Exact distributions of cover times for N independent random walkers in one dimension

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We study the probability density function (PDF) of the cover time t_c of a finite interval of size L by N independent one-dimensional Brownian motions, each with diffusion constant D. The cover time t_c is the minimum time needed such that each point of the entire interval is visited by at least one of the N walkers. We derive exact results for the full PDF of t_c for arbitrary $N \geqslant 1$ for both reflecting and periodic boundary conditions. The PDFs depend explicitly on N and on the boundary conditions. In the limit of large N, we show that t_c approaches its average value of $\langle t_c \rangle \approx L^2/(16D \ln N)$ with fluctuations vanishing as $1/(\ln N)^2$. We also compute the centered and scaled limiting distributions for large N for both boundary conditions and show that they are given by nontrivial N independent scaling functions.

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I. INTRODUCTION

Stochastic search processes are ubiquitous in nature [1]. These include animals foraging for food [2–4], various biochemical reactions [5,6], such as proteins searching for specific DNA sequences to bind [7–10] or sperm cells searching for an oocyte to fertilize [11,12]. Several of these stochastic search processes often are modeled by a single searcher performing a simple random walk (RW) [1,6]. In many situations, the search takes place in a confined domain as the targets typically are scattered over the entire domain. Finding all these targets therefore requires an exhaustive exploration of this confined domain. In this context, an important observable that characterizes the efficiency of the search process is the cover time t_c, i.e., the minimum time needed by the RW to visit all sites of this domain at least once [13]. The cover time of a single random walker has also an important application in computer science, for instance, for generating random spanning trees (with uniform measure) on an arbitrary connected and undirected graph G [14,15].

Computing analytically the statistics of t_c for a given confined domain has remained an outstanding challenge in RW theory. Most previous studies focused on calculating the mean cover time on regular lattices, graphs, and networks [16–22]. Obtaining analytical results for the full distribution of t_c is a notoriously hard task. In Ref. [23] the authors derived formal expressions for the distribution of t_c on an arbitrary finite graph from which it remains however extremely difficult to extract explicit results for large systems. For RW on the d-dimensional regular lattice with periodic boundary conditions (PBCs) and in high dimensions $d > 2$, it was shown rigorously in Ref. [24] that the distribution of t_c, properly centered and scaled, converges in the limit of a large system to a Gumbel distribution. It was then shown that the same conclusion holds for the RW on the fully connected graph [25], corresponding formally to the limit $d \to \infty$. Note that on the d-dimensional regular graph with $d > 2$, the RW is *transient*, i.e., the walker escapes to infinity with a nonzero probability in the unbounded domain. In fact, very recently, Chupeau *et al.* studied the full distribution of the cover time on a finite graph by an arbitrary *transient* RW and found that the aforementioned results are quite robust [13]. Indeed, for such transient RWs, they showed [13] that the distribution of t_c, appropriately centered and scaled, indeed approaches a Gumbel distribution, irrespective of the topology of the graph.

An important exception to this class of transient walkers is a RW in one or two dimensions where the walker is recurrent (i.e., starting from a given site, it comes back to it with probability one). It is thus natural to investigate the distribution of t_c for a RW in one or two dimensions. In particular, on a finite segment in $d = 1$, is the scaled distribution of t_c still given by a Gumbel law or is it something completely different? This question is clearly relevant for any process modeled by a one-dimensional (1D) RW in a finite domain, for instance, for proteins searching for a binding site on a DNA strand [7,9]. Another important question concerns the role of the boundary conditions on the confined domain. How sensitive is the distribution of t_c to the boundary conditions in the limit of a large domain? In $d = 1$, although the mean cover time $\langle t_c \rangle$ is known exactly for a RW on a finite interval of size L, for both reflecting and periodic boundary conditions, computing the full distribution for these two boundary conditions has remained an outstanding challenge.

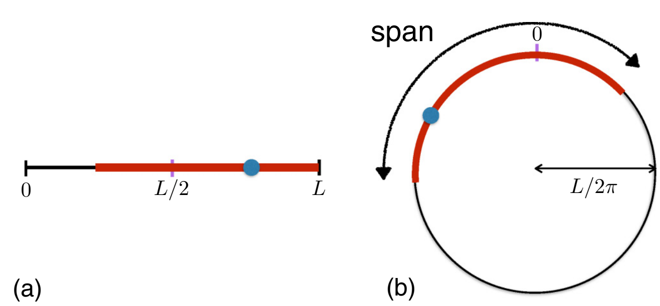

FIG. 1. (a) Brownian motion (BM) with RBCs at $x = 0$ and $x = L$. The thick (red) region indicates the space already covered by the walker up to time t, starting at $x_0 = L/2$. The circle denotes the current position at time t of the walker. (b) The same walker on a ring, i.e., with PBCs, starting at 0. The thick (red) region, indicating the covered space up to time t, is equivalent to the span $\mathscr{S}(t)$ (the spatial extent of the visited region) of a walker on the infinite line. In both cases, the cover time t_c is the first time at which the entire domain becomes red.

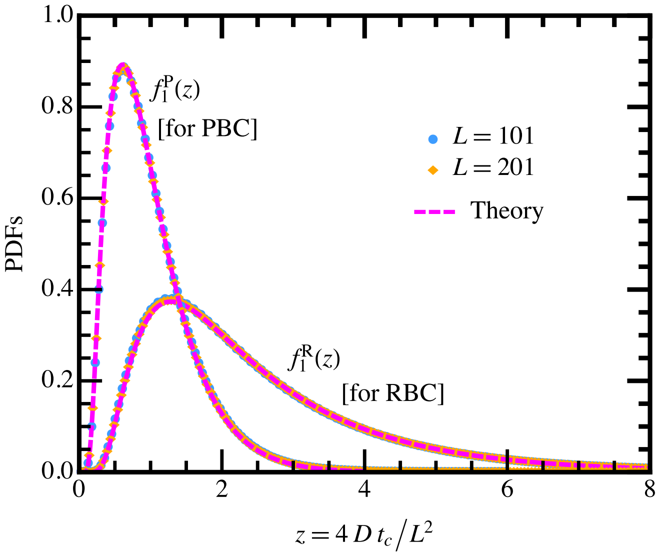

FIG. 2. The PDFs of the scaled cover time for a single lattice RW with RBCs and PBCs for sizes $L = 101$ (blue) and $L = 201$ (orange). The collapsed scaling data are compared to theoretical scaling functions (dashed lines) in the Brownian limit in Eq. (1) with $D = 1/2$, where $f_1^{R|P}(z)$'s are given in Eqs. (22) and (26), respectively.

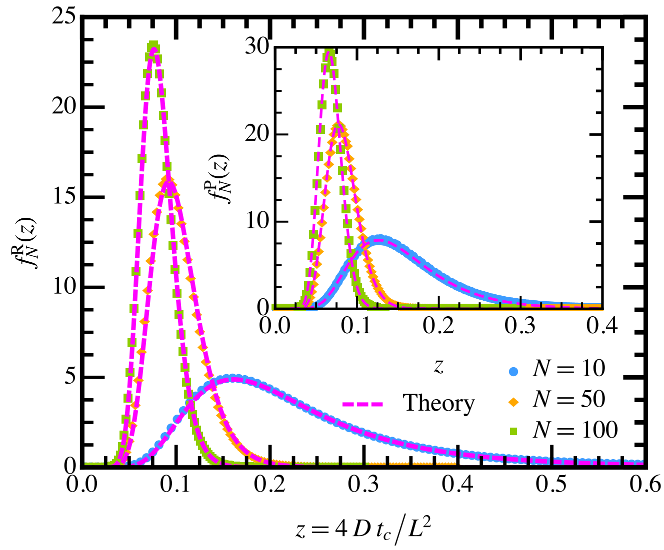

FIG. 3. Main: The PDFs of the scaled cover time (30) for different N's with RBCs. For each N, the numerical results were obtained for lattice RW of size $L = 201$ as in Fig. 2. The (magenta) dashed lines are the exact theoretical scaling functions in the Brownian limit with $D = 1/2$ and are given by $f_N^R(z)$ in Eq. (31). The inset: Same data as shown in the main panel but for PBCs. The (magenta) dashed lines correspond to the exact theoretical results given by $f_N^P(z)$ in (38).

In this paper, we present exact results for the full distribution of t_c in $d = 1$ for a RW in the Brownian limit (i.e., the long-time scaling limit of a discrete-time RW on a lattice) on a finite interval of size L for both reflecting boundary conditions (RBCs) and PBCs (see Fig. 1). In the case of the PBC, the RW takes place on a ring of size L, and evidently the distribution of t_c is independent of the starting point, whereas it depends explicitly on the starting point $x_0 \in [0,L]$ in the case of the RBC. In the latter case, for simplicity, we present the results only when the walker starts at the center of the interval, i.e., at $x_0 = L/2$. We show that, in the Brownian limit (with a diffusion constant D), the probability density function (PDF) of t_c is given by

$$\text{Prob.}[t_c = t|L] = \frac{4D}{L^2} f_1^{R|P}\left(\frac{4Dt}{L^2}\right), \tag{1}$$

where $R|P$ denotes the RBC and the PBC, respectively. The exact scaling functions $f_1^R(x)$ and $f_1^P(x)$ are given in Eqs. (22) and (26), respectively, along with their asymptotics in Eqs. (23) and (29). Plots of these two scaling functions are shown in Fig. 2.

Another interesting question concerns the statistics of the cover time t_c where there are N independent walkers. This problem of multiple independent random walkers naturally arises in various search problems where there is a team of N independent searchers as opposed to a single searcher. Various observables associated with this multiple random walker process have been studied over the past few decades, such as the first passage time to the origin [26–29], the number of distinct and common sites visited by these walkers [30–34], the statistics of the maximum displacement [35–38], the statistics of records [39], etc. For N walkers, the cover time t_c is the minimum time needed for all sites to be visited at least once by at least one of the walkers. In the literature, only the mean cover time was computed and that too only for $N = 2$ with the PBC in 1D [20]. It is evident that the average cover time will decrease with increasing N, but how does it decrease for large N? In this paper, we generalize our result for the cover time distribution for one walker to arbitrary N walkers in 1D, both for RBCs and PBCs (for a plot of these distributions for different N's, see Fig. 3). In particular, we show that the mean cover time for both boundary conditions decreases for large N as

$$\frac{4D\langle t_c\rangle}{L^2} \approx \frac{1}{4\ \ln\ N}. \tag{2}$$

However, it turns out that the fluctuations around the mean are sensitive to the boundary conditions. Indeed we show that for large N, the random variable t_c approaches

$$\frac{4Dt_c}{L^2} \approx \frac{1}{4\ \ln\ N} - \frac{1}{4(\ln\ N)^2}\chi_{R,P}, \tag{3}$$

where χ_R and χ_P are two N independent distinct random variables with nontrivial PDFs, respectively, given by [with $x \in (-\infty, +\infty)$]

$$\text{Prob.}[\chi_R = x] = g_R(x) = 2e^{-x-e^{-x}}\left(1 - e^{-e^{-x}}\right) \tag{4}$$

plotted in Fig. 4, and

$$\text{Prob.}[\chi_P = x] = g_P(x) = 4e^{-2x} K_0(2e^{-x}), \tag{5}$$

where K_0 is the modified Bessel function. The function $g_P(x)$ is plotted in Fig. 4.

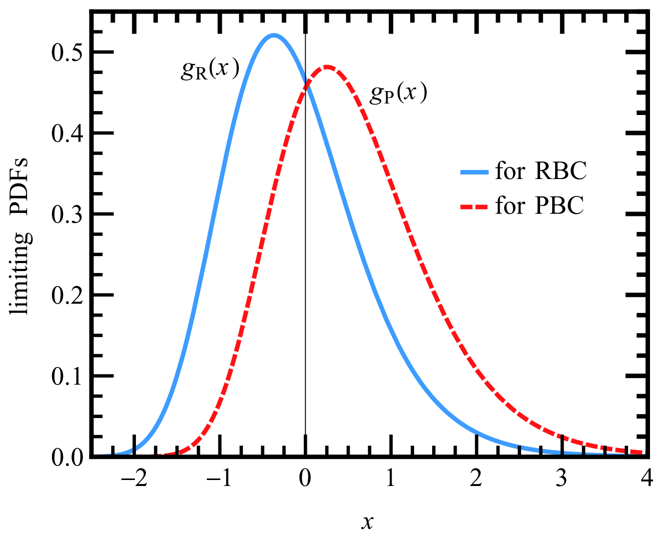

FIG. 4. Plot of the limiting PDFs of t_c (for N RWs in the limit of large N) $g_R(x)$ (for RBC) and $g_P(x)$ (for PBC) given in Eqs. (4) and (5), respectively.

II. SINGLE WALKER (REFLECTING CASE)

Let us start by first considering the case of a single Brownian motion on the interval $[0,L]$ starting at x_0 with the RBCs at $x=0$ and $x=L$. The cover time t_c in this case is clearly the first time when the walker has hit both boundaries at $x=0$ and $x=L$. It is useful to consider the cumulative distribution Prob.$[t_c>t|x_0]$. If $t_c>t$, this means that at time t one of the boundaries has not been hit up to time t. This means that Prob.$[t_c>t|x_0]=$ Prob.[L is unhit up to time t] + Prob.[0 is unhit up to time t]$-$Prob.[both are unhit up to time t]. All three probabilities can be computed by solving the standard backward Fokker-Planck equation for the survival probability $S(x_0,t)$ (x_0 being the starting position of the walker),

$$\frac{\partial S(x_0,t)}{\partial t}=D\frac{\partial^2 S(x_0,t)}{\partial x_0^2}, \tag{6}$$

with appropriate boundary conditions at $x_0=0$ and $x_0=L$. For example, Prob.[L is unhit up to time t] $=S_{\rm AR}(x_0,t)$ where the subscript A indicates an absorbing boundary condition at $x_0=L$ [i.e., $S(x_0=L,t)=0$], whereas the subscript R refers to the reflecting boundary condition at $x_0=0$, [i.e., $\partial_{x_0}S(x_0,t)|_{x_0=0}=0$] [26,27,40]. Hence we have

$$\text{Prob.}[t_c>t|x_0]=S_{\rm AR}(x_0,t)+S_{\rm RA}(x_0,t)-S_{\rm AA}(x_0,t), \tag{7}$$

where the subscripts refer to the boundary conditions. These survival probabilities can be computed exactly from Eq. (6) using standard methods [26,27,41], as follows.

This Eq. (6) holds for $x_0\in[0,L]$ with the initial condition $S(x_0,0)=1$ for $0<x_0<L$. By taking Laplace transform $\tilde S(x_0,\lambda)=\int_0^\infty e^{-\lambda t}S(x_0,t)dt$ in Eq. (6) and using the initial condition $S(x_0,0)=1$ yields an ordinary differential equation,

$$D\frac{d^2\tilde S(x_0,\lambda)}{dx_0^2}-\lambda\tilde S(x_0,\lambda)=-1. \tag{8}$$

This differential equation can be solved trivially with the appropriate boundary conditions at $x_0=0$ and $x_0=L$ as needed here (7). For convenience, we will choose $x_0=L/2$ for which by symmetry $S_{\rm AR}(x_0=L/2,t)=S_{\rm RA}(x_0=L/2,t)$. In this case we obtain

$$S_{\rm AA}(L/2,t)=S_1\left(\frac{4Dt}{L^2}\right), \tag{9}$$

$$S_{\rm AR}(L/2,t)=S_{\rm RA}(L/2,t)=S_2\left(\frac{4Dt}{L^2}\right), \tag{10}$$

where

$$\int_0^\infty S_1(z)e^{-\lambda z}dz=\frac{1}{\lambda}\left[1-\frac{1}{\cosh\sqrt{\lambda}}\right], \tag{11}$$

$$\int_0^\infty S_2(z)e^{-\lambda z}dz=\frac{1}{\lambda}\left[1-\frac{\cosh\sqrt{\lambda}}{\cosh 2\sqrt{\lambda}}\right]. \tag{12}$$

These Laplace transforms can be inverted by using the standard Bromwich contour on the complex λ plane and calculating the residues at the poles. This gives

$$S_1(z)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}e^{-(2n+1)^2\pi^2 z/4}, \tag{13}$$

and

$$S_2(z)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\cos[(2n+1)\pi/4]}{(2n+1)}e^{-(2n+1)^2\pi^2 z/16}. \tag{14}$$

These series representations are very useful for calculating the large z asymptotics where only the $n=0$ term gives the leading contribution, yielding

$$S_1(z)\underset{z\to\infty}{\approx}\frac{4}{\pi}e^{-\pi^2 z/4}, \tag{15}$$

$$S_2(z)\underset{z\to\infty}{\approx}\frac{4}{\pi\sqrt{2}}e^{-\pi^2 z/16}. \tag{16}$$

However, for small z, it is harder to compute the tail from the series representations in (13) and (14). In that case, one can use an alternative representation that can be obtained via the Poisson summation formula [42]. Equivalently, we can derive it by inserting the following identity:

$$\frac{1}{\cosh(p\sqrt{\lambda})}=2\sum_{n=0}^{\infty}(-1)^n e^{-(2n+1)p\sqrt{\lambda}}, \tag{17}$$

in Eqs. (11) and (12) and then inverting the Laplace transform term by term. This gives, after straightforward algebra,

$$S_1(z)=1-2\sum_{n=0}^{\infty}(-1)^n\text{erfc}\left(\frac{2n+1}{2\sqrt{z}}\right), \tag{18}$$

$$S_2(z)=1-\sum_{n=0}^{\infty}\left(\sin\frac{n\pi}{2}+\cos\frac{n\pi}{2}\right)\text{erfc}\left(\frac{2n+1}{2\sqrt{z}}\right), \tag{19}$$

where $\text{erfc}(x)=(2/\sqrt{\pi})\int_x^\infty e^{-u^2}du$. Note that $\text{erfc}(x)\approx e^{-x^2}/(x\sqrt{\pi})$ as $x\to\infty$. Consequently, for small z, keeping only terms up to $n=1$ in the sums in Eqs. (18) and (19) (higher order terms only give subleading corrections), gives

$$S_1(z) \underset{z\to 0}{\approx} 1 - \frac{4\sqrt{z}}{\sqrt{\pi}} e^{-(1/4z)} + \frac{4\sqrt{z}}{3\sqrt{\pi}} e^{-(9/4z)}, \quad (20)$$

$$S_2(z) \underset{z\to 0}{\approx} 1 - \frac{2\sqrt{z}}{\sqrt{\pi}} e^{-(1/4z)} - \frac{2\sqrt{z}}{3\sqrt{\pi}} e^{-(9/4z)}. \quad (21)$$

Therefore, from Eqs. (7) and (9), we have Prob.$[t_c > t|L] = F_1(4Dt/L^2)$ where the scaling function $F_1(z) = 2S_2(z) - S_1(z)$. Taking the derivative with respect to z yields the PDF announced in Eq. (1) with the scaling function $f_1^R(z)$ given explicitly by

$$f_1^R(z) = S_1'(z) - 2S_2'(z), \quad (22)$$

where $S_1(z)$ and $S_2(z)$ are given in Eqs. (13) and (14). The tails of the scaling function are obtained easily from the asymptotic behaviors given in Eqs. (15)– (21) as

$$f_1^R(z) \sim \begin{cases} (6/\sqrt{\pi}) z^{-3/2} e^{-9/(4z)} & \text{as } z \to 0, \\ \pi/(2\sqrt{2}) e^{-\pi^2 z/16} & \text{as } z \to \infty. \end{cases} \quad (23)$$

Note that, for small z, the leading term $\propto e^{-1/(4z)}$ in Eqs. (20) and (21) actually cancels, yielding the behavior in Eq. (23). A plot of this scaling function $f_1^R(z)$ is shown in Fig. 2 where it also is compared to the simulation results. Simulations were performed for a RW on a lattice of $L = 101$ and $L = 201$ sites with reflecting boundary conditions, which in the long-time limit collapses to the Brownian scaling function in Eqs. (1) and (22).

III. SINGLE WALKER (PERIODIC CASE)

We now consider the cover time for a single RW on a ring of length L. In this case, the distribution of t_c is independent of x_0, which we take to be at 0 [see Fig. 1(b)]. We first show that the cumulative probability Prob.$[t_c > t|L]$ on the ring can be mapped exactly onto the cumulative distribution of the span $\mathscr{S}(t)$ of the walker at time t on an infinite line—the span being the length of the covered region by the walker up to time t. The probability that $t_c > t$ indicates that at time t the ring has not been covered by the walker [see Fig. 1(b)]. Since the ring has not been traversed fully at time t, the walker does not realize that it is on a ring. Thus one can think of the walk taking place on an infinite line and Prob.$[t_c > t|L]$ on the ring is just the probability that the span $\mathscr{S}(t)$ of the walker on the infinite line is less than L, i.e., one has the exact relation [see Fig. 1(b)],

$$\text{Prob.}[t_c > t|L] = \text{Prob.}[\mathscr{S}(t) < L]. \quad (24)$$

The PDF of the span $\mathscr{S}(t)$ on the infinite line is known [43], Prob.$[\mathscr{S}(t) = s] = (1/\sqrt{4Dt}) h_1(s/\sqrt{4Dt})$ where

$$h_1(y) = \frac{8}{\sqrt{\pi}} \sum_{m=1}^{\infty} (-1)^{m+1} m^2 e^{-m^2 y^2}. \quad (25)$$

Therefore, taking the derivative of Eq. (24) with respect to t, we obtain the PDF of the cover time on a ring as in Eq. (1) where the scaling function $f_1^P(z) = 1/(2z^{3/2}) h_1(1/\sqrt{z})$. Using the explicit expression of $h_1(y)$ in Eq. (25) we then get

$$f_1^P(z) = \frac{4}{\sqrt{\pi} z^{3/2}} \sum_{m=1}^{\infty} (-1)^{m+1} m^2 e^{-m^2/z}. \quad (26)$$

This formula is useful to extract the small z asymptotics of $f_1^P(z)$. Indeed, keeping the $m = 1$ term in (26) gives $f_1^P(z) \sim (4/\sqrt{\pi}) z^{-3/2} e^{-1/z}$. However, this representation is not very convenient to derive the large z asymptotics. For this, we can use the following identity:

$$1 + 2\sum_{m=1}^{\infty} (-1)^m e^{-m^2 x} = 2\sqrt{\frac{\pi}{x}} \sum_{n=0}^{\infty} e^{-(\pi^2/x)(n+1/2)^2}, \quad x > 0, \quad (27)$$

which can easily be derived from the Poisson summation formula [42]. Taking the derivative with respect to x on both sides of Eq. (27) and setting $x = 1/z$ one obtains

$$f_1^P(z) = \sum_{n=0}^{\infty} [(2n+1)^2 \pi^2 z - 2] e^{-\pi^2 (n+1/2)^2 z}. \quad (28)$$

For large z, the $n = 0$ term provides the most dominant contribution $f_1^P(z) \sim \pi^2 z e^{-\pi^2 z/4}$. Finally, the tails of this function can be summarized as

$$f_1^P(z) \sim \begin{cases} (4/\sqrt{\pi}) z^{-3/2} e^{-1/z}, & \text{as } z \to 0, \\ \pi^2 z e^{-\pi^2 z/4}, & \text{as } z \to \infty. \end{cases} \quad (29)$$

For a plot of this scaling function, see Fig. 2.

IV. MULTIPLE WALKERS (REFLECTING CASE)

Here we consider, for simplicity, N independent walkers all starting at the same point x_0. Using the mutual independence of the N walkers, the cumulative cover time distribution for N walkers is clearly given by Prob.$[t_c > t|x_0, N] = [S_{\rm AR}(x_0,t)]^N + [S_{\rm RA}(x_0,t)]^N - [S_{\rm AA}(x_0,t)]^N$. Choosing as before $x_0 = L/2$, we find that

$$\text{Prob.}[t_c = t|x_0 = L/2, N] = \frac{4D}{L^2} f_N^R\left(\frac{4Dt}{L^2}\right), \quad (30)$$

with the superscript R denoting the RBC and the scaling function given by

$$f_N^R(z) = -F_N'(z), \quad \text{where } F_N(z) = 2[S_2(z)]^N - [S_1(z)]^N, \quad (31)$$

where $S_{1,2}(z)$ are given in Eqs. (13) and (14). A plot of this function for different values of N is shown in the main panel of Fig. 3 where it is compared to simulations with an excellent agreement. The asymptotic behaviors of $f_N^R(z)$ are obtained from Eqs. (15)– (21), and they are given by (for $N \geqslant 2$)

$$f_N^R(z) \sim \begin{cases} 2N(N-1)/(\pi z) e^{-1/(2z)}, & \text{as } z \to 0, \\ (N\pi^2/8)(2\sqrt{2}/\pi)^N e^{-N\pi^2 z/16}, & \text{as } z \to \infty. \end{cases} \quad (32)$$

Note that the small z asymptotics of $f_N^R(z)$ are quite different for $N = 1$ (23) and $N \geqslant 2$ (32).

One naturally wonders whether there exists a limiting distribution of t_c for large N. We first estimate the mean cover time $4D\langle t_c\rangle/L^2 = \int_0^\infty F_N(z)dz$, where $F_N(z)$ is the cumulative scaling function given in Eq. (31). For large N, one expects that $\langle t_c\rangle$ is small. Hence, the integral $\int_0^\infty F_N(z)dz$ is dominated by the small z behavior of $F_N(z)$. For small z, one has from Eqs. (20) and (21) that $S_1(z) \sim 1 - 4\sqrt{z/\pi} e^{-1/(4z)}$ and $S_2(z) \sim 1 - 2\sqrt{z/\pi} e^{-1/(4z)}$. Substituting these behaviors in Eq. (31) and exponentiating for large N we get

$$F_N(z) \approx 2e^{-u_N(z)} - e^{-2u_N(z)}, \quad u_N(z) = \frac{2\sqrt{z}}{\sqrt{\pi}} N e^{-1/(4z)}. \tag{33}$$

Therefore $F_N(z) \sim 1$ as long as $u_N(z) \ll 1$ [which happens for $z < 1/(4 \ln N)$], whereas $F_N(z)$ is exponentially small in N for $z > 1/(4 \ln N)$. Hence, to leading order for large N, $4D\langle t_c\rangle/L^2 = \int_0^\infty F_N(z)dz \approx 1/(4 \ln N)$ as announced in Eq. (2). In addition, we can also compute the limiting distribution from Eq. (33) by expanding around $z = 1/(4 \ln N)$. We set $z = 1/(4 \ln N) - x/[4(\ln N)^2]$ where we assume that the scaled fluctuation x is of order $O(1)$. Substituting this z in $u_N(z)$ in Eq. (33) and expanding for large N, one gets to leading order $u_N(z) \approx e^{-x}$. Hence, in this limit, one obtains $F_N(z) \to 2e^{-e^{-x}} - e^{-2e^{-x}}$. Taking the derivative with respect to x gives the limiting PDF of t_c as announced in Eq. (4).

V. MULTIPLE WALKERS (PERIODIC CASE)

We now consider N independent walkers on a ring of size L, all starting at the same point 0. As in the $N = 1$ case discussed earlier, the cumulative cover time distribution is exactly related to the cumulative distribution of the span $\mathcal{S}_N(t)$ of N walkers on an infinite line, all starting at the same point, via the relation,

$$\text{Prob.}[t_c > t|L,N] = \text{Prob.}[\mathcal{S}_N(t) < L]. \tag{34}$$

The study of the PDF of $\mathcal{S}_N(t)$ was initiated in Ref. [30] and recently was computed exactly for all N in Ref. [33]. It was shown in Ref. [33] that $\text{Prob.}[\mathcal{S}_N(t) = s] = (1/\sqrt{4Dt})h_N(s/\sqrt{4Dt})$ where the N dependent scaling function $h_N(y)$ is given by

$$h_N(y) = \int_0^\infty dl_1 \int_0^\infty dl_2 \delta(y - l_1 - l_2)\frac{\partial^2 g^N}{\partial l_1 \partial l_2}. \tag{35}$$

Here $g(l_1,l_2)$ is the scaled cumulative joint distribution of the maximum and the minimum of a single BM, starting at the origin on an infinite line and is given by [33],

$$g(l_1,l_2) = \frac{4}{\pi}\sum_{n=0}^\infty \frac{1}{2n+1}\sin\left[\frac{(2n+1)\pi l_2}{l_1+l_2}\right] \times \exp\left\{-\left[\frac{(2n+1)\pi}{2(l_1+l_2)}\right]^2\right\}. \tag{36}$$

As in the case of $N = 1$, taking derivative of Eq. (34) with respect to t, we get

$$\text{Prob.}[t_c = t|L,N] = \frac{4D}{L^2} f_N^P\left(\frac{4Dt}{L^2}\right), \tag{37}$$

with the superscript P denoting the PBC. The scaling function $f_N^P(z)$ is given by

$$f_N^P(z) = \frac{1}{2z^{3/2}} h_N\left(\frac{1}{\sqrt{z}}\right), \tag{38}$$

where $h_N(y)$ is given in Eq. (35). In the inset of Fig. 3, we show a plot of $f_N^P(z)$ for different N's and compare it to numerical results. The asymptotic tails of $f_N^P(z)$ for $N \geqslant 2$ can be obtained from the tails of $h_N(y)$ computed in Ref. [33] with the result,

$$f_N^P(z) \sim \begin{cases} \sqrt{2}N(N-1)/(\sqrt{\pi}z^{3/2})e^{-1/(2z)}, & \text{as } z \to 0, \\ (a_N z/2)e^{-N\pi^2 z/4}, & \text{as } z \to \infty, \end{cases} \tag{39}$$

where a_N is given by

$$a_N = 4\pi^{3/2}N(N-1)(4/\pi)^{N-2}\Gamma([N-1]/2)/\Gamma(N/2). \tag{40}$$

In particular one can check that $a_1 = 2\pi^2$, in agreement with Eq. (29). As in the reflecting case (32), the behavior for $z \to 0$ is quite different for $N = 1$ and $N \geqslant 2$.

We now turn to the limiting distribution of t_c for large N for the PBC. In the context of the span distribution, the limiting form of the scaling function $h_N(y)$ was already analyzed for large N in Ref. [33], and it was found that

$$h_N(y) \approx 2\sqrt{\ln N}\mathcal{D}[2\sqrt{\ln N}(y - 2\sqrt{\ln N})], \tag{41}$$

where the function $\mathcal{D}(s) = 2e^{-s}K_0(2e^{-s/2})$ was obtained as a convolution of two Gumbel laws. Substituting this result (41) in Eq. (38) one finds that this function $f_N^P(z)$ has a sharp peak at $z = 1/(4 \ln N)$. To analyze the large N scaling limit of $f_N^P(z)$, we set $z = 1/(4 \ln N) - x/[4(\ln N)^2]$ as in the reflecting case. Expanding for large N, we get $f_N^P(z) \approx 8(\ln N)^2\mathcal{D}(2x)$. Using $dz = dx/[4(\ln N)^2]$, one immediately obtains the results for the periodic case announced in Eqs. (3) and (5).

VI. CONCLUSION

We have obtained the full PDF of the cover time t_c for N independent Brownian motions in one dimension, both for reflecting and for periodic boundary conditions. Previously, only the first moment of t_c was known in 1D for $N = 1$ and $N = 2$. Our results provide an instance of exact cover time distributions for recurrent random walks, demonstrating clearly that this is different from a Gumbel law found recently for transient (i.e., nonrecurrent) walks [13,24]. In addition, we have shown that in the limit of large N, the random variable t_c approaches its average value of $\langle t_c\rangle \approx L^2/(16D \ln N)$ with fluctuations decaying as $1/(\ln N)^2$. The centered and scaled distributions converge to two distinct and nontrivial N independent scaling functions $g_R(x)$ and $g_P(x)$ given in Eqs. (4) and (5), respectively, and plotted in Fig. 4. Another instance of recurrent RW is in $d = 2$ for which the average value of t_c has been well studied [16,17,19,21,22]. However, determining its full PDF in $d = 2$ for one or multiple ($N \geqslant 2$) walkers remains an outstanding challenge.

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