hered}_L(U) = \mathcal{O}\left(\frac{1}{c^3}\right). \tag{3.16b}$$

Other current-type moments V_L contributing to 3PN waveform are given as

$$V_L = V^{\rm inst}_L + V^{\rm hered}_L, \tag{3.17}$$

where finally $V^{\rm inst}_L$ and $V^{\rm hered}_L$ in terms of source multipole moments read

$$V^{\rm inst}_L(U) = J^{(\ell)}_L(U) + \mathcal{O}\left(\frac{1}{c^3}\right), \tag{3.18a}$$

$$V^{\rm hered}_L(U) = \mathcal{O}\left(\frac{1}{c^3}\right). \tag{3.18b}$$

B. Source multipole moments in general dynamical variables

What we need next are expressions for various source multipole moments with the PN accuracy sufficient for the present computation. Expressions for various multipole moments presented here are generalizations of related circular orbit expressions presented in [40,69] to the case of general orbits and have been computed using the methods presented in [65,66]. We skip all the details of the computation and list the final expressions for the source multipole moments related to a source composed of two nonspinning compact objects moving in general orbits.

The only moment required here with 3PN accuracy is the mass quadrupole, I_{ij}, which for CCBs in general orbit was computed in Ref. [66] and listed in Ref. [56] in standard harmonic (SH) coordinates.[7] As was argued in Ref. [56], though the use of SH coordinate is useful in performing algebraic checks on PN computations, quantities when expressed in these coordinates involve some *gauge-dependent* logarithmic terms and are not suitable for numerical calculations. It was suggested in Ref. [56] that such logarithms can be

[7]Note that Ref. [56] lists explicit expressions for all the source multipole moments for binaries in general orbits needed for computing 3PN energy flux.

transformed away by using some coordinate transformations. They showed how the use of a modified harmonic (MH) coordinate system (or alternatively an ADM coordinate system) removes these logarithms. We skip the details related to those transformations and directly write the expression for the mass quadrupole moment in MH coordinates. In MH coordinates, to 3PN accuracy, I_{ij} reads

$$I_{ij} = \nu m\left\{\left[A_1 - \frac{24}{7}\frac{\nu}{c^5}\frac{G^2m^2}{r^2}\dot{r}\right]x_{\langle i}x_{j\rangle} + \left[A_2\frac{r\dot{r}}{c^2} + \frac{48}{7}\frac{\nu}{c^5}\frac{G^2m^2}{r}\right]x_{\langle i}v_{j\rangle} + A_3\frac{r^2}{c^2}v_{\langle i}v_{j\rangle}\right\} + \mathcal{O}\left(\frac{1}{c^7}\right), \tag{3.19}$$

where

$$\begin{aligned}A_1 = {}& 1 + \frac{1}{c^2}\left[v^2\left(\frac{29}{42} - \frac{29\nu}{14}\right) + \frac{Gm}{r}\left(-\frac{5}{7} + \frac{8\nu}{7}\right)\right] + \frac{1}{c^4}\left[\frac{Gm}{r}v^2\left(\frac{2021}{756} - \frac{5947\nu}{756} - \frac{4883\nu^2}{756}\right)\right.\\
&+ \frac{Gm}{r}\dot{r}^2\left(-\frac{131}{756} + \frac{907\nu}{756} - \frac{1273\nu^2}{756}\right) + \frac{G^2m^2}{r^2}\left(-\frac{355}{252} - \frac{953\nu}{126} + \frac{337\nu^2}{252}\right) + v^4\left(\frac{253}{504} - \frac{1835\nu}{504}\right.\\
&\left.\left.+ \frac{3545\nu^2}{504}\right)\right] + \frac{1}{c^6}\left[v^6\left(\frac{4561}{11088} - \frac{7993\nu}{1584} + \frac{117067\nu^2}{5544} - \frac{328663\nu^3}{11088}\right) + \frac{G^2m^2}{r^2}\dot{r}^2\left(-\frac{8539}{20790}\right.\right.\\
&\left.+ \frac{52153\nu}{4158} - \frac{4652\nu^2}{231} - \frac{54121\nu^3}{5544}\right) + \frac{Gm}{r}\dot{r}^4\left(\frac{2}{99} - \frac{1745\nu}{2772} + \frac{16319\nu^2}{5544} - \frac{311\nu^3}{99}\right) + \frac{Gm}{r}v^4\\
&\times\left(\frac{307}{77} - \frac{94475\nu}{4158} + \frac{218411\nu^2}{8316} + \frac{299857\nu^3}{8316}\right) + v^2\left(\frac{G^2m^2}{r^2}\left(\frac{187183}{83160} - \frac{605419\nu}{16632} + \frac{434909\nu^2}{16632}\right.\right.\\
&\left.\left.- \frac{37369\nu^3}{2772}\right) + \frac{Gm}{r}\dot{r}^2\left(-\frac{757}{5544} + \frac{5545\nu}{8316} - \frac{98311\nu^2}{16632} + \frac{153407\nu^3}{8316}\right)\right) + \frac{G^3m^3}{r^3}\left(\frac{6285233}{207900}\right.\\
&\left.\left.+ \frac{15502\nu}{385} - \frac{3632\nu^2}{693} + \frac{13289\nu^3}{8316} - \frac{428}{105}\log\left[\frac{r}{r_0}\right]\right)\right],\end{aligned} \tag{3.20a}$$

$$\begin{aligned}A_2 = {}& -\frac{4}{7} + \frac{12\nu}{7} + \frac{1}{c^2}\left[v^2\left(-\frac{26}{63} + \frac{202\nu}{63} - \frac{418\nu^2}{63}\right) + \frac{Gm}{r}\left(-\frac{155}{54} + \frac{4057\nu}{378} + \frac{209\nu^2}{54}\right)\right]\\
&+ \frac{1}{c^4}\left[\frac{Gm}{r}v^2\left(-\frac{2839}{693} + \frac{237893\nu}{8316} - \frac{188063\nu^2}{4158} - \frac{58565\nu^3}{2079}\right) + \frac{Gm}{r}\dot{r}^2\left(\frac{305}{2772} + \frac{3233\nu}{2772}\right.\right.\\
&\left.- \frac{8611\nu^2}{2772} - \frac{895\nu^3}{77}\right) + \frac{G^2m^2}{r^2}\left(-\frac{12587}{20790} + \frac{406333\nu}{8316} - \frac{2713\nu^2}{198} + \frac{4441\nu^3}{1386}\right) + v^4\left(-\frac{457}{1386}\right.\\
&\left.\left.+ \frac{6103\nu}{1386} - \frac{13693\nu^2}{693} + \frac{40687\nu^3}{1386}\right)\right],\end{aligned} \tag{3.20b}$$

$$\begin{aligned}A_3 = {}& \frac{11}{21} - \frac{11\nu}{7} + \frac{1}{c^2}\left[\frac{Gm}{r}\left(\frac{106}{27} - \frac{335\nu}{189} - \frac{985\nu^2}{189}\right) + \dot{r}^2\left(\frac{5}{63} - \frac{25\nu}{63} + \frac{25\nu^2}{63}\right) + v^2\left(\frac{41}{126}\right.\right.\\
&\left.\left.- \frac{337\nu}{126} + \frac{733\nu^2}{126}\right)\right] + \frac{1}{c^4}\left[v^4\left(\frac{1369}{5544} - \frac{19351\nu}{5544} + \frac{45421\nu^2}{2772} - \frac{139999\nu^3}{5544}\right) + \frac{Gm}{r}\dot{r}^2\left(\frac{79}{77}\right.\right.\\
&\left.- \frac{5807\nu}{1386} + \frac{515\nu^2}{1386} + \frac{8245\nu^3}{693}\right) + v^2\left(\dot{r}^2\left(\frac{115}{1386} - \frac{1135\nu}{1386} + \frac{1795\nu^2}{693} - \frac{3445\nu^3}{1386}\right) + \frac{Gm}{r}\left(\frac{587}{154}\right.\right.\\
&\left.\left.- \frac{67933\nu}{4158} + \frac{25660\nu^2}{2079} + \frac{129781\nu^3}{4158}\right)\right) + \frac{G^2m^2}{r^2}\left(-\frac{40716}{1925} - \frac{10762\nu}{2079} + \frac{62576\nu^2}{2079} - \frac{24314\nu^3}{2079}\right.\\
&\left.\left.+ \frac{428}{105}\log\left[\frac{r}{r_0}\right]\right)\right].\end{aligned} \tag{3.20c}$$

In the above, x_i and v_i denote the binary's relative separation and relative velocity, respectively, whereas $\dot{r}$ denotes the radial velocity. As we see, the above expression still has a dependence on some logarithms ($\log[r_0]$), where the quantity r_0 is related to the arbitrary constant τ_0 appearing in tail integrals by $\tau_0 = r_0/c$. It has been argued and shown that it disappears from all the physical quantities like the radiation field at infinity and the far-zone energy flux [40,41,56,70].

The expression for mass octopole, I_{ijk}, at 2.5PN order reads

$$I_{ijk} = -\nu m\Delta\left\{\left[B_1 - \frac{56}{9}\frac{\nu}{c^5}\frac{G^2m^2}{r^2}\dot{r}\right]x_{\langle ijk\rangle} + \left[B_2\frac{r\dot{r}}{c^2} + \frac{\nu r}{c^5}\left(\frac{232}{15}\frac{G^2m^2}{r^2} - \frac{12}{5}\frac{Gm}{r}v^2\right)\right]x_{\langle ij}v_{k\rangle} + B_3\frac{r^2}{c^2}x_{\langle i}v_{jk\rangle} + B_4\frac{r^3\dot{r}}{c^4}v_{\langle ijk\rangle}\right\} + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{3.21}$$

where

$$B_1 = 1 + \frac{1}{c^2}\left[v^2\left(\frac{5}{6} - \frac{19\nu}{6}\right) + \frac{Gm}{r}\left(-\frac{5}{6} + \frac{13\nu}{6}\right)\right] + \frac{1}{c^4}\left[\frac{Gm}{r}v^2\left(\frac{3853}{1320} - \frac{14257\nu}{1320} - \frac{17371\nu^2}{1320}\right) + \frac{G^2m^2}{r^2}\left(-\frac{47}{33} - \frac{1591\nu}{132} + \frac{235\nu^2}{66}\right) + v^4\left(\frac{257}{440} - \frac{7319\nu}{1320} + \frac{5501\nu^2}{440}\right) + \frac{Gm}{r}\dot{r}^2\left(-\frac{247}{1320} + \frac{531\nu}{440} - \frac{1347\nu^2}{440}\right)\right], \tag{3.22a}$$

$$B_2 = -(1-2\nu) + \frac{1}{c^2}\left[v^2\left(-\frac{13}{22} + \frac{107\nu}{22} - \frac{102\nu^2}{11}\right) + \frac{Gm}{r}\left(-\frac{2461}{660} + \frac{8689\nu}{660} + \frac{1389\nu^2}{220}\right)\right], \tag{3.22b}$$

$$B_3 = 1 - 2\nu + \frac{1}{c^2}\left[\frac{Gm}{r}\left(\frac{1949}{330} + \frac{62\nu}{165} - \frac{483\nu^2}{55}\right) + v^2\left(\frac{61}{110} - \frac{519\nu}{110} + \frac{504\nu^2}{55}\right) + \dot{r}^2\left(-\frac{1}{11} + \frac{4\nu}{11} - \frac{3\nu^2}{11}\right)\right], \tag{3.22c}$$

$$B_4 = \left(\frac{13}{55} - \frac{52\nu}{55} + \frac{39\nu^2}{55}\right). \tag{3.22d}$$

The remaining mass-type source multipole moments with PN accuracy required in the present work are

$$\begin{aligned}I_{ijkl} = \nu m\Bigg\{ & x_{\langle ijkl\rangle}\Bigg[1 - 3\nu + \frac{1}{c^2}\left[\frac{Gm}{r}\left(-\frac{10}{11} + \frac{61\nu}{11} - \frac{105\nu^2}{11}\right) + v^2\left(\frac{103}{110} - \frac{147\nu}{22} + \frac{279\nu^2}{22}\right)\right] \\ & + \frac{1}{c^4}\left[v^4\left(\frac{3649}{5720} - \frac{50191\nu}{5720} + \frac{112357\nu^2}{2860} - \frac{325687\nu^3}{5720}\right) + \frac{G^2m^2}{r^2}\left(-\frac{15549}{10010} - \frac{9457\nu}{715} + \frac{7961\nu^2}{143} - \frac{5829\nu^3}{286}\right)\right. \\ & \left. + \frac{Gm}{r}v^2\left(\frac{11049}{3575} - \frac{152489\nu}{7150} + \frac{15124\nu^2}{715} + \frac{46934\nu^3}{715}\right) + \frac{Gm}{r}\dot{r}^2\left(-\frac{659}{3575} + \frac{12619\nu}{7150} - \frac{10557\nu^2}{1430} + \frac{9617\nu^3}{715}\right)\right]\Bigg] \\ & + x_{\langle ijk}v_{l\rangle}\frac{r\dot{r}}{c^2}\left[-\frac{72}{55} + \frac{72\nu}{11} - \frac{72\nu^2}{11} + \frac{1}{c^2}\left[\frac{Gm}{r}\times\left(-\frac{15463}{3575} + \frac{98374\nu}{3575} - \frac{25606\nu^2}{715} - \frac{18839\nu^3}{715}\right)\right.\right. \\ & \left.\left. + v^2\left(-\frac{476}{715} + \frac{1228\nu}{143} - \frac{23512\nu^2}{715} + \frac{25796\nu^3}{715}\right)\right]\right] + x_{\langle ij}v_{kl\rangle}\frac{r^2}{c^2}\left[\frac{78}{55} - \frac{78\nu}{11} + \frac{78\nu^2}{11} + \frac{1}{c^2}\left[v^2\left(\frac{553}{715} - \frac{6913\nu}{715} + \frac{25994\nu^2}{715}\right.\right.\right. \\ & \left.\left.\left. - \frac{28207\nu^3}{715}\right) + \frac{Gm}{r}\left(\frac{27818}{3575} - \frac{72474\nu}{3575} - \frac{17202\nu^2}{715} + \frac{27568\nu^3}{715}\right) + \dot{r}^2\left(-\frac{4}{13} + \frac{28\nu}{13} - \frac{56\nu^2}{13} + \frac{28\nu^3}{13}\right)\right]\right] \\ & + x_{\langle i}v_{jkl\rangle}\frac{r^3\dot{r}}{c^4}\left[\frac{304}{715} - \frac{2128\nu}{715} + \frac{4256\nu^2}{715} - \frac{2128\nu^3}{715}\right] + v_{\langle ijkl\rangle}\frac{r^4}{c^4}\times\left[\frac{71}{715} - \frac{497\nu}{715} + \frac{994\nu^2}{715} - \frac{497\nu^3}{715}\right]\Bigg\} + \mathcal{O}\left(\frac{1}{c^5}\right),\end{aligned} \tag{3.23a}$$

$$\begin{aligned}I_{ijklm} = -\nu m\Delta\Bigg\{ & x_{\langle ijklm\rangle}\left[1 - 2\nu + \frac{1}{c^2}\left[\frac{Gm}{r}\left(-\frac{25}{26} + \frac{139\nu}{26} - \frac{109\nu^2}{13}\right) + v^2\left(\frac{79}{78} - \frac{511\nu}{78} + \frac{137\nu^2}{13}\right)\right]\right] \\ & + x_{\langle ijkl}v_{m\rangle}\frac{r\dot{r}}{c^2}\left[-\frac{20}{13} + \frac{80\nu}{13} - \frac{60\nu^2}{13}\right] + x_{\langle ijk}v_{lm\rangle}\frac{r^2}{c^2}\left[\frac{70}{39} - \frac{280\nu}{39} + \frac{70\nu^2}{13}\right]\Bigg\} + \mathcal{O}\left(\frac{1}{c^4}\right),\end{aligned} \tag{3.23b}$$

$$I_{ijklmn} = \nu m\bigg\{x_{\langle ijklmn\rangle}\bigg[1-5\nu+5\nu^2+\frac{1}{c^2}\bigg[v^2\bigg(\frac{15}{14}-\frac{21\nu}{2}+33\nu^2-\frac{63\nu^3}{2}\bigg)-\frac{Gm}{r}(1-9\nu+27\nu^2-26\nu^3)\bigg]\bigg] - x_{\langle ijklm}v_{n\rangle}\frac{12}{7}\frac{r\dot{r}}{c^2}(1-7\nu+14\nu^2-7\nu^3)+x_{\langle ijkl}v_{mn\rangle}\frac{15}{7}\frac{r^2}{c^2}(1-7\nu+14\nu^2-7\nu^3)\bigg\}+\mathcal{O}\bigg(\frac{1}{c^4}\bigg), \tag{3.23c}$$

$$I_{ijklmno} = -\nu m\Delta(1-4\nu+3\nu^2)x_{\langle ijklmno\rangle}+\mathcal{O}\bigg(\frac{1}{c^2}\bigg), \tag{3.23d}$$

$$I_{ijklmnop} = \nu m(1-7\nu+14\nu^2-7\nu^3)x_{\langle ijklmnop\rangle}+\mathcal{O}\bigg(\frac{1}{c^2}\bigg). \tag{3.23e}$$

The current quadrupole moment is needed at 2.5PN order and is given as

$$J_{ij} = -\nu m\Delta\bigg\{\bigg[C_1-\frac{62}{7}\frac{\dot{r}\nu}{c^5}\frac{G^2m^2}{r^2}\bigg]\epsilon_{ab\langle i}x_{j\rangle a}v_b+\bigg[C_2\frac{r\dot{r}}{c^2}+\frac{r\nu}{c^5}\frac{Gm}{r}\bigg(\frac{216}{35}\frac{Gm}{r}-\frac{4}{5}v^2\bigg)\bigg]\epsilon_{ab\langle i}v_{j\rangle b}x_a\bigg\}+\mathcal{O}\bigg(\frac{1}{c^6}\bigg), \tag{3.24}$$

where

$$C_1 = 1+\frac{1}{c^2}\bigg[v^2\bigg(\frac{13}{28}-\frac{17\nu}{7}\bigg)+\frac{Gm}{r}\bigg(\frac{27}{14}+\frac{15\nu}{7}\bigg)\bigg]+\frac{1}{c^4}\bigg[\frac{Gm}{r}v^2\bigg(\frac{671}{252}-\frac{1297\nu}{126}-\frac{121\nu^2}{12}\bigg) + \frac{Gm}{r}\dot{r}^2\bigg(-\frac{5}{252}-\frac{241\nu}{252}-\frac{335\nu^2}{84}\bigg)+\frac{G^2m^2}{r^2}\bigg(-\frac{43}{252}-\frac{1543\nu}{126}+\frac{293\nu^2}{84}\bigg)+v^4\bigg(\frac{29}{84}-\frac{11\nu}{3}+\frac{505\nu^2}{56}\bigg)\bigg], \tag{3.25a}$$

$$C_2 = \frac{5}{28}(1-2\nu)+\frac{1}{504}\frac{1}{c^2}\bigg[\frac{Gm}{r}(824+1348\nu-1038\nu^2)+75v^2(1-7\nu+12\nu^2)\bigg]. \tag{3.25b}$$

Other current-type source multipole moments with PN accuracies sufficient for present calculations read

$$\begin{aligned}J_{ijk} = \nu m\epsilon_{ab\langle i}\bigg\{&x_{jk\rangle a}v_b\bigg[1-3\nu+\frac{1}{c^2}\bigg[\frac{Gm}{r}\bigg(\frac{14}{9}-\frac{16\nu}{9}-\frac{86\nu^2}{9}\bigg)+v^2\bigg(\frac{41}{90}-\frac{77\nu}{18}+\frac{185\nu^2}{18}\bigg)\bigg] \\ &+\frac{1}{c^4}\bigg[v^4\bigg(\frac{1349}{3960}-\frac{4159\nu}{792}+\frac{52409\nu^2}{1980}-\frac{171539\nu^3}{3960}\bigg)+\frac{G^2m^2}{r^2}\bigg(-\frac{45}{44}-\frac{988\nu}{99}+\frac{9925\nu^2}{198}-\frac{8099\nu^3}{396}\bigg) \\ &+\frac{Gm}{r}\dot{r}^2\bigg(-\frac{23}{396}-\frac{637\nu}{990}-\frac{1861\nu^2}{990}+\frac{32221\nu^3}{1980}\bigg)+\frac{Gm}{r}v^2\bigg(\frac{1597}{660}-\frac{19381\nu}{990}+\frac{6307\nu^2}{198}+\frac{21127\nu^3}{396}\bigg)\bigg]\bigg] \\ &+x_jv_{k\rangle b}x_a\frac{r\dot{r}}{c^2}\bigg[\frac{2}{9}-\frac{10\nu}{9}+\frac{10\nu^2}{9}+\frac{1}{c^2}\bigg[v^2\bigg(\frac{73}{495}-\frac{841\nu}{495}+\frac{3002\nu^2}{495}-\frac{3151\nu^3}{495}\bigg) \\ &+\frac{Gm}{r}\bigg(\frac{133}{66}-\frac{81\nu}{55}-\frac{3914\nu^2}{165}+\frac{3089\nu^3}{330}\bigg)\bigg]\bigg]+v_{jk\rangle b}x_a\frac{r^2}{c^2}\bigg[\frac{7}{45}(1-5\nu+5\nu^2) \\ &+\frac{1}{c^2}\bigg[v^2\bigg(\frac{119}{990}-\frac{259\nu}{198}+\frac{2219\nu^2}{495}-\frac{4529\nu^3}{990}\bigg)+\dot{r}^2\bigg(\frac{14}{165}-\frac{98\nu}{165}+\frac{196\nu^2}{165}-\frac{98\nu^3}{165}\bigg) \\ &+\frac{Gm}{r}\bigg(\frac{751}{495}-\frac{1792\nu}{495}-\frac{227\nu^2}{99}+\frac{427\nu^3}{99}\bigg)\bigg]\bigg]\bigg\}+\mathcal{O}\bigg(\frac{1}{c^5}\bigg),\end{aligned} \tag{3.26a}$$

$$\begin{aligned}J_{ijkl} = -\nu m\Delta\epsilon_{ab\langle i}\bigg\{&x_{jkl\rangle a}v_b\bigg[1-2\nu+\frac{1}{c^2}\bigg[\frac{Gm}{r}\bigg(\frac{15}{11}+\frac{35\nu}{44}-\frac{185\nu^2}{22}\bigg)+v^2\bigg(\frac{5}{11}-\frac{95\nu}{22}+\frac{195\nu^2}{22}\bigg)\bigg]\bigg] \\ &+\frac{5}{22}x_{jk}v_{l\rangle b}x_a\frac{r\dot{r}}{c^2}(1-4\nu+3\nu^2)+\frac{4}{11}x_jv_{kl\rangle b}x_a\frac{r^2}{c^2}(1-4\nu+3\nu^2)\bigg\}+\mathcal{O}\bigg(\frac{1}{c^4}\bigg),\end{aligned} \tag{3.26b}$$

$$J_{ijklm} = \nu m \epsilon_{ab\langle i}\bigg\{x_{jklm\rangle a}v_b\bigg[1-5\nu+5\nu^2+\frac{1}{c^2}\bigg[v^2\bigg(\frac{83}{182}-\frac{161\nu}{26}+\frac{317\nu^2}{13}-\frac{707\nu^3}{26}\bigg) + \frac{Gm}{r}\bigg(\frac{81}{65}-\frac{138\nu}{65}-\frac{210\nu^2}{13}+\frac{339\nu^3}{13}\bigg)\bigg]\bigg] + \frac{20}{91}x_{jkl}v_{m\rangle b}x_a\frac{r\dot{r}}{c^2}(1-7\nu+14\nu^2-7\nu^3) + \frac{54}{91}x_{jk}v_{lm\rangle b}x_a\frac{r^2}{c^2}(1-7\nu+14\nu^2-7\nu^3)\bigg\} + \mathcal{O}\bigg(\frac{1}{c^4}\bigg), \tag{3.26c}$$

$$J_{ijklmn} = -\nu m\Delta\epsilon_{ab\langle i}x_{jklmn\rangle}x_a v_b(1-4\nu+3\nu^2) + \mathcal{O}\bigg(\frac{1}{c^2}\bigg), \tag{3.26d}$$

$$J_{ijklmno} = \nu m\epsilon_{ab\langle i}x_{jklmno\rangle}x_a v_b(1-7\nu+14\nu^2-7\nu^3) + \mathcal{O}\bigg(\frac{1}{c^2}\bigg). \tag{3.26e}$$

The required gauge moments, the monopolar moment W, and two dipolar moments W_i and Y_i are finally given by

$$W = \frac{1}{3}\nu m r\dot{r} + \mathcal{O}\bigg(\frac{1}{c^2}\bigg), \tag{3.27a}$$

$$W_i = \frac{1}{10}\nu m\Delta r^2\bigg[v^i - 3\frac{\dot{r}}{r}x^i\bigg] + \mathcal{O}\bigg(\frac{1}{c^2}\bigg), \tag{3.27b}$$

$$Y_i = \frac{1}{5}\nu m\Delta\bigg[\frac{1}{2}\frac{Gm}{r}x^i + \frac{1}{2}v^2x^i - \frac{3}{2}r\dot{r}v^i\bigg] + \mathcal{O}\bigg(\frac{1}{c^2}\bigg). \tag{3.27c}$$

C. The post-Newtonian compact binary dynamics

Since relations connecting the radiative multipole moment to the source multipole moment involve time derivatives of the source multipole moments, computations of various modes will require a knowledge of the equations of motion (EOM) with the PN accuracy with which one wants to compute various modes. Before we write expressions for the EOM, with the PN accuracy required for the present work, let us recall the definitions of various dynamical variables as well as some other related results which will be used in calculations performed here. The binary's relative separation x^i is given by

$$x^i = y_1^i - y_2^i = rn^i \tag{3.28}$$

with $r = |\mathbf{x}|$, where $\mathbf{x}$ is the relative separation vector. Here, y_1^i and y_2^i are position vectors of the individual components of the binary and n^i are components of the unit vector $\hat{\mathbf{n}}$ along the relative separation vector.

For the relative velocity and relative acceleration, we have

$$v^i = \frac{\mathrm{d}x^i}{\mathrm{d}t} \quad \text{and} \quad \dot{r} = \hat{\mathbf{n}}\cdot\mathbf{v}, \tag{3.29}$$

$$a_i = \frac{\mathrm{d}v^i}{\mathrm{d}t} = \frac{\mathrm{d}^2x^i}{\mathrm{d}^2t}. \tag{3.30}$$

In addition, we also would need expressions for $\dot{v}$ and $\ddot{r}$, which can be given as

$$\dot{v} = \frac{\mathbf{a}\cdot\mathbf{v}}{v}, \tag{3.31}$$

$$\ddot{r} = \frac{1}{r}[(v^2-\dot{r}^2) + \mathbf{a}\cdot\mathbf{x}], \tag{3.32}$$

with $v = |\mathbf{v}|$.

While computing the time derivatives of the source multipole moments, whenever quantities like a^i or $\dot{v}$ or $\ddot{r}$ appear, they are consistently replaced by their expressions in terms of variables related to the position and velocities $(r, \dot{r}, v)$. Computations of various modes at the 3PN order would require the knowledge of the 3PN EOM governing the compact binary dynamics. EOM associated with SH coordinates at 3PN order for a system of two compact objects moving in general orbits is available in the literature [64,67,71–73] and are given in terms of the variables related to the position and the velocity of individual constituents of the binary. Since we are using expressions for the source multipole moments in the center-of-mass frame of the system, we need the EOM reduced to the CM frame. The 3PN accurate expression for relative acceleration, reduced to the CM frame, in SH coordinates, were obtained in [74]. However, as discussed in the previous section (about using MH or ADM coordinates instead of SH coordinates), we wish to use EOM reduced to CM frame associated with MH coordinate which is given in [75] and takes the following form,

$$a^i = -\frac{Gm}{r^2}\bigg\{\bigg[P_1 - \frac{\nu}{c^5}\bigg(\frac{136}{15}\frac{G^2m^2}{r^2}\dot{r} + \frac{24}{5}\frac{Gm}{r}\dot{r}v^2\bigg)\bigg]n^i + \bigg[P_2\frac{\dot{r}}{c^2} + \frac{\nu}{c^5}\bigg(\frac{24}{5}\frac{G^2m^2}{r^2} + \frac{8}{5}\frac{Gm}{r}v^2\bigg)\bigg]v^i\bigg\}, \tag{3.33}$$

where

$$
\begin{aligned}
P_1 = {} & 1+\frac{1}{c^2}\left[\frac{Gm}{r}(-4-2\nu)-\frac{3\dot{r}^2\nu}{2}+v^2(1+3\nu)\right]+\frac{1}{c^4}\left[\frac{G^2m^2}{r^2}\left(9+\frac{87\nu}{4}\right)+\dot{r}^4\left(\frac{15\nu}{8}-\frac{45\nu^2}{8}\right)\right.\\
&\left.+v^4(3\nu-4\nu^2)+\frac{Gm}{r}\dot{r}^2(-2-25\nu-2\nu^2)+v^2\left(\frac{Gm}{r}\left(-\frac{13\nu}{2}+2\nu^2\right)+\dot{r}^2\left(-\frac{9\nu}{2}+6\nu^2\right)\right)\right]\\
&+\frac{1}{c^6}\left[\frac{G^3m^3}{r^3}\left(-16-\frac{1399\nu}{12}+\frac{41\pi^2\nu}{16}-\frac{71\nu^2}{2}\right)+\frac{Gm}{r}\dot{r}^4\left(79\nu-\frac{69\nu^2}{2}-30\nu^3\right)\right.\\
&+\dot{r}^6\left(-\frac{35\nu}{16}+\frac{175\nu^2}{16}-\frac{175\nu^3}{16}\right)+\frac{G^2m^2}{r^2}\dot{r}^2\left(1+\frac{22717\nu}{168}+\frac{615\pi^2\nu}{64}+\frac{11\nu^2}{8}-7\nu^3\right)\\
&+v^6\left(\frac{11\nu}{4}-\frac{49\nu^2}{4}+13\nu^3\right)+v^4\left(\dot{r}^2\left(-\frac{15\nu}{2}+\frac{237\nu^2}{8}-\frac{45\nu^3}{2}\right)+\frac{Gm}{r}\left(\frac{75\nu}{4}+8\nu^2-10\nu^3\right)\right)\\
&\left.+v^2\left(\frac{G^2m^2}{r^2}\left(-\frac{20827\nu}{840}-\frac{123\pi^2\nu}{64}+\nu^3\right)+\frac{Gm}{r}\dot{r}^2(-121\nu+16\nu^2+20\nu^3)+\dot{r}^4\left(\frac{15\nu}{2}-\frac{135\nu^2}{4}+\frac{255\nu^3}{8}\right)\right)\right],
\end{aligned}
\tag{3.34a}
$$

$$
\begin{aligned}
P_2 = {} & -4+2\nu+\frac{1}{c^2}\left[v^2\left(-\frac{15\nu}{2}-2\nu^2\right)+\dot{r}^2\left(\frac{9\nu}{2}+3\nu^2\right)+\frac{Gm}{r}\left(2+\frac{41\nu}{2}+4\nu^2\right)\right]\\
&+\frac{1}{c^4}\left[\dot{r}^4\left(-\frac{45\nu}{8}+15\nu^2+\frac{15\nu^3}{4}\right)+v^4\left(-\frac{65\nu}{8}+19\nu^2+6\nu^3\right)\right.\\
&+\frac{G^2m^2}{r^2}\left(-4-\frac{5849\nu}{840}-\frac{123\pi^2\nu}{32}+25\nu^2+8\nu^3\right)+\frac{Gm}{r}\dot{r}^2\left(\frac{329\nu}{6}+\frac{59\nu^2}{2}+18\nu^3\right)\\
&\left.+v^2\left(\dot{r}^2\left(12\nu-\frac{111\nu^2}{4}-12\nu^3\right)+\frac{Gm}{r}(-15\nu-27\nu^2-10\nu^3)\right)\right].
\end{aligned}
\tag{3.34b}
$$

We now have all the inputs which are needed to compute the instantaneous expressions for various spherical harmonic modes ($h^{\ell m}$) associated with 3PN gravitational waveforms of GW signals from CCBs moving in general orbits. With this motivation, we shall proceed towards the next section where we shall present our results.

IV. INSTANTANEOUS TERMS IN THE 3PN GRAVITATIONAL WAVEFORM FOR CCBS IN GENERAL ORBITS

Combing Eq. (2.2) and Eq. (2.7), we can write the instantaneous part of various modes as

$$
h^{\ell m}_{\rm inst} = -\frac{G}{\sqrt{2}Rc^{\ell+2}}\frac{4}{\ell!}\sqrt{\frac{(\ell+1)(\ell+2)}{2\ell(\ell-1)}}\alpha_L^{\ell m}U_L^{\rm inst} \quad \text{if } \ell+m \text{ is even,} \tag{4.1}
$$

$$
h^{\ell m}_{\rm inst} = -\frac{iG}{\sqrt{2}Rc^{\ell+3}}\frac{8}{\ell!}\sqrt{\frac{\ell(\ell+2)}{2(\ell+1)(\ell-1)}}\alpha_L^{\ell m}V_L^{\rm inst} \quad \text{if } \ell+m \text{ is odd.} \tag{4.2}
$$

Relations connecting the instantaneous part of STF radiative moments ($U_L^{\rm inst}$ and $V_L^{\rm inst}$) to the source multipole moments have been listed in the previous section. This allows one to write the instantaneous part of various modes ($h^{\ell m}_{\rm inst}$) in terms of the source multipole moments. With expressions for the source multipole moments for CCBs moving in general orbits and their relevant time derivatives, one can write expressions for various modes in terms of dynamical variables related to the position and velocity (r, $\dot{r}$, ϕ, v).[8]

Again, since $v^2=\dot{r}^2+r^2\dot{\phi}^2$, we can write various modes of the waveform in terms of the dynamical variables, namely, the radial separation (r), radial velocity ($\dot{r}$), orbital phase (ϕ), and angular velocity ($\dot{\phi}$) . The structure of $h^{\ell m}$ reads

$$
h^{\ell m}_{\rm inst} = \frac{4Gm\nu}{c^4R}\sqrt{\frac{\pi}{5}}e^{-im\phi}\hat{H}^{\ell m}_{\rm inst}. \tag{4.3}
$$

For the dominant mode ($\ell=2, m=2$), with 3PN accuracy, various PN pieces of the coefficient $\hat{H}^{22}_{\rm inst}$ read

$$
(\hat{H}^{22}_{\rm inst})_{\rm Newt} = \frac{Gm}{r}+r^2\dot{\phi}^2+2ir\dot{r}\dot{\phi}-\dot{r}^2, \tag{4.4a}
$$

[8]Alternatively, one can also compute various modes associated with the gravitational waveform using polarization waveforms (see Sec. II and IX of Ref. [40] for the details).

$$(\hat{H}^{22}_{\rm inst})_{\rm 1PN} = \frac{1}{c^2}\left[\frac{G^2m^2}{r^2}\left(-5+\frac{\nu}{2}\right)+\frac{Gm\dot{r}^2}{r}\left(-\frac{15}{14}-\frac{16\nu}{7}\right)+\left(-\frac{9}{14}+\frac{27\nu}{14}\right)\dot{r}^4+r\left(\frac{9i}{7}-\frac{27i\nu}{7}\right)\dot{r}^3\dot{\phi}\right.$$
$$\left.+Gmr\left(\frac{11}{42}+\frac{26\nu}{7}\right)\dot{\phi}^2+r^4\left(\frac{9}{14}-\frac{27\nu}{14}\right)\dot{\phi}^4+\dot{r}\left(Gm\left(\frac{25i}{21}+\frac{45i\nu}{7}\right)\dot{\phi}+r^3\left(\frac{9i}{7}-\frac{27i\nu}{7}\right)\dot{\phi}^3\right)\right], \quad (4.4b)$$

$$(\hat{H}^{22}_{\rm inst})_{\rm 2PN} = \frac{1}{c^4}\left[\frac{G^3m^3}{r^3}\left(\frac{757}{63}+\frac{181\nu}{36}+\frac{79\nu^2}{126}\right)+\left(-\frac{83}{168}+\frac{589\nu}{168}-\frac{1111\nu^2}{168}\right)\dot{r}^6+r\left(\frac{83i}{84}-\frac{589i\nu}{84}+\frac{1111i\nu^2}{84}\right)\dot{r}^5\dot{\phi}\right.$$
$$+G^2m^2\left(-\frac{11891}{1512}-\frac{5225\nu}{216}+\frac{13133\nu^2}{1512}\right)\dot{\phi}^2+Gmr^3\left(\frac{835}{252}+\frac{19\nu}{252}-\frac{2995\nu^2}{252}\right)\dot{\phi}^4$$
$$+r^6\left(\frac{83}{168}-\frac{589\nu}{168}+\frac{1111\nu^2}{168}\right)\dot{\phi}^6+\dot{r}^4\left(\frac{Gm}{r}\left(-\frac{557}{168}+\frac{83\nu}{21}+\frac{214\nu^2}{21}\right)+r^2\left(-\frac{83}{168}+\frac{589\nu}{168}-\frac{1111\nu^2}{168}\right)\dot{\phi}^2\right)$$
$$+\dot{r}^3\left(Gm\left(\frac{863i}{126}-\frac{731i\nu}{63}-\frac{211i\nu^2}{9}\right)\dot{\phi}+r^3\left(\frac{83i}{42}-\frac{589i\nu}{42}+\frac{1111i\nu^2}{42}\right)\dot{\phi}^3\right)$$
$$+\dot{r}^2\left(\frac{G^2m^2}{r^2}\left(\frac{619}{252}-\frac{2789\nu}{252}-\frac{467\nu^2}{126}\right)+Gmr\left(\frac{11}{28}-\frac{169\nu}{14}-\frac{58\nu^2}{21}\right)\dot{\phi}^2+r^4\left(\frac{83}{168}-\frac{589\nu}{168}+\frac{1111\nu^2}{168}\right)\dot{\phi}^4\right)$$
$$+\dot{r}\left(\frac{G^2m^2}{r}\left(-\frac{773i}{189}-\frac{3767i\nu}{189}+\frac{2852i\nu^2}{189}\right)\dot{\phi}+Gmr^2\left(\frac{433i}{84}+\frac{103i\nu}{12}-\frac{1703i\nu^2}{84}\right)\dot{\phi}^3\right.$$
$$\left.\left.+r^5\left(\frac{83i}{84}-\frac{589i\nu}{84}+\frac{1111i\nu^2}{84}\right)\dot{\phi}^5\right)\right], \quad (4.4c)$$

$$(\hat{H}^{22}_{\rm inst})_{\rm 2.5PN} = \frac{1}{c^5}\left[-\frac{122G^2m^2\nu\dot{r}^3}{35r^2}-\frac{468iG^3m^3\nu\dot{\phi}}{35r^2}+\frac{184iG^2m^2\nu\dot{r}^2\dot{\phi}}{35r}-\frac{316}{35}iG^2m^2r\nu\dot{\phi}^3+\dot{r}\left(\frac{2G^3m^3\nu}{105r^3}-\frac{121}{5}G^2m^2\nu\dot{\phi}^2\right)\right], \quad (4.4d)$$

$$(\hat{H}^{22}_{\rm inst})_{\rm 3PN} = \frac{1}{c^6}\left[\frac{G^4m^4}{r^4}\left(-\frac{512714}{51975}+\left(-\frac{1375951}{13860}+\frac{41\pi^2}{16}\right)\nu+\frac{1615\nu^2}{616}+\frac{2963\nu^3}{4158}-\frac{214}{105}\log\left[\frac{r}{r_0}\right]\right)\right.$$
$$+\left(-\frac{507}{1232}+\frac{6101\nu}{1232}-\frac{12525\nu^2}{616}+\frac{34525\nu^3}{1232}\right)\dot{r}^8+r\left(\frac{507i}{616}-\frac{6101i\nu}{616}+\frac{12525i\nu^2}{308}-\frac{34525i\nu^3}{616}\right)\dot{r}^7\dot{\phi}$$
$$+\frac{G^3m^3}{r}\left(\frac{42188851}{415800}+\left(\frac{190703}{3465}-\frac{123\pi^2}{64}\right)\nu-\frac{18415\nu^2}{308}+\frac{281473\nu^3}{16632}-\frac{214}{15}\log\left[\frac{r}{r_0}\right]\right)\dot{\phi}^2$$
$$+G^2m^2r^2\left(\frac{328813}{55440}-\frac{374651\nu}{33264}+\frac{249035\nu^2}{4158}-\frac{1340869\nu^3}{33264}\right)\dot{\phi}^4$$
$$+Gmr^5\left(\frac{12203}{2772}-\frac{36427\nu}{2772}-\frac{13667\nu^2}{1386}+\frac{49729\nu^3}{924}\right)\dot{\phi}^6+r^8\left(\frac{507}{1232}-\frac{6101\nu}{1232}+\frac{12525\nu^2}{616}-\frac{34525\nu^3}{1232}\right)\dot{\phi}^8$$
$$+\dot{r}^4\left(\frac{G^2m^2}{r^2}\left(-\frac{92567}{13860}+\frac{7751\nu}{396}+\frac{400943\nu^2}{11088}+\frac{120695\nu^3}{3696}\right)\right.$$
$$\left.+Gmr\left(-\frac{42811}{11088}+\frac{6749\nu}{1386}+\frac{19321\nu^2}{693}-\frac{58855\nu^3}{1386}\right)\dot{\phi}^2\right)+\dot{r}^6\left(\frac{Gm}{r}\left(-\frac{5581}{1232}+\frac{4694\nu}{231}-\frac{3365\nu^2}{462}-\frac{1850\nu^3}{33}\right)\right.$$
$$\left.+r^2\left(-\frac{507}{616}+\frac{6101\nu}{616}-\frac{12525\nu^2}{308}+\frac{34525\nu^3}{616}\right)\dot{\phi}^2\right)+\dot{r}^5\left(Gm\left(\frac{17233i}{1848}-\frac{31532i\nu}{693}+\frac{65575i\nu^2}{2772}+\frac{85145i\nu^3}{693}\right)\dot{\phi}\right.$$
$$\left.+r^3\left(\frac{1521i}{616}-\frac{18303i\nu}{616}+\frac{37575i\nu^2}{308}-\frac{103575i\nu^3}{616}\right)\dot{\phi}^3\right)$$

$$
\begin{aligned}
&+\dot{r}^3\left(\frac{G^2m^2}{r}\left(\frac{39052i}{3465}-\frac{154114i\nu}{2079}-\frac{246065i\nu^2}{4158}-\frac{365725i\nu^3}{4158}\right)\dot{\phi}\right.\\
&+Gmr^2\left(\frac{13867i}{792}-\frac{191995i\nu}{2772}-\frac{8741i\nu^2}{5544}+\frac{52700i\nu^3}{231}\right)\dot{\phi}^3+r^5\left(\frac{1521i}{616}-\frac{18303i\nu}{616}+\frac{37575i\nu^2}{308}-\frac{103575i\nu^3}{616}\right)\dot{\phi}^5\Big)\\
&+\dot{r}^2\left(\frac{G^3m^3}{r^3}\left(\frac{913799}{29700}+\left(\frac{174679}{2310}+\frac{123\pi^2}{32}\right)\nu-\frac{158215\nu^2}{2772}-\frac{12731\nu^3}{4158}-\frac{428}{105}\log\left[\frac{r}{r_0}\right]\right)\right.\\
&+G^2m^2\left(\frac{20191}{18480}-\frac{3879065\nu}{33264}-\frac{411899\nu^2}{8316}-\frac{522547\nu^3}{33264}\right)\dot{\phi}^2\\
&+Gmr^3\left(\frac{381}{77}-\frac{101237\nu}{2772}+\frac{247505\nu^2}{5544}+\frac{394771\nu^3}{5544}\right)\dot{\phi}^4+r^6\left(\frac{507}{616}-\frac{6101\nu}{616}+\frac{12525\nu^2}{308}-\frac{34525\nu^3}{616}\right)\dot{\phi}^6\Big)\\
&+\dot{r}\left(\frac{G^3m^3}{r^2}\left(-\frac{68735i}{378}+\left(-\frac{57788i}{315}+\frac{123i\pi^2}{32}\right)\nu-\frac{701i\nu^2}{27}+\frac{11365i\nu^3}{378}+\frac{428}{21}i\log\left[\frac{r}{r_0}\right]\right)\dot{\phi}\right.\\
&+G^2m^2r\left(\frac{91229i}{13860}+\frac{97861i\nu}{4158}+\frac{919811i\nu^2}{8316}-\frac{556601i\nu^3}{8316}\right)\dot{\phi}^3+Gmr^4\left(\frac{6299i}{792}-\frac{68279i\nu}{5544}-\frac{147673i\nu^2}{2772}+\frac{541693i\nu^3}{5544}\right)\dot{\phi}^5\\
&+r^7\left(\frac{507i}{616}-\frac{6101i\nu}{616}+\frac{12525i\nu^2}{308}-\frac{34525i\nu^3}{616}\right)\dot{\phi}^7\Big)\Big].
\end{aligned}
\tag{4.4e}
$$

As mentioned in Sec. I, since expressions for various modes are very large and would run over several pages, we have decided to provide an additional file (Hlm-GenOrb.m) containing expressions for all the modes which will be made available along with the paper. Finally, circular-orbit limit of the instantaneous $h_{\ell m}$ can be obtained by replacing related expressions for $\dot{\phi}(=\omega)$, $\dot{r}$ and r given in Sec. IV of [40].

Note that $h_{\ell m}$ can directly be used to write the polarization waveforms (h_+, $h_\times$) using the standard decomposition of h_+ and $h_\times$ in terms of spherical harmonic modes of spin weight -2 given in Ref. [40,41],

$$h_+ - ih_\times = \sum_{\ell=2}^{+\infty}\sum_{m=-\ell}^{\ell} h^{\ell m} Y_{-2}^{\ell m}(\Theta,\Phi), \tag{4.5}$$

where $Y_{-2}^{\ell m}$'s (the spin-weighted spherical harmonics of weight -2) are functions of the spherical angles (Θ,Φ) defining the binary's location and given as

$$Y_{-2}^{\ell m} = \sqrt{\frac{2\ell+1}{4\pi}}\, d_2^{\ell m}(\Theta) e^{im\Phi}, \tag{4.6a}$$

$$d_2^{\ell m} = \sum_{k=k_1}^{k_2} \frac{(-)^k}{k!}\frac{\sqrt{(\ell+m)!(\ell-m)!(\ell+2)!(\ell-2)!}}{(k-m+2)!(\ell+m-k)!(\ell-k-2)!} \times \left(\cos\frac{\Theta}{2}\right)^{2\ell+m-2k-2}\left(\sin\frac{\Theta}{2}\right)^{2k-m+2}, \tag{4.6b}$$

with $k_1 = \max(0, m-2)$ and $k_2 = \min(\ell+m, \ell-2)$.

Further, the polarization waveform can be used to write the transverse-traceless part of the radiation field (h_{ij}) by using the following relation [40,41]

$$h_{ij}^{\mathrm{TT}} = 2(h_+ e_{ij}^+ + h_\times e_{ij}^\times), \tag{4.7}$$

where

$$e_{ij}^+ = \frac{1}{2}(P_iP_j - Q_iQ_j), \tag{4.8a}$$

$$e_{ij}^\times = \frac{1}{2}(P_iQ_j + P_jQ_i). \tag{4.8b}$$

Here **P** and **Q** are the two unit polarization vectors, and they have been chosen following the convention used in [40].

V. QUASI-KEPLERIAN REPRESENTATION

In the previous section, we presented general orbit expressions for the dominant mode of the gravitational waveform from a compact binary system. In this section we aim to specialize to the case of compact binary systems in quasielliptical orbits. Expressions for various $h_{\ell m}$ describing the radiation from binaries in quasielliptical orbits can simply be obtained by using relations connecting generic dynamical variable r, $\dot{r}$, ϕ and $\dot{\phi}$ to a set of parameters associated with elliptical orbits in general orbit expressions for various modes. Such relations can be established by using the generalized quasi-Keplerian (QK) representation of the conservative dynamics of the binary moving in eccentric orbits, which indeed is available to us due to the work of Memmesheimer, Gopakumar, and Schäfer (hereafter MGS) [54].

The QK representation was first introduced by Damour and Deruelle [76] and dealt with the binary dynamics at 1PN order. The generalized QK representation at 2PN order in ADM-type coordinates was given in Refs. [77–79]. MGS provides the 3PN generalized QK representation in both the ADM and MH coordinates, which involve expressions of the orbital elements associated with the orbit of the binary in terms of the conserved energy and orbital angular momentum of the binary. Before we get into the details of the parametrization, we first summarize equations describing the radial and angular motion of the binary in terms of various orbital elements associated with elliptical orbits (see Refs. [53,56] for details). In the parametric form, the radial separation, r, is given by

$$r = a_r(1 - e_r \cos u), \tag{5.1}$$

where a_r is the semi-major axis of the orbit and e_r is the eccentricity of the orbit (both labeled after the radial coordinate, r). The quantity u is called eccentric anomaly and at the 3PN order it is related to the mean anomaly (l) by the relation

$$l = u - e_t \sin u + f_t \sin V + g_t[V - u] + i_t \sin 2V + h_t \sin 3V. \tag{5.2}$$

The orbital phase, ϕ, at the 3PN order reads

$$\phi = \phi_P + K[V + f_\phi \sin 2V + g_\phi \sin 3V + i_\phi \sin 4V + h_\phi \sin 5V], \tag{5.3}$$

where ϕ_P is the initial phase at the first passage of the periastron and V is the true anomaly that takes the form

$$V \equiv V(u, e_\phi) = 2 \arctan\left[\sqrt{\frac{1+e_\phi}{1-e_\phi}} \tan\left(\frac{u}{2}\right)\right]. \tag{5.4}$$

Also, the mean anomaly, l, is related to the time as

$$l = n(t - t_P), \tag{5.5}$$

where t_P is the instant of the first passage at the periastron and $n = 2\pi/P$ is the mean motion with P being the orbital period. In addition to this, expressions for the radial and angular velocity can be given as

$$\dot{r} = a_r e_r \sin u \left(\frac{\partial l}{\partial u}\right)^{-1} \frac{\partial l}{\partial t}, \tag{5.6a}$$

$$\dot{\phi} = K(1 + 2f_\phi \cos 2V + 3g_\phi \cos 3V + 4i_\phi \cos 4V + 5h_\phi \cos 5V) \frac{\partial V}{\partial u} \left(\frac{\partial l}{\partial u}\right)^{-1} \frac{\partial l}{\partial t}. \tag{5.6b}$$

It may be seen from Eqs. (5.2), (5.4), and (5.5) that

$$\frac{\partial l}{\partial t} = n, \tag{5.7a}$$

$$\frac{\partial l}{\partial u} = 1 - e_t \cos u + f_t \cos V \frac{\partial V}{\partial u} + g_t\left(\frac{\partial V}{\partial u} - 1\right) + 2i_t \cos 2V \frac{\partial V}{\partial u} + 3h_t \cos 3V \frac{\partial V}{\partial u}. \tag{5.7b}$$

$$\frac{\partial V}{\partial u} = \frac{(1 - e_\phi^2)^{1/2}}{(1 - e_\phi \cos u)}. \tag{5.7c}$$

In the above, e_ϕ and e_t denote eccentricities related to the coordinates ϕ and t, respectively. K is related to the advance of the periastron per orbit and is given by $K = \Phi/(2\pi)$, where Φ is the angle of return to the periastron. In this parametrization, f_t, f_ϕ, g_t, and g_ϕ contribute both at 2PN and 3PN order, whereas i_t, i_ϕ, h_t, and h_ϕ contribute only at the 3PN order (see Ref. [56] for related details).

Once we have written equations connecting the generic dynamical variables (r, $\dot{r}$, ϕ, and $\dot{\phi}$) to the orbital elements of the elliptical orbit, we can use inputs from MGS to express them in terms of a suitable set of parameters of our choice. The main result of MGS is that, it provides 3PN accurate expressions for various orbital elements ($a_r, e_t, e_r, e_\phi \ldots$) associated with the elliptical orbits in terms of the corresponding conserved energy per unit reduced mass (E) and the parameter h, related to the reduced angular momentum (J), by $h = J/Gm$.[9] Using these relations, one can express the dynamical variables r, $\dot{r}$, ϕ, and $\dot{\phi}$ in terms of E, h, and u. Here one should note that this is not the only way in which the orbital dynamics can be parametrized. In fact, one can reexpress E and h in terms of any of the two orbital elements to write equations describing the orbital motion of the binary; however, a parametrization involving gauge invariant parameters is sometimes preferred as such parametrization is suitable for making comparisons with related numerical results. This led MGS to use a parametrization involving n and $K = \Phi/(2\pi)$ (both are independent of the coordinate system used when expressed in terms of E and h in order to describe the orbital motion of the binary in elliptical

[9] In Ref. [56], which uses results obtained in MGS, orbital elements are expressed in terms of the parameters $\{\epsilon, j\}$ instead of $\{E, h\}$, where ϵ and j are defined as $\epsilon = -2E/c^2$ and $j = -2Eh^2$.

orbits).[10] Here, n is the mean motion and $K = \Phi/(2\pi)$ denotes the angle of the advance of the periastron per orbital revolution. In a work related to the phasing of the GWs from inspiralling compact binary in elliptical orbit due to Damour, Gopakumar, and Iyer [52], the orbital dynamics has been described using n and e_t as parametrizing variables. Following the conventions of [52], Königsdöfer and Gopakumar [53] provided the 3PN accurate expressions for r, $\dot{r}$, ϕ, and $\dot{\phi}$ in terms of n, e_t, and u.[11] Reference [56] makes an alternative choice of parametrization in terms of variables x and e_t, where x is related to the orbital frequency ω by $x = (Gm\omega/c^3)^{2/3}$ and is independent of the choice of the coordinate system used.[12] The choice $\{x, e_t\}$ as parametrizing variables leads to expressions which can be reduced to those related to the circular orbit case ($e_t \to 0$) which uses x as the expansion parameter. In addition to this, in another related work Hinder *et al.* [39] compared the two parameterizations and concluded that the choice of x as compared to n provides better agreement with NR results. They separately discuss these two PN models (based on the choice of parametrization as x or n) and call them x-model and n-model. Note that although [39] uses 3PN QK representation describing the conserved dynamics, it used only 2PN accurate dissipative dynamics and, hence, when used to obtain quasicircular orbit limit of the orbital phase, it shows some deviations from related standard results as was pointed out in Ref. [80]. As discussed below, our results must be coupled with 3PN evolution equations for QK variables given in Ref. [57] and, hence, would be more suitable for obtaining quasicircular limits of the amplitude and phase of various modes.

Following the arguments presented above, we choose to parametrize the dynamical variables $\{r, \dot{r}, \phi, \dot{\phi}\}$ in terms of the QK variables $\{x, e_t, u\}$ which can further be used to write expressions for various $h_{\ell m}$ in terms of the QK variables. Note that Ref. [39] already lists the 3PN expressions for r and $\dot{\phi}$ in terms of $\{x, e_t, u\}$; however, they numerically differentiate r to obtain $\dot{r}$ and perform numerical integration of $\dot{\phi}$ to obtain ϕ in order to avoid the use of long and complicated expressions for $\dot{r}$ and ϕ in their numerical code. As mentioned above, Ref. [53] lists 3PN accurate expressions for $\{r, \dot{r}, \phi, \dot{\phi}\}$ in terms of the QK variables $\{n, e_t, u\}$ in MH coordinates (see Eqs. (23)–(27) there). In fact, they use a variable ξ related to n by $\xi = (Gmn/c^3)$. Hence, to obtain related expressions parametrized in terms of $\{x, e_t, u\}$, all we need to know is how ξ is related to x and e_t. The relation between ξ and our QK variables x and e_t with 3PN accuracy in MH coordinates is given as

$$\xi = \frac{x^{3/2}}{(1-e_t^2)^3}\Bigg\{1 - 3e_t^2 + 3e_t^4 - e_t^6 + x(-3 + 6e_t^2 - 3e_t^4) + x^2\left[-\frac{9}{2} + 7\nu + \left(-\frac{33}{4} - \frac{\nu}{2}\right)e_t^2 + \left(\frac{51}{4} - \frac{13\nu}{2}\right)e_t^4\right] + x^3\Bigg[\frac{3}{2} + \nu\left(\frac{457}{4} - \frac{123\pi^2}{32}\right) - 7\nu^2 + \left(-\frac{267}{4} + \nu\left(\frac{279}{2} - \frac{123\pi^2}{128}\right) - 40\nu^2\right)e_t^2 + \left(-\frac{39}{2} + \frac{55\nu}{4} - \frac{65\nu^2}{8}\right)e_t^4 + \sqrt{1-e_t^2}(-15 + 6\nu + (-30 + 12\nu)e_t^2)\Bigg]\Bigg\}. \tag{5.8}$$

Using the above in Eqs. (23)–(27) of Ref. [53], one can write expressions for r, $\dot{r}$, ϕ, and $\dot{\phi}$ in terms of the variables $\{x, e_t, u\}$ and then use them to obtain the expressions for the spherical harmonic modes of the waveform for quasielliptical orbits.

We now are in a position to use the expressions for r, $\dot{r}$, ϕ, and $\dot{\phi}$ in terms of the variables $\{x, e_t, u\}$ to reexpress the instantaneous part of spin-weighted spherical harmonic modes ($h^{\ell m}_{\rm inst}$) in terms of parameters x, e_t, and u. Just like the expressions for various $h_{\ell m}$ in general orbits we find that the expressions for various modes in QK representation is too large to be given in the main text of the paper. Hence, we will just provide the expression for the dominant mode here and list all the relevant modes in an additional file (Hlm-EllOrb.m). In addition, since even the h_{22} expression runs over many pages, we provide the PN structure of h_{22} in the main text of the paper and list explicit expressions for various PN pieces in Appendix A to maintain the flow of discussion in the main text of the paper.

The structure of various modes, $h^{\ell m}$, remains the same as in Eq. (4.3); however, now the coefficients $\hat{H}^{\ell m}_{\rm inst}$ and ϕ are functions of the parameters $\{x, e_t, u\}$,

$$h^{\ell m}_{\rm inst} = \frac{4Gm\nu x}{c^2 R}\sqrt{\frac{\pi}{5}}e^{-im\phi}\hat{H}^{\ell m}_{\rm inst}. \tag{5.9}$$

The instantaneous part of the dominant mode ($h^{22}_{\rm inst}$) reads

[10] In fact, MGS uses $x_{\rm MGS} = (Gmn/c^3)^{2/3}$ and the parameter $k' = (K-1)/3$.

[11] In fact, [53] is the extension of [52] and discusses the 3PN conservative dynamics of the binaries in elliptical orbits.

[12] Parameters n and ω are related by $n = K\omega$, and K has been defined above.

$$h^{22}_{\rm inst} = \frac{4Gm\nu x}{c^2 R}\sqrt{\frac{\pi}{5}}e^{-2i\phi}\hat{H}^{22}_{\rm inst}, \tag{5.10}$$

with

$$H^{22}_{\rm inst} = H^{22}_{\rm Newt} + H^{22}_{\rm 1PN} + H^{22}_{\rm 2PN} + H^{22}_{\rm 2.5PN} + H^{22}_{\rm 3PN}, \tag{5.11}$$

where various PN pieces appearing in Eq. (5.11) are listed in Appendix A. Before we move to the concluding section, we would like to make a few remarks about the results presented here.

We observe logarithmic dependences on the arbitrary length scale r_0 in Eq. (A1e) through the quantity x_0 which is related to r_0 by $x_0 = (Gm/c^2 r_0)$. This dependence is due to the presence of terms involving logarithms of r_0 in the expression for the mass quadrupole moment (I_{ij}) given by Eqs. (3.19)–(3.20). As was discussed in Sec. III B we would expect such dependences to disappear from final expression for various modes (for instance, see Ref. [40] which lists $h_{\ell m}$ for binaries in quasicircular orbits). As has been observed in Refs. [40,41,56,70], it turns out the hereditary contribution has equal and opposite dependences on the arbitrary length scale r_0 and cancels out from the final expression. Since we do not provide hereditary contributions in this paper, such cancellation cannot be shown here. However, we explicitly show this cancellation in [61] which deals with computation of hereditary effects in various modes at 3PN order.

In this paper we used the 3PN accurate QK representation which describes the conserved dynamics of CCBs in eccentric orbits to obtain various modes in terms of QK variables (x, e_t). However, it should be noted that the parameters x, e_t evolve with time over radiation reaction time scales and these secular effects start to show at 2.5PN order [53,57]. Hence, in order to correctly account for the reactive dynamics of the binary at 3PN order our results should always be coupled with equations describing secular time-evolution of x and e_t. Equations describing the secular evolution of orbital elements with relative 3PN accuracies were presented in Ref. [57] (see Sec. VI there). Evolution equations (due to instantaneous terms in energy and angular momentum loss) for orbital frequency ($d\omega/dt$ (related to x by $\omega = (c^3 x^{3/2}/Gm)$) and for time-eccentricity parameter (de_t/dt) in terms of x and e_t have been listed as Eqs. (6.14)–(6.15) and Eqs. (6.18)–(6.19), respectively, in Ref. [57].

VI. SUMMARY AND CONCLUDING REMARKS

In this paper we presented computations of the instantaneous contributions to all the relevant modes of the 3PN accurate gravitational waveform of the GW signal from nonspinning coalescing compact binaries in general orbits. The expression for the instantaneous part of the dominant mode ($h^{\ell m}_{\rm inst}$), in terms of the variables r, $\dot{r}$, ϕ, and $\dot{\phi}$, has been given by Eqs. (4.3)–(4.4) above, whereas expressions for other subdominant modes (along with the dominant mode) have been listed in a separate file (Hlm-GenOrb.m) that is being made available along with the paper. Next, we specialized to the case of CCBs in quasielliptical orbits using the 3PN quasi-Keplerian representation of the conserved dynamics of compact binaries in eccentric orbits in Sec. V. Related 3PN accurate expressions for the instantaneous part of the dominant mode ($h^{\ell m}_{\rm inst}$), in terms of the variables, namely, the time-eccentricity e_t, a PN parameter x and eccentric anomaly u, is given by Eqs. (5.10)–(5.11) and Eq. (A1). The expressions for other sub-dominant modes (along with the dominant mode) have been listed in a separate file (Hlm-EllOrb.m) that is being made available along with the paper.

We once again remind the readers that the results presented here only account for the contributions from the instantaneous terms in the waveform which must be complemented by computations accounting for the hereditary effects. Our investigations suggest that it is not possible to provide closed-form analytical expressions for the hereditary terms for binaries moving in general orbits. Moreover, even for the special case of CCBs in quasielliptical orbits, it may not be possible to have closed-form analytical expressions for the hereditary terms valid for systems with arbitrary eccentricities. However, we find that such computations can be performed assuming an expansion in the eccentricity parameter (e_t) [61]. Unlike the results presented in this paper, which can be applied to a binary with arbitrary eccentricity, results of [61] can only be applied to systems with small eccentricities. However, the positive side of the work is that we shall have complete 3PN analytical expression for the waveform for binaries in quasielliptical orbits that can be used for comparison with related numerical relativity results, which is one of the motivations for high PN order computations of the gravitational waveforms. In addition, with complete waveforms at hand, one would be able to write complete polarization waveforms (as discussed in Sec. IV) which would be useful for data analysis purposes.

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APPENDIX: VARIOUS PN PIECES OF THE COEFFICIENT $\hat{H}^{22}$ ASSOCIATED WITH THE DOMINANT MODE h^{22}

$$(\hat{H}^{22}_{\rm inst})_{\rm Newt} = \frac{2}{(1-e_t\cos u)^2}\left\{1-e_t^2-\frac{1}{2}(e_t\cos u)+\frac{1}{2}(e_t\cos u)^2+i(e_t\sin u)\sqrt{1-e_t^2}\right\}, \tag{A1a}$$

$$\begin{aligned}(\hat{H}^{22}_{\rm inst})_{\rm 1PN} = -\frac{x}{42(1-e_t^2)(1-e_t\cos u)^3}\Big\{&214-110\nu+e_t^2(64+46\nu)+e_t^4(-278+64\nu)+(e_t\cos u)(-405+123\nu\\&+e_t^2(207-89\nu)+e_t^4(114-34\nu))+(e_t\cos u)^2(54+34\nu+e_t^2(114-34\nu))+(e_t\cos u)^3(-27-17\nu\\&+e_t^2(-57+17\nu))+i(e_t\sin u)\sqrt{1-e_t^2}(-20-38\nu+e_t^2(272-46\nu)\\&+(e_t\cos u)(-138+50\nu+e_t^2(-114+34\nu)))\Big\},\end{aligned} \tag{A1b}$$

$$\begin{aligned}(\hat{H}^{22}_{\rm inst})_{\rm 2PN} = \frac{x^2}{3024(1-e_t^2)^2(1-e_t\cos u)^5}\{&-38932-17836\nu+8188\nu^2+e_t^2(182850-92982\nu-3966\nu^2)\\&+e_t^4(-196098+212448\nu-30360\nu^2)+e_t^6(53374-133790\nu+39866\nu^2)\\&+e_t^8(-1194+32160\nu-13728\nu^2)+(e_t\cos u)(-4628+84514\nu-30202\nu^2\\&+e_t^2(-121926-34230\nu+65946\nu^2)+e_t^4(199848-80970\nu-41286\nu^2)+e_t^6(-35494+3470\nu+5542\nu^2))\\&+(e_t\cos u)^2(14904+7158\nu+18558\nu^2+e_t^2(-151740+4032\nu-27288\nu^2)+e_t^4(4320+86670\nu-1098\nu^2)\\&+e_t^6(-18684+11004\nu+9828\nu^2))+(e_t\cos u)^3(104628-84714\nu-9786\nu^2\\&+e_t^2(140196-58056\nu+19248\nu^2)+e_t^4(-23604-11202\nu-9138\nu^2)+e_t^6(5580-9324\nu-324\nu^2))\\&+(e_t\cos u)^4\times(-102888+57384\nu-648\nu^2+e_t^2(-59472+70128\nu+1296\nu^2)\\&+e_t^4(11160-18648\nu-648\nu^2))+(e_t\cos u)^5(25722-14346\nu+162\nu^2+e_t^2(14868-17532\nu-324\nu^2)\\&+e_t^4(-2790+4662\nu+162\nu^2))+i(e_t\sin u)(75600-30240\nu+e_t^2(-196560+78624\nu)\\&+e_t^4(120960-48384\nu)+(e_t\cos u)(-136080+54432\nu+e_t^2(408240-163296\nu)\\&+e_t^4(-272160+108864\nu))+(e_t\cos u)^2(45360-18144\nu+e_t^2(-226800+90720\nu)\\&+e_t^4(181440-72576\nu))+(e_t\cos u)^3(15120-6048\nu+e_t^2(15120-6048\nu)+e_t^4(-30240+12096\nu)))\\&+\sqrt{1-e_t^2}(30240-12096\nu+e_t^2(-151200+60480\nu)+e_t^4(120960-48384\nu)\\&+(e_t\cos u)(-37800+15120\nu+e_t^2(309960-123984\nu)+e_t^4(-272160+108864\nu))\\&+(e_t\cos u)^2(37800-15120\nu+e_t^2(-219240+87696\nu)+e_t^4(181440-72576\nu))\\&+(e_t\cos u)^3(-98280+39312\nu+e_t^2(128520-51408\nu)+e_t^4(-30240+12096\nu))\\&+(e_t\cos u)^4(83160-33264\nu+e_t^2(-83160+33264\nu))+(e_t\cos u)^5(-15120+6048\nu+e_t^2(15120-6048\nu))\\&+i(e_t\sin u)(-93152+50476\nu+4844\nu^2+e_t^2(105604-154688\nu-4624\nu^2)\\&+e_t^4(-47816+142684\nu-10612\nu^2)+e_t^6(5124-44520\nu+7368\nu^2)+(e_t\cos u)\\&\times(153308-10456\nu-9488\nu^2+e_t^2(-81832+21920\nu+25024\nu^2)+e_t^4(19244+6680\nu-6464\nu^2))\\&+(e_t\cos u)^2(-82596+18648\nu-1464\nu^2+e_t^2(-26904-11664\nu-2256\nu^2)+e_t^4(18780-25128\nu-5352\nu^2))\\&+(e_t\cos u)^3(17316-14148\nu-1260\nu^2+e_t^2(18504+10872\nu+3960\nu^2)\\&+e_t^4(-5580+9324\nu+324\nu^2))))\},\end{aligned} \tag{A1c}$$

$$(\hat{H}^{22}_{\rm inst})_{\rm 2.5PN} = \frac{ix^{5/2}\nu}{105(1-e_t\cos u)^5}\Big\{\sqrt{1-e_t^2}(-2352+1500e_t^2+1404(e_t\cos u)-552(e_t\cos u)^2) + i(e_t\sin u)(2539-2175e_t^2+2(e_t\cos u)-366(e_t\cos u)^2)\Big\}, \tag{A1d}$$

$$(\hat{H}^{22}_{\rm inst})_{\rm 3PN} = \frac{x^3}{1277337600(1-e_t^2)^3(1-e_t\cos u)^8}\Big\{186870371328-20826685440\log(1-e_t\cos u) + 20826685440\log\left(\frac{x}{x_0}\right)+15657907200\nu-13806489600\nu^2+2934656000\nu^3 + e_t^2\bigg(-319951363584+75496734720\log(1-e_t\cos u)-75496734720\log\left(\frac{x}{x_0}\right) +\nu(-572456762880+18002476800\pi^2)+128784537600\nu^2-2970944000\nu^3\bigg) + e_t^4\bigg(7924875264-101530091520\log(1-e_t\cos u)+101530091520\log\left(\frac{x}{x_0}\right) +\nu(1455622894080-43369603200\pi^2)-240132902400\nu^2+5117260800\nu^3\bigg) + e_t^6\bigg(292424159232+59876720640\log(1-e_t\cos u)-59876720640\log\left(\frac{x}{x_0}\right) +\nu(-1579426229760+34368364800\pi^2)+359519040000\nu^2-32956160000\nu^3\bigg) + e_t^8\bigg(-171895234560-13016678400\log(1-e_t\cos u)+13016678400\log\left(\frac{x}{x_0}\right) +\nu(873551592960-9001238400\pi^2)-368015424000\nu^2+53854515200\nu^3\bigg) + e_t^{10}(4469829120-197763148800\nu+146426457600\nu^2-32062809600\nu^3) + e_t^{12}(157363200+4813747200\nu-12775219200\nu^2+6083481600\nu^3) + (e_t\cos u)\bigg(-815811686400+65083392000\log(1-e_t\cos u)-65083392000 \times\log\left(\frac{x}{x_0}\right)+\nu(-107770229760-2454883200\pi^2)+70743936000\nu^2 -21185804800\nu^3+e_t^2(1636480788480-234300211200\log(1-e_t\cos u) +234300211200\log\left(\frac{x}{x_0}\right)+\nu(1487299445760-49915958400\pi^2) -420029798400\nu^2+30425203200\nu^3\bigg)+e_t^4\bigg(-691922419200+312400281600 \log(1-e_t\cos u)-312400281600\log\left(\frac{x}{x_0}\right)+\nu(-3463303150080 +135836870400\pi^2)+230527872000\nu^2+22174886400\nu^3\bigg)+e_t^6\bigg(-529844974080 -182233497600\log(1-e_t\cos u)+182233497600\log\left(\frac{x}{x_0}\right)+\nu(3240401410560$$

$$- 106378272000\pi^2) - 90015513600\nu^2 - 31134348800\nu^3\Big) + e_t^8\Big(500039447040$$
$$+39050035200\log(1 - e_t\cos u) - 39050035200\log\left(\frac{x}{x_0}\right) + \nu(-1788108142080$$
$$+27003715200\pi^2) + 486709862400\nu^2 - 26111232000\nu^3\Big) + e_t^{10}(-10221719040$$
$$+397006502400\nu - 243769420800\nu^2 + 31914777600\nu^3) + e_t^{12}(-157363200$$
$$-4813747200\nu + 12775219200\nu^2 - 6083481600\nu^3)) + (e_t\cos u)^2\Big(1161019760640$$
$$-65083392000\log(1 - e_t\cos u) + 65083392000\log\left(\frac{x}{x_0}\right) + \nu(1194968125440$$
$$-14729299200\pi^2) - 260655360000\nu^2 + 53078886400\nu^3 + e_t^2(-2802117381120$$
$$+234300211200\log(1 - e_t\cos u) - 234300211200\log\left(\frac{x}{x_0}\right) + \nu(-2357003520000$$
$$+66281846400\pi^2) + 972179174400\nu^2 - 130893849600\nu^3\Big) + e_t^4\Big(1596715430400$$
$$-312400281600\log(1 - e_t\cos u) + 312400281600\log\left(\frac{x}{x_0}\right) + \nu(3961710581760$$
$$-171023529600\pi^2) - 332844249600\nu^2 + 78350246400\nu^3\Big) + e_t^6\Big(-86515975680$$
$$+182233497600\log(1 - e_t\cos u) - 182233497600\log\left(\frac{x}{x_0}\right) + \nu(-1923370721280$$
$$+117834393600\pi^2) - 784967232000\nu^2 + 19523494400\nu^3\Big) + e_t^8\Big(-495126120960$$
$$-39050035200\log(1 - e_t\cos u) + 39050035200\log\left(\frac{x}{x_0}\right) + \nu(1028316948480$$
$$-27003715200\pi^2) - 47218214400\nu^2 - 15916761600\nu^3\Big) + e_t^{10}(6089771520$$
$$-229606041600\nu + 124910784000\nu^2 - 4142016000\nu^3)) + (e_t\cos u)^3\Big(-334437120000$$
$$+13016678400\log(1 - e_t\cos u) - 13016678400\log\left(\frac{x}{x_0}\right) + \nu(-3817315046400$$
$$+84284323200\pi^2) + 457420492800\nu^2 - 48936947200\nu^3 + e_t^2(2153965992960$$
$$-52066713600\log(1 - e_t\cos u) + 52066713600\log\left(\frac{x}{x_0}\right) + \nu(2622913935360$$
$$-87557500800\pi^2) - 969149184000\nu^2 + 146756108800\nu^3\Big) + e_t^4\Big(-713296212480$$
$$+78100070400\log(1 - e_t\cos u) - 78100070400\log\left(\frac{x}{x_0}\right) + \nu(-3996933972480$$
$$+139110048000\pi^2) + 791266867200\nu^2 - 150936691200\nu^3\Big) + e_t^6\Big(564367726080$$
$$-52066713600\log(1 - e_t\cos u) + 52066713600\log\left(\frac{x}{x_0}\right) + \nu(239607559680$$
$$-58917196800\pi^2) + 810237081600\nu^2 + 61642892800\nu^3\Big) + e_t^8\Big(189541040640$$

$$+ 13016678400 \log(1 - e_t \cos u) - 13016678400 \log\left(\frac{x}{x_0}\right) + \nu(-103681282560$$
$$+9001238400\pi^2) - 76422144000\nu^2 - 12815411200\nu^3) + e_t^{10}(-337881600$$
$$+30362688000\nu - 27567820800\nu^2 + 4290048000\nu^3)\Big) + (e_t \cos u)^4 \Big(-687858478080$$
$$+13016678400 \log(1 - e_t \cos u) - 13016678400 \log\left(\frac{x}{x_0}\right) + \nu(5807370984960$$
$$-153021052800\pi^2) - 455413900800\nu^2 + 17977395200\nu^3 + e_t^2(-1490127552000$$
$$-39050035200 \log(1 - e_t \cos u) + 39050035200 \log\left(\frac{x}{x_0}\right) + \nu(-384485191680$$
$$+90830678400\pi^2) - 203492160000\nu^2 - 53794060800\nu^3\Big) + e_t^4\Big(-664152491520$$
$$+39050035200 \log(1 - e_t \cos u) - 39050035200 \log\left(\frac{x}{x_0}\right) + \nu(2723950126080$$
$$-99013622400\pi^2) - 465812467200\nu^2 + 53517811200\nu^3\Big) + e_t^6\Big(-224123312640$$
$$-13016678400 \log(1 - e_t \cos u) + 13016678400 \log\left(\frac{x}{x_0}\right) + \nu(255670333440$$
$$+18002476800\pi^2) - 514787366400\nu^2 - 17563020800\nu^3\Big) + e_t^8(-33410741760$$
$$-27429388800\nu - 3469593600\nu^2 - 138124800\nu^3)) + (e_t \cos u)^5 \Big(787721912832$$
$$-5206671360 \log(1 - e_t \cos u) + 5206671360 \log\left(\frac{x}{x_0}\right) + \nu(-5142719623680$$
$$+140746636800\pi^2) + 348355737600\nu^2 - 7195404800\nu^3 + e_t^2(1833256917504$$
$$+15620014080 \log(1 - e_t \cos u) - 15620014080 \log\left(\frac{x}{x_0}\right) + \nu(-2284122493440$$
$$-38459836800\pi^2) + 1005667968000\nu^2 + 20515392000\nu^3\Big) + e_t^4\Big(477180919296$$
$$-15620014080 \log(1 - e_t \cos u) + 15620014080 \log\left(\frac{x}{x_0}\right) + \nu(-597097059840$$
$$+45824486400\pi^2) - 7296998400\nu^2 - 18373747200\nu^3\Big) + e_t^6\Big(-10724639232$$
$$+5206671360 \log(1 - e_t \cos u) - 5206671360 \log\left(\frac{x}{x_0}\right) + \nu(-375209879040$$
$$-4909766400\pi^2) + 288255360000\nu^2 + 3982937600\nu^3\Big) + e_t^8(12237465600$$
$$+24072192000\nu + 7993420800\nu^2 + 1070822400\nu^3)) + (e_t \cos u)^6(-411239646720$$
$$+\nu(2829529175040 - 74464790400\pi^2) - 207633139200\nu^2 + 3495795200\nu^3$$
$$+ e_t^2(-1455591651840 + \nu(2185925806080 - 4091472000\pi^2) - 724992192000\nu^2$$
$$-10543577600\nu^3) + e_t^4(9839024640 + \nu(-146180229120 - 7364649600\pi^2)$$

$$
\begin{aligned}
&+15925593600\nu^2 + 10655961600\nu^3) + e_t^6(-1425415680 + 162493286400\nu \\
&-69507648000\nu^2 - 3664371200\nu^3) + e_t^8(-1385856000 - 6721920000\nu \\
&+422092800\nu^2 + 56192000\nu^3)) + (e_t\cos u)^7(132690700800 + \nu(-909674841600 \\
&+22912243200\pi^2) + 71153510400\nu^2 - 196672000\nu^3 + e_t^2(518098291200 \\
&+\nu(-814416422400 + 5728060800\pi^2) + 246203596800\nu^2 + 590016000\nu^3) \\
&+ e_t^4(-26003980800 + 72602611200\nu + 9760665600\nu^2 - 590016000\nu^3) \\
&+e_t^6(-4850496000 - 23526720000\nu + 1477324800\nu^2 + 196672000\nu^3)) \\
&+ (e_t\cos u)^8(-18955814400 + \nu(129953548800 - 3273177600\pi^2) - 10164787200\nu^2 \\
&+28096000\nu^3 + e_t^2(-74014041600 + \nu(116345203200 - 818294400\pi^2) \\
&-35171942400\nu^2 - 84288000\nu^3) + e_t^4(3714854400 - 10371801600\nu \\
&-1394380800\nu^2 + 84288000\nu^3) + e_t^6(692928000 + 3360960000\nu - 211046400\nu^2 \\
&-28096000\nu^3)) + i(e_t\sin u)(74511360000 + \nu(-225308160000 + 2727648000\pi^2) \\
&+13199155200\nu^2 + e_t^2(-387763200000 + \nu(693492940800 - 7091884800\pi^2) \\
&-37407744000\nu^2) + e_t^4(417871872000 + \nu(-487101542400 + 4364236800\pi^2) \\
&+729907200\nu^2) + e_t^6(-104620032000 + 18916761600\nu + 23478681600\nu^2) \\
&+ (e_t\cos u)(-241173504000 + \nu(1018828800000 - 13092710400\pi^2) - 56932761600\nu^2 \\
&+ e_t^2(1711176192000 + \nu(-3424055500800 + 36004953600\pi^2) + 194885222400\nu^2) \\
&+ e_t^4(-1894109184000 + \nu(2447865446400 - 22912243200\pi^2) - 12043468800\nu^2) \\
&+e_t^6(424106496000 - 42638745600\nu - 125908992000\nu^2)) + (e_t\cos u)^2(196314624000 \\
&+\nu(-1765980057600 + 24548832000\pi^2) + 89961062400\nu^2 + e_t^2(-3063785472000 \\
&+\nu(6872258764800 - 73646496000\pi^2) - 423163699200\nu^2) + e_t^4(3526820352000 \\
&+\nu(-5087179468800 + 49097664000\pi^2) + 59122483200\nu^2) + e_t^6(-659349504000 \\
&-19099238400\nu + 274080153600\nu^2)) + (e_t\cos u)^3(136249344000 + \nu(1381795430400 \\
&-21821184000\pi^2) - 54499737600\nu^2 + e_t^2(2898643968000 + \nu(-7118338867200 \\
&+76374144000\pi^2) + 491957452800\nu^2) + e_t^4(-3523627008000 + \nu(5612094259200 \\
&-54552960000\pi^2) - 130653388800\nu^2) + e_t^6(488733696000 + 124449177600\nu \\
&-306804326400\nu^2)) + (e_t\cos u)^4(-292571136000 + \nu(-358171545600+8182944000\pi^2) - 4926873600\nu^2 \\
&+ e_t^2(-1608532992000 + \nu(3959624908800-40914720000\pi^2) - 324626227200\nu^2) \\
&+ e_t^4(2082972672000 + \nu(-3485337292800+32731776000\pi^2) + 146711347200\nu^2) \\
&+ e_t^6(-181868544000 - 116116070400\nu+182841753600\nu^2)) \\
&+ (e_t\cos u)^5(149935104000 - 107418009600\nu + 18977587200\nu^2 \\
&+ e_t^2(546974208000 + \nu(-1093948416000 + 9819532800\pi^2) + 116055244800\nu^2) \\
&+ e_t^4(-739031040000 + \nu(1166391705600 - 9819532800\pi^2) - 82114560000\nu^2) \\
&+e_t^6(42121728000 + 34974720000\nu - 52918272000\nu^2)) \\
&+ (e_t\cos u)^6(-23265792000 + \nu(56253542400 - 545529600\pi^2) - 5778432000\nu^2
\end{aligned}
$$

$$
\begin{aligned}
&+e_t^2(-96712704000+\nu(110966169600-545529600\pi^2)-17700249600\nu^2)\\
&+e_t^4(129102336000+\nu(-166733107200+1091059200\pi^2)+18247680000\nu^2)\\
&+e_t^6(-9123840000-486604800\nu+5231001600\nu^2)))\\
&+\sqrt{1-e_t^2}(-39536640000+\nu(-37022515200+1091059200\pi^2)-4866048000\nu^2\\
&+e_t^2(-183389184000+\nu(472087756800-5455296000\pi^2)-25911705600\nu^2)\\
&+e_t^4(332107776000+\nu(-469492531200+4364236800\pi^2)+12773376000\nu^2)\\
&+e_t^6(-109181952000+34427289600\nu+18004377600\nu^2)\\
&+(e_t\cos u)(285956352000+\nu(94893004800-4637001600\pi^2)\\
&+26793676800\nu^2+e_t^2(809740800000+\nu(-2359414886400+27549244800\pi^2)+142483968000\nu^2)\\
&+e_t^4(-1500187392000+\nu(2355425740800-22912243200\pi^2)-65813299200\nu^2)\\
&+e_t^6(442810368000-106231910400\nu-103464345600\nu^2))\\
&+(e_t\cos u)^2(-702383616000+\nu(-158623027200+8728473600\pi^2)-32176742400\nu^2\\
&+e_t^2(-1736266752000+\nu(5106816000000-57826137600\pi^2)-358262784000\nu^2)\\
&+e_t^4(2858955264000+\nu(-4921064755200+49097664000\pi^2)+151394918400\nu^2)\\
&+e_t^6(-688545792000+80168140800\nu+239044608000\nu^2))\\
&+(e_t\cos u)^3(714016512000+\nu(650230732800-12819945600\pi^2)-68824166400\nu^2\\
&+e_t^2(2664617472000+\nu(-6534899712000+67372905600\pi^2)+540040089600\nu^2)\\
&+e_t^4(-3083629824000+\nu(5509679155200-54552960000\pi^2)-189593395200\nu^2)+e_t^6(509718528000\\
&+53100748800\nu-281622528000\nu^2))+(e_t\cos u)^4(-163012608000\\
&+\nu(-1666905292800+19093536000\pi^2)+237828096000\nu^2+e_t^2(-3119440896000\\
&+\nu(5801890406400-51825312000\pi^2)-524012544000\nu^2)+e_t^4(2129504256000\\
&+\nu(-3504101990400+32731776000\pi^2)+111006720000\nu^2)+e_t^6(-188255232000\\
&-94401331200\nu+175177728000\nu^2))+(e_t\cos u)^5(-234862848000\\
&+\nu(2089780070400-21548419200\pi^2)-284815872000\nu^2+e_t^2(2485790208000\\
&+\nu(-3852277862400+31367952000\pi^2)+325021593600\nu^2)+e_t^4(-952300800000\\
&+\nu(1192592332800-9819532800\pi^2)+12165120000\nu^2)+e_t^6(42577920000\\
&+33423667200\nu-52370841600\nu^2))+(e_t\cos u)^6(180195840000+\nu(-1346993049600\\
&+14183769600\pi^2)+171953971200\nu^2+e_t^2(-1212558336000+\nu(1821503692800\\
&-15274828800\pi^2)-124084224000\nu^2)+e_t^4(236763648000+\nu(-152134963200\\
&+1091059200\pi^2)-53100748800\nu^2)+e_t^6(-9123840000-486604800\nu+5231001600\nu^2))\\
&+(e_t\cos u)^7(-44934912000+\nu(424780646400-4637001600\pi^2)-52218777600\nu^2
\end{aligned}
$$

$$
\begin{aligned}
&+ e_t^2(338950656000 + \nu(-520930713600 + 4637001600\pi^2) + 28435968000\nu^2) \\
&+ e_t^4(-25774848000 - 11146291200\nu + 23782809600\nu^2)) + (e_t \cos u)^8(4561920000 \\
&+\nu(-50140569600 + 545529600\pi^2) + 6325862400\nu^2 + e_t^2(-47443968000 \\
&+\nu(65225318400 - 545529600\pi^2) - 3710361600\nu^2) + e_t^4(4561920000 + 243302400\nu \\
&-2615500800\nu^2)) + i(e_t \sin u)\bigg(-328593346560 + 26033356800 \log(1 - e_t \cos u) \\
&-26033356800 \log\left(\frac{x}{x_0}\right) + 71427240960\nu - 11877196800\nu^2 + 2718080000\nu^3 \\
&+ e_t^2\bigg(962564305920 - 78100070400 \log(1 - e_t \cos u) + 78100070400 \log\left(\frac{x}{x_0}\right) \\
&+\nu(-85968921600 - 3273177600\pi^2) + 106786867200\nu^2 - 6687820800\nu^3\bigg) \\
&+ e_t^4\bigg(-1161059174400 + 78100070400 \log(1 - e_t \cos u) - 78100070400 \log\left(\frac{x}{x_0}\right) \\
&+\nu(455668669440 + 4909766400\pi^2) - 231924172800\nu^2 + 11614080000\nu^3\bigg) \\
&+ e_t^6\bigg(483224340480 - 26033356800 \log(1 - e_t \cos u) + 26033356800 \log\left(\frac{x}{x_0}\right) \\
&+\nu(-538960911360 - 3273177600\pi^2) + 319443148800\nu^2 - 12469836800\nu^3\bigg) \\
&+ e_t^8(-22426122240 + 229909708800\nu - 173743948800\nu^2 + 7364044800\nu^3) \\
&+e_t^{10}(294220800 - 9305395200\nu + 23887411200\nu^2 - 1261209600\nu^3) + (e_t \cos u) \\
&\times \bigg(925176069120 - 78100070400 \log(1 - e_t \cos u) + 78100070400 \log\left(\frac{x}{x_0}\right) \\
&+\nu(-458849817600 + 8182944000\pi^2) - 11032934400\nu^2 - 9615590400\nu^3 \\
&+ e_t^2(-2356066314240 + 234300211200 \log(1 - e_t \cos u) - 234300211200 \log\left(\frac{x}{x_0}\right) \\
&-300753884160\nu - 62224051200\nu^2 + 13476403200\nu^3\bigg) + e_t^4(3159065625600 \\
&-234300211200 \log(1 - e_t \cos u) + 234300211200 \log\left(\frac{x}{x_0}\right) + \nu(-338253788160 \\
&-8182944000\pi^2) - 32412595200\nu^2 - 11630284800\nu^3) + e_t^6\bigg(-1380322575360 \\
&+78100070400 \log(1 - e_t \cos u) - 78100070400 \log\left(\frac{x}{x_0}\right) + \nu(811123860480
\end{aligned}
$$

$$+9819532800\pi^2) - 319331097600\nu^2 + 5767065600\nu^3\Big) + e_t^8(48416071680$$
$$-459194112000\nu + 253455436800\nu^2 - 6922828800\nu^3) + e_t^{10}(-294220800$$
$$+9305395200\nu - 23887411200\nu^2 + 1261209600\nu^3)) + (e_t\cos u)^2\Big(-901804861440$$
$$+78100070400\log(1 - e_t\cos u) - 78100070400\log\left(\frac{x}{x_0}\right) + \nu(1567758735360$$
$$-34368364800\pi^2) - 4901222400\nu^2 + 16293504000\nu^3 + e_t^2(1178660367360$$
$$-234300211200\log(1 - e_t\cos u) + 234300211200\log\left(\frac{x}{x_0}\right) + \nu(638531481600$$
$$+26185420800\pi^2) + 85797964800\nu^2 - 23253580800\nu^3\Big) + e_t^4\Big(-2558005079040$$
$$+234300211200\log(1 - e_t\cos u) - 234300211200\log\left(\frac{x}{x_0}\right) + \nu(-627219118080$$
$$-6546355200\pi^2) + 636385766400\nu^2 + 18039628800\nu^3\Big) + e_t^6\Big(1322237813760$$
$$-78100070400\log(1 - e_t\cos u) + 78100070400\log\left(\frac{x}{x_0}\right) + \nu(-12701306880$$
$$-9819532800\pi^2) - 123584793600\nu^2 + 10020940800\nu^3\Big) + e_t^8\Big(-31024880640$$
$$+275186073600\nu - 105116083200\nu^2 - 1940428800\nu^3)) + (e_t\cos u)^3(452697154560$$
$$-26033356800\log(1 - e_t\cos u) + 26033356800\log\left(\frac{x}{x_0}\right) + \nu(-2649367956480$$
$$+58917196800\pi^2) + 75336499200\nu^2 - 10847257600\nu^3 + e_t^2(1093220843520$$
$$+78100070400\log(1 - e_t\cos u) - 78100070400\log\left(\frac{x}{x_0}\right) + \nu(5716638720$$
$$-49097664000\pi^2) - 361501670400\nu^2 + 20455526400\nu^3\Big) + e_t^4\Big(211920660480$$
$$-78100070400\log(1 - e_t\cos u) + 78100070400\log\left(\frac{x}{x_0}\right) + \nu(602037273600$$
$$+19639065600\pi^2) - 554518809600\nu^2 - 36066355200\nu^3\Big) + e_t^6\Big(-442958069760$$
$$+26033356800\log(1 - e_t\cos u) - 26033356800\log\left(\frac{x}{x_0}\right) + \nu(-367892106240$$
$$+3273177600\pi^2) + 163837209600\nu^2 - 587878400\nu^3\Big) + e_t^8(5034931200$$

$$-45901670400\nu + 25404595200\nu^2 + 1499212800\nu^3)) + (e_t\cos u)^4(-287513978880$$
$$+\nu(2242649057280 - 52370841600\pi^2) - 56454220800\nu^2 - 971289600\nu^3$$
$$+ e_t^2(-1158390312960 + \nu(-333825269760 + 39278131200\pi^2) + 302949196800\nu^2$$
$$+1949107200\nu^3) + e_t^4(427268290560 + \nu(-205133137920 - 11456121600\pi^2)$$
$$+280446028800\nu^2 + 20456678400\nu^3) + e_t^6(28699361280 + 137865216000\nu$$
$$-38359372800\nu^2 - 2274432000\nu^3)) + (e_t\cos u)^5(169541452800$$
$$+\nu(-933238149120 + 24548832000\pi^2) + 9044275200\nu^2 + 2341017600\nu^3$$
$$+ e_t^2(328893465600 + \nu(69430671360 - 16365888000\pi^2) - 74749132800\nu^2$$
$$-5811609600\nu^3) + e_t^4(-90193536000 + \nu(163341803520 + 1636588800\pi^2)$$
$$-128144793600\nu^2 - 3793766400\nu^3) + e_t^6(-12266726400 - 36156672000\nu$$
$$-1583001600\nu^2 - 399667200\nu^3)) + (e_t\cos u)^6(-29502489600 + \nu(159620889600$$
$$-4909766400\pi^2) - 115200000\nu^2 + 81536000\nu^3 + e_t^2(-48882355200 + \nu(6869283840$$
$$+3273177600\pi^2) + 2940825600\nu^2 - 128025600\nu^3) + e_t^4(11003212800 - 50441702400\nu$$
$$+30168576000\nu^2 + 1380019200\nu^3) + e_t^6(1385856000 + 6721920000\nu - 422092800\nu^2$$
$$-56192000\nu^3))))\bigg\}. \tag{A1e}$$

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