Constraint algebra in Smolin's $G \to 0$ limit of 4D Euclidean gravity

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(Received 23 February 2018; published 14 May 2018)

Smolin's generally covariant $G_{\mathrm{Newton}} \to 0$ limit of 4d Euclidean gravity is a useful toy model for the study of the constraint algebra in loop quantum gravity (LQG). In particular, the commutator between its Hamiltonian constraints has a metric dependent structure function. While a prior LQG-like construction of nontrivial anomaly free constraint commutators for the model exists, that work suffers from two defects. First, Smolin's remarks on the inability of the quantum dynamics to generate propagation effects apply. Second, the construction only yields the action of a single Hamiltonian constraint together with the action of its commutator through a continuum limit of corresponding discrete approximants; the continuum limit of a product of two or more constraints does not exist. Here, we incorporate changes in the quantum dynamics through structural modifications in the choice of discrete approximants to the quantum Hamiltonian constraint. The new structure is motivated by that responsible for propagation in an LQG-like quantization of paramatrized field theory and significantly alters the space of physical states. We study the off shell constraint algebra of the model in the context of these structural changes and show that the continuum limit action of multiple products of Hamiltonian constraints is (a) supported on an appropriate domain of states, (b) yields anomaly free commutators between pairs of Hamiltonian constraints, and (c) is diffeomorphism covariant. Many of our considerations seem robust enough to be applied to the setting of 4d Euclidean gravity.

DOI: 10.1103/PhysRevD.97.106007

I. INTRODUCTION

The construction of a physically viable quantum dynamics for loop quantum gavity (LQG) (see for e.g. [1–5] and the references therein) constitutes a key open problem. Two desirable features of such a dynamics are its compatibility with general covariance and its ability to propagate perturbations [6]. Here, we focus on the issue of general covariance in the context of Smolin's novel weak coupling limit of Euclidean gravity [7]. General covariance is expected to be encoded in a representation of the algebra of Hamiltonian and spatial diffeomorphism constraints [8]. Accordingly, we construct a domain of quantum states for the model together with the action of constraint operator products thereon in such a way that the resulting algebra of constraints exhibits anomaly free constraint commutators. The model shares several structural aspects with canonical general relativity and we expect our considerations here to serve as essential inputs in the construction of a generally covariant dynamics for LQG.

On the other hand, propagation properties of quantum dynamics in LQG-like quantizations seem to be related to certain structural properties of the Hamiltonian constraint [9]. While we defer an analysis of propagation properties of the dynamics of this model to future work [10], we note that the general structural properties believed to be connected with propagation effects in our study of parametrized field theory [9] play a key role in our demonstration of an anomaly free constraint algebra here.

We initiated our study of the quantum constraint algebra of the model in [11,12]. The phase space of the system consists of a triplet of Abelian connections and conjugate electric fields, its dynamics is driven by Hamiltonian and diffeomorphism constraints with a Poisson Bracket algebra isomorphic to that of (Euclidean) gravity, and the LQG-like quantum theory supports a representation of operators consisting of holonomies of connections around spatial loops and electric fluxes through spatial surfaces. While the quantum theory supports a unitary representation of spatial diffeomorphisms, the action of the Hamiltonian constraint operator is defined in an indirect manner via a continuum limit of appropriate discrete approximants. The reason, as in LQG, is as follows. The classical constraint depends on the curvature of the connection. While the classical curvature can be defined via a "shrinking loop" limit of an approximant constructed out of classical holonomies, the corresponding quantum holonomy operator limit does not exist because the background independent quantum theory cannot distinguish between a bigger loop and its smaller shrinking versions. However, following [13], it *is* nevertheless possible to construct a classical approximant to the Hamiltonian constraint through a suitable conglomeration of such discrete approximants in such a way that the limit of the action of the corresponding conglomeration of

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operators can be defined despite individual operator limits being ill defined. Since the limit involves shrinking of "discrete regulating labels" such as loops and graphs, it is referred to as a "continuum limit" and the approximants are referred to as discrete approximants.

The work in Ref. [11] constructs the continuum limit of the action of a single Hamiltonian constraint and an anomaly free continuum limit action of the commutator between two Hamiltonian constraints from suitably defined discrete approximants. The work in Ref. [12] improves upon the single Hamiltonian constraint action so as to render it spatially covariant thus ensuring an anomaly free commutator of the single Hamiltonian constraint action with the spatial diffeomorphism constraint. This is achieved while maintaining the anomaly free nature of the commutator between a pair of Hamiltonian constraints. It is important to note that the work in [11,12] constructs the continuum limit of a discrete approximant to the commutator between a pair of Hamiltonian constraints rather than the commutator between continuum limit products. More in detail, the product of the action of 2 discrete approximant single Hamiltonian constraints is constructed, the commutator of this product is evaluated *first* and *then* the continuum limit is taken. Instead, a better implementation of the commutator between the quantum constraints would be to *first* take the continuum limit of the product of a pair of discrete single Hamiltonian constraint actions and then take the commutator of this product. However it turns out that with the choice of discrete approximants used in [11,12], while the continuum limit of the discrete commutator action is well defined, the limit of the discrete product action is not. This is because certain terms with divergent continuum limits in the discrete product action drop out when commutation is performed before continuum limit.

Here we significantly improve upon the analysis of [11,12] as follows. We construct the continuum limit action of multiple products of Hamiltonian constraints, each such constraint smeared by a "c-number" lapse i.e. we are able to compute the action of a string of Hamiltonian constraint operators $\hat{C}(N_1)..\hat{C}(N_m)$.[1] From this action we can compute the action of the operator obtained by replacing, in this operator string, any number of pairs of successive smeared Hamiltonian constraint operators by their commutators, i.e. we can compute actions of operator products of the type

$$\hat{C}(N_1)..\hat{C}(N_{i_1-1})[\hat{C}(N_{i_1}),\hat{C}(N_{i_1+1})]\hat{C}(N_{i_1+2})..\hat{C}(N_{i_2-1})[\hat{C}(N_{i_2}),\hat{C}(N_{i_2+1})]\hat{C}(N_{i_2+2})\ldots$$
$$\ldots.\hat{C}(N_{i_j-1})[\hat{C}(N_{i_j}),\hat{C}(N_{i_j+1})]\hat{C}(N_{i_j+2})..\hat{C}(N_m). \quad (1.1)$$

We show that each of the commutators in this string is anomaly free in the sense that each can be replaced by the operator correspondent of the corresponding classical Poisson bracket (this operator correspondent, as in general relativity, is itself *not* a Hamiltonian constraint smeared by a c-number lapse because of the occurrence of structure functions in the Poisson bracket algebra). We are also able to show that the continuum limit action of multiple products of smeared Hamiltonian constraints is diffeomorphism covariant and that the group of finite spatial diffeomorphisms is implemented in an anomaly free manner. This is almost but not quite the same as what is conventionally referred to as the implementation of the constraint algebra without anomalies in that we do not concern ourselves with higher order commutators of the type $[\ldots.[[\hat{C}(N_1),\hat{C}(N_2)],\hat{C}(N_3)],\ldots.,\hat{C}(N_j)]$. We shall return to this point in the concluding section of this work. Till then we shall refer to our results as an *anomaly free single commutator implementation of the constraint algebra*.

While our basic strategy is the same as in Refs. [11,12] (referred to here on as P1, P2 respectively), its implementation here is more complex than in those works. A brief summary of the strategy, as implemented here, follows. As in P1, P2 we deal with the Hamiltonian constraint of density $4/3$ smeared with a density weight $-1/3$ lapse as this seems essential for *nontrivial* anomaly free commutators (see, for e.g. Sec. IX in [14] and Chap. 2 of [5]). For reasons explained above, we first define the action of suitable discrete approximants to this constraint and then take the continuum limit. As for LQG [13,15], the action of these discrete approximants on a charge network state[2] receives contributions only from vertices of the charge net. As in P1, P2, we confine our attention to the case of charge nets with a single contributing vertex. Since the lapse function has a nontrivial density weight the action of a discrete approximant to the constraint (henceforth referred to as the *discrete action of the constraint*) can only be computed with the aid of a coordinate patch around the contributing vertex. This action on such a charge net state generates deformations of the state and the "size" of these

[1] Specifically, we are able to define the action of up to $k-1$ products of these constraints where we use the C^k semianalytic category and k can be chosen to be arbitrarily large. Note this is similar to the fact that for C^k vector fields one can only define up to k nested commutators and this is the analog of the Lie algebra for the group of C^{k+1} diffeomorphsims.

[2] Charge network states are the Abelian analog of the spin network basis states of LQG [16] each such state being labeled by a spatial graph whose edges are labeled by integer valued "charges."

deformations is measured, in a precise sense, by the coordinate patch associated with the charge net being acted upon. The continuum limit action then involves shrinking the size of these deformations away. Thus, the constraint action depends on a choice of "regulating"' coordinate patches, one for (the contributing vertex of) each charge net.

While the discrete action is defined on any charge network state, the continuum limit of this discrete action can only be defined on distributional states which are non-normalizable infinite sums over charge network states and which lie in the algebraic dual to the finite span of charge networks states.[3] In this work, as in P1, P2 we restrict our attention to the case where the coefficents in these sums are nonvanishing only for "single vertex" charge nets of the type described above. The coefficients in this sum are determined through the specification of a density weighted function and a Riemmanian metric on the 3d Cauchy slice. This is in contrast to the specification of the scalar "vertex smooth" function [17] of P1, P2. Due to the density weight of the function and the tensorial nature of the metric, the evaluation of these coefficients also requires a choice of coordinate patches at vertices of the charge network states they multiply. We choose these coordinate patches used to evaluate these coefficients to be the same as the regulating coordinate patches chosen above to define the discrete action of the Hamiltonian constraint. This choice of coordinate patches then allows the coefficients to be evaluated and, consequently, the distributional states which support the continuum limit constraint action to be specified. It is on this set of distributional states that anomaly freedom is verified. Each such state will be called an "anomaly free state" and the set of states will be referred to as the "anomaly free domain."

The requirement of anomaly free single commutators is phrased in terms of an identity (2.11) discovered in P1 which expresses the Poisson bracket between a pair of classical Hamiltonian constraints in terms of Poisson brackets between certain phase space functions known as electric diffeomorphism constraints (this name derives from their construction as smearings of the diffeomorphism constraint with electric field dependent vector fields). Anomaly freedom is the requirement that this identity holds between the commutator between a pair of Hamiltonian constraints and the (continuum limit of the) corresponding electric diffeomorphism commutators. Since the electric fields in quantum theory are not smooth, the deformations corresponding to electric diffeomorphisms are "singular" versions of smooth diffeomorphisms, and, hence, *distinct* from the latter. This enables us to focus first on the construction of an anomaly free single commutator implementation of the algebra of Hamiltonian constraints and analyze spatial diffeomorphism covariance of our constructions in a second step as follows.

Classical diffeomorphism covariance is encoded in the Poisson brackets between the diffeomorphism constraint and the Hamiltonian constraint and between the diffeomorphism constraints themselves. The diffeomorphism constraint generates the action of infinitesimal diffeomorphisms on the connection and electric fields. In contrast, in LQG-like representations the natural operators are those which implement *finite* diffeomorphisms. It is possible to encode the content of the Poisson brackets involving the diffeomorphism constraint in terms of the action of finite diffeomorphisms. The Poisson bracket between the diffeomorphism constraints is encoded in the requirement that the group of finite diffeomorphisms connected to identity is represented faithfully. The Poisson brackets between the diffeomorphism constraint and the Hamiltonian constraint are encoded in the requirement that the action of the Hamiltonian constraint be appropriately diffeomorphism covariant [see Eq. (12.4)]. Since LQG-like representations provide a unitary representation of the group of finite diffeomorphisms, we need to concentrate only on the diffeomorphism covariance of the Hamiltonian constraint action on states in the anomaly free domain. It is here that the *metric dependence* of states in the anomaly free domain allows, relative to P2, a qualitatively *new* mechanism for the implementation of diffeomorphism covariance of the continuum limit action of the Hamiltonian constraint.

Recall that this continuum limit action arises as the limit of the action of discrete approximants to the constraint. Also recall that this discrete action underlying the continuum limit action requires, for its definition, the choice of a regulating coordinate patch around the contributing vertex of the charge net being acted upon. Hitherto (see P2), these coordinate patches (and hence the corresponding discrete deformations generated by the discrete approximant to the constraint) were chosen once and for all independent of the choice of the anomaly free state. The new ingredient in this work is to tie the choice of these structures to the metric label of the state as follows. Smooth diffeomorphisms are represented unitarily on the space of charge network states. Hence they have a well-defined dual action on any anomaly free state. Consider one such state with metric label h_{ab}. Then it turns out that the dual action of a finite diffeomorphism ϕ on this state maps the state to a new anomaly free state with metric label $\phi^* h_{ab}$ which is the push forward of h_{ab} by ϕ. Let the choice of coordinate patch around the contributing vertex v of the charge net state c when the anomaly free state has metric label h_{ab} be $\{x\}$. Similar to the case of LQG spin nets, the unitary action of the diffeomorphism ϕ on c yields the charge net c_ϕ with contributing vertex $\phi(v)$. Then the idea is to choose the coordinate patch around the contributing vertex of the

[3] The algebraic dual comprises of linear mappings from this finite span to the complex numbers; its elements may be thought of as (in geneneral non-normalizable) sums of charge network bras.

charge net state c_ϕ when the anomaly free state has metric label $\phi^* h_{ab}$ to be $\phi^*\{x\}$.

As we shall see in the main body of the paper, tying the choice of regulating coordinate patches to the metric label of the state in this "diffeomorphism covariant" manner results in an elegant and immediate implementation of diffeomorphism covariance of the continuum limit action of the Hamiltonian constraint. To summarize, we have a tight formalism wherein the label of the anomaly free distributional state dictates the choice of discrete approximant to the Hamiltonian constraint which in turn defines a discrete action whose continuum limit is diffeomorphism covariant. This implementation of diffeomorphism covariance seems to us to be a robust and beautiful phenomenon with possible applicability to full blown LQG. This concludes our summary of the strategy employed in this paper.

Our considerations in the main body of the paper are based on the contents of P1 and P2. While we shall aim at a self-contained presentation, the reader interested in a complete understanding is urged to establish some familiarity with P1, P2 especially Secs. 2, 4, 5 and Appendix C4 of P1 and Secs. 3.2 and 3.3 and 5.5 of P2. The reader interested in only a bird's eye view of our results may peruse Secs. II, III, XII and XIII.. Before we proceed to a description of the layout of the paper, we note that this model was first studied in an LQG representation in [18] wherein the authors focussed on the case of 3 dimensions. The model was studied in 4d in [11,12]. An attempt was made to apply the lessons learned from these studies, together with a remarkable identity discovered by Ashtekar [19] (see also [20] where this identity is reproduced) and earlier pioneering work by Bruegman [21], to 4d Euclidean gravity in [20].

The layout of the paper is as follows. In Sec. II we briefly review the model and the derivation of the discrete approximants used in P2. In Sec. III we briefly review the structural lessons learned from the study of propagation in parametrized field theory [9] and show how to incorporate these lessons into a modified choice of discrete approximants for the action of the Hamiltonian and the electric diffeomorphism constraint on a certain restricted class of states. The modifications, though seemingly minor, are responsible for an anomaly free single commutator implementation of the constraint algebra. Due to the nature of the modifications it turns out that the set of restricted states considered in Sec. III are not large enough for our purposes because the action of the constraints maps these states out of this set. Hence it is necessary to define the discrete constraint action on a slightly larger set. We develop this for a restricted class of elements of this larger set in Sec. IV and lift this restriction in Sec. V, wherein we display our detailed choice for the action on elements of this larger set (called the ket set in Sec. VI).

In Sec. VI we construct the discrete action of products of constraint operators. This action derives from multiple applications of actions each of the type specified in Sec. V. The specification in Sec. V on elements of the ket set is not complete in that the coordinate patches underlying the constraint action remain unspecified. In Sec. VI we remedy this and provide a complete construction of the action corresponding to discrete approximants to products of constraints on elements of the ket set. Finally, we also indicate as to how the constraints act on states outside this larger set. It turns out that for our purposes, this action on the complement of this set does not need to be specified in great detail; any action which maps the complement to itself suffices.

In Sec. VII we construct the anomaly free domain of quantum states. As mentioned earlier the quantum states in the anomaly free domain are obtained as non-normalizable sums over kinematic states with certain coefficients. Since it is mathematically more precise to think of these states as residing in a dual space, the sum is over "bras" rather than kets. The set of bras being summed over is referred to as the bra set. As in P1, P2, for simplicity, we restrict attention to a bra set in which each bra has a single nontrivial vertex at which the constraints act. These bras are "bra" correspondents of states of the type encountered in Sec. V. Every state in the anomaly free domain is labeled by a density weighted function and a Riemmanian metric on the Cauchy slice. The coefficient which multiplies a bra in the bra set is evaluated from the structure of the graph underlying the bra together with the density weighted function and metric associated with the anomaly free state. As mentioned earlier, the continuum limit action of discrete approximant is defined through the contraction of the discrete deformations generated by the approximant. The dual action of the discrete approximant on an anomaly free state transfers this contraction behavior to the contraction behavior of coefficients which characterize the anomaly free state. We analyze this behavior in Sec. VIII and Appendixes F, G. 2 as a necessary prerequisite to the computation of the continuum limit action. In Sec. IX we evaluate the continuum limit action of a product of two Hamiltonian constraints on an anomaly free state. This defines the action of its commutator. Next, we compute the continuum limit action of the appropriate commutator between two electric diffeomorphism constraints and demonstrate equality with the Hamiltonian constraint commutator, thus showing that the action of a product of two Hamiltonian constraints is well defined and anomaly free. In Sec. X we extend this result to the action of higher order products of constraints so as to show that the commutators in (1.1) are anomaly free. In Sec. XI we show that the action of the constraint products of Sec. X is also diffeomorphism covariant. We briefly summarize and display our results in Sec. XII. Section XIII is devoted to discussion.

Notation and conventions.— We set the speed light to be unity but retain factors of $\hbar$. The analog of spin net states in LQG are called charge network states here. We refer to a charge network state as c or $|c\rangle$ depending on our convenience, even changing from one to the other in the course of a single calculation. The symbol c is used for the charge network label (see Sec. II) underlying a charge net state. We work with the C^k semianalytic category [3,22]. Due to the finite number of English alphabets, the letter k may occasionally refer to objects other than the differentiability degree; however, the context should make the usage clear. The Cauchy slice Σ is semianalytic, oriented, connected and compact without boundary. All semianalytic charts used are right handed. The pushforward action of a C^k semianalytic diffeomorphism ϕ is denoted by ϕ^* and its pullback action by ϕ_* so that $\phi^*\phi_* = \mathbf{1}$.

II. REVIEW OF ESSENTIAL BACKGROUND FROM P1, P2

Almost all the material below is contained in P1. The only part of P2 we allude to is in the choice of conical deformations at the end of Sec. II. C below. The only new material not from P1, P2 is in the last two paragraphs of Sec. II. B wherein we describe our choice of the inverse metric determinant operator.

A. Classical description of the model

The phase space variables $(A_a^i, E_i^a, i = 1, 2, 3)$ are a triplet of $U(1)$ connections and conjugate density weight one electric fields on the Cauchy slice Σ so that the phase space is that of a $U(1)^3$ gauge theory. We define the density weight 2 contravariant metric $qq^{ab} := \sum_i E_i^a E_i^b$, q being the determinant of the corresponding covariant metric q_{ab}. The phase space functions,

$$G[\Lambda] = \int d^3x \Lambda^i \partial_a E_i^a, \tag{2.1}$$

$$D[\vec{N}] = \int d^3x N^a (E_i^b F_{ab}^i - A_a^i \partial_b E_i^b), \tag{2.2}$$

$$H[N] = \frac{1}{2} \int d^3x N q^{-1/3} \epsilon^{ijk} E_i^a E_j^b F_{ab}^k, \tag{2.3}$$

are the Gauss law, diffeomorphism, and Hamiltonian constraints of the theory, and where $F_{ab}^i := \partial_a A_b^i - \partial_b A_a^i$. The Poisson brackets between the constraints are

$$\{G[\Lambda], G[\Lambda']\} = \{G[\Lambda], H[N]\} = 0, \tag{2.4}$$

$$\{D[\vec{N}], G[\Lambda]\} = G[\pounds_{\vec{N}} \Lambda], \tag{2.5}$$

$$\{D[\vec{N}], D[\vec{M}]\} = D[\pounds_{\vec{N}} \vec{M}], \tag{2.6}$$

$$\{D[\vec{N}], H[N]\} = H[\pounds_{\vec{N}} N], \tag{2.7}$$

$$\{H[N], H[M]\} = D[\vec{\omega}] + G[A \cdot \vec{\omega}], \quad \omega^a := q^{-2/3} E_i^a E_i^b (M \partial_b N - N \partial_b M). \tag{2.8}$$

The last Poisson bracket (between the Hamiltonian constraints) exhibits structure functions just as in gravity.

It is useful to define the electric shifts N_i^a by

$$N_i^a = N E_i^a q^{-1/3} \tag{2.9}$$

and the electric diffeomorphism constraints $D(\vec{N}_i)$ by

$$D[\vec{N}_i] = \int d^3x N_i^a E_j^b F_{ab}^j. \tag{2.10}$$

Assuming the Gauss law constraint is satisfied, a key identity derived in P1 is

$$\{H[N], H[M]\} = (-3) \sum_{i=1}^{3} \{D[\vec{N}_i], D[\vec{M}_i]\}. \tag{2.11}$$

B. Quantum kinematics

The basic functions of interest are $U(1)^3$ holonomies associated with oriented closed graphs colored by representations of $U(1)^3$ and electric fluxes through surfaces. Colored graphs are labeled by charge network labels. A charge network label c is the collection $(\gamma, \vec{q}_I, I = 1, , N)$ where γ is an oriented graph with N edges, the Ith edge e_I colored with a triplet of $U(1)$ charges $(q_I^1, q_I^2, q_I^3) \equiv \vec{q}_I$. The holonomy associated with c is h_c,

$$h_c := \prod_{I=1}^{N} e^{i\kappa\gamma q_I^j \int_{e_I} A_a^j dx^a}. \tag{2.12}$$

Here κ is a fixed constant with dimensions ML^{-1} and γ is a dimensionless Immirzi parameter. In what follows we shall choose units such that $\kappa\gamma = 1$.

h_c is $U(1)^3$ gauge invariant if the total $U(1)^3$ charge flowing into every vertex is the same as that flowing out of the vertex, where "into" and "out of" corresponds to whether the edge in question is incoming or outgoing at the vertex. In the rest of this paper we restrict our attention exclusively to gauge invariant charge net labels. The gauge invariant electric flux through a surface S is $E_i(S)$,

$$E_i(S) := \int_S \eta_{abc} E_i^a. \tag{2.13}$$

where η_{abc} is the coordinate 3-form. The holonomy flux Poisson bracket algebra is closed and represented on the space of charge network states. Each charge network state $|c\rangle$ is labeled by a charge network label c. Holonomies act by multiplication and electric flux operators count the discrete electric flux corresponding to the weighted sum of the charge carried by edges of c which intersect S_i with the weights being $\pm 1, 0$ depending on the orientation and placement of the intersecting edges relative to the (oriented) surface S.

Next consider the electric shift operator

$$\hat{N}^a_i = N\hat{E}^a_i q^{-1/3} \tag{2.14}$$

corresponding to the classical expression (2.9). It turns out that this operator only has a nontrivial action at vertices of charge net states and to compute its explicit action we need a regulating coordinate patch at the vertex in question (see P1). The final expression for the operator action at a vertex v of the charge net $|c\rangle$ is

$$\hat{N}^a_i(v)|c\rangle = N^a_i(v)|c\rangle \coloneqq \sum_{I_v} N^a_{I_v i}|c\rangle,$$
$$N^a_{I_v i} \coloneqq \frac{3}{4\pi} N(x(v))\nu_v^{-2/3} q^i_{I_v}\hat{e}^a_{I_v}. \tag{2.15}$$

Here I_v refers to the I_vth edge at v, and $\hat{e}^a_{I_v}$ to the unit I_vth edge tangent vector, unit with respect to the coordinates $\{x\}$ at v and $N(x(v))$ denotes the evaluation of the density weighted lapse N at v in this coordinate system. $\nu_v^{-2/3}$ is proportional to the eigenvalue of the $\hat{q}^{-1/3}$ operator in equation (2.14). Specifically, a regulated version of this operator acting at the vertex v of the charge net state $|c\rangle$ can be defined. It has the eigenvalue $\nu^{-2/3}\epsilon^2$ where ϵ^3 is the coordinate size of a small regulating region around v so that $\hat{q}^{-1/3}(v)|c\rangle \coloneqq (\nu_v^{-2/3}\epsilon^2)|c\rangle$. In P1 this regulated version of $\hat{q}^{-1/3}$ is defined through a Thiemann trick [3,13].

In this work we use a slightly different definition of $\hat{q}^{-1/3}$ as follows. From P1, we have that the regulated metric determinant operator $\hat{q}$ acts at v as $\hat{q}(v) = \epsilon^{-6}\hat{q}_{\rm loc}(v)|c\rangle$ where, again, ϵ^3 is the coordinate size of a small regulating region around v and where the operator $\hat{q}_{\rm loc}(v)$ is defined through

$$\hat{q}_{\rm loc}(v)|c\rangle = \frac{1}{48}\hbar^3\left(\left|\sum_{IJK}\epsilon^{IJK}\epsilon_{ijk}q^i_I q^j_J q^k_K\right|\right)|c\rangle \eqqcolon \hbar^3(\nu_v)^2|c\rangle \tag{2.16}$$

where each of the three sums (over I, J, K) extends over the valence of v, with I, J, K labeling (outgoing) edges e_I, e_J, e_K emanating from v. $\epsilon^{IJK} = 0, +1, -1$ depending on whether the tangents of e_I, e_J, e_K are linearly dependent, define a right-handed frame (with respect to the orientation of the underlying manifold), or define a left-handed frame, respectively. We define $\hat{q}^{-1/3}(v)$ by spectral decomposition of $\hat{q}(v)$ on states with nonzero eigenvalues for $\hat{q}_{\rm loc}(v)$ so that on such states $\nu_v^{-2/3}$ is given by the $-2/3$rd power of ν_v in (2.16). The vertex v for such states will be referred to as a *nondegenerate* vertex.[4] On the zero eigenvalue subspace we define it through the Thiemann trick employed in P1. The result pertinent to the rest of this work is that for the type of zero eigenvalue states of $\hat{q}_{\rm loc}$ encountered in this work; the Thiemann trick returns a vanishing eigenvalue for $\hat{q}^{-1/3}(v)$. This is similar to the definitions of inverse metric operators employed in the loop quantum cosmology context of Ref. [23].

C. Discrete Hamiltonian constraint from P1

The action of the discrete approximant to the Hamiltonian constraint operator of P1 is motivated through the following heuristics. Given a charge net label, define the charge net coordinate $c^{ai}(x)$:

$$c^{ai}(x) = c^{ai}(x; \{e_I\}, \{q_I\})$$
$$= \sum_{I=1}^{M} \mathrm{i} q^i_I \int \mathrm{d}t_I \delta^{(3)}(e_I(t_I), x)\dot{e}^a_I(t_I). \tag{2.17}$$

The associated holonomy h_c can then be written as $h_c = \exp\left(\int \mathrm{d}^3x c^a_i A^i_a\right)$. A charge net state can be thought of heuristically as a wave function of the connection $c(A) = h_c(A)$. Holonomy operators then act by multiplication and the electric field operator by functional differentiation so that $\hat{E}^a_i(x) = -i\hbar\frac{\delta}{\delta A^i_a(x)}$.

The Hamiltonian constraint in terms of the electric shift is

$$H[N] = \frac{1}{2}\int_\Sigma \mathrm{d}^3x\epsilon^{ijk}N^a_i F^k_{ab}E^b_j + \frac{1}{2}\int_\Sigma \mathrm{d}^3x N^a_i F^i{}_{ab}E^b_i$$
$$= \frac{1}{2}\int_\Sigma \mathrm{d}^3x\left(-\epsilon^{ijk}(\pounds_{\vec{N}_j}A^k_b)E^b_i + \sum_i(\pounds_{\vec{N}_i}A^i_b)E^b_i\right). \tag{2.18}$$

Here the second term on the right-hand side of the first line vanishes classically and the second line is obtained using the identity $N^a_i F^k_{ab} = \pounds_{\vec{N}_i}A^k_b - \partial_b(N^c_i A^i_c)$.

The quantum analog of (2.18) acts on a charge net wave function. For simplicity, we restrict our attention to charge nets with a single nondegenerate vertex. The electric shift is then replaced by its operator analog (2.14) which is, in turn, replaced by its eigenvalue $N^a_i(v)$ (2.15) to yield

$$\hat{C}[N]c(A) = -\frac{\hbar}{2\mathrm{i}}c(A)\int_\Sigma \mathrm{d}^3x A^i_a(\epsilon^{ijk}\pounds_{\vec{N}_j}c^a_k + \pounds_{\vec{N}_i}c^a_i). \tag{2.19}$$

[4]It turns out that this notion of nondegeneracy is appropriate for the Grot-Rovelli (GR) vertices of P1, P2 and Sec. III. We shall encounter a different type of vertex in Sec. IV of this work called a "CGR" vertex and will discuss the notion of nondegeneracy for such a vertex in Sec. IV. A

We refer to $N_i^a(v)$ as the quantum shift. While $N_i^a(v)$ is nonzero only at the point v on the Cauchy slice Σ, we shall think of some regulated version thereof which is of small compact support $\Delta_\delta(v)$ of coordinate size δ^3 about v (in the coordinates we used to define the quantum shift). Expanding the quantum shift into its edge components (2.15) yields

$$\hat{C}[N]c(A) = \sum_{I_v} -\frac{\hbar}{2\mathrm{i}}c(A)\int_{\triangle_{\delta(v)}} \mathrm{d}^3 x A_a^i(\epsilon^{ijk}\pounds_{\vec{N}_j^{I_v}}c_k^a + \pounds_{\vec{N}_i^{I_v}}c_i^a). \tag{2.20}$$

Next, we approximate the Lie derivative by the difference of a small diffeomorphism and the identity as follows:

$$(\pounds_{\vec{N}_i^I}c_j^a)A_a^k = -\frac{3}{4\pi}N(x(v))\nu_v^{-2/3}q_{I_v}^i\frac{\varphi(\vec{e}_I,\delta)^* c_j^a A_a^k - c_j^a A_a^k}{\delta} + O(\delta), \tag{2.21}$$

where we imagine extending the edge tangents $\vec{e}_I$ to $\Delta_\delta(v)$ in some smooth compactly supported way and define $\varphi(\vec{e}_I,\delta)$ to be the finite diffeomorphism corresponding to translation by an affine amount δ along this edge tangent vector field. Using the replacement (2.21) and using the compact support property of the edge tangent vector field to replace the integration domain $\Delta_\delta(v)$ by Σ yields

$$\hat{C}[N]c(A) = \frac{1}{\delta}\frac{\hbar}{2\mathrm{i}}c(A)\frac{3}{4\pi}N(x(v))\nu_v^{-2/3}\sum_{I_v} q_{I_v}^i\int_\Sigma \mathrm{d}^3 x[\cdots]_\delta^{I_v,i} + O(\delta), \tag{2.22}$$

$$\begin{aligned}
[\cdots]_\delta^{I_v,1} &= [(\varphi c_2^a)A_a^3 - c_2^a A_a^3] + [(\varphi\bar{c}_3^a)A_a^2 - \bar{c}_3^a A_a^2] + [(\varphi c_1^a)A_a^1 - c_1^a A_a^1]\\
[\cdots]_\delta^{I_v,2} &= [(\varphi c_3^a)A_a^1 - c_3^a A_a^1] + [(\varphi\bar{c}_1^a)A_a^3 - \bar{c}_1^a A_a^3] + [(\varphi c_2^a)A_a^2 - c_2^a A_a^2]\\
[\cdots]_\delta^{I_v,3} &= [(\varphi c_1^a)A_a^2 - c_1^a A_a^2] + [(\varphi\bar{c}_2^a)A_a^1 - \bar{c}_2^a A_a^1] + [(\varphi c_3^a)A_a^3 - c_3^a A_a^3],
\end{aligned} \tag{2.23}$$

where we have written $\bar{c}_i^a \equiv -c_i^a$ and where we have suppressed the edge label I_v and set $\varphi c_j^a \equiv \varphi(\vec{e}_{I_v},\delta)^* c_j^a$.

The integral in (2.22) is of order δ and we approximate by its exponential minus the identity to get our final expression:

$$\hat{C}[N]c(A) = \frac{\hbar}{2\mathrm{i}}c(A)\frac{3}{4\pi}N(x(v))\nu_v^{-2/3}\sum_{I_v}\sum_i q_{I_v}^i\frac{\mathrm{e}^{\int_\Sigma[\cdots]_\delta^{I_v,i}} - 1}{\delta} + O(\delta). \tag{2.24}$$

For each fixed (I_v, i) the exponential term is a product of edge holonomies corresponding to the charge net labels specified through (2.23). This product may be written as $h^{-1}_{c_{(i,\mathrm{flip})}}h_{c_{(i,\mathrm{flip},I_v,\delta)}}$ where $c_{(i,\mathrm{flip},I_v,\delta)}$ is the deformation of $c_{(i,\mathrm{flip})}$ by $\varphi(\vec{e}_I,\delta)$ and $c_{i,\mathrm{flip}}$ has the same graph as c but "flipped" charges. To see what these charges are, fix $i=1$ and some edge I_v corresponding to the first line of (2.23). In $c_{(1,\mathrm{flip})}$, the connection A_a^3 corresponding to the third copy of $U(1)$ is multiplied by the charge net c_2^a corresponding to the second copy of $U(1)$. This implies that in the holonomy $h_{c_{(1,\mathrm{flip})}}$ the charge label in the third copy of $U(1)$ for any edge is exactly the charge label in the second copy of $U(1)^3$ of the same edge in c i.e. in obvious notation $q^3|_{c_{(1,\mathrm{flip})}} = q^2|_c$ where we have suppressed the edge label. A similar analysis for all the remaining terms in (2.23) indicates that charges ${}^{(i)}q^j$, $j=1, 2, 3$ on any edge of $c_{(i,\mathrm{flip})}$ are given by the following "i flipping" of the charges on the same edge of c,

$${}^{(i)}q^j = \delta^{ij}q^j - \sum_k \epsilon^{ijk}q^k. \tag{2.25}$$

The exact nature of the deformed charge net $c_{(i,\mathrm{flip},I_v,\delta)}$ depends on the definition of the deformation. Since the deformation is of compact support around v, the combination $h^{-1}_{c_{(i,\mathrm{flip})}}h_{c_{(i,\mathrm{flip},I_v,\delta)}}$ is just identity except for a small region around v. From (2.24), this term multiplies $c(A)$. We call the resulting charge net $c_{(i,I_v,\delta)}$. Our final expression as derived in P1 for the discrete approximant to the Hamiltonian constraint then reads

$$\hat{C}[N]_\delta c(A) = \frac{\hbar}{2\mathrm{i}}\frac{3}{4\pi}N(x(v))\nu_v^{-2/3}\sum_{I_v}\sum_i q_{I_v}^i\frac{c_{(i,I_v,\delta)} - c}{\delta}. \tag{2.26}$$

An identical analysis for the action of the electric diffeomorphism constraint yields the result

$$\hat{D}_\delta[\vec{N}_i]c = \frac{\hbar}{\mathrm{i}}\frac{3}{4\pi}N(x(v))\nu_v^{-2/3}\sum_{I_v}q_{I_v}^i\frac{1}{\delta}(c_{(I_v,\delta)} - c) \tag{2.27}$$

where $c_{(I_v,\delta)}$ is obtained from c only by deformation without any charge flipping so that

$$(c_{(I_v,\delta)})_i^a(x) \coloneqq \varphi(\vec{\hat{e}}_{I_v},\delta)^* c_i^a(x). \tag{2.28}$$

It remains to specify the deformation $\varphi(\vec{\hat{e}}_{I_v},\delta)$. From the discussion above this deformation must distort the graph underling c in the vicinity of its vertex v in such a way that its vertex is displaced by a coordinate distance δ along the I_vth edge direction to leading order in δ. Due to the vanishing of the quantum shift except at v, this regulated deformation is visualized to abruptly pull the vertex structure at v in the direction of the I_v the edge. In P1 this was achieved by moving the vertex *almost* along the edge by an amount δ but not exactly along it so that the displaced vertex lay in a δ^q, $q>1$ vicinity of the edge. The edges connected to the original vertex v were then pulled along the direction of the displaced vertex. Due to the "abrupt" pulling, the original edges developed certain kinks signaling the point from which they were suddenly pulled. The reader is urged to consult the figures in P1 detailing this. The final picture of the distortion is one in which the off-edge displaced vertex is connected to a kink on the I_vth edge by an edge which almost coincides with the original I_vth, and is connected to the kinks on the remaining edges by edges which point almost exactly opposite to the I_vth one, the structure in the vicinity of the displaced vertex resembling (and in P2 being exactly that of) the latter set of edges lying along a "downward" cone with the former edge being upward along the cone axis. This completes our summary of discrete constraint action as developed in P1, P2.

III. MODIFIED DISCRETE CONSTRAINT ACTION

In Sec. III. A we recall some of the structures responsible for propagation in parametrized field theory [9], discuss their analogs in the context of the $U(1)^3$ model studied here and argue that constraint actions in P1, P2 do not display these structural analogs.

In Sec. III. B we indicate how these structural features can be incorporated into a modified constraint action which we display in Eqs. (3.10) and (3.11). We shall focus on the case in which the charge net being acted upon has a single GR vertex where (as in P1, P2) a GR vertex is defined as one which has valence greater than 3 and at which no triple of edge tangents is linearly dependent. In addition we shall restrict our attention to *linear* GR vertices; a vertex will be said to be linear iff there exists a neighborhood of the vertex equipped with a coordinate patch such that the entire set of edges at this vertex in this neighborhood are straight lines in this coordinate patch.[5] The constraints generate displacements and deformations of the vertex structure around the linear GR vertex. The deformed vertex structure takes the form of a cone, this conical structure being defined in terms of the coordinates associated with the linear structure of the GR vertex. For pedagogical reasons we shall focus on "downward" conical deformations in this section. It turns out that it is also necessary to consider "upward" conical deformations and that the choice of upward or downward conicality is linked to the positivity properties of the edge charge labels at the GR vertex. A complete treatment will be presented in Sec. V.

In Sec. III. C we show the existence of an alternate choice of charge flips to that defined by Eq. (2.25); as we shall see later both choices of flips are needed to obtain the crucial minus sign on the right-hand side of (2.11). In Sec. III. D we summarize our results. We remind the reader that as mentioned in Sec. II, all charge nets encountered in the remainder of the paper are $U(1)^3$ gauge invariant.

[5]A further technicality which may be ignored for now is that we also restrict the charge nets here to be "primordial" in the language of Sec. VI. B.

A. Structures responsible for propagation

Our comments in this section will be very brief as our main focus in this work is the construction of an anomaly free constraint algebra rather than an analysis of propagation. We intend to analyze the issue of propagation in this model in future work [10].

Smolin [6] argued that LQG methods necessarily yield discrete constraint actions whose repeated application on spin network states create nested structures around the original vertices of the spin net. These nested deformations are created independently for each different vertex. As a result, a deformation near one vertex cannot have any bearing on that near another vertex and in this sense no information can propagate from the vicinity of one of the original vertices of the spin net to another. In Ref. [9], we showed that while Smolin's observations are indeed valid, propagation should be viewed as a property of physical states lying in the kernel of the constraints rather than as a property of repeated actions of the discrete approximants to the constraint on kinematical states. Propagation can be viewed in terms of the structure of a given physical state as follows. A physical state is a (in general, kinematically non-normalizable) sum of kinematic states. We may then view the physical state as one which encodes propagation effects if kinematic states in this sum are related by propagation [9]. Since physical states are solutions of the quantum constraints, their structure depends on that of the constraints which in turn derives from the structure of the chosen discrete approximants. It was argued in Ref. [9] that one of the features responsible for propagation was the $\frac{\hat{O}-1}{\delta}$ of these discrete approximants, where $\hat{O}$ is some kinematic operator which has a finite well-defined action on any spin net state. Roughly speaking, this structure together with requirement that a continuum limit exists, ensures that the sum over kinematic states which represents any physical state must have a structure such that if the "offspring" state $\hat{O}|s\rangle$ is in this sum then the "parent" state $|s\rangle$ must also be in the sum. While at first sight, Eqs. (2.26) and (2.27) seem to have this structure, a more careful perusal of these equations shows that due to gauge invariance $\sum_{I_v} q^i_{I_v} = 0$ so that the $-\mathbf{1}$ term is absent.

Secondly, in the simple context of [9] the analog of spin network states live on one-dimensional graphs so that any two successive vertices are connected by an edge. It is this connection which provides a path for putative propagation effects i.e. a deformation from one vertex can putatively propagate to another along this "conducting" edge. In contrast (2.26) and (2.27) generate deformations which move *off* the edges of the graph (see the material at the end of the Sec. II. C) and this feature is preserved by repeated actions of the type (2.26) and (2.27).

In view of these remarks we shall modify the discrete action (2.26) and (2.27) so that (i) there is a nontrivial $-\mathbf{1}$ term in the expression for the discrete constraint action, and (ii) the displaced vertex $\varphi(\vec{\hat{e}}_{I_v}, \delta) \cdot v$ is along the I_vth edge of the graph rather than off it.

B. Modified action for linear GR vertices

We implement (i) in Sec. III. B. 1 and (ii) in Sec. III. B. 2. As mentioned above we shall restrict our considerations to the context of linear GR vertices. Recall that a linear vertex is one equipped with a coordinate patch in its neighborhood with respect to which the edges at the vertex in this neighborhood appear as straight lines. The vertex will be said to be linear with respect to such a coordinate patch. In what follows the coordinate patch used to specify the deformations generated by constraints is assumed to be one with respect to which the vertex is linear. The detailed choice of these coordinates will be discussed in Sec. VI.

1. Addressing the -1 issue

We refer the reader to Eq. (2.21). Let us scale the (regulated, compact supported in $\triangle_{\delta(v)}$) vector field $\vec{\hat{e}}_{I_v}$ by its charge label $q^i_{I_v}$ and define $\varphi(q^i_I \vec{\hat{e}}_{I_v}, \delta)$ to be the small diffeomorphism generated by the resulting vector field $q^i_I \vec{\hat{e}}_{I_v}$. If we use this diffeomorphism to approximate the Lie derivative on the left-hand side of (2.21), we obtain the equation

$$(\pounds_{\vec{N}^I_i} c^a_j) A^k_a = -\frac{3}{4\pi} N(x(v)) \nu_v^{-2/3} \frac{\varphi(q^i_{I_v} \vec{\hat{e}}_I, \delta)^* c^a_j A^k_a - c^a_j A^k_a}{\delta} + O(\delta). \quad (3.1)$$

Using Eq. (3.1) as our starting point instead of Eq. (2.21) and repeating the subsequent argumentation and steps of Sec. II. C, we see that the q^i_I factor in (2.24) now disappears by virtue of the replacement of $\varphi(\vec{\hat{e}}_I, \delta)$ by $\varphi(q^i_I \vec{\hat{e}}_I, \delta)$. As a result, the holonomy $h_{c_{(i,\text{flip},I_v,\delta)}}$ is replaced by $h_{c_{(i,\text{flip},q^i_{I_v},I_v,\delta)}}$, where $c_{(i,\text{flip},q^i_{I_v},I_v,\delta)}$ is the image of $c_{(i,\text{flip})}$ by $\varphi(q^i_{I_v} \vec{\hat{e}}_I, \delta)^*$:

$$(c_{(i,\text{flip},q^i_{I_v},I_v,\delta)})^a_j(x) \coloneqq \varphi(q^i_I \vec{\hat{e}}_I, \delta)^* (c_{(i,\text{flip})})^a_j(x). \quad (3.2)$$

Consequently, the deformed charge net $c_{(i,I_v,\delta)}$ in (2.26) is replaced by the charge net $c_{(i,q^i_{I_v},I_v,\delta)}$ which is obtained by the action of the holonomy $h^{-1}_{c_{(i,\text{flip})}} h_{c_{(i,\text{flip},q^i_{I_v},I_v,\delta)}}$ on c. This leads us to the constraint action

$$\hat{C}[N]_\delta c(A) = \frac{\hbar}{2\mathrm{i}} \frac{3}{4\pi} N(x(v)) \nu_v^{-2/3} \sum_{I_v} \sum_i \frac{c_{(i,q^i_{I_v},I_v,\delta)} - c}{\delta}. \quad (3.3)$$

An identical analysis for the action of the electric diffeomorphism constraint yields the result

$$\hat{D}_\delta[\vec{N}_i] c = \frac{\hbar}{\mathrm{i}} \frac{3}{4\pi} N(x(v)) \nu_v^{-2/3} \sum_{I_v} \frac{1}{\delta} (c_{(q^i_{I_v},I_v,\delta)} - c), \quad (3.4)$$

where $c_{(q^i_{I_v},I_v,\delta)}$ is obtained from c only by the action of $\varphi(q^i_I \vec{\hat{e}}_I, \delta)$ without any charge flipping so that

$$(c_{(q^i_{I_v},I_v,\delta)})^a_j(x) \coloneqq \varphi(q^i_{I_v} \vec{\hat{e}}_{I_v}, \delta)^* c^a_j(x). \quad (3.5)$$

Clearly this addresses issue (i) of Sec. III. A.

2. Addressing the conducting edge issue

Instead of the off edge placement of the displaced vertex by $\varphi(\vec{\hat{e}}_{I_v}, \delta)$ as in P1, we place the vertex on the edge e_{I_v}. In view of the considerations of Sec. III. B. 1, the action of $\varphi(q^i_{I_v} \vec{\hat{e}}_{I_v}, \delta)$ is defined to displace the vertex v by a coordinate distance $q^i_{I_v}$ along the I_vth edge. Denote the displaced vertex by $v_{q^i_{I_v},I_v,\delta}$. The remaining edges $e_{J_v \neq I_v}$ are dragged along in the direction of the I_vth edge so as to form a downward pointing cone in the vicinity of the cone vertex at $v_{q^i_{I_v},I_v,\delta}$ where "upward" refers to the direction of the edge e_{I_v} and where, as in, P1, P2, all edges at $v_{q^i_{I_v},I_v,\delta}$ are taken to point outwards from $v_{q^i_{I_v},I_v,\delta}$. These remaining edges develop kinks at the points $\tilde{v}_{J_v}$ at which the edge tangents are discontinuous. As in P1, P2 we refer to these kinks as C^0 kinks (for a formal definition see Appendix A.

An explicit construction of the relevant deformation is provided in Appendix B where the linear GR condition is used.[6] The deformations based on the construction of

[6]More precisely, as we shall see in Sec. VII, the deformation constructed in Appendix B is diffeomorphic to that discussed here. Hence all diffeomormphism invariant properties of the latter are identical to that of the former.

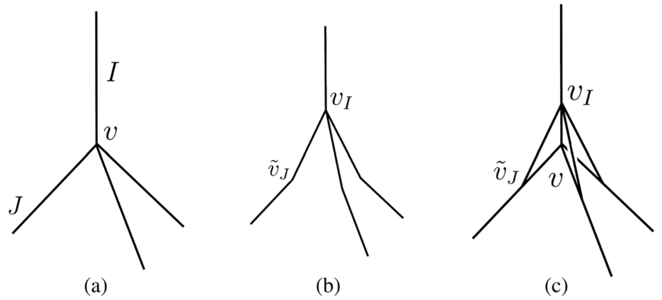

FIG. 1. (a) Undeformed GR vertex v of a charge net c with its Ith and Jth edges as labeled. The vertex is deformed along its Ith edge in (b) wherein the displaced vertex v_I and the C^0 kink, $\tilde{v}_J$ on the Jth edge are labeled. (c) Result of a Hamiltonian type deformation obtained by multiplying the charge net holonomies obtained by coloring the edges of (b) by flipped images of charges on their counterparts in c, (a) by negative of these flipped charges and (a) by the charges on c. If the edges of (b) are colored by the charges on their counterparts in c then one obtains an electric diffemorphism deformation.

Appendix B are displayed in Fig. 1. We shall summarize the content of this figure in Sec. III. D.

The downward conical deformations of Appendix B displace the vertex v upward along the I_vth edge. This is clearly appropriate only if $q^i_{I_v}$ is *positive*. If $q^i_{I_v}$ is negative it is necessary to consider deformations which displace v in the *opposite* direction. This, in turn, requires the further construction of an *extension* of the edge I_v together with an upward conical deformation of the vertex structure around v. We shall defer a discussion of such upward conical deformations and graph extensions to Sec. V in the interests of pedagogy. Hence the deformations described above are only valid for deformations along edges for which the charges labels are *positive*.

In view of the discussion in Sec. III. A, we refer to the edge along which the vertex is displaced in the deformed charge net as the *conducting* edge in the deformed charge net. The remaining edges at the displaced vertex in the deformed charge net which connect the displaced vertex with C^0 kinks will be called *nonconducting* edges. In the case of Hamiltonian constraint type deformations, the conducting edge at the displaced vertex of the deformed charge net $c_{(i,q^i_{I_v},I_v,\delta)}$ splits into 2 parts, a lower conducting edge which connects the displaced vertex with the vertex v (i.e. with the vertex of c) and an upper part beyond the displaced vertex.

C. Charge flips

Note that in Sec. II. C we could equally have started with a minus sign in front of the second term in (2.18) since that term is nonvanishing. Let us do this. This leads to the replacement of equation (2.19) by

$$\hat{C}[N]c(A) = \frac{\hbar}{2\mathrm{i}} c(A) \int_\Sigma \mathrm{d}^3x A^i_a (-\epsilon^{ijk} \pounds_{\vec{N}_j} c^a_k + \pounds_{\vec{N}_i} c^a_i). \quad (3.6)$$

Repeating the subsequent argumentation, we are lead to define the charge net $c(-i, \mathrm{flip})$ instead of $c(i, \mathrm{flip})$, with $-i$ flipped charges ${}^{(-i)}q^j$ instead of the i flipped charges of Eq. (2.25), with these $-i$ flipped charges defined as

$$^{(-i)}q^j = \delta^{ij} q^j + \sum_k \epsilon^{ijk} q^k. \quad (3.7)$$

The exponential term in Eq. (2.24) is then replaced, in obvious notation, by $h^{-1}_{c_{(-i,\mathrm{flip})}} h_{c_{(-i,\mathrm{flip},I_v,\delta)}}$ and we are lead to, instead of Eq. (2.26), the expression

$$\hat{C}[N]_\delta c(A) = -\frac{\hbar}{2\mathrm{i}} \frac{3}{4\pi} N(x(v)) \nu_v^{-2/3} \sum_{I_v} \sum_i q^i_{I_v} \frac{c_{(-i,I_v,\delta)} - c}{\delta} \quad (3.8)$$

where $c_{(-i,I_v,\delta)}$ is exactly the same as $c_{(i,I_v,\delta)}$ of (2.26) except that the i flipped charges of equation (2.25) are replaced by their $-i$ flipped version in Eq. (3.7). Repeating the considerations of Sec. III. B. 1 we are lead to the final equation:

$$\hat{C}[N]_\delta c(A) = -\frac{\hbar}{2\mathrm{i}} \frac{3}{4\pi} N(x(v)) \nu_v^{-2/3} \sum_{I_v} \sum_i \frac{c_{(-i,q^i_{I_v},I_v,\delta)} - c}{\delta} \quad (3.9)$$

where, once again in obvious notation, $c_{(-i,q^i_{I_v},I_v,\delta)}$ is exactly the same as $c_{(i,q^i_{I_v},I_v,\delta)}$ except that the role of i flipping is replaced by that of $-i$ flipping.

To summarize, we are able to generate an overall minus sign in the expression (3.9) relative to (3.3) by changing the charge flip from an i flip to a $-i$ flip. Putting everything together (and using the notation $c_{(+i,q^i_{I_v},I_v,\delta)} \equiv c_{(i,q^i_{I_v},I_v,\delta)}$ we are lead to two possible discrete actions of the Hamiltonian constraint:

$$\hat{C}[N]_\delta c(A) = \pm \frac{\hbar}{2\mathrm{i}} \frac{3}{4\pi} N(x(v)) \nu_v^{-2/3} \sum_{I_v} \sum_i \frac{c_{(\pm i,q^i_{I_v},I_v,\delta)} - c}{\delta}. \quad (3.10)$$

As no charge flipping is involved, the expression for the electric diffeomorphism constraint remains the same:

$$\hat{D}_\delta[\vec{N}_i]c = \frac{\hbar}{\mathrm{i}} \frac{3}{4\pi} N(x(v)) \nu_v^{-2/3} \sum_{I_v} \frac{1}{\delta} (c_{(q^i_{I_v},I_v,\delta)} - c). \quad (3.11)$$

In view of the considerations of Sec. III. B. 2 the deformations in Eqs. (3.10) and (3.11) are of the on edge, conical type. We slightly abuse notation and continue to use the notation $\varphi(q^i_I \vec{\hat{e}}_I, \delta)$ of Sec. III. B. 1 for the deformation map corresponding to the modified deformations of Sec. III. B. 2. In Sec. VII we shall find it necessary to use both the versions of discrete Hamiltonian action described in (3.10).

Finally, as emphasized in Sec. III. B, the deformations along the I_vth edge constructed therein are valid only if

$q^i_{I_v} > 0$. For $q^i_{I_v} < 0$, we shall define the deformed states $c_{(\pm i, q^i_{I_v}, I_v, \delta)}$, $c_{(q^i_{I_v}, I_v, \delta)}$ in Eqs. (3.10) and (3.11), in Sec. V.

D. Summary

For the case that $q^i_{I_v} > 0$, we display the deformed charge net $c_{(\pm i, q^i_{I_v}, I_v, \delta)}$ of (3.10) in Fig. 1(c). This charge net can be visualized as the product of following three holonomies:

(i) a holonomy labeled by the deformed charge net colored with flipped charges, $h_{c_{(-i,\mathrm{flip}, I_v, \delta)}}$, shown in Fig. 1(a);

(ii) a holonomy labeled by an undeformed charge net based on the same graph [see Fig. 1(a)] as c and colored with the negative of the flipped charges $h^{-1}_{c_{(-i,\mathrm{flip})}}$, the negative sign coming from the inverse;

(iii) the original charge net holonomy based on the graph shown in Fig. 1(a).

As a result, the charge carried by the undeformed counterparts of the nonconducting edges at v in $c_{(\pm i, q^i_{I_v}, I_v, \delta)}$ (namely the edges which connect v to the C^0 kinks) have vanishing ith component. By gauge invariance the charge along the (lower) conducting edge passing through v in $c_{(\pm i, q^i_{I_v}, I_v, \delta)}$ also has vanishing ith component. It is then straightforward to see that, similar to P1, P2, the vertex v is *degenerate* in $c_{(\pm i, q^i_{I_v}, I_v, \delta)}$. Also note that each nonconducting edge in (i) carries flipped versions of the charges carried by its undeformed counterpart in c. Hence, using gauge invariance at the displaced vertex in $c_{(\pm i, q^i_{I_v}, I_v, \delta)}$, we have the following remark:

Remark 0.—The *difference* between the outgoing and incoming charges along the conducting edge at the deformed vertex in $c_{(\pm i, q^i_{I_v}, I_v, \delta)}$ is the $\pm i$ flipped version of the charge along the I_vth edge in c.Finally, recall that vertex structure in a sufficiently small vicinity of the displaced vertex when viewed in terms of the coordinates associated with the linear vertex v in c takes the following form. All edges are straight lines. The conducting edge in $c_{(\pm i, q^i_{I_v}, I_v, \delta)}$ is split into two parts by the displaced vertex. The remaining (nonconducting) edges at the displaced vertex form a downward cone. With respect to the downward direction of the cone, the conducting edge splits into an upper conducting edge and a lower conducting edge.

The deformed charge net $c_{(q^i_{I_v}, I_v, \delta)}$ of (3.11) is based on the same deformed graph as that in (i) above; the only difference is that the charge labels are unflipped i.e. each deformed edge in $c_{(q^i_{I_v}, I_v, \delta)}$ has the same charge as its undeformed counterpart in c.

IV. MODIFIED ACTION: LINEAR CGR VERTICES

In the last section we restricted our attention to linear GR vertices. The action of the Hamiltonian constraint (3.10) displaces such a vertex along a conducting edge so that the conducting edge splits into an incoming and outgoing part at the displaced vertex and the incoming and outgoing conducting edge tangents comprise a linearly dependent pair at the displaced vertex [see Fig. 1(c)]. Hence any triple of edge tangents which contains the incoming and outgoing conducting edge tangents is no longer linearly independent and the displaced vertex is not strictly GR. Due to the role played by the conducting edge in altering the (linear) GR structure of such a vertex, we shall call it a (linear) conducting edge-altered GR vertex or a CGR vertex.[7]

In Sec. IV. A we isolate the structure in the vicinity of such a vertex, discuss it in detail and define modified discrete constraint actions for states with such a vertex. As in the previous section the coordinates with respect to which the deformations generated by these constraints actions are defined will be assumed to be ones with respect to which the vertex is linear. The detailed choice of these coordinates will be discussed in Sec. VI. In Sec. IV. D we define a single notation which succinctly describes the deformed states produced by the modified constraint actions both for the GR and the CGR cases.

A. Linear CGR vertices: Definition and constraint action

From Sec. III. D, we define a (linear) CGR vertex as follows. A vertex v of a charge net c will be said to be linear CGR if

(i) there exists a coordinate patch around v such that all edges at v are straight lines;

(ii) the union of two of the edges at v form a single straight line so that v splits this straight line into two parts;

(iii) the set of remaining edges together with any one of the two edges in (i) constitute a GR vertex in the following sense. Consider, at v, the set of outgoing edge tangents to each of the remaining edges together with the outgoing edge tangent to one of the two edges in (i). Then any triple of elements of this set is linearly dependent.

We shall call the edges other than those in (ii) as nonconducting in c and the two edges in (ii) as upper

[7]Note that the transition from a GR vertex to a CGR vertex by the Hamiltonian constraint action is *not* generated by the action of the deformation map $\varphi(q^i_{I_v} \hat{e}_{I_v}, \delta)$. Indeed, the graph underlying the deformed charge net created by the action of the deformation map on c displays a single GR vertex as shown in Fig. 1(b). Rather, the CGR property stems from the fact that $c_{(\pm i, q^i_{I_v}, I_v, \delta)}$ is constructed not only from the deformed charge net of Fig. 1(b) but also the undeformed ones based on the graph shown in Fig. 1(a). Indeed, the electric diffeomorphism constraint action (3.11) retains the GR nature of the vertex acted upon as displayed in Fig. 1(b).

and lower conducting edges in c and refer to the union of the conducting edges as the conducting line in c.[8] Let the upper conducting edge and the nonconducting edges be assigned an outward pointing orientation from v in c and let the lower conducting edge be assigned an incoming orientation at v in c so that the conducting line acquires a natural well-defined orientation induced from the conducting edges. Let the number of nonconducting edges be $N-1$. Hence there are $N+1$ edges at v but these edges define only N distinct oriented straight lines passing through v in c, one of them being the conducting line and the remaining $N-1$ being the nonconducting edges. Let $J_v = 1, .., N$ be an index which numbers these straight lines. Let the conducting line be the K_vth one. It follows that the nonconducting edges are assigned indices $\{J_v, J_v \neq K_v\}$. Denote such a nonconducting edge by e_{J_v} for some $J_v \neq K_v$ and its outgoing charge by $q^i_{J_v}$. Denote the upper conducting edge with outward orientation by $e_{K_v,\mathrm{out}}$, the lower conducting edge with incoming orientation by $e_{K_v,\mathrm{in}}$ and their respective outgoing and incoming charges by $q^i_{K_v,\mathrm{out}}$ and $q^i_{K_v,\mathrm{in}}$.

We turn now to a derivation of modified constraint actions on a state c with a linear CGR vertex using the notation discussed above. We shall convert the situation into one in which the lower conducting edge is absent at v and the upper conducting edge acquires a charge $q^i_{K_v,\mathrm{out}} - q^i_{K_v,\mathrm{in}}$. The vertex v then becomes GR and we may then use the deformations described in Appendix B. 1. In this section we shall restrict our attention to the case where the net conducting charge $q^i_{K_v,\mathrm{out}} - q^i_{K_v,\mathrm{in}}$ is *positive*. This restriction is for pedagogical reasons which are identical to those which underlie the applicability of the downward conical deformations of Sec. III. B to the case of $q^i_{I_v} > 0$ (see the discussion at the end of Sec. III. B). The general case involving charges with no positivity restrictions together with the consideration of upward conical deformations will be discussed in Sec. V.

We are interested in the discrete action of the constraints at small enough discretization parameter δ where δ is measured by the coordinate system in (i). Consider a loop l made up of two edges l_1, l_2 so that $l = l_1 \circ l_2$. Let l_1 be a segment of the conducting line running between two of its points p_1 and p_2 equidistant from v, where p_1 is below v and p_2 is above v. Let p_1 and p_2 be chosen such that the coordinate length of l_1 is $C\delta, C > 16 q_{\max}$[9] where

[8]Here we assume that we are given a specification of which of the two edges is upper and which is lower; how this specification arises will be discussed in Sec. V.

[9]See (a)–(c), Sec. V. A. 2 for the reason for this choice of C.

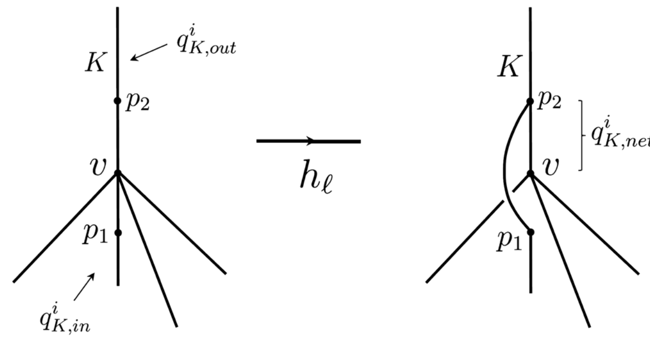

FIG. 2. Left: vertex structure at the CGR vertex v. The conducting edges are the Kth ones. Right: effect of multiplication by the intervening holonomy h_l on this vertex structure. The lower conducting edge at v is removed and the upper conducting edge is charged with the net conducting charge.

$$q_{\max} = \max_{(i=1,2,3),(I_v=1,\ldots,N)} |q^i_{I_v}|. \tag{4.1}$$

Further, let l_1 be oriented so as to run from p_1 to p_2. Let l_2 be a semicircular arc connecting p_2 with p_1 such that its diameter is $C\delta$. Let l lie in a coordinate plane P_l such that no nonconducting edge lies in P_l. Define the holonomy h_l to run along l with charge equal to $-q_{K_v,\mathrm{in}}$ i.e. h_l is charged with the negative of the incoming charge at v carried by the incoming lower conducting edge. Note that for any smooth connection A^j_a,

$$h_l \coloneqq \exp i\left(-\sum_{j=1}^{3} q^j_{K_v,\mathrm{in}} \int_l A^j_a dx^a\right) \sim 1 + O(\delta^2). \tag{4.2}$$

Since the classical holonomy h_l is unity to order δ^2, multiplication of an approximant to a constraint by h_l continues to yield an acceptable approximant. Accordingly, we first multiply c by h_l. Clearly, this yields the charge net c_l in which, as mentioned above, the lower conducting edge of c is absent from p_1 to v, the upper conducting edge acquires a charge $q^i_{K_v,\mathrm{out}} - q^i_{K_v,\mathrm{in}}$ between v and p_2 and the nonconducting edges are untouched. As shown in Fig. 2, the vertex v in c_l then becomes GR and we may then act on the result by the discrete approximant to the constraint of interest as in Sec. III, the vertex structure deformations of c_l being constructed along the lines described in Appendix B. 1.

We act on the result by $\hat{h}_l^{-1}$. Since the deformation of Appendix B. 1 is confined to within a ball of radius $2q_{\max}\delta$ about v [see (4.1) for the definition of $q_{\max}$], the semicircular arc l_2 does not touch the deformed structures, and due to its placement does not touch the undeformed structure (for small enough δ) except at p_1, p_2. Hence the action of $\hat{h}_l^{-1}$ simply removes the extra segment l_2 from the charge nets generated hitherto and restores the missing part of the conducting line, so that we have

$$\hat{C}[N]_\delta c(A) = \pm \hat{h}_l^{-1} \frac{\hbar}{2\mathrm{i}} \frac{3}{4\pi} N(x(v)) \nu_v^{-2/3} \sum_{I_v} \sum_i \frac{c_{l(\pm i, q^i_{I_v}, I_v, \delta)} - c_l}{\delta}$$
$$= \pm \frac{\hbar}{2\mathrm{i}} \frac{3}{4\pi} N(x(v)) \nu_v^{-2/3} \left(\sum_{I_v \neq K_v} \sum_i \frac{c_{(\pm i, q^i_{I_v}, I_v, \delta)} - c}{\delta} + \sum_i \frac{c_{(\pm i, q^i_{K_v,\mathrm{out}} - q^i_{K_v,\mathrm{in}}, K_v, \delta)} - c}{\delta} \right). \quad (4.3)$$

In the second and third lines we have used ν_v to denote the volume eigenvalue of c_l at its GR vertex v. Note that this is *not* the same as the volume eigenvalue for c.[10] The fact that a nontrivial constraint action is only possible if v is nondegenerate in c_l (rather than in c) suggests that we define our notion of nondegeneracy for a CGR vertex to be tied to that of the corresponding GR vertex obtained by modifying the CGR one through the intervention of the holonomy h_l. We shall formalize this definition in Secs. IV. B and V.

The deformed charge net $c_{(\pm i, q^i_{I_v}, I_v, \delta)}$ for $I_v \neq K_v$ and for the case $q^i_{I_v} > 0$[11]is shown in Fig. 3(c).

It may be viewed as the product of three holonomies: one of which is deformed and has flipped charges as shown in Fig. 3(b), a second which is based on the undeformed graph of Fig. 3(a) with negative of the flipped charges and the last which is just the holonomy corresponding to c. Due to the deformations of the GR vertex structure of c_l, each of the edges of $c_{l(\pm i, q^i_{I_v}, I_v, \delta)}$ at its nondegenerate vertex other than the I_vth one meet their undeformed counterparts in C^0 kinks. Since there is no lower conducting edge at the vertex v of $c_{l(\pm i, q^i_{I_v}, I_v, \delta)}$, the subsequent multiplication by $\hat{h}_l^{-1}$ results in a restoration of this "missing" part of $e_{K_v,\mathrm{in}}$ without any further kink. Thus the deformed graph structure underlying $c_{(\pm i, q^i_{I_v}, I_v, \delta)}$ obtained by first intervening with $\hat{h}_l$ then deforming the resulting GR structure and finally intervening with $\hat{h}_l^{-1}$ is to (besides generating the displaced vertex and its attendant vertex structure) deform the graph underlying c so as to generate a C^0 kink on each nonconducting edge of c other than the I_vth one and to generate a single C^0 kink on the conducting line of c, this kink lying on the upper conducting edge of c with the lower conducting edge having no kink.

Note that the lower conducting edge of c between p_1 and v does not intersect the deformed edges of $c_{l(\pm i, q^i_{I_v}, I_v, \delta)}$. To see this proceed as follows. Note that the deformation in Appendix B is constructed first out of straight lines and then the straight lines at the displaced vertex are "conically" deformed in a sufficiently small neighborhood of the displaced vertex. Clearly this neighborhood can always be chosen to be small enough that the lower conducting edge is in its complement. Hence if we show that if this edge does not intersect the initial construction of the deformation in terms of exclusively straight lines, it does not intersect their conical deformation. For the initial part of the construction in Appendix B. 1 (a)–(c) below hold:

(a) Consider the deformation of the upper conducting edge in c which connects a kink vertex on the upper

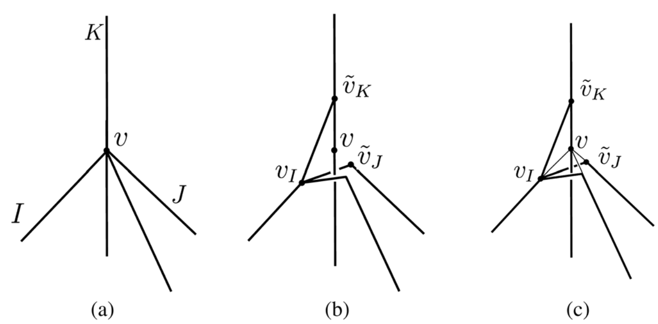

FIG. 3. (a) Undeformed CGR vertex v of a charge net c with its Kth conducting edge and Ith and Jth nonconducting edges as labeled. (b) Vertex structure of (a) is deformed along its Ith edge and the displaced vertex v_I and the C^0 kinks $\tilde{v}_J$, $\tilde{v}_K$ on the Jth, Kth edges are as labeled. (c) Result of a Hamiltonian type deformation. To obtain this result (i) in (b) color the edge from v_I to $\tilde{v}_K$ with the flipped image of the *net* conducting charge in c, that from v to $\tilde{v}_K$ with the flipped image of the lower conducting charge at c and the remaining edges with the flipped images of the charges on their undeformed counterparts in c; (ii) color the edges of (a) by the negative of the flipped charges on c; (iii) color the edges of (a) by the charges on c; (iv) multiply the holonomies corresponding to (i), (ii), (iii). In (b), if the edge from v_I to $\tilde{v}_K$ is colored with the *net* conducting charge in c, that from v to $\tilde{v}_K$ by the lower conducting charge in c and the remaining edges by the charges on their counterparts in c one obtains the result of an electric diffemorphism deformation.

[10]From (2.16), it follows that the volume eigenvalue is sensitive only to the structure of c in a small vicinity of v. If we replace this structure by one which has identical colored nonconducting edges, no lower conducting edge and an upper conducting edge which has charge $q^i_{K_v,\mathrm{out}} + q^i_{K_v,\mathrm{in}}$, the volume eigenvalue for this structure is the same as that for c. This differs from that for c_l because the vertex structure there has the upper conducting edge charge as $q^i_{K_v,\mathrm{out}} - q^i_{K_v,\mathrm{in}}$.

[11]We will tackle the $q^i_{I_v} < 0$ case in Sec. V. Hence the deformed charge nets $c_{l(\pm i, q^i_{I_v}, I_v, \delta)}$ for $q^i_{I_v} < 0$ will be constructed in detail only in that section.

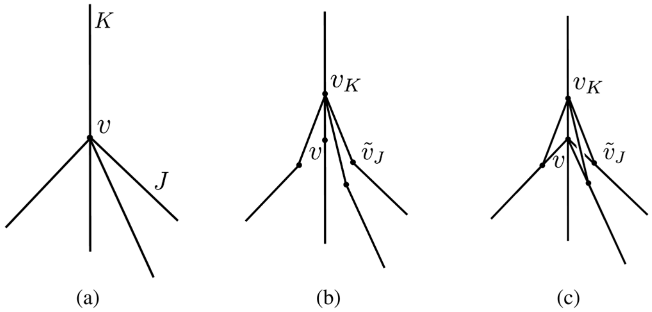

FIG. 4. (b) Vertex structure of (a) is deformed along its Kth edge and the displaced vertex v_K and the C^0 kink $\tilde{v}_J$ on the Jth edge are as labeled. (c) Result of a Hamiltonian type deformation obtained by multiplying the three charge net holonomies obtained by coloring the edges of (b) by the flipped images of the charges on their counterparts in c, the edges of (a) by the negative of these flipped charges and the edges of (a) by the charges on c. If the edges of (b) are colored by the charges on their counterparts in c then one obtains an electric diffemorphism deformation.

conducting edge in c to the displaced vertex in $c_{l(\pm i,q^i_{I_v},I_v,\delta)}$ which lies along the I_vth edge of c at a position distinct from v. This deformed edge cannot intersect the lower conducting edge because two distinct straight lines can intersect at most at a single point.

(b) Clearly the lower conducting edge of c does not intersect the I_vth (upper conducting and lower conducting) edge in $c_{l(\pm i,q^i_{I_v},I_v,\delta)}$ except at v, once again because two distinct straight lines can intersect at most at a single point.

(c) Consider the J_vth nonconducting edge in c with $J_v \neq I_v$. Its deformation connects a kink vertex on the J_vth edge to the displaced vertex. From Appendix B. 1 this deformed edge lies in a plane containing the I_vth and the J_vth edges. The lower conducting edge can only intersect this plane at v by virtue of the fact that v is CGR in c.

From (a)–(c) it follows as claimed that the lower conducting edge between p_1 and v does not intersect the deformed edges of $c_{l(\pm i,q^i_{I_v},I_v,\delta)}$. It then follows that the multiplication by $\hat{h}_l^{-1}$ in Eq. (4.3) simply restores this part of the lower conducting edge without creating any more intersections.

For the case that $I_v = K_v$, the deformed charge net $c_{(\pm i,q^i_{K_v,\mathrm{out}}-q^i_{K_v,\mathrm{in}},K_v,\delta)}$ is displayed in Fig. 4(c). This charge net can be thought of as the product of three holonomies [see Figs. 4(a) and 4(b)]. Once again it is easy to see that the deformed edges of $c_{l(\pm i,q^i_{K_v},K_v,\delta)}$ do not intersect the lower conducting edge in c from the fact that two distinct lines can intersect at most at a point. Hence once again the multiplication by $\hat{h}_l^{-1}$ simply restores this part of the lower conducting edge without creating any more intersections.

Similarly, we have

$$\hat{D}_\delta[\vec{N}_i]c = \hat{h}_{l,\vec{q}_{K,\mathrm{in}}}\frac{\hbar}{\mathrm{i}}\frac{3}{4\pi}N(x(v))\nu_v^{-2/3}\left(\sum_{I_v}\frac{1}{\delta}(c_{l(q^i_{I_v},I_v,\delta)}-c_l)\right)$$
$$=\frac{\hbar}{\mathrm{i}}\frac{3}{4\pi}N(x(v))\nu_v^{-2/3}\left(\sum_{I_v\neq K_v}\frac{1}{\delta}(c_{(q^i_{I_v},I_v,\delta)}-c)\right.$$
$$\left.+\frac{1}{\delta}(c_{(q^i_{K_v,\mathrm{out}}-q^i_{K_v,\mathrm{in}},K_v,\delta)}-c)\right). \tag{4.4}$$

The charge net which is obtained through a deformation of c along an edge which is nonconducting in c looks identical to that in Fig. 3(b) except that the charge labels are identical to their counterparts in c.[12] Similarly, the charge net which is obtained through a deformation of c along an edge which is conducting in c looks identical to that in Fig. 4(b) except that the charge labels are identical to their counterparts in c.

B. The net conducting charge: Remarks

We define the difference between the outgoing upper and incoming lower conducting charges at a CGR vertex to be the *net conducting charge* at that vertex. The following remarks highlight the significance of this difference of conducting charges.

In the case of the action of the Hamiltonian constraint (4.3) we have that:

Remark 1.—The deformed K_vth edge in $c_{(\pm i,q^i_{I_v},I_v,\delta)}$ carries the *difference* between the flipped charges of the outgoing upper and incoming lower conducting edges in c.

Remark 2.—The displaced vertex in the deformed charge net $c_{(\pm i,q^i_{K_v,\mathrm{out}}-q^i_{K_v,\mathrm{in}},K_v,\delta)}$ is displaced by an amount $|q^i_{K_v,\mathrm{out}}-q^i_{K_v,\mathrm{in}}|\delta$ from v.

Remark 3.—The difference between the charges on the outgoing upper and incoming lower conducting edges at the nondegenerate vertex of $c_{(\pm i,q^i_{K_v,\mathrm{out}}-q^i_{K_v,\mathrm{in}},K_v,\delta)}$ is the $\pm i$ flipped image of the difference between the charges on the outgoing upper and incoming lower conducting edges at the nondegenerate vertex of c.

In the case of the electric diffeomorphism constraint action (4.4), we have that:

Remark 4.—The deformed K_vth edge in $c_{(q^i_{I_v},I_v\neq K_v,\delta)}$ carries the *difference* between the charges of the upper and lower conducting edges in c.

Remark 5.—The displaced vertex in the deformed charge net $c_{(q^i_{K_v,\mathrm{out}}-q^i_{K_v,\mathrm{in}},K_v,\delta)}$ is displaced by an amount $(q^i_{K_v,\mathrm{out}}-q^i_{K_v,\mathrm{in}})\delta$ from v.

Remark 6.—The difference between the charges on the outgoing upper and incoming lower conducting edges at

[12]Here and below, similar to footnote 11, our comments only apply to those deformed charge nets $c_{(q^i_{I_v},I_v,\delta)}$ for which $q^i_{I_v} > 0$. The deformed charge nets in (4.4) for which this condition does not apply will be defined in Sec. V.

the nondegenerate vertex of $c_{(q^i_{K_v,\mathrm{out}}-q^i_{K_v,\mathrm{in}},K_v,\delta)}$ is equal to the difference between the charges on the outgoing upper and incoming lower conducting edges at the nondegenerate vertex of c.

Remark 7.—Were it not for the intervention by the holonomy around the small loop l, this difference in Remarks (2) and (5) would be replaced by the *sum* because the heuristics of Secs. II and III. B indicate a displacement of the vertex by $\delta(q^i_{K_v,\mathrm{out}}\vec{e}_{K_v,\mathrm{out}} + q^i_{K_v,\mathrm{in}}\vec{e}_{K_v,\mathrm{in}})$ with the outgoing upper conducting edge tangent $\vec{e}_{K_v,\mathrm{out}}$ being equal to the ingoing lower conducting edge tangent $\vec{e}_{K_v,\mathrm{in}}$. As will be apparent in Secs. X and XI this "difference of charges associated with the conducting edge" plays a key role in anomaly freedom.

As we have noted in Sec. IV. A, we may obtain this intervention for the Hamiltonian constraint by starting from (2.18) and putting in factors of the holonomy around l and its inverse and then proceeding along the lines of the subsequent heuristics of Sec. II. C. Since classically, the holonomy and its inverse cancel (and since, furthermore, the classical holonomy is unity to higher order terms in δ than the leading order required by the putative approximant), the intervention leads to an equally acceptable discrete action. Similar heuristics hold for the electric diffeomorphism constraint.

C. Nondegeneracy of CGR vertices

From Figs. 3 and 4, and our discussion above it follows that the displaced vertices in the deformed charge nets generated by (4.3) and (4.4) are CGR or GR.[13] While the notion of nondegeneracy of a GR vertex is just the nonvanishing of the volume eigenvalue at the vertex, in the case of a CGR vertex, the action of the constraints (4.3) and (4.4) is sensitive to the nondegeneracy of the (GR) vertex in c_l rather than the (CGR) vertex in c. Accordingly, we define the notion of nondegeneracy of a CGR vertex as follows:

Definition 1: Nondegeneracy of a CGR vertex.—A CGR vertex of a charge net c will be said to be nondegenerate iff the corresponding GR vertex in the charge net c_l is nondegenerate. If the vertex in c_l is degenerate we shall say that the CGR vertex in c is degenerate.[14]

With the definition of nondegeneracy above, the original parent CGR vertex v is degenerate in the deformed charge nets generated by (4.3). To see this, recall that the deformed charge nets $c_{(\pm i,q^i_{I_v},I_v\neq K_v,\delta)}$, $c_{(q^i_{K_v,\mathrm{out}}-q^i_{K_v,\mathrm{in}},K_v,\delta)}$ in that equation are obtained from the action of h_l^{-1} on $c_{l(\pm i,q^i_{I_v},I_v\neq K_v,\delta)}$, $c_{l(q^i_{K_v,\mathrm{out}}-q^i_{K_v,\mathrm{in}},K_v,\delta)}$. The latter are obtained by the Hamiltonian constraint action on c_l at its GR vertex and hence, as noted in Sec. III. D, the charges on the edges at the vertex v in these deformed and i flipped charge nets have vanishing ith component. In particular the edges in $c_{l(\pm i,q^i_{I_v},I_v\neq K_v,\delta)}$, $c_{l(q^i_{K_v,\mathrm{out}}-q^i_{K_v,\mathrm{in}},K_v,\delta)}$ which connect v to the C^0 kinks have charges with vanishing ith component. Since the action of h_l^{-1} does not affect the charges on the edges at v which connect v to the C^0 kinks, this is also true for these edges in the charge nets $c_{(\pm i,q^i_{I_v},I_v\neq K_v,\delta)}$, $c_{(q^i_{K_v,\mathrm{out}}-q^i_{K_v,\mathrm{in}},K_v,\delta)}$. Gauge invariance implies that the *net* conducting charge at v in these charge nets also has vanishing ith component. Now, *independent* of which part of the conducting edge at v we assign as upper/lower, it is straightforward to check that the appropriate intervention on $c_{(\pm i,q^i_{I_v},I_v\neq K_v,\delta)}$, $c_{(q^i_{K_v,\mathrm{out}}-q^i_{K_v,\mathrm{in}},K_v,\delta)}$ yields charge nets each of which has the leftover upper conducting edge at the (now GR) vertex v colored with the net conducting charge at v. The other edges at v retain their charges so that all the edge charges at v now have vanishing ith componet which implies that the volume eigenvalue after the intervention vanishes. Hence using the definition of nondegeneracy above, we see that the CGR vertex v in $c_{(\pm i,q^i_{I_v},I_v\neq K_v,\delta)}$ and in $c_{(q^i_{K_v,\mathrm{out}}-q^i_{K_v,\mathrm{in}},K_v,\delta)}$ is degenerate.

In the case of deformations generated by (4.4), the vertex v is bivalent in the deformed charge nets $c_{(q^i_{I_v},I_v\neq K_v,\delta)}$, $c_{(q^i_{K_v,\mathrm{out}}-q^i_{K_v,\mathrm{in}},K_v,\delta)}$ and hence degenerate.

D. Convenient notation

Given a charge net c with a single nondegenerate linear GR or CGR vertex v, its deformations by the discrete action of the Hamiltonian constraint in Eqs. (3.10) and (4.3) can be specified through the following[15]:

(a) the edge e_{I_v} along which the deformation occurs and its associated charge label. If v is GR this is just $q^i_{I_v}$ and the specification is denoted by $(I_v, q^i_{I_v})$. If v is CGR and the deformation is along the conducting line in c the appropriate conducting line index K_v must be specified together with the difference between the upper and lower conducting edge charges $q^i_{K_v,\mathrm{out}} - q^i_{K_v,\mathrm{in}}$. If v is CGR but the deformation is along an edge e_{I_v}, $I_v \neq K_v$, the specification is, as for the GR case, $(I_v, q^i_{I_v})$.

[13] Note that in Fig. 4(c), the displaced vertex is generically CGR; however, it is possible for the charge values to conspire so that the charge at the lower conducting edge at the displaced vertex vanishes in which case the displaced vertex would be GR.

[14] This notion of (non)degeneracy requires the intervention by h_l, which in turn is fixed by the specification of which part of the conducting edge is upper and which is lower. A unique specification will be given in Sec. V. Such a specification then makes the notion of (non)degeneracy of a CGR vertex a well-defined one.

[15] While we have only explicitly defined deformed charge nets for deformations along edges of c which have positive charges, it turns out that the specifications below also extend to the general case tackled in Sec. V.

(b) the charge flip involved which is specified by a sign $\pm$ and a $U(1)^3$ index i [which is the same as that of the charge labels in (a)].
(c) the coordinate patch around v and the nature of the deformation it specifies including the size of the deformation parameter δ measured by it.

In Sec. VIII we will see that the coordinate patch is uniquely specified for every c as is the nature of the deformation given the value of the deformation parameter δ and the information in (a), (b). The information in (a), (b) is known given the charge net label c (which includes all its edges and charges), the deformation edge/line index I_v, the $U(1)^3$ index i and a parameter β which takes values $+1$ or -1 corresponding to a $+i$ or $-i$ charge flip. Hence, suppressing the (unique) specification of the coordinate patch associated with c, we denote the deformed charge nets $c_{(\pm i, q^i_{I_v}, I_v, \delta)}$ in (3.10) and (4.3) and $c_{(\pm i, q^i_{K_v,\mathrm{out}} - q^i_{K_v,\mathrm{in}}, K_v, \delta)}$ in (4.3) by the symbol $c_{(i,I,\beta,\delta)}$ where we have suppressed the v subscript as we shall need this notation only for states with a single nondegenerate (linear GR or CGR) vertex.

Similarly we denote the charge nets $c_{(q^i_{I_v}, I_v, \delta)}$ in (3.11) and (4.4) and $c_{(q^i_{K_v,\mathrm{out}} - q^i_{K_v,\mathrm{in}}, K_v, \delta)}$ in (4.4) by the symbol $c_{(i,I,0,\delta)}$ where 0 signifies that the deformation is of the electric deformation type. By allowing β to range over 0 in addition to ± 1, we refer to the deformed charge nets in (3.10), (4.3), (3.11), and (4.4) by the single symbol $c_{(i,I,\beta,\delta)}$ and say that $c_{(i,I,\beta,\delta)}$ is the (i, I, β, δ) deformed child of the parent c. In terms of this notation, Eqs. (3.10) and (4.3) take the form

$$\hat{C}[N]_\delta c(A) = \beta \frac{\hbar}{2\mathrm{i}} \frac{3}{4\pi} N(x(v)) \nu_v^{-2/3} \sum_I \sum_i \frac{c_{(i,I,\beta,\delta)} - c}{\delta}, \tag{4.5}$$

with $\beta = +1$ or $\beta = -1$, and Eqs. (3.11) and (4.4) take the form

$$\hat{D}_\delta[\vec{N}_i]c = \frac{\hbar}{\mathrm{i}} \frac{3}{4\pi} N(x(v)) \nu_v^{-2/3} \sum_I \frac{1}{\delta} (c_{(i,I,\beta=0,\delta)} - c). \tag{4.6}$$

V. LINEAR GR AND CGR VERTICES: THE GENERAL CASE

In Secs. III and IV the explicit downward conical deformations considered were applicable only for those outgoing edges at the vertex of interest which had charges with certain positivity properties. The positivity property for GR vertices was that the outgoing charge had to be positive and for CGR vertices that the outgoing charge for a nonconducting edge had to be positive and that the outgoing net conducting charge had to be positive. The associated downward conicality of the deformation was defined with respect to an assignation of upward direction, this direction coinciding with the outgoing edge direction for GR vertices[16] and being arbitrarily prescribed for the CGR case. Here we shall lift the positivity restrictions on charges and also remove the arbitrariness in the definition of upward and downward directions in the CGR case. In what follows we shall, as in Secs. III and IV, appeal to the constructions of Appendix B. 1. However, in addition, we shall also find it necessary to embellish these constructions with an appropriate placement of kinks through the constructions of Appendix B. 2.

We proceed as follows. First in Sec. V. A we formalize the definitions of upward and downward conical deformations for GR and CGR vertices. As we shall see, these deformations will be defined to be downward or upward conical with respect to an edge orientation determined by the kink structure in the vicinity of the vertex rather than with respect to the outward pointing edge tangent. Next, in Secs. V. B and V. C we tie the choice of downward or upward conical deformation for GR and CGR vertices to the sign of the charge labels on the edges at the vertex, with the definition of upward and downward fixed by the kink structure in the vicinity of the vertex as in Sec. V. A. The intricacy of these choices plays a key role in the emergence of anomaly free commutators in the continuum limit. Had we not been guided by the anomaly free requirement, it would have been difficult to home in on these choices. In Secs. V. B and V. C we also show how each of these choices is implemented through a corresponding choice of discrete approximants to the action of the Hamiltonian and electric diffeomorphism constraints. We summarize our results in Sec. V. E. In what follows we use the notion of a C^m kink $m = 0, 1, 2$ as defined in Appendix A.

A. Upward and downward conically deformed states

1. Linear GR vertex

Let v be a linear GR vertex of the charge net c. Let the coordinates around v with respect to which v is linear be $\{x\}$. In this section we shall construct upward and downward conically deformed states obtained by subjecting the graph underlying c to upward and downward conical deformations. These deformed states are the analogs of the deformed charge nets depicted in Fig. 1.

A conical deformation of c along the edge e_I at the vertex v of c is one in which the deformed state c_I has a vertex v_I displaced with respect to v along the straight line determined by e_I, deformations of the edges $e_{J\neq I}$ which connect the edges e_J in c to v_I, these deformations being straight lines in the vicinity of v_I which form a regular cone

[16] This choice of upward direction made in Sec. III, even with the positivity restrictions therein, coincides with the choice outlined in this section only for special cases of GR vertices, an example being those which are "primordial" in the language of Sec. VI. We had pointed out this further restriction of the considerations of Sec. III to such vertices in footnote 5.

around the line joing v to v_I. To characterize the conical deformation as downward or upward it is necessary to specify which direction is up. Accordingly, let $\vec{V}_I$ be a tangent vector at v which points either parallel to the outward pointing edge tangent to the edge e_I or antiparallel to the outward pointing edge tangent to the edge e_I. Given a choice of $\vec{V}_I$, the direction along $\vec{V}_I$ is defined to be upward and the direction opposite to that of $\vec{V}_I$ is defined to be downward. A conical deformation of c at v will be called *downward* with respect to $\vec{V}_I$ if:

(a) the deformed edges (other than the Ith one) form a downward cone around the upward direction defined by $\vec{V}_I$ so that the angle between this upward axis and any such edge as measured by $\{x\}$ is greater than $\frac{\pi}{2}$, and
(b) there is a specific kink structure in the vicinity of the displaced vertex in the deformed state which is consistent with the choice of $\vec{V}_I$ in a sense which we shall describe as we go along.

In particular, if $\vec{V}_I$ is specified as being parallel to the outward pointing edge tangent $\vec{\hat{e}}_I$ at v in c then the deformations described in Sec. III are *downward* pointing because the cone is downward pointing. In addition we use the construction of Appendix B. 2 to place kinks around the displaced vertex v_I as follows. Using the terminology of Sec. IV. A, the displaced vertex v_I lies on the conducting line passing through v. We place a C^2 kink at a point $v_{I,2}$ on this conducting line "beyond" v_I so that the part of the conducting line *from* v_I *to* $v_{I,2}$ is oriented *parallel* to $\vec{V}_I$. We also place a C^1 kink at a point $v_{I,1}$ on the part of the conducting line between v and v_I so that the part of the conducting line *from* v_I *to* $v_{I,1}$ is oriented *antiparallel* to $\vec{V}_I$. It follows that the upward direction $\vec{V}_I$ can be *inferred* from the position of these kinks from the orientation of the straight lines (with respect to $\{x\}$) from the displaced vertex v_I to these kinks. This is what we mean by the consistency of the kink placement with the specification of the choice of $\vec{V}_I$ in (b).

Similarly an *upward* conical deformation of c at v with respect to $\vec{V}_I$ is a conical deformation in which the deformed edges (other than the Ith one) point upwards so that the angle between any such edge and $\vec{V}_I$ is acute and such that there is an appropriately defined kink structure which is consistent with the choice of $\vec{V}_I$. As an example of an upward conical deformation, consider the case where, once again, $\vec{V}_I$ is specified as being parallel to the outgoing edge tangent $\vec{\hat{e}}_I$ at v in c. We define the *upward* conical deformation of c along e_I at v as follows. First we describe the deformation of the graph underlying c so as to obtain the analog of Fig, 1(b). Recall that v is linear with respect to $\{x\}$. Extend the (straight line) edge e_I linearly past v in the ingoing direction opposite to $\vec{V}_I$. Let the extension, $e_I^{(-,\tau)}$ be of coordinate length τ with τ small enough that $e_I^{(-,\tau)}$ does not intersect any part of c other than v.[17] Let us consider the altered vertex structure at v when we include this extension as an edge at v. Clearly, the addition of this edge to the existing set of edges at v converts v into a linear CGR vertex. The deformation of this CGR vertex structure is similar to that for CGR vertices in section IV with $e_I^{(-,\tau)}$ playing the role of the upper conducting edge, and is as follows. We (a) displace the vertex v by an amount $\epsilon = \frac{\tau}{2}$ along $e_I^{(-,\tau)}$ to the point v_I, (b) connect v_I to the edges $e_{J\neq I}$ at the C^0 kinks $\tilde{v}_J$ by straight lines as described in Appendix B. 1 and Sec. IV. A, and (c) deform the resulting vertex structure in a small enough vicinity of v_I along the lines of Appendix B. 1 so as to obtain a regular conical structure in this vicinity. The deformed graph is then obtained by removing the parts of the edges of the original graph between v and the C^0 kinks $\{\tilde{v}_J\}$ as well as the part of the extension $e_I^{(-,\tau)}$ beyond v_I so that v_I is now a GR vertex. We emphasize here that the deformation detailed through (a) to (c) does not require any holonomy intervention of the sort provided by h_l and its inverse in Sec. IV. That (a)–(c) can be implemented without the creation of any further unwanted intersections follows from an argumentation similar to that in Sec. IV. A using the properties of straight lines and the small compactly supported nature of the transformations of the type detailed in Appendix B. 1 which render the conical structure regular.

Next, if the deformation is of the "Hamiltonian constraint" type, the deformed graph is colored with appropriate (β, i) flipped charges and the displacement ϵ of the displaced vertex v_I is chosen to be $|q_I^i|\delta$ where δ is the discretization parameter associated with the Hamiltonian constraint action and q_I^i is the charge of the outgoing edge e_I in c at v. The holonomy corresponding to this deformed charge net is multiplied by the inverse charge net holonomy with (β, i) flipped charges on the graph underlying c together with the holonomy corresponding to c. The product of these three yield a deformed charge net generated by the Hamiltonian constraint. We show this in Fig. 5.

If the deformed charge net is generated by the electric diffeomorphism constraint at discretization parameter value δ, its edges bear the same charges as their counterparts in c and we have, once again, that $\epsilon = |q_I^i|\delta$. The graph underlying the deformed charge net is the one shown in Fig. 5(b).

Finally, we apply a construction of the type detailed in Appendix B. 2 so as to introduce a C^2 kink at a point $v_{I,2}$ between v_I and v on the remaining part of $e_I^{(-,\tau)}$. From the arguments of Sec. IV. A and Appendix B, it follows that the

[17]That such a small enough extension exists follows from the linear GR nature of the vertex; the linear GR property implies that the edges $e_{J\neq I}$ of c in the vicinity of their vertex v are straight lines, none of which are parallel to e_I.

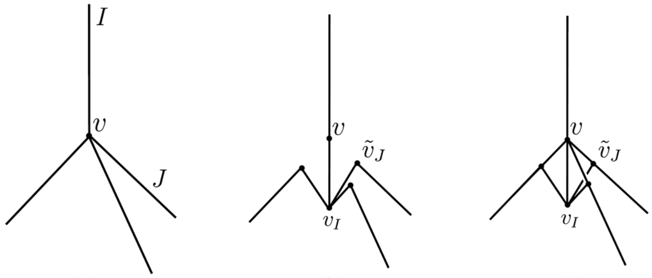

FIG. 5. (a) Undefromed GR vertex v of a charge net c with its Ith and Jth edges as labeled. The Ith edge is extended beyond v and the vertex is displaced along this extended edge in (b) wherein the displaced vertex v_I and the C^0 kink, $\tilde{v}_J$ on the Jth edge are labeled. (c) Result of a Hamiltonian type deformation (i, I, β, δ) obtained by multiplying the charge net holonomies obtained by coloring the edges of (b) by (β, i) flipped images of charges on their counterparts in c, (a) by negative of these (β, i) flipped charges and (a) by the charges on c. If the edges of (b) are colored by the charges on their counterparts in c then one obtains an electric diffemorphism deformation.

deformed structure does not intersect c except at the points $\{v, \tilde{v}_J, J \neq I\}$ and that the deformed edges form an *upward* cone with respect to the specified upward direction $\vec{V}_I$. Further, the kink structure in the vicinity of v_I is, once again, such that the oriented line from v_I to the C^2 kink $v_{I,2}$ is in the direction of $\vec{V}_I$ Note that in this case there is no lower conducting edge beyond v_I and hence no C^1 kink placement.

Next consider the case where $\vec{V}_I$ is antiparallel to the outgoing edge tangent $\vec{\hat{e}}_I$ at v in c. The *downward* conical deformation of c along e_I at v with respect to this choice of $\vec{V}_I$ is exactly the same as the *upward* conical deformation with the opposite choice of direction of $\vec{V}_I$ which we sketched immediately above, except that the C^2 kink is replaced by a C^1 kink so that, once again, this placement is consistent with $\vec{V}_I$ in the sense that the oriented line from v_I to the C^1 kink $v_{I,1}$ is in the direction opposite to that of $\vec{V}_I$.

Finally consider the case where $\vec{V}_I$ is antiparallel to the outgoing edge tangent $\vec{\hat{e}}_I$ at v in c and conical deformation is *upward* of c along e_I at v with respect to this choice of $\vec{V}_I$. This is exactly the same as the *downward* conical deformation with the opposite choice of direction of $\vec{V}_I$ which we discussed as our first example (and which we have encountered in Sec. III), except for the placement of the kinks. In this case, relative to our first example, the location of the C^2, C^1 kinks are interchanged so that once again, this placement is consistent with $\vec{V}_I$. Thus the oriented line from v_I to the C^2 kink $v_{I,2}$ is in the direction of $\vec{V}_I$ where as that from v_I to the C^1 kink $v_{I,1}$ is in the direction opposite to $\vec{V}_I$ and $v_{I,2}$ is placed between v and v_I whereas $v_{I,1}$ is placed on the other side of v_I on e_I.

2. Linear CGR vertex

We extend the considerations of Sec. V. A. 1 to the case where v is a linear CGR vertex of c with linear coordinate patch $\{x\}$ and conducting line e_K. Let the prescribed upward direction for the deformation along any nonconducting edge e_I be $\vec{V}_I$ and let the prescribed upward direction for the deformation along the conducting line be $\vec{V}_K$.

Recall from Sec. IV that the deformations of the CGR vertex constructed there involved the conversion of this vertex to a GR one through the intervention of the holonomy h_l. The loop l has a part which runs along the conducting line at v in the direction of its upper conducting edge. Here we use exactly the same intervention with this straight line part of l oriented along the direction $\vec{V}_K$ i.e. we use $\vec{V}_K$ to identify the upper and lower conducting edges.

Accordingly, let the net conducting charge at v (namely the *sum* of the *outgoing* charges along the two edges at v which comprise the conducting line through v) be $q^i_{K,\mathrm{net}}$:

$$q^i_{K,\mathrm{net}} \coloneqq q^i_{K,1} + q^i_{K,2} \tag{5.1}$$

where both $e_{K,1}, e_{K,2}$ are taken to be outward pointing at v in c so that $q^i_{K,1}, q^i_{K,2}$ are the outward edge charges.[18] Without loss of generality, let us designate the outward pointing edge $e_{K,2}$ to be parallel to $\vec{V}_K$. Let the intervening holonomy h_l run around the loop l with l constructed as in Sec. IV. Let the orientation of l be such that the straight line part of l runs upward (i.e. in the direction parallel to $\vec{V}_K$). Let l with this orientation be charged with $q^i_{K,1}$. Multiplication by h_l converts the CGR vertex into a GR vertex and the resulting charge net is called, as in Sec. IV, c_l. Note that the Kth edge of c_l has charge $q^i_{K,\mathrm{net}}$.[19]

Since the nonconducting edges are unaffected by this intervention, we assign the Ith edge of c_l $(I \neq K)$ the same upward direction $\vec{V}_I$ as for the same edge in c. Similarly for the Kth edge of c_l we assign the same upward direction $\vec{V}_K$ as for the Kth (i.e. conducting) line of c so that $\vec{V}_K$ is parallel to the outgoing Kth edge of c_l at v. Thus the assignments $\{\vec{V}_I\}$ for the edges at v in c, induce (the same) assignments for the corresponding edges in c_l. The upward and downward conical deformations of this GR vertex along the Ith edge of c_l with respect to $\vec{V}_I$ are then constructed as in Sec. V. A. 1 except for the placement of the kinks. Note that the deformations are small enough that they are restricted to a coordinate ball whose diameter is smaller than the length of the straight line part of l and is

[18] Note that this is exactly the same as the *difference* between the outgoing and *incoming* charges which we used in Sec. IV.

[19] As in Definition 1, Sec. IV. C, the notion of degeneracy of the CGR vertex in c relevant to the action of the constraints is that of the corresponding GR vertex in c_l.

also small enough that the ball does not intersect the curved part of l. To see this recall that:

(a) for downward deformations, replacing δ in Appendix B. 1 by $|q_I^i|\delta$ for $I \neq K$ and by $|q_{K,\mathrm{net}}^i|\delta$ for $I = K$, the deformation is confined to within ball of size $2|q_I^i|\delta$ around v for $I \neq K$ and within a ball of size $2|q_{K,\mathrm{net}}^i|\delta$ for $I = K$.

(b) for upward deformations also (a) is true; this follows from the construction of such deformations as detailed in Sec. V. A. 1. Further the length of the extension $e_I^{-,\tau}$ of the graph underlying the single GR vertex state c_l (see Sec. V. A. 1) is chosen to be twice that of the displacement of the vertex v to its displaced position so that $\tau = 2|q_I^i|\delta, I \neq K$ and $2|q_{K,\mathrm{net}}^i|\delta$ for $I = K$.

(c) the length $|l_1|$ is chosen to be larger than $16 q_{\max}$ [see (4.1)] so that $|l_1| > \tau$.

In the case of Hamiltonian deformations, the colorings of the deformed graph and its multiplication by the two graph holonomies based on the undeformed graph underlying c_l are as in Sec. V. A. 1. For the electric diffeomorphism case as well we follow Sec. V. A. 1 applied to c_l instead of c.

Subsequent to this, as in Sec. IV we multiply the result by the inverse holonomy h_l^{-1} which removes the curved part of l from the deformed charge nets $c_{l(i,I,\beta,\delta)}$. Finally we use constructions similar to that in Step 2 of Appendix B to place a C^1 kink and C^2 kink around the displaced vertex so that this placement is consistent with $\vec{V}_I$ in the sense described in Sec. V. A. 1. Thus the straight line from the displaced vertex v_I to the C^2 kink is parallel to $\vec{V}_I$ and that from v_I to the C^1 kink is opposite to $\vec{V}_I$.

This completes our discussion of the linear CGR vertex case.

B. Choices of deformation: Linear GR vertex

1. Choice of conical deformation type

Let v be a nondegenerate linear GR vertex of c. We are interested in making a choice of upward or downward deformation at v when the deformation is specified as (i, I, β, δ) where similar to Sec. IV. D, $\beta \neq 0$ specifies a deformation with (β, i) flipped charges along the edge e_I with parameter δ and where $\beta = 0$ specifies a deformation with unflipped charges along e_I with parameter δ.

Let the outgoing tangent at v along e_I be $\vec{\hat{e}}_I$. We define the *nearest* vertex on e_I to be the first C^0, C^1 or C^2 vertex which is encountered on e_I as e_I is traversed in the outward direction from v in c. From our considerations in Secs. III, IV and V. A, in the C^1, C^2 cases the vertex is bivalent and in the C^0 case the vertex can be bi- or trivalent.

In all cases of interest, if the outgoing charge $q_I^i > 0$ the deformation is chosen to *downward* conical and if $q_I^i < 0$ the deformation is chosen to be *upward* conical. In both cases the displaced vertex is at a distance $|q_I^i|\delta$ from v. It turns out that for future purposes, only the following cases are of interest:

(1) The nearest vertex is C^0: Then $\vec{V}_I$ is chosen parallel to $\vec{\hat{e}}_I$.
(2) The nearest vertex is C^1: $\vec{V}_I$ is chosen antiparallel to $\vec{\hat{e}}_I$.
(3) The nearest vertex is C^2: $\vec{V}_I$ is chosen parallel to $\vec{\hat{e}}_I$.
(4) There is no nearest vertex: $\vec{V}_I$ is chosen parallel to $\vec{\hat{e}}_I$.

2. Choice of discrete approximant to constraint

In this section we describe the choice of discrete approximants to the constraints for which the ensuing discrete action implements (1)–(4) of Sec. V. B. 1.

In cases (1), (3), (4) of Sec. V. B. 1 the heuristics of Secs. II and III can be repeated to conclude that these deformations are generated by the diffeomorphsim $\varphi(q_I^i \vec{\hat{e}}_{I_v}, \delta)$ of Sec. III. B. 1 because $\vec{V}_I$ is in the direction of $\vec{\hat{e}}_I$ and, from the initial part of Sec. V. B. 1, the positive or negative character of q_I^i then dictates whether the displaced vertex is displaced in the direction of $\vec{\hat{e}}_I$ or opposite to it. If the displacement is in the direction of $\vec{\hat{e}}_I$ then the deformation corresponding to Eq. (3.5) is downward conical and if the displacement is in the direction opposite to $\vec{\hat{e}}_I$ the deformation is defined to be upward conical.

In all these three cases, in accordance with the heuristics of Secs. II and III, if the deformation is generated by the Hamiltonian constraint, the deformed graph is colored with appropriate (β, i)- flipped charges and the displacement ϵ of the displaced vertex v_I in Sec. V. A. 1 is chosen to be $|q_I^i|\delta$, where δ is the discretization parameter associated with the Hamiltonian constraint action. The holonomy corresponding to this deformed charge net is multiplied, as in Figs. 1(c) and 5(c) by the inverse charge net holonomy with (β, i) flipped charges on the graph underlying c together with the holonomy corresponding to c. The product of these three yield a deformed charge net generated by the Hamiltonian constraint. Any deformed charge net generated by the electric diffeomorphism constraint at discretization parameter value δ bears the same charges on each of its edges as on the counterpart of this edge in c [see Figs. 1(b) and 5(b)] and we have that $\epsilon = |q_I^i|\delta$. Finally, using the constructions of Appendix B. 2, C^1 or C^2 kinks are placed at appropriate positions around the displaced vertex v_I in a manner consistent with the specification of $\vec{V}_I$ at v in the sense described in Sec. V. A. 1. We use the notation of Sec. IV. D to denote the deformed charge nets generated in this way by $c_{(i,I,\beta,\delta)}$ and $c_{(i,I,\beta=0,\delta)}$.

In case (2) of Sec. V. B. 1, the vertex displacement corresponds to that generated by $\varphi(-q_I^i \vec{\hat{e}}_{I_v}, \delta)$ due to the fact that $\vec{V}_I$ is opposite to $\vec{\hat{e}}_I$. In order to remove this conflict with the considerations of Sec. III. B. 1 [see Eq. (3.5)], it is necessary to introduce an intervention of the type used in Sec. IV. Accordingly, we first multiply the state c by a holonomy $h_{\bar{l}}$ around a loop $\bar{l}$ made up of two edges $\bar{l}_1, \bar{l}_2$ so that $\bar{l} = \bar{l}_1 \circ \bar{l}_2$. Let $\bar{l}_1$ run from $\bar{p}_1$ to $\bar{p}_2$. Here $\bar{p}_1$, $\bar{p}_2$ are equidistant from v, with $\bar{p}_1$ on the linear extension of e_I past v and $\bar{p}_2$ on e_I. Let $\bar{p}_1$ and $\bar{p}_2$ be chosen such that the coordinate length of l_1 is $C\delta, C = 8q_{\max}$. Let $\bar{l}_2$ be a semicircular arc connecting $\bar{p}_2$ with $\bar{p}_1$ such that its diameter is $C\delta$. Let $\bar{l}$ lie in a coordinate plane $P_{\bar{l}}$ such that no nonconducting edge lies in $P_{\bar{l}}$. Define the holonomy $h_{\bar{l}}$ to run along $\bar{l}$ with charge equal to $-q^i_I$. Multiplication of c by this holonomy yields the state $c_{\bar{l}}$ with a GR vertex. The Ith outgoing edge of $c_{\bar{l}}$ has outgoing charge q^i_I and the outgoing tangent to this edge is parallel to $\vec{V}_I$. We now act with an approximant of the type underlying the action of Sec. III. B. 1 on $c_{\bar{l}}$. As discussed in the first paragraph of this section, the deformation generated by this approximant is upward (or downward) with respect to $\vec{V}_I$ if q^i_I is negative (or positive). At this stage we refrain from placing any C^1 or C^2 kinks. Next, we multiply the result by the inverse holonomy $h^{-1}_{\bar{l}}$.[20] Finally we place a C^1 or a C^2 kink between the displaced vertex and v in a manner consistent with $\vec{V}_I$, this placement being achieved through multiplication by a holonomy which is classically close to identity similar to that employed in Step 2 of Appendix B. Clearly the end result is equivalent to deforming c as indicated in Sec. V. B. 1. It turns out that for future purposes the situation of interest in this case [i.e. Case (2)], is one in which the other edges at c conform to Case (1). Hence in this situation, the action of the constraints needs no further intervention beyond that of $h_{\bar{l}}$ and its inverse.

C. Choices of deformation: Linear CGR vertex

1. Choice of conical deformation type

Let v be a linear GR vertex of c. Let the deformation of interest be (i, I, β, δ).

Let the conducting edge in c be e_K so that v seperates e_K into two parts $e_{K,1}$ and $e_{K,2}$. Let us first consider the case where $I = K$ so that the deformation is along the conducting edge. We first need to determine the vector $\vec{V}_K$. It turns out that the cases of interest are such that $e_{K,1}$ has a nearest kink which is C^1 and $e_{K,2}$ has a nearest kink which is C^2 or *vice versa*. In each case we apply the appropriate criteria [i.e. one of (2),(3) of Sec. V. B. 1 to either the edge $e_{K,1}$ oriented in the outgoing direction from v or to the edge $e_{K,2}$, also oriented in the outgoing direction from v] to obtain $\vec{V}_K$. It is easy to check that irrespective of whether the criteria are applied to $e_{K,1}$ or to $e_{K,2}$, the same choice of $\vec{V}_K$ ensues. Next, we base our choice of upward or downward deformation with respect to $\vec{V}_K$ on the sign of the net conducting charge $q^i_{K,\mathrm{net}}$ [see (5.1)]. If $q^i_{K,\mathrm{net}} > 0$ we choose the deformation $(i, I = K, \beta, \delta)$ of c to be downward with respect to $\vec{V}_K$ and if $q^i_{K,\mathrm{net}} < 0$ we choose this deformation of c to be upward with respect to $\vec{V}_K$. The deformations corresponding to these choices are constructed as in Sec. V. A. 2.

Next consider the case where $I \neq K$. It turns out that the case of interest is then such that e_I has a nearest kink which is C^0. In this case we apply criterion (1) of Sec. V. B. 1 i.e. we choose $\vec{V}_I$ to be along the outgoing edge direction. We then choose the deformation to be upward with respect to $\vec{V}_I$ if the outgoing charge $q^i_I > 0$ and downward if $q^i_I < 0$. The deformation is then implemented as in Sec. V. A. 2.

2. Choice of discrete approximant to constraint

The choice of discrete approximants which implement the choices described in Sec. V. C. 1 is then as follows. First, as in Sec. V. A. 2, we apply the intervention h_l with l chosen in accord with $\vec{V}_K$ as described in that section. For l of small enough area the classical holonomy h_l is a good approximant to identity and for small enough δ, the straight line part of l does not overlap with any nearest kinks on e_K. The intervention yields the state c_l with a GR vertex at v.

We then use the appropriate choice of approximant detailed in Sec. V. B. 2 to generate the chosen (upward or downward) deformation of c_l (according to the assignment $\{\vec{V}_I\}$ induced from c to c_l as explained in Sec. V. A. 2)[21] except that we refrain from placing the desired C^2, C^1 kinks i.e. we do not implement the analog of Step 2, Appendix B. Since this placement is implemented via multiplication by a holonomy whose classical correspondent is a good approximant to the identity, the postponement of this implementation does not affect the viability of the approximant used. We then multiply the resulting deformed charge net by the inverse holonomy h^{-1}_l.

Finally we use the analog of Step 2, Appendix B to place kinks consistent with the choice of $\{\vec{V}_I\}$. Accordingly, when $I = K$, the conducting line of the deformed charge net is also labeled by K and we place C^2, C^1 kinks consistent with the specification of $\vec{V}_K$ for c. When $I \neq K$, the conducting line in the deformed charge net is along the Ith nonconducting edge (or its extension) of the undeformed charge net c and we place C^2, C^1 kinks around

[20]Note that we have chosen the size of the loop $\bar{l}$ slightly smaller than that of l in Sec. IV. Nevertheless, $\bar{l}$ is still large enough that an argumentation similar to (a)–(c) of Sec. V. A. 2 shows that no unwanted intersections ensue due to this intervention.

[21]Note that no edge of c_l satisfies criterion (2). The only possibility is an edge along the conducting line in c; however only the *upper* conducting edge is retained in c_l, its outward orientation coinciding with the upward direction.

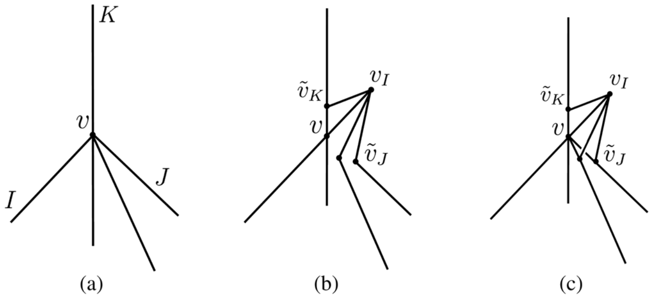

FIG. 6. (a) Undeformed CGR vertex v of a charge net c with its Kth conducting edge and Ith and Jth nonconducting edges as labeled. (b) Vertex structure of (a) is deformed along an extension of Ith edge past v and the displaced vertex v_I and the C^0 kinks $\tilde{v}_J$, $\tilde{v}_K$ on the Jth, Kth edges are as labeled. With charge colorings similar to those described in Fig. 3, (c) shows the result of a Hamiltonian type deformation (i, K, β, δ) and (b) the result of an electric diffeomorphism deformation. The parental vertex v is doubly CGR in (c) and is 4 valent and planar in (b).

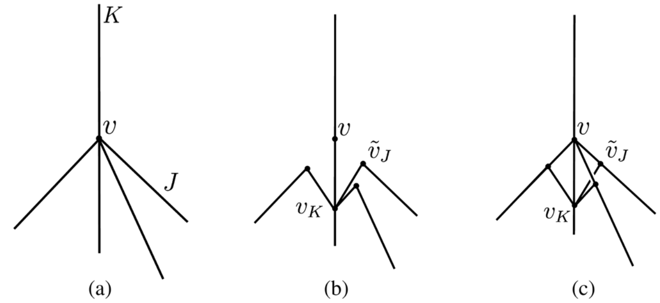

FIG. 7. (a) Undeformed CGR vertex v of a charge net c with its Kth conducting edge and Jth nonconducting edge as labeled. (b) Vertex structure of (a) is deformed along an extension of Kth edge past v and the displaced vertex v_K and the C^0 kink $\tilde{v}_J$ on the Jth edge are as labeled. With charge colorings similar to those described in Fig. 4, (c) shows the result of a Hamiltonian type deformation (i, K, β, δ) and (b) the result of an electric diffeomorphism deformation. These deformations are isomorphic to those in Fig. 4 as can be ascertained by viewing them upsidedown.

the displaced vertex in a manner consistent with the specification of $\vec{V}_I$ at v in c.

D. (Non)degeneracy of vertex types

Given a GR vertex, constraint operators act nontrivially at this vertex only if it is nondegenerate, its nondegeneracy being defined as the nonvanishing of its volume eigenvalue ν (2.16). At a CGR vertex, the action of a constraint is sensitive to the (non)degeneracy of the same (but now GR) vertex in its image by intervention described in Sec. V. A. 2. It is useful to formalize this notion of degeneracy as a definition identical to Definition 1, Sec. IV. C. Before doing so it is useful to catalog the kinds of vertices which are generated by the deformations of GR and CGR vertices described in Secs. V. A–V. C with a view to analyzing their possible nondegeneracy. Since the C^1, C^2 vertices are always bivalent and hence degenerate, and since their placement does not affect the vertex structures at other vertices, we need only analyze the vertex types generated prior to their placement.

An exhaustive analysis of such vertex structures is provided in Appendix C, the catalog of vertex types being those encountered in Cases 1a, 1b, 2a.1, 2a.2, 2b.1, 2b.2, 3 therein. Figures pertinent to Cases 1a and 1b are Figs. 1 and 5 and to Cases 2a.1 and 2a.2 are Figs. 3, 4. Figures 6 and 7, pertinent to Cases 2b.1 and 2b.2 are displayed below.[22] From the discussion in Appendix C, the figures for Case 3 may be obtained by setting the upper conducting charge equal to zero in Figs. 4 and 7.

As discussed in Appendix C, and as seen in the relevant figures, Cases 1a, 1b, 2a.1, 2a.2 and 2b.2 do not present any new potentially nondegenerate vertices of types other than GR and CGR. However, as seen in Fig. 6 and discussed in Appendix C, Case 2b.1 presents two new vertex types, both associated with parental vertices in deformed children. These are the four valent vertex of Fig. 6(b) and the $N+2$ valent vertex of Fig. 6(c). The former is a planar vertex and hence degenerate. The latter is a *linear doubly CGR vertex* where we define such a vertex as follows.

Definition 3: Linear doubly CGR vertex.—An $N+2$ valent vertex v of a charge net c will be said to be linear doubly CGR if:

(i) There exists a coordinate patch around v such that in a small enough neighborhood of v all edges at v are straight lines.

(ii) There are two sets of two edges such that the union of the two edges in each set forms a straight line so that v splits this line into two parts and such that the two straight lines corresponding to each of these two sets have a single isolated intersection at v. Each of these lines will be called conducting lines, each conducting line consisting of a pair of conducting edges.

(iii) The set of the remaining $N-2$ edges (called nonconducting edges) together with any one of the two edges in each pair of (ii) constitute a GR vertex in the following sense. Consider, at v, the set of out going edge tangents to each of the remaining edges together with each of the outgoing edge tangents to one of the two edges in each pair in (ii). Then any triple of elements of this set is linearly dependent.

We now formalize the definition of (non)degeneracy of CGR and doubly CGR vertices.

Definition 4: Nondegeneracy of a CGR vertex.—A CGR vertex of a charge net c will be said to be nondegenerate iff the corresponding GR vertex in the charge net c_l is

[22]These figures are schematic and show the edge intersection structure at vertices of interest. They do not faithfully reproduce the deformations of Sec. B. 2 which result in regular conicality of the deformed vertex, nor do they show the C^1, C^2 kinks.

nondegenerate. If the vertex in c_l is degenerate we shall say that the CGR vertex in c is degenerate.

This definition provides a unique definition of (non) degeneracy for the kind of CGR vertices we encounter. These vertices correspond to the following two cases. In the first case the CGR vertex in the state c is generated through a conical deformation of a parent state c_p as specified in Secs. V. A–V. C. In this case, the choice of upward and downward directions at its displaced vertex and hence the choice of any intervention if required, is uniquely defined and Definition 3 may be applied unambiguously to this vertex. The second case corresponds to a conical deformation of the parent state c_p at its vertex v_p such that this vertex is CGR in c and c_p; here we are interested in the application of Definition 3 to this vertex in c. In this case we interpret Definition 3 applied to the vertex v_p in c to mean that the degeneracy of this vertex is well defined iff it is *independent* of which part of the edge passing through v_p in c is chosen to be upper and lower. Since in our considerations, such a state c is obtained through a Hamiltonian constraint type β, i flipped deformation of c_p, it follows from Appendix C that the net charges at v_p in c have vanishing ith component so that v_p is degenerate independent of this choice and hence independent of the corresponding choice of intervention.

Next, note that a doubly CGR vertex can be rendered GR through two holonomy interventions $h_{l^i}, i = 1, 2$ with l^i chosen to be "semicircular" with the straight line parts of l^i being along the ith conducting line defined by the ith set of edges in (ii), Definition 4. These interventions leave the $N - 2$ edges in (iii), Definition 4, unaffected and remove one of the conducting edges from each conducting line in (ii). The remaining conducting edge in each line is colored with the net conducting charge corresponding to that conducting line. For our purposes the following definition suffices:

Definition 5: Degeneracy of a doubly CGR vertex.—A doubly CGR vertex will be said to be degenerate if the GR vertex obtained by any choice of interventions is degenerate.

Since the edges in the parental vertex of the deformed charge net discussed above and in (2b.1), Appendix C are such that the nonconducting charges and the net conducting charges all have vanishing ith component, this doubly CGR vertex is degenerate.

E. Summary and discussion

From our discussion in Sec. V. D and Conclusions 1 and 2, Appendix C, it follows that the only possibly nondegenerate vertices which are generated by the action of the constraints on a nondegnerate linear GR or CGR vertex are also GR or CGR. Sections V. A–V. C specify the deformation of charge nets with such vertices provided the vertex structures are characterized by the kink structures discussed in Secs. V. B. 1 and V. C. 1. As we shall see in Sec. VI, the charge nets of interest will have a single nondegenerate linear GR or CGR vertex with a kink structure of the type discussed. Denoting such a charge net of interest with such a vertex v by c and its deformed child by the deformation (i, I, β, δ) by $c_{(i,I,\beta,\delta)}$ where the deformed charge nets $c_{(i,I,\beta,\delta)}$ for all choices of I, i, β and sufficiently small δ have been constructed in Secs. V. A–V. C, the action of discrete approximants to the Hamiltonian and electric diffeomorphism constraints is expressed in Eqs. (4.5) and (4.6). We shall continue to refer to these two equations with the understanding that they implement the detailed choices discussed in Secs. V. A–V. C.

The reason we use criteria (1)–(4) rather than simply choose $\vec{V}_I$ to be in the direction of the outgoing tangent vector $\vec{\hat{e}}_I$ is that the former choice yields anomaly free continuum limit commutators whereas the latter does not. To see this requires a detailed study of double deformations of a charge net by two constraint actions which will be done in Secs. VI–X. Nevertheless we attempt to provide a brief explanation here for the choice of $\vec{V}_I$ as opposed to $\vec{\hat{e}}_I$. The reader is urged to peruse this explanation once again after reading the entire paper as it may, at this stage, be quite opaque. Each double deformation generated by two discrete constraint actions on a charge net is composed of a pair of single conical deformations. Each such single deformation is along some edge of a parent state and yields deformed offspring which are conically deformed along a cone whose axis is determined by the direction of the parental edge. The continuum limit involves shrinking two of these single deformations away from "grandchild" to "immediate parent" to "grandparent." It turns out that for certain delicate recombinations of terms to occur as a result of this process so as to generate an anomaly free result, the edge directions of the parent and the grandparent must be correlated (and, as will be seen, in a precise sense, identical). To ensure that this happens we must ensure a consistent choice of edge tangent directions in the child-parent-grandparent genealogy. This choice, it turns out, is exactly that of $\vec{V}_I$ (which clearly depends on the genetic trace provided by the C^1, C^2 kink placement), as opposed to the choice of outgoing edge tangent $\vec{\hat{e}}_I$ (which would be a purely local choice independent of lineage). Finally, note that the use of (1)–(4) is tantamount to the replacement of $q^i_I\vec{\hat{e}}_I$ by $q^i_{I,\mathrm{net}}\vec{\hat{V}}_I$ [with $q^i_{I,\mathrm{net}}$ being the net outgoing charge along the edge I, see Eq. (5.1) above] in the heuristically motivated Eq. (3.5). Thus there is a tension; we require the choice of $\vec{V}_I$ with the net outgoing charge for anomaly free commutators but the argumentation of Secs. II and III imply that we must use the outgoing tangent vectors with the outgoing charges. In order to remove this tension it is necessary to use the intervention $h_{\bar{l}}$ of Sec. V. B. 2 so as to ensure that criteria (1)–(4) are implemented through the use of valid approximants to the constraints.

VI. DISCRETE ACTION OF CONSTRAINT OPERATOR PRODUCTS

In the last two sections we did not specify the choice of coordinates with respect to which the deformations generated by the discrete action of the constraints were defined. In this section we specify these coordinates as well as the action of constraint operator products of interest along the lines sketched in Sec. I. It turns out that in view of the single vertex anomaly free states studied in this paper, the detailed specification of this action only needs to be made for a certain set of kets, which we shall refer to as the ket set. This set corresponds to all the kets which are obtained by multiple actions of the type (4.5) and (4.6) on certain primordial kets which themselves are not generated by any such action on any other state. In Sec. VI. A we generalize the notation of Sec. IV. D to describe the multiply deformed kets which are generated by such multiple actions. In Sec. VI. B we define the ket set. In Sec. VI. C we choose a reference ket in each diffeomorphism class of kets in the ket set and a set of reference diffeomorphisms such that each distinct ket in the diffeomorphism class of a reference ket is the image of the reference ket by a unique reference diffeomorphism. We also define certain key structures known as contraction diffeomorphisms which play a crucial role in defining the continuum limit by "contracting" the deformations away.

In Sec. VI. D we define the discrete action of products of contraint operators on any ket in the ket set through multiple applications of Eqs. (4.5) and (4.6). These multiple actions generate multiply deformed kets as discussed in Sec. VI. A. It remains to specify the coordinates with respect to which these deformations are defined. We do so through slightly involved manipulations of the structures developed in Secs. VI. A, VI. B and VI. C. The end result of these manipulations is a specification of the coordinates with respect to which the deformations are defined together with a definition of the discrete action of products of constraint operators on any ket in the ket set for arbitrarily small values of the discretization parameters. The corresponding dual action can then be defined on states in the algebraic dual space. The continuum limit of this action on anomaly free states (which reside in the algebraic dual space to the space of finite linear combinations of charge nets) will be evaluated in Secs. X and XI.

A. Notation for multiply deformed states

Let c be a state with a single nondegenerate vertex v, this vertex being either a linear GR or linear CGR vertex with respect to some choice of coordinates around v.[23]

[23]In addition, as shall become clear in Secs. VI. B–VI. D, the kink structure of the state c under consideration as well as of the states generated from c via multiple applications of (4.5) and (4.6) conforms to those alluded to in Sec. V.

The action of a single discrete constraint operator at discretization parameter δ on c is given by (4.5) and (4.6). From Conclusion 2 of Appendix C it follows that the deformed states $c_{(i,I,\beta,\delta)}$ have at most a single nondegenerate GR or CGR vertex and that this corresponds to the displaced vertex in each of these states. We shall assume that c is such that the displaced vertex in each of the deformed states $c_{(i,I,\beta,\delta)}$ is nondegenerate and that our choice of coordinates around each displaced vertex is such that the vertex is linear with respect to this choice. Hence these "singly" deformed states $c_{(i,I,\beta,\delta)}$ are all single nondegenerate linear GR or CGR vertex states. The action (4.5) and (4.6) on c yields these singly deformed states as well as c itself.

The action (4.5) and (4.6) on each of *these* states (namely $c_{(i,I,\beta,\delta)}$, c) is then well defined because each of these states is a single linear GR or CGR state. Since the actions (4.5) and (4.6) correspond to the discrete action of a single constraint, it follows that an action of one of (4.5) or (4.6) followed by a second action of either (4.5) or (4.6) on c corresponds to that of a discrete approximant to the product of two constraints and creates "doubly deformed" states, singly deformed states and the undeformed state. From Sec. V it follows that each of the doubly deformed states has a (doubly displaced) vertex which is once again, either GR or CGR. We shall assume that this vertex is nondegenerate and that the associated coordinate system is such that this vertex is linear; from Conclusion 2, the doubly deformed states are then again single, nondegenerate, linear GR or CGR vertex states. As a result the action (4.5) and (4.6) is well defined on *these* states as well. In this manner any combination of three actions of the type (4.5) or (4.6) yields triply deformed states, doubly deformed, singly deformed states and the undeformed state.

Continuing on and making an appropriate nondegeneracy and linearity assumption at every stage we find that the action of a product of n constraint operators can be approximated as n applications of the type (4.5) or (4.6) and that this results in states which are m deformed, $m = 0, 1, .., n$ with $m = 0$ corresponding to the undeformed state c. The continuum limit involves contracting these deformations away; it turns out that mth deformation is contracted away first, then the $m-1$th one and so on all the way to the first deformation. Hence we shall be interested in multiple deformations such that the parameter associated with the size of each successive deformation is smaller than its predecessors.

We now develop appropriate notation and genealogical language related to multiply deformed states. We shall refer to c as a parent state. As noted in Sec. IV. D any deformation of c can be specified through the information (i, I, β, δ) where, as in that section, for the reasons explained there, we have suppressed information about the coordinate patch used to define the deformation. In the language of Sec. IV. D, this deformation yields the (i, I, β, δ) deformed child $c_{(i,I,\beta,\delta)}$. Generalizing this notation, we can specify a sequence of m deformations by

$$[(i_{m-1},I_{m-1},\beta_m,\epsilon_m),(i_{m-2},I_{m-2},\beta_{m-1},\epsilon_{m-1}),.....,(i_1,I_1,\beta_2,\epsilon_2),(i,I,\beta_1,\epsilon_1)]\epsilon_i<\epsilon_j<\text{for } i>j \tag{6.1}$$

and denote the resulting "mth generation" child of the parent c by

$$c_{[(i_{m-1},I_{m-1},\beta_m,\epsilon_m),(i_{m-2},I_{m-2},\beta_{m-1},\epsilon_{m-1}),.....,(i,I,\beta_1,\epsilon_1)]}. \tag{6.2}$$

This signifies that the child is obtained from the parent c through the sequence

$$c \xrightarrow{i,I,\beta_1,\epsilon_1} c_{i,I,\beta_1,\epsilon_1} \xrightarrow{i_1,I_1,\beta_2,\epsilon_2} c_{[(i_1,I_1,\beta_2,\epsilon_2),(i,I,\beta_1,\epsilon_1)]}$$
$$.........\xrightarrow{i_{m-1},I_{m-1},\beta_m,\epsilon_m} c_{[(i_{m-1},I_{m-1},\beta_m,\epsilon_m),(i_{m-2},I_{m-2},\beta_{m-1},\epsilon_{m-1}),.....,(i,I,\beta_1,\epsilon_1)]}. \tag{6.3}$$

Above, we have assumed that the displaced vertex in each ith generation child with $1\le i\le m$ is nondegenerate and that the specification of the coordinate system around this vertex is such that the edges there appear as straight lines, so that the vertex is linear as well. Further, we have chosen to enumerate the edges of each charge net in this sequence in such a way that the enumeration of edges in a child and in its immediate parent are related as follows. Consider the charge net $c_{(i,I,\beta_1,\epsilon_1)}$ obtained by deforming its immediate parent c. Each nonconducting edge emanating from the nondegenerate vertex in $c_{(i,I,\beta_1,\epsilon_1)}$ is obtained by deforming, a corresponding edge in c which emanates from the nondegenerate vertex of c, the two edges meeting at a C^0 kink. We assign these corresponding edges the same number i.e. if $e_{I_1}\in c_{(i,I,\beta_1,\epsilon_1)}$ is the deformation of the edge e_I in c then we have $I_1=I$. Since there are $N-1$ such pairs of edges, one in the parent and one in the child, the remaining edge in the parent and in the child also bear the same number. Clearly, we can extend this enumeration scheme so that the numbering of edges of any child and immediate parent in (6.3) are so related. This immediately implies that given the sequence (6.3), the enumeration scheme of any charge net in the sequence is uniquely fixed by the enumeration scheme in c.[24]

Finally, where it creates no confusion, we will find it convenient to use the notation

$$[i,I,\beta,\epsilon]_m \coloneqq [(i_m,I_m,\beta_m,\epsilon_m),(i_{m-1},I_{m-1},\beta_{m-1},\epsilon_{m-1}), ...,(i_1,I_1,\beta_1,\epsilon_1)] \tag{6.4}$$

so that the state in (6.2) can be written as $c_{[i,I,\beta,\epsilon]_m}$.

B. Primordial states and the ket set

We think of primordial states as being states which cannot be obtained by a Hamiltonian or electric diffeomorphism type deformation of any state. Rather than provide a precise definition, we shall work with concrete examples and leave a more precise and complete definition of primordiality to future work. Consider any state with a single nondegenerate GR N valent vertex. Let all other vertices of the (coarsest) graph underlying the state be degenerate and let no such vertex have valence $2,3,4$, $N,N+1$ or $N+2$. Let there exist some coordinate patch around the nondegenerate vertex with respect to which the edges at this vertex are straight lines in a small neighborhood of the vertex i.e. let the vertex be linear with respect to some choice of coordinates. Such a state cannot be created by the discrete action of a constraint because, notwithstanding that we have defined this action in detail only on a restricted class of states, we visualize the action of the deformation maps (see Sec. II of this paper as well as P1, P2) to only create vertices of valence $2,3,4,N,N+1$, $N+2$. We shall call such a state as primordial state provided it is subject to four additional restrictions described below.

First, we restrict our attention to the case where the nondegenerate vertex is N valent for some fixed *even* integer N. This is for certain technical reasons. Note that GR and nondgeneracy restrictions imply that $N\ge 4$. We shall return to this point in our final section. In order to articulate our second (mild) restriction, consider the $U(1)^3$ charge obtained by the action of a $(\beta=\beta_1, i=i_1)$ flip of the $U(1)^3$ edge charge label $\vec{q}_I\coloneqq(q_I^1,q_I^2,q_I^3)$ in c. We may subject the flipped charge set to yet another flip (β_2,i_2). Let us denote the charge obtained by m such flips $[(\beta_m,i_m),(\beta_{m-1},i_{m-1}),\ldots,(\beta_1,i_1)]$ as $\vec{q}_{[\beta,i]_m,I}$ with $\beta_j\in\{-1,1\}$. Using this notation we require that the charges on each edge of c satisfy

$$\sum_{k=1}^{3} q_I^k \ne 0, \qquad \sum_{k=1}^{3} q^k_{[\beta,i]_m,I}\ne 0 \quad \forall[\beta,i]_m, \quad \forall I=1,..,N \tag{6.5}$$

so that the sum of the 3 $U(1)$ charges on each edge as well as the sum of any multiply flipped image of these charges is nonvanishing. We also require that

[24] In the case of CGR vertices we use this numbering for nonconducting edges and the conducting *line*; as seen in (3.10) and (3.11), we do not need to count the upper and lower conducting edges separately so that this correspondence continues to hold and the indices I_m for all m as well as the index I, all run from $1,..,N$.

$$q_I^k \neq 0, \qquad q^k_{[\beta,i]_m,I} \neq 0 \quad \forall [\beta,i]_m, \quad \forall I = 1,..,N, \quad \forall k = 1,2,3 \tag{6.6}$$

It is convenient to extend the notation for (β, i) flips to the case that $\beta = 0$. Consistent with the fact that no flipping is associated with an electric diffeorphism type transformation, we define a (β, i) flip to be the identity operation when $\beta = 0$. In this case the index i is redundant but we retain it for convenience in articulating the following definitions which will be useful for future purposes:

$$q_{\min} = {}_{I,[\beta,i]_m} \min_{\forall\, m,I,[\beta,i]_m} \left| \sum_{k=1}^{3} q^k_{[\beta,i]_m,I} \right|, \tag{6.7}$$

$$q_{\min,1} = {}_{I,[\beta,i]_m} \min_{\forall\, m,I,k,[\beta,i]_m} |q^k_{[\beta,i]_m,I}|, \tag{6.8}$$

and

$$q_{\max}^{\text{primordial}} = \max_{(i=1,2,3),(I=1,..,N)} |q_I^i|. \tag{6.9}$$

Since there are only a finite number of flipped images of the charges on each edge, Eqs. (6.5) and (6.6) are well defined and imply that $q_{\min}, q_{\min,1} > 0$. Note also that since the charges are integers we have that $q_{\min}, q_{\min,1} \geq 1$. Finally, note that while (6.9) seems identical to (4.1), these two definitions are in general distinct in that the charges on the right-hand side of (6.9) are the edge charges on the primordial charge net at its nondegenerate vertex whereas those in the right-hand side of (4.1) are the edge charges for the edges of the (not necessarily primordial) charge net under consideration in Secs. IV and V.

Third, we restrict our attention to states which exhibit linearity with respect to a particular choice of coordinate patches as follows. Fix a point p_0 on the Cauchy slice and a chart $\{x_0\}$ in some neighborhood of p_0. We require that any state under consideration be such that it is diffeomorphic to some state which has a nondegenerate vertex at p_0 and which is linear with respect to $\{x_0\}$.[25] The coordinates $\{x_0\}$ will be referred to as *primary coordinates*.

Fourth, we restrict our attention to states which satisfy the following requirement of *eternal nondegeneracy*: From Sec. VI. A, any multiple deformation of state yields a state with a multiply displaced vertex. We require that any primordial state be such that any multiple deformation of the primordial state yields a state whose multiply displaced vertex is nondegenerate. This "eternal nondegeneracy" is a strong and nontrivial restriction. The implementation of anomaly freedom in this paper does not go through if this condition is not satisfied. The classical analog of this condition is the requirement that the determinant of the 3 metric stay nonzero throughout its evolution. Clearly, if this condition is violated at any instant (i.e. anywhere on a Cauchy slice), we cannot compute the classical constraint algebra. In the Appendix D we show that the simplest GR vertex, namely one with four edges in conical configuration, provides an example of a state which satisfies these restrictions.

Consider the entire set of states subject to the above restrictions. We shall call these states as primordial states. Clearly, the set $S_{\text{primordial}}$ of these primordial states is closed under diffeomorphisms. Next, within each diffeomorphism class of these primordial states, fix a "reference" primordial state c_{P0} which has a nondegenerate vertex at p_0 and which is linear with respect to the primary coordinates $\{x_0\}$. Consider all multiple deformations of each of these reference primordial states, these multiple deformations being a sequence of single deformations of the type discussed in Sec. VI. A. More in detail, consider first some primordial reference state c_{P0}, a neighborhood of its vertex at p_0 being covered by $\{x_0\}$. Any single deformation of c_{P0} for sufficiently small deformation parameter is chosen to be upward or downward conical according to the criteria of Sec. V. B [in this case we use (4) of Sec. V. B. 1 together with the sign of the edge charge labels as discussed in that section to deform upward or downward]. Using the detailed constructions of Appendix B, and of Secs. III. A, IV. A and V. A, these deformations are defined for all sufficiently small values of deformation parameter such that the deformation is confined to the interior of a coordinate sphere $B_{\Delta_0}(p_0)$ of some size Δ_0 with $B_{\Delta_0}(p_0)$ in the domain of $\{x_0\}$. It follows that the resulting deformed children have displaced vertices which are in the domain of the chart $\{x_0\}$. These vertices (as mentioned earlier) are GR or CGR and (by assumption) nondegenerate. They are also *linear* with respect to $\{x_0\}$ because of the straight line edge structure of the cones in the vicinity of these vertices (see Appendix B for downward conical deformations; that a similar linearity holds for upward deformations is clear from their detailed construction in Sec. V. A. 1).

It is easy to check that the criteria of Secs. V. B and V. C can be applied to these children and that *their* (appropriately chosen) upward or downward conical deformations can again be defined for small enough values of deformation parameter such that the deformation is confined to the interior of $B_{\Delta_0}(p_0)$, and that each of *their* children have a single nondgenerate GR or CGR vertex. The detailed construction of the deformation for small enough values of deformation parameter implies that the primary coordinate system $\{x_0\}$ covers a small enough neighborhood of each of these vertices in which the edges at each such vertex are straight lines so that the vertex is linear with respect to the

[25] It seems plausible to us that any state with a single nondegenerate vertex which is linear with respect to some choice of coordinate patch must be diffeomorphic to one which is linear with respect to any prescribed patch. If this is indeed true, this third restriction does not actually constitute a genuine restriction. We leave an investigation of this issue to future work.

primary coordinates. Continuing in this way one can define multiply deformed states for all sufficiently small deformation parameter sets associated with the multiple deformation such that the multiple deformation lies in the interior of $B_{\Delta_0}(p_0)$. The set of all these deformed children of c_{P0} together with c_{P0} will be said to form a primary family and each element of such a family will be called a *primary*.

By letting c_{P0} vary over the set of all distinct reference primordials we obtain the set of all primaries, S_{primary}, with the multiple deformation which generates any primary from a reference primordial being confined to the interior of $B_{\Delta_0}(p_0)$. Finally consider the set of all diffeomorphic images of all primaries. This set is the ket set S_{Ket}.

To summarize, we let c range over all reference primordial states in equation (6.3). In that equation we let m range from $1\ldots\infty$ and let the deformation sequence range over all possible choices of deformation specifications for all possible small enough deformation parameter sets such that the deformations can be defined through Appendix B and Sec. V with respect to $\{x_0\}$ and such that the nondegenerate vertex of every deformed ket in the sequence is covered by $\{x_0\}$. Our definition of primordiality ensures that the resulting set of m-deformed children is such that each child in this set has a a single nondegenerate vertex. The set of all these multiply deformed children together with their primordial reference ancestors comprise the set of primaries. The ket set S_{Ket} comprises of the set of all diffeomorphic images of all primaries.

Note that since each element of the ket set is a diffeomorphic image of some primary, its nondegenerate vertex is linear with respect to the corresponding diffeomorphic image of $\{x_0\}$. We note again that from the considerations of Sec. V it follows that this vertex is a (linear) nondegenerate GR or CGR vertex and that from Conclusion 2 of Appendix C this is the only nondegenerate vertex of that element. Finally, it is straightforward to check, using the deformations detailed in Sec. V that each element is such that the criteria of Sec. V. B. 1 and V. C. 1 can be applied so that any further deformation of this element with respect to an appropriately specified coordinate patch is well defined. We develop the specification of this coordinate patch for any given element of the ket set in Secs. VI. B–VI. D.

C. Reference states, reference diffeormorphisms and contraction diffeomorphisms

Within each diffeomorphism class of elements of S_{Ket} choose one state as a *reference* state subject to the restriction that the state must be a primary i.e. the reference state must lie in S_{primary}. A charge net label with subscript 0 indicates a reference charge net. For the case of the diffeomorphism class of primordial states we choose the reference state to be as in Sec. VI. B. Next for each distinct element c of each diffeomorphism class $[c_0]$ of a reference charge net c_0 choose a *reference* diffeomorphism α such that α maps c_0 to c i.e. in "ket" notation we have

$$\hat{U}(\alpha)|c_0\rangle = |c\rangle \tag{6.10}$$

where $\hat{U}(\alpha)$ is the unitary operator representing the action of α.

Next we define *contraction* diffeomorphisms. To do so, consider a ket c in the ket set with some linear coordinate system $\{y\}$ at its nondegenerate vertex v. Let us deform it by the deformation (i, I, β, δ_0) where the detailed nature of the deformation is as in Appendix B and Sec. V. In particular, the coordinate patch used to specify the deformation (and the deformation parameter δ_0) is $\{y\}$ (i.e. we set $\{x\} = \{y\}$ in Appendix B and in Sec. V) and the displaced vertex and the C^0 kink vertices created by the deformation are each at a coordinate distance $|q^i_{I,\text{net}}|\delta_0$ from the parent vertex [here $q^i_{I,\text{net}}$ is the net charge as defined in (5.1) and Appendix C]. We would like to "contract" the deformation away so that the displaced vertex and these kinks approach the parent vertex v in a prescribed manner. Further, we would like the cone angle for the deformation at the displaced vertex to become narrower in line with our visualization of the deformation being that of a singular pulling of these edges along the Ith edge (see Sec. II).

This contraction is achieved through the action of the contraction diffeomorphism $\Phi^{\delta,Q,L,M,p_1,p_2,p_3}_{c,\{y\}(i,I,\beta,\delta_0)}$, defined for small enough δ_0, for which the following properties hold:

(i) The contraction diffeomorphism is a semianalytic diffeomorphism connected to identity.

(ii) It moves the displaced vertex v_{i,I,δ_0} along the straight line (in the coordinates $\{y\}$) between v_{i,I,δ_0} and v to the point $v_{i,I,\delta}$ located at a coordinate distance $|q^i_{I,\text{net}}|\delta \ll |q^i_{I,\text{net}}|\delta_0$ from the parent vertex v.

(iii) (a) The C^1, C^2 kinks in $c_{(i,I,\beta,\delta_0)}$ have an area $\alpha_0^2 \ll \delta_0^2$ (see Appendix B. 2). The contraction diffeomorphism shrinks the area of these kinks to $\alpha \ll \delta^2$.

(iii) (b) It moves the C^0 kink $\tilde{v}_L, L \neq I$ along the edge e_L of c to a distance δ^{p_1} from the parent vertex v. It moves the C^0 kink $\tilde{v}_M, M \neq I, M \neq L$ along the edge e_M of c to a distance $Q\delta^{p_2}$ from v (for some $Q > 0$ which we specify later). It moves each of the remaining (N-3) kinks along its nonconducting edge to a distance δ^{p_3} from v.

(iv) In a small vicinity of the (new position of) the displaced vertex it narrows the cone angle between the edges at that vertex by a *linear* deformation generated by the diffeomorphism G defined below.

(v) It maps c to itself and maps the straight line from v to v_{i,I,δ_0} (in the coordinates $\{y\}$) to itself.

(vi) It is identity outside a sphere of size $2|q^i_{I,\text{net}}|\delta_0$ around v.

The construction of the contraction diffeomorphism is along the lines sketched in P1, P2. We proceed as follows.

For convenience let us rotate the coordinate system $\{y\}$ so that y^3 runs (and increases) along the line from the parent vertex v to the displaced vertex v_{i,I,δ_0}. Let the segment of this line between v_{i,I,δ_0} and $v_{i,I,\delta}$ be $l_{\delta_0,\delta,I}$. Let l_ϵ be a straight line which contains $l_{\delta_0,\delta,I}$ and whose end points a_ϵ, b_ϵ lie at a distance ϵ from v_{i,I,δ_0}, $v_{i,I,\delta}$ respectively, $\epsilon \ll \delta_0, \delta$. Consider a small cylinder $C_{\epsilon,\tau}$ with axis l_ϵ and radius τ, $\tau \ll \delta, \delta_0$. Consider two such cylinders with parameters ϵ_1, ϵ_2 and τ_1, τ_2 with $\epsilon_1 > \epsilon_2, \tau_1 > \tau_2$ and with ϵ_1, τ_1 small enough that C_{ϵ_1,τ_1} does not intersect any edge emanating from v apart from the Ith one between v and v_{i,I,δ_0}. Consider the vector field $\xi^a = (\frac{\partial}{\partial y_3})^a$. Let f be a function compactly supported in C_{ϵ_1,τ_1} such that it is unity in C_{ϵ_2,τ_2}. Let $\phi(f\xi, t)$ be the 1 parameter set of diffeomorphisms generated by the vector field $f\xi^a$. Clearly, for an appropriate value of $t = t_0$ the diffeomorphism $\phi(f\xi, t_0) \equiv \phi_{i,I,\delta,\delta_0}$ translates v_{i,I,δ_0} to $v_{i,I,\delta}$ so that property (ii) is achieved. This diffeomorphism also respects properties (v), (vi).

Next, note that within C_{ϵ_2,τ_2} this is a rigid translation so that the translated edges at $v_{i,I,\delta}$ are straight lines in a small neighborhood of $v_{i,I,\delta}$. Hence within a small enough neighborhood of $v_{i,I,\delta}$ we can now apply the "scrunching" diffeomorphism G of Eq. C. 8, Appendix C4, P1. From that work we have that within a small neighborhood of $V_{i,I,\delta}$ of $v_{i,I,\delta}$, G acts as

$$(y^1(G(p)) - y^1(v_{i,I,\delta})) = \delta^{q-1}(y^1(p) - y^1(v_{i,I,\delta})),$$
$$(y^2(G(p)) - y^2(v_{i,I,\delta})) = \delta^{q-1}(y^2(p) - y^2(v_{i,I,\delta})),$$
$$(y^3(G(p)) - y^3(v_{i,I,\delta})) = (y^3(p) - y^3(v_{i,I,\delta})). \tag{6.11}$$

Here $q \gg 1$, p is a point in $V_{i,I,\delta}$, $y^i(p)$ refers to the ith coordinate value at p, $G(p)$ is the image of p by G and as mentioned above we have rotated our coordinates so that y^3 runs along the line joining v to v_{i,I,δ_0}. Thus property (iv) is achieved. In addition, from P1, G is identity outside a small neighborhood of $v_{i,I,\delta}$, and in particular is identity at all the edges of c other than the Ith one at v, maps the Ith one at v to itself (if v is CGR in c and if the $J \neq I$ edges are nonconducting in c, then it maps the upper and lower Ith conducting edges to themselves) and is identity in a neighborhood of v. In addition, from 5., Appendix C, P1 the vector field generating G (a) is supported only in an small neighborhood of $v_{i,I,\delta}$ and (b) when restricted to the straight line from v to v_{i,I,δ_0} always points along this line wherever it is nonvanishing. Hence G respects properties (v), (vi).

Property (iii) (b) can be achieved in a similar way as (ii) by considering ξ_J to be along the appropriate edge e_J, $J \neq I$ of c, constructing suitable neighborhoods of segments of this edge, smearing ξ_J with suitable functions of compact support and using the finite diffeomorphisms ϕ_J generated by the resulting vector field to achieve the required result. Clearly these diffeomorphisms also respect properties (v), (vi). Property (iii) (a) can be achieved through the action of a diffeomorphism ϕ_α which we shall construct at the end of this section. The product of all the semianalytic diffeomorphisms $(\prod_{J\neq I}\phi_J)G\phi_{i,I,\delta,\delta_0}\phi_\alpha$ yields the required semianalytic diffeomorphism satisfying (i)–(v). so that we have

$$\Phi^{\delta,Q,L,M,p_1,p_2,p_3}_{c,\{y\}(i,I,\beta,\delta_0)} = \left(\prod_{J\neq I}\phi_J\right)G\phi_{i,I,\delta,\delta_0}\phi_\alpha, \tag{6.12}$$

$$|c_{(i,\beta,\delta)}\rangle = \hat{U}(\Phi^{\delta,Q,L,M,p_1,p_2,p_3}_{c,\{y\}(i,I,\beta,\delta_0)})|c_{(i,\beta,\delta_0)}\rangle. \tag{6.13}$$

Before we construct ϕ_α, it is useful for future purposes to derive Eq. (6.15) below. First note that from Appendix B and Sec. V, the displaced vertex is in a region covered by the coordinates $\{y\}$. Next, consider the coordinate system $\{y^{(\delta)}\}$ obtained by the pushforward of the coordinates $\{y\}$ by the contraction diffeomorphism:

$$\{y^{(\delta)}\} = (\Phi^{\delta,Q,L,M,p_1,p_2,p_3}_{c,\{y\}(i,I,\beta,\delta_0)})^*\{y\}. \tag{6.14}$$

From (6.13) it follows that $\{y^{(\delta)}\}$ provides a coordinate patch around the displaced vertex of $c_{(i,I,\beta,\delta)}$. From the fact that ϕ_α is the identity in a neighborhood of the vertex v_{i,I,δ_0} (see the end of this section) together with the fact that the displaced vertex $v_{i,I,\delta}$ and its immediate vicinity is obtained by a rigid translation followed by the linear transformation (6.11), we have that the Jacobian between the $\{y^{(\delta)}\}$ and $\{y\}$ coordinates at the displaced vertex is

$$\left.\frac{\partial y^{(\delta)\mu}(p)}{\partial y^\nu(p)}\right|_{p=v_{i,I,\delta}} = \delta^\mu_\nu\delta^{-(q-1)} \quad \mu = 1,2$$
$$= \delta^\mu_\nu \quad \mu = 3. \tag{6.15}$$

Recall that the $\{y\}$ coordinates at v are such that the Ith edge at v in c runs along the third coordinate direction. To free us from this assumption let the coordinates at v be $\{x\}$ with $\{x\}$ related to $\{y\}$ by a rotation R_c which points y^3 along the Ith edge. Then it is straightforward to see that the Jacobian between the coordinates $\{x^{(\delta)}\} \coloneqq (\Phi^{\delta,Q,L,M,p_1,p_2,r}_{c,\{y\}(i,I,\beta,\delta_0)})^*\{x\}$ and the coordinates $\{x\}$ is

$$\left.\frac{\partial x^{(\delta)\mu}(p)}{\partial x^\nu(p)}\right|_{p=v_{i,I,\delta}} = (R_cGR_c^{-1})^\mu{}_\nu \tag{6.16}$$

where

$$G^\mu{}_\nu = \delta^\mu_\nu\delta^{-(q-1)} \quad \mu = 1,2$$
$$= \delta^\mu_\nu \quad \mu = 3. \tag{6.17}$$

Note that from property (v) and from the fact that the y^3 coordinate direction coincides with the straight line joining v to v_{i,I,δ_0} (in the $\{y\}$ coordinates), it follows that when restricted to this straight line, the third coordinate of the coordinate system $\{y^{(\delta)}\}$ also points along this line [indeed for the subset of this line lying within $V_{i,I,\delta}$, this fact can be explicitly verified from (6.11).)].

Finally, we construct ϕ_α. Let a C^1 or C^2 kink nearest to v_{i,I,δ_0} in $c_{(i,I,\beta,\delta_0)}$ be located at some $\bar{v}$. Consider two small spheres of radii $3\alpha_0, 2\alpha_0$ around this kink and a semianalytic function which vanishes outside the larger sphere and is unity inside the smaller sphere. Smear the dilatation vector field $\sum_{i=1}^{3}(y^i - y^i(\bar{v}))(\frac{\partial}{\partial y^i})^a$ with this function and exponentiate the action of this vector field to obtain a 1 parameter family of semianalytic diffeomorphisms. Clearly for an appropriate parameter value the size of the kink can be shrunk as required in (iii) (a) to α. Similarly shrink the second C^1 or C^2 kink if present. Let the diffeomorphism which shrinks these kinks be ψ_α. It is straightforward to see that the application of this diffeomorphism confines the departure from linearity, of the edge carrying the kink, to a sphere of radius 2α around the kink. We shall choose α as required below.

Next, we need to ensure that the action of $\phi_{i,I,\delta,\delta_0}$ (which acts immediately after ϕ_α in the contraction process) preserves the size of these kinks. Clearly we need only focus on any such kink if it is present at some $\bar{v}$ between $v_{i,I\delta_0}$ and v.[26] If such a kink is present we use a construction similar to that for $\phi_{i,I,\delta,\delta_0}$ to move the kink to a distance $\bar{\delta} \ll \epsilon_2$ from $v_{i,I\delta_0}$ through a rigid translation along the straight line joining v to v_{i,I,δ_0}, where ϵ_2 has been defined above in the construction of $\phi_{i,I,\delta,\delta_0}$.

More in detail, let $\bar{v}_1$ be on the straight line segment from v to $\bar{v}$ at a distance 3α from $\bar{v}$ with $\alpha \ll \bar{\delta}$.[27] Let v_1 be at a distance $\bar{\delta} - 3\alpha$ from $v_{(i,I,\delta_0)}$ on the straight line segment from v to v_{i,I,δ_0}. Let the straight line segment from $\bar{v}_1$ to v_1 be l_1. Let $l_{1\bar{\epsilon}}$ be a straight line which contains l_1 and whose end points $\bar{a}_{\bar{\epsilon}}, \bar{b}_{\bar{\epsilon}}$ lie at a distance $\bar{\epsilon}$ from $\bar{v}_1, v_1$ respectively. Consider a small cylinder $C_{\bar{\epsilon},\bar{\tau}}$ with axis $l_{1\bar{\epsilon}}$ and radius $\bar{\tau}$. Consider two such cylinders with parameters $\bar{\epsilon}_1, \bar{\epsilon}_2$ and $\bar{\tau}_1, \bar{\tau}_2$ with $\bar{\epsilon}_1 > \bar{\epsilon}_2, \bar{\tau}_1 > \bar{\tau}_2$. We shall further restrict $\bar{\epsilon}_1 \ll \alpha \ll \bar{\tau}_2 \ll \tau_2$. Choose $\bar{\epsilon}_1, \bar{\tau}_1$ to be small enough that $C_{\bar{\epsilon}_1,\bar{\tau}_1}$ does not intersect the graph underlying $c_{(i,I,\beta,\delta_0)}$ except along its edge from v to $v_{(i,I,\beta,\delta_0)}$. Consider the vector field $\xi^a = (\frac{\partial}{\partial y_3})^a$. Let $\bar{f}$ be a function compactly supported in $C_{\bar{\epsilon}_1,\bar{\tau}_1}$ such that it is unity in $C_{\bar{\epsilon}_2,\bar{\tau}_2}$. Let $\phi(\bar{f}\xi, t)$ be the 1 parameter set of diffeomorphisms generated by the vector field $f\xi^a$. Clearly, for an appropriate value of $t = t_0$ the diffeomorphism $\phi(\bar{f}\xi, t_0) \equiv \bar{\phi}$ translates the kink to its desired position. We set $\phi_\alpha \coloneqq \bar{\phi} \circ \psi_\alpha$.

[26] Here we assume that we have chosen ϵ_1, α small enough that any nearest C^2 or C^1 kink beyond v_{i,I,δ_0} does not intersect the cylinder C_{ϵ_1,τ_1} which is used to define $\phi_{i,I,\delta,\delta_0}$.

[27] We choose $\bar{\delta}$ to be much smaller than the distance between the kink $\bar{v}$ and v_{i,I,δ_0}.

D. Discrete action of product of operators

1. Action on elements of the ket set

Consider the operator product $\prod_{\mathrm{i}=1}^{n} \hat{O}_{\mathrm{i}}(N_{\mathrm{i}})$ where $\hat{O}^{\mathrm{i}}(N_{\mathrm{i}})$ is either a Hamiltonian constraint operator or electric diffeomorphism constraint operator smeared with Lagrange multiplier N_{i}, with operators ordered such that $\hat{O}^{\mathrm{i}}(N_{\mathrm{i}})$ is to the left of $\hat{O}^{\mathrm{i}}(N_{\mathrm{j}})$ if $\mathrm{i} < \mathrm{j}$ in the string of operators corresponding to the product. We are interested in the action of a discrete approximant to this operator product on a state c in the ket set. The discrete approximant we use is $\prod_{\mathrm{i}=1}^{n} \hat{O}_{\mathrm{i},\delta_{\mathrm{i}}}(N_{\mathrm{i}})$ where the discretization parameters δ_{i} are such that $\delta_{\mathrm{i}} < \delta_{\mathrm{j}}$ for $\mathrm{i} < \mathrm{j}$ and the action of $\hat{O}_{\mathrm{i},\delta_{\mathrm{i}}}(N_{\mathrm{i}})$ is given by (4.5) or (4.6) depending on whether $\hat{O}_{\mathrm{i}}(N_{\mathrm{i}})$ is a Hamiltonian or electric diffeomorphism constraint. Recall that we did not adequately specify the coordinates with respect to which these individual discrete actions were defined. Here we shall do so indirectly through a number of steps. At the end of this multistep procedure we shall have a complete definition of

$$\left(\prod_{\mathrm{i}=1}^{n} \hat{O}_{\mathrm{i},\delta_{\mathrm{i}}}(N_{\mathrm{i}})\right)|c\rangle \tag{6.18}$$

including a specification of the coordinates used.

Step 1: (6.18) as a weighted sum of deformed states.—We define each discrete operator action in the product through (4.5) or (4.6) keeping the choice of coordinates as yet unspecified. Clearly the result is a weighted sum of deformed kets. Our task in this step is to find these weights. We shall use the notation for deformed kets developed in Sec. VI. A. Any deformed ket takes the form of an mth generation child of c, $1 \leq m \leq n$. The deformation operation which produces this child from c is specified by the deformation sequence:

$$\begin{aligned} &[(i_{m-1}, I_{m-1}, \beta_{\mathrm{j}_m}, \delta_{\mathrm{j}_m})(i_{m-2}, I_{m-2}, \beta_{\mathrm{j}_{m-1}}, \delta_{\mathrm{j}_{m-1}}), \ldots.., \\ &\quad (i_1, I_1, \beta_{\mathrm{j}_2}, \delta_{\mathrm{j}_2})(i, I, \beta_{\mathrm{j}_1}, \delta_{\mathrm{j}_1})] \\ &\quad \mathrm{j}_k < \mathrm{j}_l \quad \text{iff} \quad k > l, \quad \mathrm{j}_{\mathrm{i}} \in \{1, .., n\} \end{aligned} \tag{6.19}$$

where if $m = 1$ we only have the deformation $(i, I, \beta_{\mathrm{j}_1}, \delta_{\mathrm{j}_1})$. The kth deformation in this sequence is $(i_{k-1}, I_{k-1}, \beta_{\mathrm{j}_k}, \delta_{\mathrm{j}_k})$. It corresponds to the deformation generated by the operator $\hat{O}_{\mathrm{j}_k,\delta_{\mathrm{j}_k}}(N_{\mathrm{j}_k})$ on the $k-1$th generation child:

$$c_{[(i_{k-2}, I_{k-2}, \beta_{\mathrm{j}_{k-1}}, \delta_{\mathrm{j}_{k-1}}), \ldots.., (i, I, \beta_{\mathrm{j}_1}, \delta_{\mathrm{j}_1})]} \tag{6.20}$$

where if this operator is a Hamiltonian constraint we have chosen the flip β_{j_k} to define its action. As is implicit in the discussion of Sec. VI. A, given the deformation sequence (6.19), the edges and internal charge indices at the nondegenerate vertex of the kth generation children are denoted with a subscript k and the edges and internal charge indices of the parent vertex in c by i, I. The numbering scheme used is also that discussed in Sec. VI. A so that the enumeration of edges of any child is related to that for c.

As emphasized before we have not yet made explicit our choices of coordinates with respect to which the deformations are defined. Let us see in more detail as to exactly where we need these choices to be made so as to provide a complete specification of the deformed child (6.20). Consider the deformation sequence (6.19). The sequence starts with the right most deformation $(i, I, \beta_{\mathrm{j}_1}, \delta_{\mathrm{j}_1})$ acting on the parent c. The singly deformed child it generates acts as the parent state for the next deformation $(i_1, I_1, \beta_{\mathrm{j}_2}, \delta_{\mathrm{j}_2})$. In this way proceeding from right to left, each successive deformation acts on the deformed state generated by the sequence to its right and produces a parent state for the deformation to its left. Therefore, in order to specify each of these deformations we need to specify the coordinate patch for the nondegenerate vertex of the state produced by the deformation sequence to its right. Hence in order to specify the deformed child (6.20) we need to specify coordinate patches for each of the deformed states in the "lineage" connecting (6.20) to c.

The deformed states produced in (6.18) consist of states of the form (6.2) for all choices of index sets $\{\mathrm{i}_1, \mathrm{i}_2, .., \mathrm{i}_\mathrm{m}\}$ such that the inequalities in the second line of (6.2) hold, and for all $\mathrm{m} \in \{1, .., n\}$. From our discussion in the previous paragraph, it follows that for a complete specification of all the deformed states in this set, we need a specification of a coordinate patch around each nondegenerate vertex for each mth generation child with m ranging from $1, .., n-1$. In addition we must also, of course, specify the coordinate patch around the nondegenerate vertex of c. We shall see that in the final step of our procedure (see Sec. VI. D. 1. c below), we will have a specification of all these coordinate patches.

The notation (6.19) is a cumbersome one. Hence, similar to (6.4), if there is no confusion in doing so, we will often find it convenient to abbreviate the deformation sequence in (6.2) through

$$[i, I, \beta, \delta]_m \equiv [(i_{m-1}, I_{m-1}, \beta_{\mathrm{j}_\mathrm{m}}, \delta_{\mathrm{j}_\mathrm{m}}), \ldots, (i, I, \beta_{\mathrm{j}_1}, \delta_{\mathrm{j}_1})]. \tag{6.21}$$

Thus the deformed states produced in (6.18) consist of the states $c_{[i,I,\beta,\delta]_m}$ for all choices of $[i, I, \beta, \delta]_m$ and m such that $1 \le m \le n$. Clearly, (6.18) can be expanded out as a sum over all these states so that we have

$$\left(\prod_{\mathrm{i}=1}^{n} \hat{O}_{\mathrm{i},\delta_\mathrm{i}}(N_\mathrm{i})\right)|c\rangle = \left(\prod_{i=1}^{n} \delta_\mathrm{i}\right)^{-1} \sum_{[i,I,\beta,\delta]_m, m=1,..,n} C_{[i,I,\beta,\delta]_m} |c_{[i,I,\beta,\delta]_m}\rangle + C_0|c\rangle. \tag{6.22}$$

Here the coefficients $C_{[i,I,\beta,\delta]_m}$ can be computed using (4.5) and (4.6) in (6.18). We do not need the explicit form of these coefficients here so we refrain from displaying them. Instead we restrict ourselves to a few remarks regarding their structure. Each coefficient is constructed out of the various factors which appear in each application of (4.5) or (4.6) in (6.18). In particular each coefficient has in it a product over all the n lapse functions in (6.18). Each lapse is evaluated at a vertex of one of the states in the lineage defined by the sequence using the coordinate patch specified at that vertex. Note that this is the only coordinate choice dependent feature of the coefficients. The remaining contributions come from various sign factors and overall $\hbar$ dependent numerical factors in (4.5) and (4.6). The sign factors arise from the β factors in (4.5) and from the fact that some of the actions of the constraints come from the $\mathbf{-1}$ term (see Sec. III. B. 1) in (4.5) and (4.6).

To summarize, the discrete action of the operator product of interest on any ket in the ket set can be written as a weighted sum over all its deformed children. The weights (i.e. the coefficients $C_{[i,I,\beta,\delta]_m}, C_0$) in this sum can be explicitly computed but we do not need an explicit computation in all generality for our purposes here. A complete evaluation of the coefficients and a complete specification of the deformed children requires a choice of coordinate patch around each nondegenerate vertex of each child $c_{[i,I,\beta,\delta]_m}$, $\forall\ [i, I, \beta, \delta]_m, m = 1, ..n$ as well as around the vertex of c. The coordinate choice dependence of each coefficient derives solely from its dependence on the density weighted lapse functions.

In the next step we shall define each of the deformed children of c as the image of a corresponding deformation of the reference state c_0 by the reference diffeomorphism which maps c_0 to c. Since we are interested in the continuum limit, it will suffice to define these deformations for small enough $\{\epsilon_\mathrm{i}, i = 1, .., n\}$ where $\epsilon_\mathrm{i} \ll \delta_\mathrm{i}, i = 1, ..n$ and $\epsilon_\mathrm{i} < \epsilon_\mathrm{j}$ for $\mathrm{i} < \mathrm{j}$ from which we define

$$\left(\prod_{\mathrm{i}=1}^{n} \hat{O}_{\mathrm{i},\epsilon_\mathrm{i}}(N_\mathrm{i})\right)|c\rangle = \left(\prod_{i=1}^{n} \epsilon_\mathrm{i}\right)^{-1} \sum_{[i,I,\beta,\epsilon]_m, m=1,..,n} C_{[i,I,\beta,\epsilon]_m} |c_{[i,I,\beta,\epsilon]_m}\rangle + C_0|c\rangle. \tag{6.23}$$

Step 2: Contraction of deformed reference states.—Each of the deformed states appearing on the right-hand side of (6.22) is labeled by some deformation sequence $[i, I, \beta, \delta]_m$. Replace each such deformation sequence $[i, I, \beta, \delta]_m$,

$$[i, I, \beta, \delta]_m \equiv [(i_{m-1}, I_{m-1}, \beta_{\mathrm{j}_\mathrm{m}}, \delta_{\mathrm{j}_\mathrm{m}}), \ldots, (i, I, \beta_{\mathrm{j}_1}, \delta_{\mathrm{j}_1})] \tag{6.24}$$

by the corresponding sequence:

$$[i,I,\beta,\delta_0]_m \equiv [(i_{m-1},I_{m-1},\beta_{\mathrm{j_m}},\delta_{0\mathrm{j_m}}),\ldots,(i,I,\beta_{\mathrm{j_1}},\delta_{0\mathrm{j_1}})], \tag{6.25}$$

where we have chosen the partameters $\{\delta_{0\mathrm{j}},\mathrm{j}=1,..,m\}$ to be sufficiently small in a sense to be described below. Next, let c_0 be the reference state for c. and consider the set $S_{\{\delta_{0\mathrm{i}}\},c_0}$ of all descendants of c_0 obtained by deforming c_0 by all such correspondent sequences:

$$S_{\{\delta_{0\mathrm{i}}\},c_0} \coloneqq \{c_{0[i,I,\beta,\delta_0]_m} \quad \forall\ [i,I,\beta,\delta_0]_m, m=0,1,..,n\}. \tag{6.26}$$

Here the parameters $\{\delta_{0\mathrm{j}},\mathrm{j}=1,..,m\}$ have been chosen sufficiently small that every element of $S_{\{\delta_{0\mathrm{i}}\},c_0}$ is a primary. Now, each element of the above set (apart from c_0) is some mth generation child of c_0. We define the coordinates with respect to which the multiple deformation sequence $[i,I,\beta,\delta_0]_m$ is constructed to be $\{x_0\}$. As discussed in the construction of primaries in Sec. VI. B, for sufficiently small deformation parameters $\delta_{0\mathrm{i}},\mathrm{i}=1,..n$, these deformations are well defined and the coordinate patch $\{x_0\}$ can be used as a linear coordinate patch for every nondegenerate vertex of every element of $S_{\{\delta_{0\mathrm{i}}\},c_0}$.

Next consider a set of deformation parameters $\{\epsilon_{\mathrm{i}}\}$ such that each $\epsilon_{\mathrm{i}} \ll \delta_{0\mathrm{i}}$ and $\epsilon_{\mathrm{i}} < \epsilon_{\mathrm{j}}$ for $\mathrm{i}<\mathrm{j}$. Let us fix some particular deformation sequence

$$[i,I,\beta,\delta_0]_m \equiv [(i_{m-1},I_{m-1},\beta_{\mathrm{j_m}},\delta_{0\mathrm{j_m}}),\ldots,(i,I,\beta_{\mathrm{j_1}},\delta_{0\mathrm{j_1}})] \tag{6.27}$$

and the corresponding sequence

$$[i,I,\beta,\epsilon]_m \equiv [(i_{m-1},I_{m-1},\beta_{\mathrm{j_m}},\epsilon_{\mathrm{j_m}}),\ldots,(i,I,\beta_{\mathrm{j_1}},\epsilon_{\mathrm{j_1}})]. \tag{6.28}$$

We now construct a contraction diffeomorphism which maps $c_{0[i,I,\beta,\delta_0]_m}$ to $c_{0[i,I,\beta,\epsilon]_m}$. This diffeomorphism will be constructed as a product of contraction diffeomorphisms of the type defined in Sec. IV. C. We shall use the index notation as explained in Step 1 so that the subscript k attached to the edge index signifies that the edge in question is one which is obtained as a result of k successive deformations; similary this subscript attached to the internal index of a $U(1)^3$ charge signifies that the charge in question labels such a generation k edge. Additionally we shall refer to the part of the deformation sequence from the 1st deformation to the kth one within the specific deformation sequence (6.27) as $[i,I,\beta,\delta_0]_m^k$ so that

$$[i,I,\beta,\delta_0]_m^k \equiv [(i_{k-1},I_{k-1},\beta_{\mathrm{j_k}},\delta_{0\mathrm{j_k}}),\ldots,(i,I,\beta_{\mathrm{j_1}},\delta_{0\mathrm{j_1}})], \tag{6.29}$$

with the kth generation child produced in this sequence from the ancestor c_0 denoted as

$$c_{0[i,I,\beta,\delta_0]_m^k} \equiv c_{0[(i_{k-1},I_{k-1},\beta_{\mathrm{j_k}},\delta_{0\mathrm{j_k}}),\ldots,(i,I,\beta_{\mathrm{j_1}},\delta_{0\mathrm{j_1}})]}. \tag{6.30}$$

The states $c_{0[i,I,\beta,\delta_0]_m^k}, k=1,..,m$ will be said to form the lineage for the sequence (6.27). The states $c_{0[i,I,\beta,\delta_0]_m^{k-1}}$, $c_{0[i,I,\beta,\delta_0]_m^k}$ will be called "successive" with $c_{0[i,I,\beta,\delta_0]_m^{k-1}}$ being the immediate parent of $c_{0[i,I,\beta,\delta_0]_m^k}$. We shall use a similar notation and language in relation to the deformation sequence (6.28).

Finally we introduce a hatted index notation as follows. Consider the kth transition $(i_{k-1},I_{k-1},\beta_{j_k},\delta_{0j_k})$ which produces the child $c_{0[i,I,\beta,\delta_0]_m^k}$ from the parent $c_{0[i,I,\beta,\delta_0]_m^{k-1}}$. The edge indices at the nondegenerate vertex of the child are distinguished by the subscript k and at that of the parent by $k-1$. The parental edge along which the transition occurs is the I_{k-1}th one. Consider the edges e_{J_k} in the child with $J_k \neq I_{k-1}$ (recall that the numbering of the edges of the child is correlated with that of the parent as described in Sec. VI. A). We shall denote such indices with a hat so that $\hat{J}_k$ signifies $\hat{J}_k \neq I_{k-1}$ in the above transition. Clearly, hatted indices index those edges which are nonconducting in $c_{0[i,I,\beta,\delta_0]_m^k}$ and any such edge connects the nondegenerate vertex of the child with a C^0 kink.

Next, fix some $p \gg 1$, recall that $q \gg 1$ is defined in equation (6.11), and proceed iteratively as follows:

(i) First consider the contraction diffeomorphism $\Phi^{\delta,Q,L,M,p_1,p_2,p_3}_{c,\{y\}(i,I,\beta,\delta_0)}$ and perform the replacements

$$\begin{aligned}
&\delta \to \epsilon_{\mathrm{j_1}}, \qquad c \to c_0,\\
&\{y\} \to \{x_0\} \quad L,M \to \hat{J}_1,\hat{K}_1,\\
&p_1 \to \frac{2}{3}(q-1)\mathrm{j}_1 p, \qquad p_2 \to \frac{2}{3}(q-1)\mathrm{j}_1(p+1),\\
&p_3 \to \frac{2}{3}(q-1)\mathrm{j}_1(p+1) + \frac{4}{3}(q-1)\mathrm{j}_1.
\end{aligned} \tag{6.31}$$

We shall specify the factor Q as we go along. Q will depend on

(a) the operator sequence $S_{\mathrm{j_1}} = \{\hat{O}_1(N_1),..,\hat{O}_{\mathrm{j_1}}(N_{\mathrm{j_1}}\}$ starting from the first leftmost operator in the operator product (6.18) and terminating at the j_1th one,

(b) the charges of the child $c_{0(i,I,\beta_{\mathrm{j_1}},\epsilon_{\mathrm{j_1}})} \equiv c_{0[i,I,\beta,\epsilon]_m^1}$ at its nondegenerate vertex.

(c) the charges of the parent c_0 at its nondegenerate vertex.

Denoting c_0 by $c_{[i,I,\beta,\epsilon]_m^0}$ we denote this dependence through

$$Q \equiv Q(c_{0[i,I,\beta,\epsilon]_m^{0,1}}, S_{\mathrm{j_1}},). \tag{6.32}$$

Accordingly, we replace the label Q by the label S_{j_1} and rewrite the contraction diffeomorphism as

$$\Phi^{\epsilon_{j_1},\{x_0\},\hat{J}_1,\hat{K}_1}_{c_{0[i,I,\beta,\epsilon]^0_m},(i,I,\beta,\delta_0),S_{j_1}}. \tag{6.33}$$

The notation indicates that (a) the parent $c_0 \equiv c_{0[i,I,\beta,\epsilon]^0_m}$ is deformed through (i, I, β, δ_0) and the deformation parameter δ_0 of the resulting child is contracted to the value ϵ_{j_1}, both parameters being measured by the parental coordinates system $\{x_0\}$, (b) the Q factor is that determined by the charges of this child, its parent and the operator sequence S_{j_1} (c) the kinks at $\hat{J}_1, \hat{K}_1$ are placed in accordance with the values of p_1, p_2 in (6.31). Here to avoid notational clutter we have suppressed the labels p_1, p_2, p_3.

Note that the parental coordinate system $\{x_0\}$ covers a neighborhood of the displaced vertex of the uncontracted child $c_{0[i,I,\beta,\delta_0]^1_m}$. The diffeomorphism (6.33) maps this displaced vertex to its counterpart in the contracted child $c_{0[i,I,\beta,\epsilon]^1_m}$. Hence the push forward of the parental coordinate system $\{x_0\}$ yields a coordinate system around the displaced vertex in the child $c_{(i,I,\beta_{j_1},\epsilon_{j_1})} \equiv c_{[i,I,\beta,\epsilon]^1_m}$. Suppressing various dependences to avoid notational clutter and keeping in mind that we are discussing the contraction of the *specific* deformation sequence (6.27) to that in (6.28) we denote this coordinate system by $\{x_0^{\epsilon_{j_1}}\}$

$$\{x_0^{\epsilon_{j_1}}\} \coloneqq \left(\Phi^{\epsilon_{j_1},\{x_0\},\hat{J}_1,\hat{K}_1}_{c_{0[i,I,\beta,\epsilon]^0_m},(i,I,\beta,\delta_0)S_{j_1}}\right)^*\{x_0\}. \tag{6.34}$$

We shall use this coordinate system associated with this first generation child to define the next transition in which this child acts as the parent for a second generation child in (ii) below.

(ii) In the contraction diffeomorphism $\Phi^{\delta,Q,L,M,p_1,p_2,p_3}_{c,\{y\}(i,I,\beta,\delta_0)}$ replace

$$\begin{aligned}
&\delta \to \epsilon_{j_2}, \qquad c \to c_{0[i,I,\beta,\epsilon]^1_m},\\
&\{y\} \to \{x_0^{\epsilon_{j_1}}\} \quad L, M \to \hat{J}_2, \hat{K}_2,\\
&p_1 \to \frac{2}{3}(q-1)j_2 p, \qquad p_2 \to \frac{2}{3}(q-1)j_2(p+1),\\
&p_3 \to \frac{2}{3}(q-1)j_2(p+1) + \frac{4}{3}(q-1)j_2.
\end{aligned} \tag{6.35}$$

Similar to (i) Q depends on the operator sequence $S_{j_2} = \{\hat{O}_1(N_1), .., \hat{O}_{j_2}(N_{j_2}\}$ starting from the first leftmost operator in the operator product (6.18) and terminating at the j_2th one, as well on the charges of the $[i, I, \beta, \epsilon]^2_m$ and $[i, I, \beta, \epsilon]^1_m$ children of c_0 at their nondegenerate vertices so that $Q \equiv Q(c_{0[i,I,\beta,\epsilon]^{2,1}_m}, S_{j_2})$ and we rewrite the contraction diffeomorphism as

$$\Phi^{\epsilon_{j_2},\{x_0^{\epsilon_{j_1}}\},\hat{J}_2,\hat{K}_2}_{c_{0[i,I,\beta,\epsilon]^1_m},(i_1,I_1,\beta_{j_2},\delta_{0j_2}),S_{j_2}}. \tag{6.36}$$

From the substitution $\{y\} \to \{x_0^{\epsilon_{j_1}}\}$ in (6.35) above, it follows that the coordinate system with respect to which the discretization parameters $\delta_{0j_2}, \epsilon_{j_2}$ are measured in the transition from the parent $c_{0[i,I,\beta,\epsilon]^1_m}$ to the child $c_{0[i,I,\beta,\epsilon]^2_m}$ is the parental coordinate system $\{x_0^{\epsilon_{j_1}}\}$ defined in (6.34).

More in detail, consider the uncontracted images of this parent and child; these are the states $c_{0[i,I,\beta,\delta_0]^1_m}$, $c_{0[i,I,\beta,\delta_0]^2_m}$ of Eq. (6.30). Consider the image of *both* of these states by the contraction diffeomorphism Eq. (6.33). The image of the parent simply yields the parent $c_{0[i,I,\beta,\epsilon]^1_m}$ at parameter ϵ_{j_1} as described in (i). Clearly, by virtue of the properties of diffeomorphic images, the image of the child $c_{0[i,I,\beta,\delta_0]^2_m}$ by this diffeomorphism defines a state which bears the same relation to its parent $c_{0[i,I,\beta,\epsilon]^1_m}$ as $c_{0[i,I,\beta,\delta_0]^2_m}$ bears to *its* parent $c_{0[i,I,\beta,\delta_0]^1_m}$. It follows that the image of $c_{0[i,I,\beta,\delta_0]^2_m}$ by (6.33) is a child at parameter δ_{0j_1} of $c_{0[i,I,\beta,\epsilon]^1_m}$ where the parameter δ_{0j_1} is now measured by *the pushforward coordinate system* $\{x_0^{\epsilon_{j_1}}\}$ of (6.34). It follows that this child is obtained through the deformation $(i_1, I_1, \beta_{j_2}, \delta_{0j_2})$ of its parent $c_{0[i,I,\beta,\epsilon]^1_m}$ with respect to the coordinates $\{x_0^{\epsilon_{j_1}}\}$. The contraction diffeomorphism (6.36) acts on this child, contracts the parameter value δ_{0j_2} and produces the child $c_{0[i,I,\beta,\epsilon]^2_m}$ at parameter value ϵ_{j_2} with $\delta_{0j_2}, \epsilon_{j_2}$ measured by the parental coordinates $\{x_0^{\epsilon_{j_1}}\}$ associated with the parent $c_{0[i,I,\beta,\epsilon]^1_m}$.

Clearly one can now define a coordinate around the nondegenerate vertex of $c_{0[i,I,\beta,\epsilon]^2_m}$ as the pushforward of this coordinate patch $\{x_0^{\epsilon_{j_1}}\}$ by the contraction diffeomorphism of (6.36) i.e. we define

$$\begin{aligned}
&\{x_0^{\epsilon_{j_1}\epsilon_{j_2}}\}\\
&\coloneqq \left(\Phi^{\epsilon_{j_2},\{x_0^{\epsilon_{j_1}}\},\hat{J}_2,\hat{K}_2}_{c_{0[i,I,\beta,\epsilon]^1_m},(i_1,I_1,\beta_{j_2},\delta_{0j_2}),S_{j_2}}\right)_*\{x_0^{\epsilon_{j_1}}\}\\
&= \left(\Phi^{\epsilon_{j_2},\{x_0^{\epsilon_{j_1}}\},\hat{J}_2,\hat{K}_2}_{c_{0[i,I,\beta,\epsilon]^1_m},(i_1,I_1,\beta_{j_2},\delta_{0j_2}),S_{j_2}}\Phi^{\epsilon_{j_1},\{x_0\},\hat{J}_1,\hat{K}_1}_{c_{0[i,I,\beta,\epsilon]^0_m},(i,I,\beta,\delta_0),S_{j_1}}\right)^*\{x_0\}.
\end{aligned} \tag{6.37}$$

This coordinate patch in turn is used to define the next transition in the sequence. We can then iterate this procedure. The structure obtained at the kth step is described in (iii).

(iii) At the kth step the arguments of $\Phi^{\delta,Q,L,M,p_1,p_2,p_3}_{c,\{y\}(i,I,\beta,\delta_0)}$ are replaced as

$$\delta \to \epsilon_{\mathrm{j_k}}, \qquad c \to c_{0[i,I,\beta,\epsilon]_m^{k-1}}, \qquad \{y\} \to \{x_0^{\epsilon_{\mathrm{j_1}} \cdots \epsilon_{\mathrm{j_{k-1}}}}\} \quad L, M \to \hat{J}_k, \hat{K}_k,$$
$$p_1 \to \frac{2}{3}(q-1)\mathrm{j_k} p, \qquad p_2 \to \frac{2}{3}(q-1)\mathrm{j_k}(p+1), \qquad p_3 \to \frac{2}{3}(q-1)\mathrm{j_k}(p+1) + \frac{4}{3}(q-1)\mathrm{j_k} \tag{6.38}$$

with Q depending on the operator sequence $S_{\mathrm{j_k}} = \{\hat{O}_1(N_1), .., \hat{O}_{\mathrm{j_k}}(N_{\mathrm{j_k}})\}$ and on the charges of $c_{0[i,I,\beta,\epsilon]_m^k}$, $c_{0[i,I,\beta,\epsilon]_m^{k-1}}$ at their nondegenerate vertices so that $Q \equiv Q(c_{0[i,I,\beta,\epsilon]_m^{k,k-1}}, S_{\mathrm{j_k}})$ and we rewrite the contraction diffeomorphism as

$$\Phi^{\epsilon_{\mathrm{j_k}}, \{x_0^{\epsilon_{\mathrm{j_1}} .. \epsilon_{\mathrm{j_{k-1}}}}\}, \hat{J}_k, \hat{K}_k}_{c_{0,[i,I,\beta,\epsilon]_m^{k-1}}, (i_{k-1}, I_{k-1}, \beta_{\mathrm{j_k}}, \delta_{0\mathrm{j_k}}), S_{\mathrm{j_k}}} \tag{6.39}$$

with the coordinate patch around the nondegenerate vertex of $c_{0[i,I,\beta,\epsilon]_m^k}$ defined to be

$$\begin{aligned} \{x_0^{\epsilon_{\mathrm{j_1}} .. \epsilon_{\mathrm{j_k}}}\} &:= \left(\Phi^{\epsilon_{\mathrm{j_k}}, \{x_0^{\epsilon_{\mathrm{j_1}} .. \epsilon_{\mathrm{j_{k-1}}}}\}, \hat{J}_k, \hat{K}_k}_{c_{0,[i,I,\beta,\epsilon]_m^{k-1}}, (i_{k-1}, I_{k-1}, \beta_{\mathrm{j_k}}, \delta_{0\mathrm{j_k}}), S_{\mathrm{j_k}}}\right)^* \{x_0^{\epsilon_{\mathrm{j_1}} .. \epsilon_{\mathrm{j_{k-1}}}}\} \\ &= \left(\Phi^{\epsilon_{\mathrm{j_k}}, \{x_0^{\epsilon_{\mathrm{j_1}} .. \epsilon_{\mathrm{j_{k-1}}}}\}, \hat{J}_k, \hat{K}_k}_{c_{0,[i,I,\beta,\epsilon]_m^{k-1}}, (i_{k-1}, I_{k-1}, \beta_{\mathrm{j_k}}, \delta_{0\mathrm{j_k}}), S_{\mathrm{j_k}}} \cdots \Phi^{\epsilon_{\mathrm{j_1}}, \{x_0\}, \hat{J}_1, \hat{K}_1}_{c_{0[i,I,\beta,\epsilon]_m^0}, (i,I,\beta,\delta_0), S_{\mathrm{j_1}}}\right)^* \{x_0\}. \end{aligned} \tag{6.40}$$

(iv) Finally after the mth step we define the desired composite contraction diffeomorphism:

$$\Phi^{\epsilon_{\mathrm{j_1}} .. \epsilon_{\mathrm{j_m}}, (\hat{J}_1, \hat{K}_1), ..., (\hat{J}_m, \hat{K}_m)}_{c_{0[i,I,\beta,\delta_0]_m}, S_{\mathrm{j_1}}} := \left(\prod_{k=2}^{m} \Phi^{\epsilon_{\mathrm{j_k}}, \{x_0^{\epsilon_{\mathrm{j_1}} .. \epsilon_{\mathrm{j_{k-1}}}}\}, \hat{J}_k, \hat{K}_k}_{c_{0,[i,I,\beta,\epsilon]_m^{k-1}}, (i_{k-1}, I_{k-1}, \beta_{\mathrm{j_k}}, \delta_{0\mathrm{j_k}}) S_{\mathrm{j_k}}}\right) \Phi^{\epsilon_{\mathrm{j_1}}, \{x_0\}, \hat{J}_1, \hat{K}_1}_{c_{0[i,I,\beta,\epsilon]_m^0}, (i,I,\beta,\delta_0), S_{\mathrm{j_1}}} \tag{6.41}$$

where the product is ordered from right to left in increasing k and where have labeled the left-hand side by the sequence $S_{\mathrm{j_1}}$ because the sequence $S_{\mathrm{j_1}}$ contains all $S_{\mathrm{j_k}}, k > 1$ so that the label $S_{\mathrm{j_1}}$ subsumes the set of labels $\{S_{\mathrm{j_k}}, k \geq 1\}$. This composite contraction diffeomorphism contracts the δ_0 parameters to their corresponding ϵ values so that we have

$$|c_{0[i,I,\beta,\epsilon]_m}\rangle := \hat{U}\left(\Phi^{\epsilon_{\mathrm{j_1}} .. \epsilon_{\mathrm{j_m}}, (\hat{J}_1, \hat{K}_1), ..., (\hat{J}_m, \hat{K}_m)}_{c_{0[i,I,\beta,\delta_0]_m}, S_{\mathrm{j_1}}}\right) |c_{0[i,I,\beta,\delta_0]_m}\rangle \tag{6.42}$$

where $\hat{U}(\Phi)$ refers to the unitary implementation of the diffeomorphism Φ. The superscripts $\epsilon_{\mathrm{j_1}} .. \epsilon_{\mathrm{j_m}}$ indicate that the deformations have been contracted down from $\delta_{0\mathrm{j_1}}, .., \delta_{0\mathrm{j_m}}$ in the deformation sequence (6.27) to $\epsilon_{\mathrm{j_1}}, .., \epsilon_{\mathrm{j_m}}$ in the deformation sequence (6.28). The action of the deformation sequence (6.28) on c_0 creates a series of C^0 kinks in $c_{0[i,I,\beta,\epsilon]_m}$, one set of $(N-1)$ kinks for each deformation. The superscript $(\hat{J}_1, \hat{K}_1), .., (\hat{J}_m, \hat{K}_m)$ in (6.42) refers to the two preferred C^0 kinks created by each such deformation. The preferred kinks $(\tilde{v}_{\hat{J}_k}, \tilde{v}_{\hat{K}_k})$ created by the kth deformation are brought to the specific coordinate distances specified by (iii) above [see also (iii) of Sec. VI. C].[28] These coordinate distances are measured by the coordinate system $\{x_0^{\epsilon_{\mathrm{j_1}} .. \epsilon_{\mathrm{j_{k-1}}}}\}$ associated with the nondegenerate vertex of the deformed state obtained by the action of the deformation $[i, I, \beta, \epsilon]_m^{k-1}$ on c_0, this vertex serving as the parent vertex for the next deformation $(i_k, I_k, \beta_{\mathrm{j_k}}, \epsilon_{\mathrm{j_k}})$ in the sequence $[i, I, \beta, \epsilon]_m$. We shall see in Sec. X and XI that the placement of these kinks plays a key role in obtaining an anomaly free algebra.

Step 3: Deformed states as images of contracted reference states.—The reference state c_0 of Step 2 above is related to the state c by the action of some reference diffeomorphism α via the Eq. (6.10). We define

$$|c_{[i,I,\beta,\epsilon]_m}\rangle := \hat{U}(\alpha) |c_{0[i,I,\beta,\epsilon]_m}\rangle. \tag{6.43}$$

This provides a definition of all the charge nets on the right-hand side of Eq. (6.23). Here the coordinate patch around the nondegenerate vertex of the state $c_{[i,I,\beta,\epsilon]_m}$ obtained through the action of the deformation sequence $[i, I, \beta, \epsilon]_m$ on c is defined to be the image of the coordinate patch

[28] The choice of these preferred kinks is made, at the moment, arbitrarily; in Sec. VIII we shall sum over these choices.

around the nondegenerate vertex of $c_{0[i,I,\beta,\epsilon]_m}$ [obtained by setting $k = m$ in (6.40)] by the diffeomorphism α.

Recall that the coefficient $C_{[i,I,\beta,\epsilon]_m}$ in (6.23) acquires a coordinate dependence solely from the dependence of this coefficient on the lapse functions. Each lapse function is evaluated at some nondegenerate vertex of one of the states which define the lineage of $c_{[i,I,\beta,\epsilon]_m}$. Since we have provided a unique choice of coordinate patches for every such vertex, every coefficient on the right-hand side of (6.23) can be evaluated.

Having provided a unique and well-defined evaluation of every coefficient in (6.23), we have a complete specification of the action of the operator product $\prod_{\mathrm{i}=1}^{n} \hat{O}_{\mathrm{i},\epsilon_{\mathrm{i}}}(N_{\mathrm{i}})$ for sufficiently small discretization parameters $\{\epsilon_{\mathrm{i}}, i = 1,..,n\}$.

2. Summary

In order to define the discrete action of multiple constraint operators of a state c in the ket set of Sec. VI. B, it is necessary to define multiple deformations of c. This is done in three stages. In the first stage, multiple deformations of the reference state c_0 are defined with respect to the reference coordinate system $\{x_0\}$ at small enough parameter values as measured by $\{x_0\}$. These deformations are built out of a sequence of single deformations each constructed in detail along the lines of Appendix B and Sec. V. A. In the second stage, these parameters and the associated deformations are contracted through the action of contraction diffeomorphisms. This process involves the iterated action of individual contraction diffeomorphisms. The deformation which yields each contracted child in a deformation sequence is then a deformation which is defined with respect to the coordinates associated with the contracted parent. In the third stage, all the contracted children, now obtained at any small enough set of parameter values, are imaged by the reference diffeomorphism connecting c_0 to c and these images define the desired multiple deformations of c.

In the second stage described in Sec. VI. D. 1, the deformation of the parental state $c_{0[i,I,\beta,\epsilon]_m^{k-1}}$ in a deformation sequence $[i, I, \beta, \epsilon]_m$ of (6.28) by the deformation $(i_{k-1}, I_{k-1}, \beta_{\mathrm{j_k}}, \epsilon_{\mathrm{j_k}})$ yields the child $c_{0[i,I,\beta,\epsilon]_m^k}$. The deformation is defined with respect to the coordinate system $\{x_0^{\epsilon_{\mathrm{j_1}},\ldots,\epsilon_{\mathrm{j_{k-1}}}}\}$ associated with the parental state. In this manner the deformation which yields any child $c_{0[i,I,\beta,\epsilon]_m}$ through the specific deformation sequence $[i, I, \beta, \epsilon]_m$ applied to c_0 is uniquely and completely well defined in terms of the sequence of coordinate systems $\{x_0^{\epsilon_{\mathrm{j_1}},\ldots,\epsilon_{\mathrm{j_k}}}\}$, $k = 1, 2, ..m - 1$, together with $\{x_0\}$. Further, the contraction procedure also results in the nondegenerate vertex of the child $c_{0[i,I,\beta,\epsilon]_m}$ being equipped with the coordinates $\{x_0^{\epsilon_{\mathrm{j_1}},\ldots,\epsilon_{\mathrm{j_m}}}\}$. As is easy to check, the procedure used in the second stage to construct these coordinate patches for any deformation of c_0 only depends on the deformation sequence which defines the deformation. Thus given c_0 and any deformation sequence, the deformed state comes equipped with a coordinate patch which is a pushforward of the reference coordinate patch $\{x_0\}$ associated with c_0 by an appropriately constructed composite contraction diffeomorphism, this diffeomorphism being uniquely fixed by the specification of the deformation sequence (including the specification of the preferred set of C^0 kinks, see footnote 28).

In the third stage the images of each of these coordinate systems by the reference diffeomorphism which maps c_0 to c yield coordinate systems which provide a clear coordinate interpretation for the deformations generated by the discrete action of the operator product $\prod_{\mathrm{i}=1}^{n} \hat{O}_{\mathrm{i},\epsilon_{\mathrm{i}}}(N_{\mathrm{i}})$. In particular this procedure yields a unique coordinate patch associated with the nondegenerate vertex of each state in the expansion (6.23). It is useful to give these coordinate patches a name to distinguish them from other coordinate patches we shall encounter. We shall refer to the coordinate patches associated with the nondegenerate vertex of each of the deformed states which occur on the right-hand side of (6.23) as *contraction coordinates* because of the role of contraction diffeomorphisms in their definition. In Sec. VIII we shall encounter a different set of coordinates which we shall call *reference coordinates*.

Recall that the only coordinate dependence of the coefficients in the expansion (6.23) arises from the evaluation of lapse functions. The occurrence of these lapse functions traces back to the dependence of the quantum shift on the lapse, this lapse being evaluated with respect to the coordinates associated with the vertex of the state on which the quantum shift operator acts. Indeed, it is these coordinates in terms of which the deformations generated by the quantum shift are defined. From this it is straightforward to see that the evaluation of such a lapse function must be done in terms of the contraction coordinates associated with its argument.

Next, let us discuss how the mapping, via contraction diffeomorphisms in Sec. VI. D. 1. a and reference diffeomorphisms in Sec. VI. D. 1. c, of deformed reference states can be interpreted as the discrete action of constraints. First consider the discrete action of the constraint product of interest on a state $c = c_0$ which is itself a reference state so that $\alpha = \mathbf{1}$. Focus on some contracted child-parent pair $c_{0[i,I,\beta,\delta]_m^{m-1,m}}$ and the corresponding "primary" child-parent pair $c_{0[i,I,\beta,\delta_0]_m^{m-1,m}}$. Recall from Secs. VI. D. 1 and VI. D. 1. b that $c_{0[i,I,\beta,\delta]_m^{m-1}}, c_{0[i,I,\beta,\delta]_m}$ are the images of $c_{0[i,I,\beta,\delta_0]_m^{m-1}}, c_{0[i,I,\beta,\delta_0]_m}$ by appropriate composite contraction diffeomorphisms of the form (6.41). We refer to these diffeomorphisms here, respectively, as ϕ_{m-1}, ϕ_m. Also recall that the actions of ϕ_m, ϕ_{m-1} are related by that of a single contraction diffeomorphism constructed in Sec. VI. C, which we denote by ϕ_1 so that $\phi_m = \phi_1 \phi_{m-1}$. We now show that the child $c_{0[i,I,\beta,\delta]_m^{m-1}}$ can be viewed as being generated by the discrete action of a constraint on the parent $c_{0[i,I,\beta,\delta]_m}$. In our arguments below we shall initially refrain from creating and placing any C^1, C^2 kinks around the vertex of the child. We shall also set $\phi_\alpha = \mathbf{1}$ in the definition of the contraction diffeomorphism [see (6.12)] which we have denoted here by ϕ_1. We shall return to a discussion of the placement of these kinks and justify this initial "switching off" of ϕ_α after we complete our argumentation below.

First, let the parental (nondegenerate) vertex be GR without any need for an intervention. That the contracted child is created by the discrete action of a constraint on its parent in this case, follows immediately from Secs. VI. D. 1. a and VI. D. 1. b and our discussion of contraction coordinates above. To see this we note the following using obvious notation:

(i) $c_{0[i,I,\beta,\delta]_m^{m-1}} = \phi_{m-1}(c_{0[i,I,\beta,\delta_0]_m^{m-1}})$ is the parent of interest.

(ii) $\phi_{m-1}(c_{0[i,I,\beta,\delta_0]_m})$ is the deformed child generated by the discrete action of the appropriate constraint (Hamiltonian, if $\beta_{j_m} \neq 0$ and electric diffeomorphism if $\beta_{j_m} = 0$) at parameter δ_{0j_m} with this parameter measured by the contraction coordinates the parent in (i).

(iii) $c_{0[i,I,\beta,\delta]_m} = \phi_m(c_{0[i,I,\beta,\delta_0]_m}) = \phi_1(\phi_{m-1}(c_{0[i,I,\beta,\delta_0]_m}))$ is the contracted child obtained by contracting the child in (ii) from parameter value δ_{0j_m} down to δ_{j_m} where these parameters are measured by the contraction coordinates of the parent in (i).

It is important to note, from the construction of the contraction diffeomorphism in Sec. VI. C that ϕ_1 preserves the parent in (i) so that the process in (iii) can be viewed as a contraction of the child while preserving the identity of the parent.

Next consider the case where an intervention is required so that the parental vertex is either CGR or GR of the type conforming to Case 2 in Sec. V. B. 1. As seen in Sec. V, the transition from primary parent to primary child now requires an intervention. Let the intervention holonomy h_{l_0} be based on the loop l_0. Then this transition unfolds through the following steps:

(a) holonomy intervention by h_{l_0} on the primary parent $c_{0[i,I,\beta,\delta_0]_m^{m-1}}$ yielding the parental state $c^{(l_0)}_{0[i,I,\beta,\delta_0]_m^{m-1}}$ with a GR vertex,

(b) generation of the child $c^{(l_0)}_{0[i,I,\beta,\delta_0]_m}$,

(c) multiplication by $h^{-1}_{l_0}$.

Recall that we want to show that the parent $c_{0[i,I,\beta,\delta]_m^{m-1}}$ and child $c_{0[i,I,\beta,\delta]_m}$ are related through the discrete action of a constraint. Such an action requires a holonomy intervention. Since the parent is the image of the primary parent by ϕ_{m-1}, it follows that the loop l labeling such an intervention must be the image of l_0 by the same diffeomorphism so that $l \coloneqq \phi_{m-1}(l_0)$. We may then view the child $c_{0[i,I,\beta,\delta]_m}$ as being generated from its parents through the following steps, analogous to (a)–(c) above:

(a') A holonomy intervention by h_l on the contracted parent $c_{0[i,I,\beta,\delta]_m^{m-1}} = \phi_{m-1}(c_{0[i,I,\beta,\delta_0]_m^{m-1}})$ with $l = \phi_{m-1}(l_0)$. This intervention yields the state $\phi_{m-1}(c^{(l_0)}_{0[i,I,\beta,\delta_0]_m^{m-1}})$ with a GR vertex.

(b'1) Generation of the δ_0 child $\phi_{m-1}(c^{(l_0)}_{0[i,I,\beta,\delta_0]_m})$ from its parent $\phi_{m-1}(c^{(l_0)}_{0[i,I,\beta,\delta_0]_m^{m-1}})$, where δ_0 is measured by the contraction coordinates $\phi^*_{m-1}\{x_0\}$ associated with the contracted parent $c_{0[i,I,\beta,\delta]_m^{m-1}}$.

(b'2) Contraction of this δ_0 child by the action of ϕ_1 resulting in the δ child $\phi_m(c^{(l_0)}_{0[i,I,\beta,\delta_0]_m})$.

(c') multiplication by the inverse holonomy h_l^{-1}.

Clearly, the steps (a')–(c') above can be viewed as corresponding to the discrete action of a constraint provided the contraction diffeomorphism ϕ_1 preserves the parental state $\phi_{m-1}(c^{(l_0)}_{0[i,I,\beta,\delta_0]_m^{m-1}})$ while contracting its child. If this is so and if ϕ_1 preserves l, it is easy to check that the steps (a')–(c') yield the child $c_{0[i,I,\beta,\delta]_m}$.[29] Both of these are ensured if we choose the "cylinder" supports of the various diffeomorphisms constructed in Sec. VI. C, whose product (6.12) yields the single contraction diffeomorphism denoted here by ϕ_1, to be small enough that ϕ_1 preserves the graph underlying $c_{0[i,I,\beta,\delta]_m^{m-1}}$ as well as the intervention loop l. It is straightforward to check that these supports can be so chosen and we so choose them.

It only remains to discuss the placement of C^1, C^2 kinks. Any such kink, if present in the child, is either on an edge between the parental and displaced vertex or "beyond" the displaced vertex. If there is a segment beyond the displaced vertex this segment must belong to the parental graph, and if an intervention is required, also belong to the straight line part of the intervention loop. The contraction diffeomorphism (specifically the diffeomorphism ψ_α of Sec. VI. C) preserves this part of the parental edge and the intervention loop. The straight line joining the parental vertex to the displaced vertex must either be present or absent in its entirety in each of the following elements: the parental graph, the straight line part of the intervening loop, the deformed graph prior to the kink placement. In each case the contraction diffeomorphism (specifically ϕ_α of Sec. VI. C) preserves this straight line. Hence we may, as above, first consider the deformations without kink placements (in which case ϕ_α behaves as if it were the identity) and then at the end place these kinks so as to mimic the result of imaging the primary child by ϕ_m. Since ϕ_1 also contracts the kink sizes, these kinks can be thought of as

[29] To see this note that $c_{0[i,I,\beta,\delta]_m} = \phi_m(c_{0[i,I,\beta,\delta_0]_m}) = \phi_m(h^{-1}_{l_0} c^{(l_0)}_{0[i,I,\beta,\delta_0]_m}) = h^{-1}_{\phi_1\phi_{m-1}(l_0)}(\phi_1\phi_{m-1}(c^{(l_0)}_{0[i,I,\beta,\delta_0]_m})) = h^{-1}_{\phi_1(l)} \times (\phi_1\phi_{m-1}(c^{(l_0)}_{0[i,I,\beta,\delta_0]_m}))$.

being placed by an appropriate holonomy which is an adequate approximant to identity to leading order in the *contracted* parameter value as measured by the parental contraction coordinates. This is why we had to demand and implement property (iii)(a) of the contraction diffeomorphism in Sec. VI. C. In this manner, the procedure of constructing a contracted child from its parent can be thought of as being implemented by a discrete constraint action. Finally, in the case that $\alpha \neq \mathbf{1}$ it is easy to see, by taking the α image of the contracted child-parent pair, that the child can be thought of being generated by the action of a discrete constraint action on the parent where the parental coordinates are the α image of the contracted parent as in Sec. VI. D. 1. a.

Note.—We have slightly abused our notation for multiple deformations. The notation was set up so that each individual deformation was defined as in Appendix B and Sec. V. A. These individual deformations do place the displaced vertex at the correct location. However the C^0 kinks are positioned differently [they lie at distances of order of the deformation paramter δ rather than at the positions detailed in (iii), Sec. VI. C]. In this section (i.e. Sec. VI) the contraction diffeomorphisms have been used not only to contract the displaced vertex to its desired poistion but to also place the kinks at their desired positions [see (iii) of Sec. VI. C] as well as to "stiffen" the cone angle [see (6.17)]. Indeed such positioning and stiffening is more in line with the picture developed in P1, P2 of the deformation map $\varphi(q_I^i\vec{\hat{e}}_I, \delta)$ [see (3.5)] as a singular diffeomorphism which pulls the edges along the Ith one. In Sec. VIII we shall augment the notation used in this section so as to include the specification of the preferred kink locations; the stiffening will be implicitly assumed without recourse to explicit notation.

E. Action of constraint operators on state not in the ket set

Since the ket set is closed under diffeomorphisms, any ket not in this set must have some diffeomorphism invariant characteristic which distinguishes it from elements of the ket set. We would like to define the action of constraint operators on such a ket so that this diffeomorphism invariant characteristic is preserved. However, since we do not explicitly know what this characteristic is given such a ket, we use a blunt and inelegant definition of the deformations generated by the constraints on such a ket. This definition deforms kets in such a way that the deformed offspring kets are in the complement of the ket set if the parent kets being deformed are also in the complement. This can be done, for example, by defining the deformation map (see last line of Sec. VI. C for a definition of the deformation map) to nontrivially knot one (or more) of the deformed edges at the offspring vertex. Another possibility would be to define the deformations to be "off edge" as in P1, P2.

In the remainder of the paper we assume that *some* such definition has been employed so as to ensure that the discrete action of constraint operators preserves the complement of the ket set.

VII. THE ANOMALY FREE DOMAIN OF STATES

A state in the anomaly free domain resides in the algebraic dual space to the space of finite linear combinations of charge nets. Such a state can be represented as a kinematically non-normalizable sum over charge net bras. The anomaly free domain will be constructed as the linear span of basis states. To each basis state we associate a set of bras such that the basis state is a sum over bras in this associated "bra set." The set of kets corresponding to this bra set is a subset of the ket set we constructed in Sec. VI. B. We discuss our choice of bra set in Sec. VII. A and we construct basis states in Sec. VII. B. In what follows we often denote the bra $\langle c|$ by c to avoid notational clutter.

A. Bra set

Let c_{P0} be the bra correspondent of some primordial reference ket in the ket set of Sec. VI. B. Consider the set of N edges at the nondegenerate vertex p_0 of c_{P0} and the (unordered) set of N $U(1)^3$ charge labels, one for each of these edges. Next consider the set $S_{\mathrm{primordial},P0}$ of all primordial reference states each of whose elements satisfy either of the restrictions below on their edge charge sets at p_0:

(i) the unordered set of edge charge labels at the vertex p_0 of each such state is identical to the corresponding set for c_{P0}.

(ii) there exists some flip $[i, \beta]_m$ such that the set of (unordered) edge charge labels at the vertex p_0 of each such state is the flipped image of the corresponding set for c_{P0} by this flip [here we have used the notation of (6.5) for charge flips].

Recall that any primordial is subject to the restrictions detailed in Sec. VI. B. Hence only those states which have the prescribed unordered edge charge sets of type (i) or (ii) *and* satisfy these restrictions are elements of $S_{\mathrm{primordial},P0}$. Next, fix an element $c_{\bar{P}0}$ of $S_{\mathrm{primordial},P0}$ and consider the set $S_{\mathrm{prim},\bar{P}0}$ of all its primaries (i.e. all its children and itself) together with all their diffeomorphic images. Consider the set B_{P0} of all elements of $S_{\mathrm{prim},\bar{P}0}$ as we vary $c_{\bar{P}0}$ over $S_{\mathrm{primordial},P0}$. This set constitutes our bra set.

The set has the following property. Let $c \in B_{P0}$ and let c_0 be its reference state (we use the bra correspondents of the reference kets of Sec. VI. B to define reference bras). Let $c_{P'0}$ be a primordial reference state such that c_0 is a multiple deformation of $c_{P'0}$. Then the property of B_{P0} alluded to is that $c_{P'0} \in S_{\mathrm{primordial},P0}$.

To see this, note the following. Since c_0 is diffeomorphically related to c, we have that c_0 is also in B_{P0}. Recall from Sec. VI. B that c_0 must be a primary because it is a reference state. Hence it must be obtained as some multiple deformation of some reference primordial in the ket set. From Appendix C, the unordered *net* edge charge set at the nondegenerate vertex of c_0 is the same as some multiply flipped image of the unordered edge charge set of any reference primordial ancestor whose multiple deformations give rise to c_0. By construction [see (i) and (ii) above], any such ancestor is in B_{P0}.

To appreciate the kind of situations covered by the proof let us suppose that we drop (i) and (ii) and choose the bra set to be composed of all diffeomorphic images of the primary family (including c_{P0}) emanating from c_{P0}. As before the reference state c_0 for any element c of this bra set must be a primary and hence obtained by some multiple deformation of some reference primordial in the ket set. Let this primordial be $c_{P'0}$. Consider the case where c_0 is obtained as a single electric diffeomorphism type deformation of $c_{P'0}$. Next note that by construction it must be the case that c_0 is diffeomorphic to a primary $c_{\rm prim}$ emanating from c_{P0}. Note also that from the kink structure of $c_{\rm prim}$, it must be the case that $c_{\rm prim}$ is also a single electric diffeomorphism deformation of c_{P0}.[30] If we could use this fact that c_0 is diffeomorphic to $c_{\rm prim}$ to conclude that $c_{P'0}$ is diffeomorphic to c_{P0} then, from the definition of (primordial) reference states, we could conclude that $c_{P'0}$ and c_{P0} are identical. Note however that because the deformation is of an electric diffeomorphism type, the nondegenerate vertex of c_{P0} as well as the vertex structure in a small vicinity of this vertex is absent from the graph underlying $c_{\rm prim}$ and, similarly, the nondegenerate vertex of $c_{P'0}$ as well as the vertex structure in a small vicinity of this vertex is absent from the graph underlying c_0. Hence we cannot directly conclude that the diffeomorphism which maps c_0 to $c_{\rm prim}$ necessarily maps $c_{P'0}$ to c_{P0}.[31]

In the context above, the property $c_{P'0} \in S_{{\rm primordial},P0}$ is crucial for the well defined-ness of the dual action of an electric diffeomorphism operator on anomaly free states associated with B_{P0}. As shall be apparent in Secs. X and XI, for this action to be well defined, it must be the case that the discrete action of this operator on any charge net c is such that either the (bra correspondents of) c and all its single electric diffeomorphism deformations are absent in B_{P0} or c and all its deformations are *all* present in B_{P0}. If in the above example involving an electric diffeomorphism, we had that $c_{P'0}$ above was not in B_{P0}, the fact that its first electric diffeomorphism deformation was in B_{P0} would then lead to an ill-defined-ness of the dual action of an electric diffeomorphism operator on a typical anomaly free state associated with B_{P0}.

More in general the restrictions (i), (ii) of the edge charge set of the primordials in B_{P0} can be used to conclude that all possible ancestors of any $c \in B_{P0}$ (by a possible ancestor we mean state on which multiple constraint actions lead to the creation of c) and all possible children of c (by which we mean all multiple deformations of c generated by multiple constraint actions) are in B_{P0} (here we freely switch between ket and bra correspondents of the state c). To see this, note that by construction (see Sec. VI) all possible ancestors and offspring of c are in the ket set. Recall again that all reference states must be primaries and consider for $c \in B_{P0}$ any ancestor c_a of c and its reference state c_{a0}. By definition of ancestry it must be true that deformations of this ancestor reference state (with respect to $\{x_0\}$) yield a state diffeomorphic to the reference state, c_0, for c. It follows from Appendix C that any reference primordial for the reference state c_{a0} of the ancestor must have an edge charge set related to that of any reference primordial ancestor of c_0 by (i) or (ii). Since any reference primordial for c_0 is in B_{P0} so must any reference primordial state for the ancestor reference state c_{a0}. It follows from the construction of B_{P0} that the ancestor reference state and, hence the ancestor, must also be in B_{P0}. Finally, note that by construction, if c is in B_{P0} then all its offspring are also in B_{P0}. This follows directly from the fact that by definition any such offspring is diffeomorphic to a primary which emanates from the same primordial reference state as one which yields the reference state c_0 for c. The fact that all possible ancestors and offspring of any element of c are necessarily in B_{P0} ensures the well defined-ness of the dual actions, on anomaly free states associated with B_{P0}, of those constraints which are necessary for a demonstration of anomaly free commutators.[32]

B. Basis states

Let f be a density weight $-\frac{1}{3}$ semianalytic function on the Cauchy slice Σ and let h_{ab} be a semianalytic Riemannian metric such that h_{ab} has no conformal symmetries. Let g be a function on $\Sigma^{m(N-1)}$ of the type specified in Appendix G. As detailed in Appendix G. 1, this function is determined by the network of geodesic distances, as defined by h_{ab}, between every pair of its arguments. Thus g is determined once h_{ab} is specified.

[30] $c_{\rm prim}$ must have $N-1$ C^0 kinks; any state with $m(N-1)$ such kinks is an m deformation of a primordial. Since $c_{\rm prim}$ has only a single vertex of valence N and none of valence $N+1$, this deformation is of electric diffeomorphism type.

[31] We do not rule out that it may be possible to do so using a more involved argument; since we have not constructed any such argument, we prefer to cover the adverse consequences, sketched below, of the possible absence of such an argument through our construction of B_{P0} in the first paragraph.

[32] Of course we could have chosen the (bra correspondent) of the entire ket set as our bra set as it would obviously satisfy the required property. However this would unnecessarily cut down on the size of the space of anomaly free states.

A basis state Ψ_{f,h_{ab},P_0} associated with the bra set B_{P0} is constructed as a sum over all the elements of B_{P0} where the coefficient of each such element $\bar{c}$ in this sum is determined by f, h_{ab} as follows. Let the reference state for $\bar{c}$ be $\bar{c}_0$ and let the reference diffeomorphism which maps $\bar{c}_0$ to $\bar{c}$ be $\bar{\alpha}$ so that

$$|\bar{c}\rangle = \hat{U}(\bar{\alpha})|\bar{c}_0\rangle. \tag{7.1}$$

Since the coordinate patch $\{x_0\}$ is associated with the nondegenerate vertex of $\bar{c}_0$, we define the coordinate patch associated with the nondegenerate vertex of $\bar{c}$ to be

$$\{x_{\bar{\alpha}}\} \coloneqq \bar{\alpha}^*\{x_0\}. \tag{7.2}$$

We shall refer to this coordinate patch as a *reference coordinate patch* to distinguish it from the *contraction coordinate patches* defined at the end of Sec. VI. D. 2.

Next, note that $\bar{c}_0$ is a primary and hence is either identical to, or obtained by, some multiple deformation of some reference primordial in B_{P0}. While it is possible, in principle, that this reference primordial is not unique,[33] the number m of deformations of any primordial ancestor which yields $\bar{c}_0$ *is* unique. To see this, note that from the nature of the deformations detailed in Secs. III, IV and V, each single deformation generates a set of $N-1$ C^0 kinks. Hence the number of C^0 kinks in $\bar{c}_0$, and hence $\bar{c}$, must be $m(N-1)$ for some whole number m which corresponds to the number of deformations of an appropriate reference primordial which yields $\bar{c}_0$ (if $m=0$, $\bar{c}$ is primordial).

Next, with respect to the reference coordinates (7.2) let us denote the (outward) unit edge tangents at the nondegenerate vertex v_m of $\bar{c}$ by $\{\hat{e}^a_{I_m}, I_m = 1,..,N\}$ where "unit" is with respect to the (reference) coordinate norm. As discussed earlier if v_m is CGR we shall count the upper and lower conducting edges as a single edge, where the notion of upper and lower is fixed from the kink structure in the vicinity of v_m as outlined in Sec. V. For the conducting edge we may choose the (outward pointing) upper conducting edge tangent.[34] Define

$$H_{I_m} \coloneqq \|\vec{\hat{e}}_{I_m}\| \coloneqq \sqrt{h_{ab}(v_m)\hat{e}^a_{I_m}\hat{e}^b_{I_m}}, \tag{7.3}$$

$$h_{I_m} = \sum_{J_m,K_m\neq I_m} \frac{\|\vec{\hat{e}}_{J_m}\|}{\|\vec{\hat{e}}_{K_m}\|}, \tag{7.4}$$

and let $f(v_m)$ be the evaluation of the density weighted function f at the vertex v_m in the reference coordinates (i.e. in the coordinate system $\{x_{\bar{\alpha}}\}$). Next, consider the $m(N-1)C^0$ kinks on $\bar{c}$. We evaluate $g_{\bar{c}}$ on the arguments corresponding to these kinks where we have defined $g_{\bar{c}}$ in Appendix G. Then the coefficient multiplying $\bar{c}$ in the sum over state representation of Ψ_{f,h_{ab},P_0} is

$$(\Psi_{f,h_{ab},P_0}|\bar{c}\rangle = g_{\bar{c}}\left(\sum_{I_m} h_{I_m}H_{I_m}\right)f(v_m) \tag{7.5}$$

where for any element of the algebraic dual Ψ, we write its action on a charge net state $|b\rangle$ as $(\Psi|b\rangle$.[35] The formal sum over states representation of the state $(\Psi_{f,h_{ab},P_0}|$ is

$$(\Psi_{f,h_{ab},P_0}| = \sum_{\langle\bar{c}|\in B_{P0}} \left(g_{\bar{c}}\left(\sum_{I_m} h_{I_m}H_{I_m}\right)f(v_m)\right)\langle\bar{c}|, \tag{7.6}$$

where we have implicitly used Eqs. (7.1) and (7.2) to evaluate the quantities $g_{\bar{c}}, h_{I_m}, H_{I_m}, f(v_m)$ on the right-hand side.

Finally we note the following *key property* of the right-hand side of (7.5):

Invariance property.—Let the coordinates appropriate to the evaluation of the right-hand side of (7.5) be defined through (7.2) i.e. let the right-hand side of (7.5) be evaluated with respect to the reference coordinates for $\bar{c}$. Consider a second coordinate system $\{y\}$ around the nondegenerate vertex v_m of $\bar{c}$ such that the Jacobian matrix $J(\{x_{\bar{\alpha}}\},\{y\})^\mu_\nu \coloneqq \frac{\partial x^\mu_{\bar{\alpha}}}{\partial y^\nu}$ be such that its evaluation at v_m is a constant times a rotation i.e.

$$J(\{x_{\bar{\alpha}}\},\{y\})^\mu_\nu)|_{v_m} = CR^\mu_\nu \tag{7.7}$$

for some $C>0$ and some $SO(3)$ matrix R. Then the evaluation of the right-hand side is the same whether the coordinates used are $\{x_{\bar{\alpha}}\}$ or $\{y\}$.

This is easily verified by inspection. It is straightforward to check that, in obvious notation,

$$f(v_m)|_{\{y\}} = C^{-1}f(v_m)|_{\{x_{\bar{\alpha}}\}}, \quad H_{I_m}|_{\{y\}} = CH_{I_m}|_{\{x_{\bar{\alpha}}\}}. \tag{7.8}$$

The first equality follows from the density $-1/3$ nature of f and the second from the fact the coordinate vector lengths are invariant under rotations of the coordinates and scale inversely with scaling of the coordinates. Further since h_{I_m} involves ratios of norms of coordinate tangents, it is invariant under such a transformation. Finally, from its definition in Appendix G, $g_{\bar{c}}$ is coordinate independent.

[33]While it may indeed be unique, we have not investigated the matter and hence must allow for this possibility.

[34]Since (7.3) depends on the edge tangent norm, the choice of these orientations does not matter; we provide the above choices for concreteness.

[35]Recall an element of the algebraic dual is a linear map from the finite span of charge net states to the complex numbers. We shall use the notation Ψ or $(\Psi|$ for such elements depending on our convenience.

VIII. CONTINUUM LIMIT: FINAL FORM AND CONTRACTION BEHAVIOR ON ANOMALY FREE DOMAIN

In Sec. VI we defined the contraction of deformations of states from larger discretization parameter to smaller ones using contraction diffeomorphisms. The contraction moves the nondegenerate vertex from a larger coordinate distance from its immediate parent vertex (in appropriate coordinates as explained in Sec. VI. D. 2) to a smaller one. However the contraction also has a "fine structure" involving the positioning of the C^0 kinks generated by the transformation which produces the state in question from its parent. Each choice of this fine structure yields an acceptable discrete approximant for the constraint action. In Sec. VIII. A we democratically sum over these fine structures and display our final choice of discrete approximant for the action of a single constraint in Eqs. (8.5) and (8.6) which replace Eqs. (4.5) and (4.6). Constraint product actions can then easily be defined based on the machinery developed in Sec. VI. In Sec. VIII. B we display the dual action of the constraint product on basis states in the anomaly free domain and define its continuum limit. The contraction of deformations of kets is then transferred to that of the bras in the bra set of Sec. VII. A and thence to the coefficients which multiply these bras (see Sec. VII. B). The evaluation of the continuum limit then depends on the contraction behavior of these coefficients. We detail this behaviour in Sec. VIII. C. A complete specification of the contraction behaviour requires a specification of the Q factors in the definition of the contraction diffeomorphism of Sec. VI. C. We discuss this in Sec. VIII. D. With this, we are ready to compute the continuum limit action of constraint products in Secs. IX and X.

A. Final form of discrete constraint action

The discrete action of the constraint product (6.23) is based on the single constraint actions (4.5) and (4.6) at sufficiently small parameter value ϵ so that the single constraint actions are

$$\hat{C}[N]_\epsilon c = \beta\frac{\hbar}{2\mathrm{i}}\frac{3}{4\pi}N(x(v))\nu_v^{-2/3}\sum_I\sum_i\frac{c_{(i,I,\beta,\epsilon)}-c}{\epsilon}, \tag{8.1}$$

$$\hat{D}_\epsilon[\vec{N}_i]c = \frac{\hbar}{\mathrm{i}}\frac{3}{4\pi}N(x(v))\nu_v^{-2/3}\sum_I\frac{1}{\epsilon}(c_{(i,I,\beta=0,\epsilon)}-c). \tag{8.2}$$

The deformed kets in (6.23) arise as a result of repeated applications of (8.1) and (8.2). These kets are defined through the contraction of their images at larger discretization parameter values as explained in Sec. VI. The contraction procedure involves a contraction of kink vertices to precisely defined locations. These locations are specified by a choice of two edges in the child, $c_{(i,I,\beta,\epsilon)}$, each of which is distinct from the edge along which the child vertex is displaced [this is reflected in the dependence of the contraction diffeomorphism on the 'hatted' indices in, for example, Eq. (6.42)]. As a result, a deformed ket $c_{(i,I,\beta,\epsilon)}$ is characterized not only by the labels (i,I,β,ϵ) which describe the main features of the deformation such as the location of the displaced vertex but also the labels $\hat{J}_1,\hat{K}_1$ which describe the fine structure of the location of the kinks.[36] Hence a more complete notation replaces the label set (i,I,β,ϵ) by $(i,I,\hat{J}_1,\hat{K}_1,\beta,\epsilon)$. Of course a complete set of labels pertinent to multiply deformed kets is, for example, that in Eq. (6.42). However, to display the single constraint actions in a more complete way than in (8.1) and (8.2) it suffices to use the abbreviated set of symbols $(i,I,\hat{J}_1,\hat{K}_1,\beta,\epsilon)$ so that Eqs. (8.1) and (8.2) take the form

$$\hat{C}[N]_\epsilon c = \beta\frac{\hbar}{2\mathrm{i}}\frac{3}{4\pi}N(x(v))\nu_v^{-2/3}\sum_I\sum_i\frac{c_{(i,I,\hat{J}_1,\hat{K}_1,\beta,\epsilon)}-c}{\epsilon}, \tag{8.3}$$

with $\beta=+1$ or $\beta=-1$, and

$$\hat{D}_\epsilon[\vec{N}_i]c = \frac{\hbar}{\mathrm{i}}\frac{3}{4\pi}N(x(v))\nu_v^{-2/3}\sum_I\frac{1}{\epsilon}(c_{(i,I,\hat{J}_1,\hat{K}_1,\beta=0,\epsilon))}-c). \tag{8.4}$$

Since each choice of hatted indices provides an acceptable discrete action which is derivable from the heuristics of Sec. II, summing over these choices also yields an acceptable discrete action. Accordingly we may repeat the considerations of Sec. VI based on the following form of single constraint actions:

$$\hat{C}[N]_\epsilon c = \beta\frac{\hbar}{2\mathrm{i}}\frac{3}{4\pi}N(x(v))\nu_v^{-2/3}\sum_I\sum_{\hat{J}_1,\hat{K}_1}\frac{1}{(N-1)(N-2)} \times\sum_i\frac{c_{(i,I,\hat{J}_1,\hat{K}_1,\beta,\epsilon)}-c}{\epsilon}, \tag{8.5}$$

with $\beta=+1$ or $\beta=-1$, and

$$\hat{D}_\epsilon[\vec{N}_i]c = \frac{\hbar}{\mathrm{i}}\frac{3}{4\pi}N(x(v))\nu_v^{-2/3}\sum_I\sum_{\hat{J}_1,\hat{K}_1}\frac{1}{(N-1)(N-2)} \times\frac{1}{\epsilon}(c_{(i,I,\hat{J}_1,\hat{K}_1,\beta=0,\epsilon)}-c). \tag{8.6}$$

The $N(N-1)$ factors stem from the choice of 2 of the $N-1$ edges which bear the C^0 kinks created in the

[36] The subscript 1 refers to the fact that the hatted indices number nonconducting edges of the child $c_{(i,I,\beta=0,\epsilon)}$ which, here, is obtained by a *single* deformation its parent c; see the discussion after Eq. (6.27) for the definition of hatted indices.

deformation of the parent. Equations (8.5) and (8.6) are the final form of the single constraint actions which we shall use. Once again, similar to Sec. VI. D we can expand the action of the constraint operator product $[\prod_{\mathrm{i}=1}^{n}\hat{O}_{\mathrm{i},\epsilon_{\mathrm{i}}}(N_{\mathrm{i}})]$ on the state c through repeated applications of (8.5) and (8.6) to obtain an expression of the form (6.23) except that the label set must now, in obvious notation, be embellished by the specification of the hatted indices so that we have

$$\left(\prod_{\mathrm{i}=1}^{n}\hat{O}_{\mathrm{i},\epsilon_{\mathrm{i}}}(N_{\mathrm{i}})\right)|c\rangle = \left(\prod_{i=1}^{n}\epsilon_{\mathrm{i}}\right)^{-1}\sum_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m,m=1,..,n} C_{[i,I,\beta,\hat{J},\hat{K},\epsilon]_m}|c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}\rangle + C_0|c\rangle \tag{8.7}$$

where we have abbreviated

$$[i,I,\hat{J},\hat{K},\beta,\epsilon]_m \equiv [(i_{m-1},I_{m-1},\beta_{\mathrm{j_m}},\hat{J}_m,\hat{K}_m,\epsilon_{\mathrm{j_m}}),\ldots,(i,I,\hat{J}_1,\hat{K}_1,\beta_{\mathrm{j_1}},\epsilon_{\mathrm{j_1}})]. \tag{8.8}$$

Each $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ is defined exactly as in Sec. VI. C. Thus, each $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ is the α image (where as before α maps the reference ket c_0 to c) of the state $c_{0[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ and each $c_{0[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ is obtained as the image of the state $c_{0[i,I,\beta,\delta_0]_m}$ through Eq. (6.42), the state $c_{0[i,I,\beta,\delta_0]_m}$ being defined by repeated conical deformations with respect to the coordinate system $\{x_0\}$, each of the type described in Appendix B and Sec. V. Note that the deformations of Appendix B and Sec. V do not have a further fine structure labeled by hatted indices so that $c_{0[i,I,\beta,\delta_0]_m}$ is defined as the result of the deformation $[i,I,\beta,\delta_0]_m$ of Eq. (6.25) applied to the reference state c_0.

The discussion of the coordinates underlying the deformed states $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ is exactly that of Secs. VI. D. 1. a and VI. D. 2. The coefficient $C_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ in (8.7) acquires a coordinate dependence solely from the dependence of this coefficient on the lapse functions. Each lapse function is evaluated at some nondegenerate vertex of one of the states which define the lineage of $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$. Since the considerations of Sec. VI. D (see especially Secs. VI. D. 1. c and VI. D. 2) have provided a unique choice of coordinate patches for every such vertex, every coefficient on the right-hand side of (8.7) can be evaluated.

B. Dual action on anomaly free domain

The dual action of $(\prod_{\mathrm{i}=1}^{n}\hat{O}_{\mathrm{i},\epsilon_{\mathrm{i}}}(N_{\mathrm{i}}))$ on a basis state Ψ_{f,h_{ab},P_0} is defined as

$$(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{n}\hat{O}_{\mathrm{i},\epsilon_{\mathrm{i}}}(N_{\mathrm{i}})\right)|c\rangle. \tag{8.9}$$

The action of the operator product $(\prod_{\mathrm{i}=1}^{n}\hat{O}_{\mathrm{i}}(N_{\mathrm{i}}))$ is then defined by the continuum limit:

$$\left(\lim_{\epsilon_n\to 0}\left(\lim_{\epsilon_{n-1}\to 0}\cdots\left(\lim_{\epsilon_1\to 0}(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{n}\hat{O}_{\mathrm{i},\epsilon_{\mathrm{i}}}(N_{\mathrm{i}})\right)|c\rangle\right)..\right)\right). \tag{8.10}$$

The continuum limit action exists if Eq. (8.10) holds for all charge net states $|c\rangle$. From Sec. VI. E, this limit vanishes for all c which are not in the ket set. Indeed, it follows from the discussion in Sec. VII. A that this limit also vanishes if (the bra correspondent of) c is not in the bra set B_{P_0} associated with the anomaly free state Ψ_{f,h_{ab},P_0}. Hence we need only analyse the continuum limit for states c (whose bra correspondents) are in the bra set B_{P_0}. For such states we expand out the discrete operator product action as in (8.7), so that we have

$$\begin{aligned}(\Psi_{f,h_{ab},P_0}|(\prod_{\mathrm{i}=1}^{n}\hat{O}_{\mathrm{i}}(N_{\mathrm{i}}))c\rangle &\lim_{\epsilon_n\to 0}\ldots(\lim_{\epsilon_1\to 0}(\Psi_{f,h_{ab},P_0}|(\prod_{\mathrm{i}=1}^{n}\hat{O}_{\mathrm{i},\epsilon_{\mathrm{i}}}(N_{\mathrm{i}}))|c\rangle)..)\\ = \lim_{\epsilon_n\to 0}(\lim_{\epsilon_{n-1}\to 0}\ldots(\lim_{\epsilon_1\to 0}(\prod_{i=1}^{n}\epsilon_{\mathrm{i}})^{-1}&\left(\sum_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m,m=1,..,n} C_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}(\Psi_{f,h_{ab},P_0}|c_{[i,I,\hat{J},\hat{K}\beta,\epsilon]_m}\rangle + C_0(\Psi_{f,h_{ab},P_0}|c\rangle))..\right).\end{aligned} \tag{8.11}$$

Clearly, in order to compute this limit we need to know the limiting behavior of the coefficients $C_{[i,I,\hat{J},\hat{K}\beta,\epsilon]_m}$ and of the "amplitudes" $(\Psi_{f,h_{ab},P_0}|c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}\rangle$. The limiting behavior of the coefficients stems from the dependence of the coefficients on the coordinate dependent lapse function evaluations, these coordinates being dependent on the ϵ parameters. The limiting behavior of the amplitudes can be computed from that of the expression (7.5) and the limiting behavior of the functions f, g and the (reference) coordinate dependent unit edge tangents. In the next section we compute this limiting "contraction" behavior of the amplitudes.

C. Contraction behavior of amplitudes

In this section we are interested in the behavior of

$$(\Psi_{f,h_{ab},P_0}|c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}\rangle \tag{8.12}$$

for small $\epsilon_{\mathrm{j_m}}$. We shall restrict attention to the particular deformation sequence $[i, I, \hat{J}, \hat{K}, \beta, \epsilon]_m$ in this section. As in Secs. VI, VIII. A the deformed state $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ will be assumed to have been generated by the discrete action of the operator product $(\prod_{\mathrm{i}=1}^n \hat{O}_{\mathrm{i},\epsilon_{\mathrm{i}}}(N_{\mathrm{i}}))$ on the state c [see (8.7)]. Hence the sizes of the contraction parameters are defined with respect to the *contraction coordinates* of Sec. VI (see the end of Sec. VI. D. 2). More in detail, the contraction coordinates which specify the magnitude of $\epsilon_{\mathrm{j_m}}$ are those associated with the immediate parent $c_{[i,I,\beta,\epsilon,\hat{J},\hat{K}]_m^{m-1}}$ of the state $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$. On the other hand, the amplitude (8.12) is evaluated using the *reference coordinates* associated with $\bar{c} \equiv c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ in (7.5).

Therefore we proceed as follows. First we transit from the reference coordinates associated with $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ to the contraction coordinates associated with this state. It turns out that the evaluation (7.5) is the same irrespective of which one of these coordinate systems we use. This is a *key* result and can be traced back to the definition of deformations developed in Sec. VI. Next, using the fact (6.40) that contraction coordinates for a deformed state and its immediate parent are related by a contraction diffeomorphism, we are able to compute the amplitude (8.12) in terms of the contraction coordinates of the immediate parent. Since the size of the parameter $\epsilon_{\mathrm{j_m}}$ is measured by these coordinates, we are able to evaluate the small $\epsilon_{\mathrm{j_m}}$ behavior of this amplitude.

As noted in Sec. VIII. B, if $c \notin B_{P_0}$ then all amplitudes on the right-hand side of (8.11) vanish. Hence hereon we focus on the nontrivial case $c \in B_{P_0}$ so that $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m} \in B_{P_0}$.

1. Step 1: Transition from reference to contraction coordinates of $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$

Let the reference ket for the state $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ be $(c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m})_0$. Let the reference diffeomorphism be $\alpha_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ so that similar to (6.10) we have

$$|c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}\rangle := \hat{U}(\alpha_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m})|(c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m})_0\rangle, \tag{8.13}$$

so that the associated reference coordinate system around the nondegenerate vertex v_m of $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ is

$$(\alpha_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m})^*\{x_0\}. \tag{8.14}$$

We now turn to the contraction coordinates for $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$. Let the reference ket for c be c_0 and let the reference diffeomorphism which maps c_0 to c be α so that (6.10) holds. Note that we have not restricted c_0 to be a primordial. The state $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ is obtained as the image of the state $c_{0[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ by α. Recall that $c_{0[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ is obtained through the action of a composite contraction diffeomorphism on the state $c_{0[i,I,\beta,\delta_0]_m}$ as in Eq. (6.42).[37] The state $c_{0[i,I,\beta,\delta_0]_m}$ is a primary which is obtained by deforming the reference state c_0 m times, each these deformations being defined with respect to the reference coordinates $\{x_0\}$ and each of these deformations being of the type detailed in Appendix B and Sec. V.[38] It follows from (iv), Sec. VI. D. 1. b that the contraction coordinates around the nondegenerate vertex of $c_{0[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ are obtained as the image of the primary coordinates $\{x_0\}$ around the nondegenerate vertex of $c_{0[i,I,\beta,\delta_0]_m}$ by the composite contraction diffeomorphism of (6.42) defined by (6.41). We denote the contraction coordinates for $c_{0[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ by $\{x_0^{\epsilon_{\mathrm{j}_1}..\epsilon_{\mathrm{j_m}}}\}$.[39] We then have that

$$\{x_0^{\epsilon_{\mathrm{j}_1}..\epsilon_{\mathrm{j_m}}}\} := \left(\Phi^{\epsilon_{\mathrm{j}_1}..\epsilon_{\mathrm{j_m}},(\hat{J}_1,\hat{K}_1),...,(\hat{J}_m,\hat{K}_m)}_{c_{[i,I,\beta,\delta_0]_m},S_{\mathrm{j}_1}}\right)^*\{x_0\} \tag{8.15}$$

and that the contraction coordinates for $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ are

$$\begin{aligned}\{x_\alpha^{\epsilon_{\mathrm{j}_1}..\epsilon_{\mathrm{j_m}}}\} &:= \alpha^*\{x_0^{\epsilon_{\mathrm{j}_1}..\epsilon_{\mathrm{j_m}}}\} \\ &= \alpha^*\left(\Phi^{\epsilon_{\mathrm{j}_1}..\epsilon_{\mathrm{j_m}},(\hat{J}_1,\hat{K}_1),...,(\hat{J}_m,\hat{K}_m)}_{c_{[i,I,\beta,\delta_0]_m},S_{\mathrm{j}_1}}\right)^*\{x_0\}. \end{aligned} \tag{8.16}$$

Our task is to compute the Jacobian between the reference coordinates (8.14) and the contraction coordinates (8.16). This is computed in the Appendix E wherein it is shown that the Jacobian between the two coordinate systems takes the form of a constant times a rotation matrix. From the invariance property of Sec. VII. B it then follows that we can as well evaluate (8.12) using the contraction coordinates (8.16).

[37] In the more complete notation introduced in Sec. VIII. A the left-hand side states in these equations would also have a hatted indice specification.

[38] These deformations do not have the additional specification of hatted indices because the placement of the associated C^0 kinks in Appendix B and Sec. V does not require this additional specification.

[39] This is similar to the notation used in (6.40). Recall that (6.40) was defined for $k < m$. Extending the notation in (6.40) for $k = m$, it can be easily checked that (6.41) together with (6.40) for $k = m - 1$ imply Eq. (6.40) for $k = m$.

2. Step 2: Transition to contraction coordinates of immediate parent

The immediate parent of $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ is $c_{[i,I,\beta,\epsilon,\hat{J},\hat{K}]_m^{m-1}}$. The contraction coordinates for this immediate parent are the α image of those for the state $c_{0[i,I,\beta,\epsilon,\hat{J},\hat{K}]_m^{m-1}}$. Accordingly, taking the α image of (6.40)) with $k = m - 1$, we have that these contraction coordinates are

$$\{x_\alpha^{\epsilon_{j_1}..\epsilon_{j_{m-1}}}\} \coloneqq \alpha^*\{x_0^{\epsilon_{j_1}..\epsilon_{j_{m-1}}}\}.$$

The relationship between the contraction coordinates of the child-parent pair $c_{0[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$, $c_{0[i,I,\beta,\epsilon,\hat{J},\hat{K}]_m^{m-1}}$ can be readily inferred from Eqs. (6.40) (with $k = m - 1$), (8.15), (6.41), (8.16) and (8.17), so that we have that

$$\{x_\alpha^{\epsilon_{j_1}..\epsilon_{j_m}}\} = \alpha^*\{x_0^{\epsilon_{j_1}..\epsilon_{j_m}}\} \tag{8.17}$$

$$= \alpha^*\left(\Phi^{\epsilon_{j_m},\{x_0^{\epsilon_{j_1}..\epsilon_{j_{m-1}}}\},\hat{J}_m,\hat{K}_m}_{c_{0[i,I,\beta,\epsilon]_m^{m-1}},(i_{m-1},I_{m-1},\beta_{j_m},\delta_{0j_m}),S_{j_m}}\right)^*\{x_0^{\epsilon_{j_1}..\epsilon_{j_{m-1}}}\}. \tag{8.18}$$

The above equation simply expresses the (α image of the) fact that the contraction coordinates of any deformed state and its immediate parent are related by the action of a contraction diffeomorphism defined in Sec. VI. C. Indeed the iterative procedure used in Sec. VI. D. 1. b implements this very fact. Next, from the fact that for any diffeomorphism γ and any coordinate systems $\{x\}$, $\{y\}$ we have that

$$\left.\frac{\partial(\gamma^* x)^\mu}{\partial(\gamma^* y)^\nu}\right|_{\gamma(p)} \eqqcolon \left.\frac{\partial x^\mu}{\partial y^\nu}\right|_{(p)}, \tag{8.19}$$

it follows that the Jacobian between the contraction coordinates of offspring and immediate parent is given exactly by that of Eq. (6.16) with the identifications

$$x^\delta \equiv x_\alpha^{\epsilon_{j_1}..\epsilon_{j_m}}, \qquad x \equiv x_\alpha^{\epsilon_{j_1}..\epsilon_{j_{m-1}}}, \qquad \delta \equiv \epsilon_{j_m}. \tag{8.20}$$

Recall from the discussion at the beginning of this subsection as well as from Sec. VI. D. 2 that the contraction parameter ϵ_{j_m} is measured with respect to the parental contraction coordinates $x_\alpha^{\epsilon_{j_1}..\epsilon_{j_{m-1}}}$. Hence the contraction behavior of the amplitude can be inferred from the behavior of the quantities $h_{I_{j_m}}, H_{I_{j_m}}, f, g_{c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}}$ (see Sec. VI) in these parental contraction coordinates. This a straightforward though lengthy exercise and we relegate it to the appendixes. Specifically, we compute the contraction behavior of the first three quantities in Appendix F using the Jacobian in Eq. (6.16) and that of the last quantity in Appendix G. 2.

D. Specification of Q factors

Recall that Q is one of the parameters specifying a contraction diffeomorphism [see (iii), Sec. VI. C]. We had briefly discussed its specification in Step 2 of Sec. VI. D. 1. b. Here we summarize its dependences [see (6.32)] in a bit more detail. Our explicit choices for Q will be displayed in Secs. IX and X wherein the rationale for these choices will be self-evident.

Recall that we are interested in a state which is produced by the action of some specific product of Hamiltonian and electric diffeomorphism constraint operators (6.18) on a state c. This state is produced through some deformation sequence applied to its ancestor c. We are interested in the contraction of this state to its immediate parent in this deformation sequence. Using the notation of Sec. VI. C, let the state be an mth generation offspring $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_m}$ where we have used the augmented notation with the hatted indices as explained in (8.8), and let its parent be $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_m^{m-1}}$ and let the parameter being contracted away be δ_{j_m}. Then Q depends on the *net* edge charges of the child and the parent at their nondegenerate vertices as well as the sequence of operators starting from the first operator to the j_mth one i.e. on the sequence

$$S_{j_m} = \prod_{i=1}^{\delta_{j_m}} \hat{O}_{i,\delta_i}(N_i). \tag{8.21}$$

Since the charges at the vertices of $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_m}$, $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_m^{m-1}}$ are the same as the charges on their (diffeomorphically related) δ_0 counterparts, we express the dependence Q for this contraction in the following equivalent notations:

$$Q(c_{[i,I,\beta,\delta]_m^{m-1,m}}, S_{j_m}) \equiv Q(c_{0[i,I,\beta,\delta_0]_m^{m-1,m}}, S_{j_m})$$
$$\equiv Q(c_{0[i,I,\beta,\delta_0]_m}, c_{0[i,I,\beta,\delta_0]_m^{m-1}}, S_{j_m}). \tag{8.22}$$

The individual constraint operators in the above sequence can be of two types, namely a Hamiltonian constraint or an electric diffeomorphism constraint in the kth $U(1)^3$ direction with $k = 1, 2, 3$. Let us denote these constraint types as h and d_k, $k = 1, 2, 3$ respectively so that the constraint type of an operator $\hat{C}(N)$ is h and that of $\hat{D}(N_k)$ is d_k. We shall say that the constraint type t_i of the operator $\hat{O}_i(N_i)$ [or of its discrete approximant $\hat{O}_{i,\delta_i}(N_i)$] is either h or d_k. It then turns out that the information in S_{j_m} relevant for the specification of Q is the sequence of *constraint types* $(t_1, \ldots t_{j_m})$. We shall redesignate the symbol S_{j_m} to denote the ordered set of constraint types $(t_1, \ldots t_{j_m})$:

$$S_{j_m} \equiv (t_1, \ldots t_{j_m}). \tag{8.23}$$

Henceforth we shall interpret S_{j_m} in (8.22) through (8.23).

Q also depends on the cone angle θ which characterizes the conical deformations of reference states, these deformations being constructed using the primary coordinates $\{x_0\}$. This cone angle is fixed for all deformations of reference states in the bra set which labels an anomaly free state. To avoid notational clutter we will suppress explicit notational reference to the dependence of Q on θ.

IX. ANOMALY FREE PRODUCT OF TWO HAMILTONIAN CONSTRAINTS

In Sec. IX. A we compute the continuum limit action of a product of two Hamiltonian constraints. In Sec. IX. B we compute the commutator between two electric diffeomorphism constraints and thereby demonstrate the anomaly free nature of the commutator between the pair of Hamiltonian constraints whose product is computed in Sec. IX. A. The computations are long but straightforward. We shall only highlight the main steps.

Note.—In the remainder of the main body of the paper, unless mentioned otherwise, all the edge charges considered will be *net* charges where, as in Appendix C we define the net charge as follows:

Definition: Net edge charge.—The *net charge* $q^i_{\mathrm{net}I}$ on a conducting edge e_I at the nondegenerate vertex of a charge net is the *sum* of the *outgoing* upper and lower conducting charges; if the edge e_I is nonconducting we shall define its lower conducting charge to be zero so that the net charge $q^i_{\mathrm{net}I}$ on such an edge is just its outgoing charge q^i_I.

In what follows we shall drop the *net* subscript; all charges henceforth, unless mentioned otherwise, will be net charges and the net charge associated with an Ith edge will be denoted simply by q^i_I.

A. Product of two Hamiltonian constraints

1. Notation

We compute the continuum limit (8.10) when $n=2$ and $\hat{O}_{\mathrm{i}}(N_{\mathrm{i}}), \mathrm{i}=1,2$ are Hamiltonian constraint operators. We restrict our attention to the case that c in (8.10) is in the bra set because, as mentioned in Sec. VIII. A, for c not in the bra set, the dual action vanishes. In Eq. (8.10), we set

$$N_1 \equiv M, \qquad \beta_1 \equiv \beta_M, \qquad N_2 \equiv N, \qquad \beta_2 \equiv \beta_N,$$
$$\epsilon_1 \equiv \bar{\delta}, \qquad \epsilon_2 \equiv \delta. \tag{9.1}$$

As we shall see, the discrete action of this Hamiltonian constraint operator product generates the doubly deformed states $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_2}$, the singly deformed states $c_{[i,I,\hat{J},\hat{K}\beta,\delta]_1}$ and $c_{(j,J,\hat{R}_1,\hat{S}_1,\beta_M,\bar{\delta})}$, and c. Here we have defined the transitions:

$$[i,I,\hat{J},\hat{K},\beta,\delta]_2 = [(i_1,I_1,\hat{J}_2,\hat{K}_2,\beta_M,\bar{\delta}),(i,I,\hat{J}_1,\hat{K}_1,\beta_N,\delta)], \tag{9.2}$$

$$[i,I,\hat{J},\hat{K},\beta,\delta]_1 = (i,I,\hat{J}_1,\hat{K}_1,\beta_N,\delta). \tag{9.3}$$

The singly deformed state $c_{(j,J,\hat{R}_1,\hat{S}_1,\beta_M,\bar{\delta})}$ is distinct from the singly deformed state (9.3) and is obtained through the deformation $(j,J,\hat{R}_1,\hat{S}_1,\beta_M,\bar{\delta})$ of c. In particular, the parameter for this transformation is $\bar{\delta}$ whereas that for (9.3) is δ.

The above transitions are exactly those described in Secs. VI. D augmented with the hatted indices of (8.8). To see this, use (9.1). It is then straightforward to see that by setting, in (8.8),

$$m=2, \quad j_2=1, \quad j_1=2, \quad \text{we obtain} \quad [i,I,\hat{J},\hat{K},\beta,\delta]_2, \tag{9.4}$$

$$m=1, \quad j_1=2, \quad \text{we obtain} \quad [i,I,\hat{J},\hat{K},\beta,\delta]_1, \tag{9.5}$$

$$m=1, \quad j_1=1, \quad i=j, \quad I=J, \quad \hat{J}_1=\hat{R}_1,$$
$$\hat{K}_1=\hat{S}_1, \quad \text{we obtain} \quad (j,J,\hat{R}_1,\hat{S}_1,\beta_M,\bar{\delta}). \tag{9.6}$$

The contraction coordinates associated with the states obtained by applying the deformations (9.4)–(9.6) are, denoted respectively (in abbreviated notation) in Sec. VI. C and in Step 2 of Sec. VIII. C by $\{x^{\epsilon_2,\epsilon_1}_\alpha\}$, $\{x^{\epsilon_1}_\alpha\}$, $\{x^{\epsilon_2}_\alpha\}$ and the coordinates for c by $\{x_\alpha\}$. Here we set

$$\{x_\alpha\} \equiv \{x\}, \qquad \{x^{\epsilon_1}_\alpha\} \equiv \{x^{\bar{\delta}}\},$$
$$\{x^{\epsilon_2}_\alpha\} \equiv \{x^{\delta}\}, \qquad \{x^{\epsilon_2,\epsilon_1}_\alpha\} \equiv \{x^{\delta,\bar{\delta}}\}. \tag{9.7}$$

The notation we use for the nondegenerate vertex of

$$c \ \text{is} \ v, \qquad c_{(j,J,\hat{R}_1,\hat{S}_1,\beta_M,\bar{\delta})} \ \text{is} \ v_{(j,J,\bar{\delta})},$$
$$c_{[i,I,\hat{J},\hat{K}\beta,\delta]_1} \ \text{is} \ v_{[i,I,\delta]_1}, \qquad c_{[i,I,\hat{J},\hat{K},\beta,\delta]_2} \ \text{is} \ v_{[i,I,\beta,\delta]_2}. \tag{9.8}$$

Wherever required explicitly, we denote the density weighted object B evaluated at point p in the coordinate system $\{y\}$ by $B(p,\{y\})$.

2. Calculation

From (8.5) we have

$$\hat{C}[N]_\delta c = \beta_N \frac{3\hbar}{8\pi \mathrm{i}} N(v,\{x\}) \nu_v^{-2/3} \sum_{i,I,\hat{J}_1,\hat{K}_1} \frac{c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1} - c}{(N-1)(N-2)\delta}, \tag{9.9}$$

$$\hat{C}[M]_{\bar{\delta}} \hat{C}[N]_\delta c = \beta_N \frac{3\hbar}{8\pi \mathrm{i}} N(v,\{x\}) \nu_v^{-2/3} \times \sum_{i,I,\hat{J}_1,\hat{K}_1} \frac{\hat{C}[M]_{\bar{\delta}} c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1} - \hat{C}[M]_{\bar{\delta}} c}{(N-1)(N-2)\delta}. \tag{9.10}$$

Using (8.5) again,

$$\hat{C}[M]_{\bar{\delta}} c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1} = \beta_M \frac{3\hbar}{8\pi \mathrm{i}} M(v_{[i,I,\delta]_1}, \{x^\delta\}) \nu^{-2/3}_{v_{[i,I,\delta]_1}} \times \sum_{i_1,I_1,\hat{J}_2,\hat{K}_2} \frac{c_{[i,I,\hat{J},\hat{K},\beta,\delta]_2} - c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}}{(N-1)(N-2)\bar{\delta}}, \tag{9.11}$$

$$\Rightarrow (\Psi_{f,h_{ab},P_0}|\hat{C}[M]_{\bar{\delta}} c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}\rangle = \beta_M \frac{3\hbar}{8\pi \mathrm{i}} M(v_{[i,I,\delta]_1}, \{x^\delta\}) \nu^{-2/3}_{v_{[i,I,\delta]_1}} \times \sum_{i_1,I_1,\hat{J}_2,\hat{K}_2} \frac{(\Psi_{f,h_{ab},P_0}|c_{[i,I,\hat{J},\hat{K},\beta,\delta]_2}\rangle - (\Psi_{f,h_{ab},P_0}|c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}\rangle}{(N-1)(N-2)\bar{\delta}}. \tag{9.12}$$

Using (7.5),

$$(\Psi_{f,h_{ab},P_0}|c_{[i,I,\hat{J},\hat{K},\beta,\delta]_2}\rangle = g_{c_{[i,I,\hat{J},\hat{K},\beta,\delta]_2}} f(v_{[i,I,\beta,\delta]_2}, \{x^{\delta,\bar{\delta}}\}) \times \left(\sum_{L_2} h_{L_2} H_{L_2}\right) \tag{9.13}$$

where we have used Steps 1 and 2, Sec. VIII. C to evaluate the amplitude with respect to the contraction coordinates at $v_{[i,I,\beta,\delta]_2}$. Next, we evaluate its contraction behavior.

From Appendix G. 2, and using $q \ll 1$, we have, as $\bar{\delta} \to 0$,

$$\sum_{\hat{J}_2,\hat{K}_2} g_{c_{[i,I,\hat{J},\hat{K},\beta,\delta]_2}} = g_{c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}} (\bar{\delta})^{\frac{2}{3}(q-1)} Q(c_{[i,I,\beta,\delta_0]_2^{2,1}}, S_1) \times h_{I_1}(1 + O(\bar{\delta}^2)) \tag{9.14}$$

where we have used (9.1) and (9.4) to set $j = 1$ in Eq. (G12).

From Appendix F, a straightforward computation yields

$$\sum_{L_2} h_{L_2} H_{L_2} = [(N-1)(N-2)(2+\cos^2\theta + (N-3)|\cos\theta|)] \times \sqrt{h_{ab}(v_{[i,I,\beta,\delta]_2}) \hat{V}^{(\delta)a}_{I_1} \hat{V}^{(\delta)b}_{I_1}} + O(\bar{\delta}^2) \tag{9.15}$$

where, as in Appendix F, $\hat{V}^{(\delta)a}_{I_1}$ is the constant extension in the chart $\{x^\delta\}$ of the unit upward direction for the I_1th edge of the immediate parent $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}$. Here $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}$ is the immediate parent of $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_2}$. This immediate parent has contraction coordinates $\{x^\delta\}$ from (9.7) and unit upward direction for its I_1th edge $\hat{V}^{(\delta)a}_{I_1}$. This vector is extended in an open neighborhood of the parental vertex $v_{[i,I,\delta]_1}$ by defining its components in the chart $\{x^\delta\}$ at any point p in this neighborhood to be the same as its components at the parental vertex, this neighborhood being large enough to contain the child vertex $v_{[i,I,\beta,\delta]_2}$. Finally we have, from (F15) that

$$f(v_{[i,I,\beta,\delta]_2}, \{x^{\delta,\bar{\delta}}\}) = (\bar{\delta})^{-\frac{2}{3}(q-1)} f(v_{[i,I,\beta,\delta]_2}, \{x^\delta\}). \tag{9.16}$$

We choose the Q factor above to be

$$Q(c_{0[i,I,\beta,\delta_0]_2^{2,1}}, S_1) := \frac{N(N-1)(N-2)}{[(N-1)(N-2)(2+\cos^2\theta + (N-3)|\cos\theta|)]}. \tag{9.17}$$

Clearly, $Q > 0$ as required. From (9.13), (9.14)–(9.17), and setting

$$\sqrt{h_{ab}(v_{[i,I,\beta,\delta]_2}) \hat{V}^{(\delta)a}_{I_1} \hat{V}^{(\delta)b}_{I_1}} \equiv \|\vec{\hat{V}}^{(\delta)}_{I_1}\|_{v_{[i,I,\beta,\delta]_2}} \tag{9.18}$$

we have

$$\sum_{i_1,I_1,\hat{J}_2,\hat{K}_2} (\Psi_{f,h_{ab},P_0}|c_{[i,I,\hat{J},\hat{K},\beta,\delta]_2}\rangle = N(N-1)(N-2) g_{c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}} \sum_{i_1,I_1} h_{I_1} \|\vec{\hat{V}}^{(\delta)}_{I_1}\|_{v_{[i,I,\beta,\delta]_2}} \times f(v_{[i,I,\beta,\delta]_2}, \{x^\delta\}) + O(\bar{\delta}^2). \tag{9.19}$$

From (7.5), the second amplitude in (9.12) is

$$(\Psi_{f,h_{ab},P_0}|c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}\rangle = g_{c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}} f(v_{[i,I,\beta,\delta]_1}, \{x^\delta\}) \left(\sum_{L_1} h_{L_1} H_{L_1}\right). \tag{9.20}$$

From (9.19) and (9.20) and the fact that $v_{[i,I,\beta,\delta]_2}$ is displaced by an amount $q^{i_1}_{I_1}\bar{\delta}$ in the direction $\vec{\hat{V}}^{(\delta)}_{I_1}$ from $v_{[i,I,\delta]_1}$, we have that

$$\sum_{i_1,I_1,\hat{J}_2,\hat{K}_2} (\Psi_{f,h_{ab},P_0}|c_{[i,I,\hat{J},\hat{K},\beta,\delta]_2}\rangle = 3N(N-1)(N-2)(\Psi_{f,h_{ab},P_0}|c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}\rangle + \bar{\delta} N(N-1)(N-2) g_{c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}} \times \sum_{i_1,I_1} h_{I_1} q^{i_1}_{I_1} \hat{V}^{(\delta)a}_{I_1} (\partial_a \|\vec{\hat{V}}^{(\delta)}_{I_1}\|_p f(p, \{x^\delta\}))|_{p=v_{[i,I,\delta]_1}} + O(\bar{\delta}^2) \tag{9.21}$$

where similar to Appendix F, we have set

$$\|\vec{\hat{V}}^{(\delta)}_{I_1}\|_p = \sqrt{h_{ab}(p)\hat{V}^{(\delta)a}_{I_1}(p)\hat{V}^{(\delta)b}_{I_1}(p)} \tag{9.22}$$

with $\hat{V}^{(\delta)a}_{I_1}(p)$ being the constant extension of $\hat{V}^{(\delta)a}_{I_1}$ at $v_{[i,I,\delta]_1}$. The partial derivative ∂_a can be taken with respect to any coordinates as its tangent space index a is contracted with that of $\hat{V}^{(\delta)a}_{I_1}$. If we take it to be the coordinate derivative with respect to $\{x^\delta\}$ then it passes through $\hat{V}^{(\delta)a}_{I_1}(p)$ and only acts on h_{ab}, f. Using (9.21) in (9.12) and taking the limit $\bar{\delta}\to 0$, we have that

$$\lim_{\bar{\delta}\to 0}(\Psi_{f,h_{ab},P_0}|\hat{C}[M]_{\bar{\delta}}c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}\rangle = \beta_M\frac{3\hbar N}{8\pi\mathrm{i}}M(v_{[i,I,\delta]_1},\{x^\delta\})\nu^{-2/3}_{v_{[i,I,\delta]_1}} \times g_{c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}}\sum_{i_1,I_1}h_{I_1}q^{i_1}_{I_1}\hat{V}^{(\delta)a}_{I_1}(\partial_a\|\vec{\hat{V}}^{(\delta)}_{I_1}\|_p f(p,\{x^\delta\}))|_{p=v_{[i,I,\delta]_1}}. \tag{9.23}$$

Using the notation of Appendix F, we may write this concisely as

$$\lim_{\bar{\delta}\to 0}(\Psi_{f,h_{ab},P_0}|\hat{C}[M]_{\bar{\delta}}c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}\rangle = \beta_M\frac{3\hbar N}{8\pi\mathrm{i}}\nu^{-2/3}_{v_{[i,I,\delta]_1}}g_{c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}}\sum_{i_1,I_1}h_{I_1}q^{i_1}_{I_1}(H^1_{I_1}(p=v_{[i,I,\delta]_1})). \tag{9.24}$$

Next consider the term $\hat{C}[M]_{\bar{\delta}}c$ in (9.10). We have, using (8.5),

$$\hat{C}[M]_{\bar{\delta}}c = \beta_M\frac{3\hbar}{8\pi\mathrm{i}}M(v,\{x\})\nu_v^{-2/3}\sum_{j,J,\hat{R}_1,\hat{S}_1}\frac{c_{(j,J,\hat{R},\hat{S},\beta_M,\bar{\delta})}-c}{(N-1)(N-2)\bar{\delta}}. \tag{9.25}$$

A similar analysis yields

$$\lim_{\bar{\delta}\to 0}(\Psi_{f,h_{ab},P_0}|\hat{C}[M]_{\bar{\delta}}c\rangle = \beta_M\frac{3\hbar N}{8\pi\mathrm{i}}M(v,\{x\})\nu_v^{-2/3}g_c\sum_{j,J}h_J q^j_J\hat{V}^a_J(\partial_a\|\vec{\hat{V}}_J\|_p f(p,\{x\}))|_{p=v} \tag{9.26}$$

which can be written in the notation of Appendix F as

$$\lim_{\bar{\delta}\to 0}(\Psi_{f,h_{ab},P_0}|\hat{C}[M]_{\bar{\delta}}c\rangle = \beta_M\frac{3\hbar N}{8\pi\mathrm{i}}\nu_v^{-2/3}g_c\sum_{j,J}h_J q^j_J(H^1_J(p=v)). \tag{9.27}$$

In the above calculation the Q factor is the same as that in (9.17):

$$Q(c_{0(j,J,\beta_M,\delta_0)},c,S_1) \coloneqq \frac{N(N-1)(N-2)}{[(N-1)(N-2)(2+\cos^2\theta+(N-3)|\cos\theta|)]}, \tag{9.28}$$

and we have used Eq. (G12) in conjunction with Eqs. (9.1) and (9.6). Note that the Q factor in (9.28) and the Q factor in (9.17) are labeled by the same sequence label $S_1 = h$ (see Sec. VIII. D for a discussion of this labeling). It follows from this fact, together with the charge independence of the Q factor in (9.28) and the discussion in Sec. VIII. D, that the Q factors in (9.28) and (9.17) *must necessarily* be identical.

Finally, we need to the compute the contraction limit $\delta\to 0$ of (9.24). From Appendix G. 2, and using $q \ll 1$, we have, as $\delta\to 0$,

$$\sum_{\hat{J}_1,\hat{K}_1}g_{c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}} = g_c(\delta)^{\frac{4}{3}(q-1)}Q(c_{0[i,I,\beta,\delta_0]_1^{0,1}},S_2)h_I(1+O(\delta^2)) \tag{9.29}$$

where we have used (9.1) and (9.5) to set $j=2$ in Eq. (G12). Using Eqs. (F14) and (F27) from Appendix F, as well as (F16), we have that as $\delta\to 0$,

$$\sum_{i_1,I_1} h_{I_1} q^{i_1}_{I_1} (H^1_{I_1}(p = v_{[i,I,\delta]_1}))$$
$$= \delta^{-\frac{4}{3}(q-1)}\{(M(v_{[i,I,\delta]_1},\{x\})\hat{V}^a_I \partial_a (f(v_{[i,I,\delta]_1},\{x\})\sqrt{h_{ab}(v_{[i,I,\delta]_1})\hat{V}^a_I\hat{V}^b_I})$$
$$\times [(N-1)(N-2)q^{i_1}_{I_1=I} + \left(\sum_{I_1\neq I} q^{i_1}_{I_1}\right)(N-2)(1+\cos\theta(1+\cos^2\theta+(N-3)|\cos\theta|)]) + O(\delta^2)\}$$
$$= \delta^{-\frac{4}{3}(q-1)}\{(M(v_{[i,I,\delta]_1},\{x\})q^{i_1}_{I_1=I}\hat{V}^a_I \partial_a (f(v_{[i,I,\delta]_1},\{x\})\sqrt{h_{ab}(v_{[i,I,\delta]_1})\hat{V}^a_I\hat{V}^b_I})$$
$$\times [(N-1)(N-2) - (N-2)(\cos\theta)(1+\cos^2\theta+(N-3)|\cos\theta|)]) + O(\delta^2)\} \tag{9.30}$$

where we have used gauge invariance applied to the *net* charges to go from the first equality to the second.

Next, we choose the Q factor in (9.29) to be

$$Q(c_{0[i,I,\beta,\delta_0]^{0,1}_1}, S_2) = \frac{\nu_v^{-2/3}}{\nu_{v_{[i,I,\delta]_1}}^{-2/3}} \frac{3N(N-1)(N-2)}{[(N-1)(N-2)-(N-2)(\cos\theta)(1+\cos^2\theta+(N-3)|\cos\theta|)]}. \tag{9.31}$$

It is straightforward to check that (using the facts that $N \geq 4$, $|\cos\theta| < 1$), as required, this Q factor is positive. Note that for Q to be well defined, we need the nondegenaracy condition $\nu_{v_{[i,I,\delta]_1}} \neq 0$ (see the relevant discussion in the beginning of Sec. VI. B).

Next, using (9.29)–(9.31) in (9.24) yields

$$\sum_{\hat{J}_1,\hat{K}_1} \lim_{\bar{\delta}\to 0} (\Psi_{f,h_{ab},P_0}|\hat{C}[M]_{\bar{\delta}} c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}\rangle$$
$$= 3(N)(N-1)(N-2)\beta_M \frac{3\hbar N}{8\pi \mathrm{i}} \nu_v^{-2/3} g_c h_I \sum_{i_1} q^{i_1}_{I_1=I}\left(M(v_{[i,I,\delta]_1},\{x\})\hat{V}^a_I\partial_a\left(f(v_{[i,I,\delta]_1},\{x\})\sqrt{h_{ab}(v_{[i,I,\delta]_1})\hat{V}^a_I\hat{V}^b_I}\right)\right)$$
$$+ O(\delta^2). \tag{9.32}$$

In the above equation note that $q^{i_1}_{I_1=I}$ refers to the charge on the $I_1 = I$th edge of $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}$. This charge is related to the charge on the Ith edge of c by an (i, β_M) flip. Hence, depending on whether $\beta_M = \pm 1$ we have from (2.25) and (3.7) that

$$q^{i_1}_{I_1=I} = {}^{(i)}q^{i_1}_I = \delta^{ii_1} q^{i_1}_I \mp \sum_k \epsilon^{ii_1k} q^k_I. \tag{9.33}$$

Two identities, *key* to the anomaly free result, follow from the above equation:

$$\sum_{i,i_1} q^{i_1}_{I_1=I} = \sum_i q^i_I \tag{9.34}$$

and

$$\sum_{i,i_1} q^{i_1}_{I_1=I} q^i_I = \sum_i (q^i_I)^2. \tag{9.35}$$

Using (9.33) in (9.32) we have

$$\sum_{\hat{J}_1,\hat{K}_1} \lim_{\bar{\delta}\to 0} (\Psi_{f,h_{ab},P_0}|\hat{C}[M]_{\bar{\delta}} c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}\rangle$$
$$= 3(N)(N-1)(N-2)\beta_M \frac{3\hbar N}{8\pi \mathrm{i}} \nu_v^{-2/3} g_c h_I \sum_{i_1} {}^{(i)}q^{i_1}_I \left(M(v_{[i,I,\delta]_1},\{x\})\hat{V}^a_I\partial_a\left(f(v_{[i,I,\delta]_1},\{x\})\sqrt{h_{ab}(v_{[i,I,\delta]_1})\hat{V}^a_I\hat{V}^b_I}\right)\right) + O(\delta^2). \tag{9.36}$$

Next, we expand the second line of (9.36) in a Taylor approximation and sum over i, I to obtain

$$\sum_I h_I \sum_i \sum_{i_1} {}^{(i)}q_I^{i_1}\Big(M(v_{[i,I,\delta]_1},\{x\})\hat{V}_I^a\partial_a\Big(f(v_{[i,I,\delta]_1},\{x\})\sqrt{h_{ab}(v_{[i,I,\delta]_1})\hat{V}_I^a\hat{V}_I^b}\Big)\Big)$$
$$= \sum_I h_I\Big\{\sum_{i,i_1} {}^{(i)}q_I^{i_1}\Big(M(v,\{x\})\hat{V}_I^a\partial_a\Big(f(v,\{x\})\sqrt{h_{ab}(v)\hat{V}_I^a\hat{V}_I^b}\Big)\Big)$$
$$+\delta\sum_{i,i_1} {}^{(i)}q_I^{i_1}(q_I^i\hat{V}_I^b\partial_b\Big(M(p,\{x\})\Big(\hat{V}_I^a\partial_a\Big(f(p,\{x\})\sqrt{h_{ab}(p)\hat{V}_I^a\hat{V}_I^b}\Big)\Big)\Big)\Big|_{p=v}\Big\} + O(\delta^2)$$
$$= \sum_I h_I\Big\{\sum_i q_I^i\Big(M(v,\{x\})\hat{V}_I^a\partial_a\Big(f(v,\{x\})\sqrt{h_{ab}(v)\hat{V}_I^a\hat{V}_I^b}\Big)\Big)$$
$$+\delta\sum_i((q_I^i)^2\hat{V}_I^b\partial_b\Big(M(p,\{x\})\Big(\hat{V}_I^a\partial_a\Big(f(p,\{x\})\sqrt{h_{ab}(p)\hat{V}_I^a\hat{V}_I^b}\Big)\Big)\Big)\Big|_{p=v}\Big\} + O(\delta^2). \tag{9.37}$$

Here we have used the identities (9.34) and (9.35) to obtain the second equality from the first.

Next, consider the dual action of (9.10) on the anomaly free state in the limit $\bar{\delta}\to 0$:

$$\lim_{\bar{\delta}\to 0}(\Psi_{f,h_{ab},P_0}|\hat{C}[M]_{\bar{\delta}}\hat{C}[N]_\delta c\rangle = \beta_N\frac{3\hbar}{8\pi \mathrm{i}}N(v,\{x\})\nu_v^{-2/3}\frac{1}{(N-1)(N-2)\delta}\Bigg(\sum_{i,I,\hat{J}_1,\hat{K}_1}\lim_{\bar{\delta}\to 0}(\Psi_{f,h_{ab},P_0}|\hat{C}[M]_{\bar{\delta}}c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}\rangle\Bigg)$$
$$-\sum_{i,I,\hat{J}_1,\hat{K}_1}(\Psi_{f,h_{ab},P_0}|\hat{C}[M]_{\bar{\delta}}c\rangle). \tag{9.38}$$

The second line of (9.38) can be evaluated using (9.37). The zeroth order term in δ in this expansion is precisely $3(N)(N-1)(N-2)$ times the right-hand side of (9.26). In the term on the third line of (9.38), the amplitude is exactly that of (9.26) and the indices $i,\hat{J}_1,\hat{K}_1$ are dummy indices for this amplitude so that the amplitude is simply multiplied by a factor of N (coming from the sum over I), $(N-1)(N-2)$ (from the sum over the hatted indices) and 3 (from the sum over i). Hence the zeroth order term of the second line cancels the contribution from the third line and this is what allows the $\delta\to 0$ limit of the left-hand side in the first line to exist.

Note.—This cancellation is precisely due to the **−1** *structure introduced in Sec. III, this structure being motivated by considerations of "propagation."*

Finally taking the $\delta\to 0$ limit of (9.38), we obtain

$$(\Psi_{f,h_{ab},P_0}|\hat{C}[M]\hat{C}[N]c\rangle = 3\beta_N\beta_M\left(\frac{3\hbar N}{8\pi \mathrm{i}}\right)^2 N(v,\{x\})\nu_v^{-4/3}g_c$$
$$\times\Bigg\{\sum_{i,I}h_I(\Big(q_I^i)^2\hat{V}_I^b\partial_b\Big(M(p,\{x\})(\hat{V}_I^a\partial_a\Big(f(p,\{x\})\sqrt{h_{ab}(p)\hat{V}_I^a\hat{V}_I^b}\Big)\Big)\Big)\Big|_{p=v}\Bigg\}. \tag{9.39}$$

Next, as in Appendix F. 3, it is convenient to define the quantity $H^l_{L_m}(N_1,N_2,..,N_3;p)$ associated with Eq. (8.9) as follows. Let c in (8.9) be in the bra set (as in this section). Let the contraction coordinate associated with the nondegenerate vertex of its mth generation descendant, $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$, be denoted by $\{z\}$ so that $\{z\}\coloneqq\{x^{\epsilon_{j_1}..\epsilon_{j_m}}\}$. Then we define

$$H^l_{L_m}(N_1,..,N_l;p)\coloneqq\prod_{i=1}^{l}N_{l-i+1}(p,\{z\})\hat{V}^{a_{l-i+1}}_{L_m}(p)\partial_{a_{l-i+1}}\Big(f(p,\{z\})\sqrt{h_{ab}(p)\hat{V}^a_{L_m}(p)\hat{V}^b_{L_m}(p)}\Big) \tag{9.40}$$

where the product is ordered from left to right in order of increasing i and the point p is in a small enough neighborhood of the nondegenerate vertex of $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ wherein the unit (with respect to $\{z\}$) upward direction $\vec{\hat{V}}_{L_m}$ associated with the L_mth edge at this vertex admits the constant extension $\vec{\hat{V}}_{L_m}(p)$ as discussed in Appendix F. 3.[40]

[40]It is straightforward to check that if we express Eq. (9.40) in terms of the notation and the right to left ordering convention for $\prod$ used in Appendix F. 3 that (9.40) takes the form of (F18).

Making contact through (9.1) with (9.40), Eq. (9.39) can be written succinctly as

$$(\Psi_{f,h_{ab},P_0}|\hat{C}[M]\hat{C}[N]c\rangle = 3\beta_N\beta_M\left(\frac{3\hbar N}{8\pi \mathrm{i}}\right)^2 \nu_v^{-4/3} g_c \times \sum_i \sum_I (q_I^i)^2 h_I H_I^2(M,N;p=v), \tag{9.41}$$

from which the commutator can be written as

$$(\Psi_{f,h_{ab},P_0}|[\hat{C}[M],\hat{C}[N]]c\rangle = 3\beta_N\beta_M\left(\frac{3\hbar N}{8\pi \mathrm{i}}\right)^2 \nu_v^{-4/3} g_c \times \sum_i \sum_I (q_I^i)^2 h_I (H_I^2(M,N; p=v) - H_I^2(N,M;p=v)). \tag{9.42}$$

B. Electric diffeomorphism commutator

1. Notation

We compute the continuum limit of the electric diffeomorphism commutator. Accordingly we consider the action of (8.9) on a state c when $n=2$ and $\hat{O}_{\mathrm{i}}(N_{\mathrm{i}}), \mathrm{i}=1,2$ are electric diffeomorphism constraint operators. We compute the commutator from the ensuing product of discrete approximants and then take the continuum limit. Similar to Sec. IX. A. 1, and for the reason articulated there we restrict attention to the case that c is in the bra set. For this section we use notation similar to that in Sec. IX. A. 1. However, the notation denotes transitions and their associated structures which are appropriate to the action of the electric diffeomorphism constraints and hence are often distinct from those appropriate to the Hamiltonian constraint in Sec. IX. A. 1.

We use the notation (9.1) in (8.9) so that once again we have

$$N_1 \equiv M, \qquad \beta_1 \equiv \beta_M, \qquad N_2 \equiv N, \qquad \beta_2 \equiv \beta_N, \qquad \epsilon_1 \equiv \bar{\delta}, \qquad \epsilon_2 \equiv \delta. \tag{9.43}$$

The discrete action of the electric diffeomorphism constraint operator product generates the doubly deformed states $c_{[i,I,\hat{J},\hat{K},\delta]_2}$, the singly deformed states $c_{[i,I,\hat{J},\hat{K},\delta]_1}$ and $c_{(i,J,\hat{R}_1,\hat{S}_1,\bar{\delta})}$ and c. Here we have defined the transitions:

$$[i,I,\hat{J},\hat{K},\delta]_2 = [(i_1,I_1,\hat{J}_2,\hat{K}_2,\beta=0,\bar{\delta}), (i,I,\hat{J}_1,\hat{K}_1,\beta=0,\delta)], \tag{9.44}$$

$$[i,I,\hat{J},\hat{K},\delta]_1 = (i,I,\hat{J}_1,\hat{K}_1,\beta=0,\delta), \tag{9.45}$$

$$(i,J,\hat{R}_1,\hat{S}_1,\bar{\delta}) = (i,J,\hat{R}_1,\hat{S}_1,\beta=0,\bar{\delta}). \tag{9.46}$$

We set in (8.8)

$$m=2, \quad j_2=1, \quad j_1=2, \quad \text{to obtain} \quad [i,I,\hat{J},\hat{K},\delta]_2, \tag{9.47}$$

$$m=1, \quad j_1=2, \quad \text{to obtain} \quad [i,I,\hat{J},\hat{K},\delta]_1, \tag{9.48}$$

$$m=1, \quad j_1=1, \quad I=J, \quad \hat{J}_1=\hat{R}_1, \quad \hat{K}_1=\hat{S}_1, \quad \text{to obtain} \quad (i,J,\hat{R}_1,\hat{S}_1,\bar{\delta}). \tag{9.49}$$

The contraction coordinates associated with the states obtained by applying the deformations (9.47)–(9.49) are, denoted respectively (in abbreviated notation) in Sec. VI. C and in Step 2 of Sec. VIII. C by $\{x_\alpha^{\epsilon_2,\epsilon_1},\}$, $\{x_\alpha^{\epsilon_2}\}$, $\{x_\alpha^{\epsilon_1}\}$ and the coordinates for c by $\{x_\alpha\}$. Similar to (9.7), we set

$$\{x_\alpha\} \equiv \{x\}, \qquad \{x_\alpha^{\epsilon_2}\} \equiv \{x^\delta\}, \qquad \{x_\alpha^{\epsilon_1}\} \equiv \{x^{\bar{\delta}}\}, \qquad \{x_\alpha^{\epsilon_2,\epsilon_1}\} \equiv \{x^{\delta,\bar{\delta}}\}. \tag{9.50}$$

The notation we use for the nondegenerate vertex of

$$c \text{ is } v, \qquad c_{(i,J,\hat{R}_1,\hat{S}_1,\bar{\delta})} \text{ is } v_{(i,J,\bar{\delta})}, \qquad c_{[i,I,\hat{J},\hat{K},\delta]_1} \text{ is } v_{[i,I,\delta]_1}, \qquad c_{[i,I,\hat{J},\hat{K},\delta]_2} \text{ is } v_{[i,I,\delta]_2}. \tag{9.51}$$

2. Calculation

Applying (8.6) to c we obtain

$$\hat{D}_\delta[\vec{N}_i]c = \frac{\hbar}{\mathrm{i}}\frac{3}{4\pi}N(x(v))\nu_v^{-2/3}\sum_I \sum_{\hat{J}_1,\hat{K}_1}\frac{1}{(N-1)(N-2)}\frac{1}{\delta}(c_{[i,I,\hat{J},\hat{K},\delta]_1} - c) \tag{9.52}$$

so that

$$\hat{D}_{\bar{\delta}}[\vec{M}_i]\hat{D}_\delta[\vec{N}_i]c = \frac{\hbar}{\mathrm{i}}\frac{3}{4\pi}N(x(v))\nu_v^{-2/3}\sum_I \sum_{\hat{J}_1,\hat{K}_1}\frac{1}{(N-1)(N-2)}\frac{1}{\delta}(\hat{D}_{\bar{\delta}}[\vec{M}_i]c_{[i,I,\hat{J},\hat{K}\delta]_1} - \hat{D}_{\bar{\delta}}[\vec{M}_i]c). \tag{9.53}$$

Using (8.6) again,

$$\hat{D}_{\bar{\delta}}[\vec{M}_i]c_{[i,I,\hat{J},\hat{K}\delta]_1} = \frac{3\hbar}{4\pi i} M(v_{[i,I,\delta]_1}, \{x^\delta\}) \nu_{v_{[i,I,\delta]_1}}^{-2/3} \sum_{I_1,\hat{J}_2,\hat{K}_2} \frac{c_{[i,I,\hat{J},\hat{K},\delta]_2} - c_{[i,I,\hat{J},\hat{K},\delta]_1}}{(N-1)(N-2)\bar{\delta}}, \tag{9.54}$$

$$\Rightarrow (\Psi_{f,h_{ab},P_0}|\hat{D}[\vec{M}_i]_{\bar{\delta}}c_{[i,I,\hat{J},\hat{K},\delta]_1}\rangle = \frac{3\hbar}{4\pi i} M(v_{[i,I,\delta]_1}, \{x^\delta\}) \nu_{v_{[i,I,\delta]_1}}^{-2/3} \sum_{I_1,\hat{J}_2,\hat{K}_2} \frac{(\Psi_{f,h_{ab},P_0}|c_{[i,I,\hat{J},\hat{K},\delta]_2}\rangle - (\Psi_{f,h_{ab},P_0}|c_{[i,I,\hat{J},\hat{K},\delta]_1}\rangle}{(N-1)(N-2)\bar{\delta}}. \tag{9.55}$$

Using (7.5),

$$(\Psi_{f,h_{ab},P_0}|c_{[i,I,\hat{J},\hat{K},\delta]_2}\rangle = g_{c_{[i,I,\hat{J},\hat{K},\delta]_2}} f(v_{[i,I,\delta]_2}, \{x^{\delta,\bar{\delta}}\}) \left(\sum_{L_2} h_{L_2} H_{L_2}\right) \tag{9.56}$$

where we have used Steps 1 and 2, Sec. VIII. C to evaluate the amplitude with respect to the contraction coordinates at $v_{[i,I,\delta]_2}$. Next, we evaluate its contraction behavior.

From Appendix G. 2, and using $q \gg 1$, we have, as $\bar{\delta} \to 0$

$$\sum_{\hat{J}_2,\hat{K}_2} g_{c_{[i,I,\hat{J},\hat{K},\delta]_2}} = g_{c_{[i,I,\hat{J},\hat{K},\delta]_1}} (\bar{\delta})^{\frac{2}{3}(q-1)} Q(c_{0[i,I,\delta_0]_2^{2,1}}, S_1) h_{I_1}(1 + O(\bar{\delta}^2)) \tag{9.57}$$

where we have used (9.43) and (9.47) to set $j = 1$ in Eq. (G12).

From Appendix F, a straightforward computation identical to that used in deriving (9.15) yields

$$\sum_{L_2} h_{I_2} H_{L_2} = [(N-1)(N-2)(2 + \cos^2\theta + (N-3)|\cos\theta|)] \sqrt{h_{ab}(v_{[i,I,\delta]_2}) \hat{V}_{I_1}^{(\delta)a} \hat{V}_{I_1}^{(\delta)b}} + O(\bar{\delta}^2) \tag{9.58}$$

where, as in Appendix F, $\hat{V}_{I_1}^{(\delta)a}$ is the constant extension in the chart $\{x^\delta\}$ of the unit upward direction for the I_1th edge of the immediate parent $c_{[i,I,\hat{J},\hat{K},\delta]_1}$. Similar to (9.16), from (F15), we have that

$$f(v_{[i,I,\delta]_2}, \{x^{\delta,\bar{\delta}}\}) = (\bar{\delta})^{-\frac{2}{3}(q-1)} f(v_{[i,I,\delta]_2}, \{x^\delta\}). \tag{9.59}$$

We choose the Q factor for this electric diffeomorphsim type transition to be identical to that of (9.17) so that

$$Q(c_{0[i,I,\delta_0]_2^{2,1}}, S_1) \coloneqq \frac{N(N-1)(N-2)}{[(N-1)(N-2)(2 + \cos^2\theta + (N-3)|\cos\theta|)]} \tag{9.60}$$

where $c_{0[i,I,\delta_0]_2^{2,1}}$ denote the δ_0 images [see (6.25) and (6.26) and Sec. VIII. D] of the electric diffeomorphism children $c_{[i,I,\hat{J},\hat{K},\delta]_2^{2,1}}$. Note that in principle the Q factors in (9.60) and (9.17) could be chosen to be distinct from each other because in the former case the sequence label S_1 corresponds to d_i whereas in the latter case $S_1 = h$.

From (9.56), (9.57)–(9.60), and setting, similar to Sec. IX. A. 2,

$$\sqrt{h_{ab}(v_{[i,I,\delta]_2}) \hat{V}_{I_1}^{(\delta)a} \hat{V}_{I_1}^{(\delta)b}} \equiv \|\vec{\hat{V}}_{I_1}^{(\delta)}\|_{v_{[i,I,\delta]_2}} \tag{9.61}$$

we have

$$\sum_{I_1,\hat{J}_2,\hat{K}_2} (\Psi_{f,h_{ab},P_0}|c_{[i,I,\hat{J},\hat{K},\delta]_2}\rangle = N(N-1)(N-2) g_{c_{[i,I,\hat{J},\hat{K},\delta]_1}} \sum_{I_1} h_{I_1} \|\vec{\hat{V}}_{I_1}^{(\delta)}\|_{v_{[i,I,\delta]_2}} f(v_{[i,I,\delta]_2}, \{x^\delta\}). \tag{9.62}$$

From (7.5), the second amplitude in (9.55) is

$$(\Psi_{f,h_{ab},P_0}|c_{[i,I,\hat{J},\hat{K},\delta]_1}\rangle = g_{c_{[i,I,\hat{J},\hat{K},\delta]_1}} f(v_{[i,I,\delta]_1},\{x^\delta\})\left(\sum_{L_1} h_{L_1} H_{L_1}\right). \tag{9.63}$$

Similar to the derivation of (9.21), from (9.62) and (9.63), we have that

$$\begin{aligned}\sum_{I_1,\hat{J}_2,\hat{K}_2}(\Psi_{f,h_{ab},P_0}|c_{[i,I,\hat{J},\hat{K},\delta]_2}\rangle &= N(N-1)N-2)(\Psi_{f,h_{ab},P_0}|c_{[i,I,\hat{J},\hat{K},\delta]_1}\rangle \\ &+ \bar{\delta}N(N-1)(N-2)g_{c_{[i,I,\hat{J},\hat{K},\delta]_1}}\sum_{I_1} h_{I_1} q_{I_1}^{i_1=i}\hat{V}_{I_1}^{(\delta)a}(\partial_a\|\vec{\hat{V}}_{I_1}^{(\delta)}\|_p f(p,\{x^\delta\}))|_{p=v_{[i,I,\delta]_1}} + O(\bar{\delta}^2)\end{aligned} \tag{9.64}$$

where

$$\|\vec{\hat{V}}_{I_1}^{(\delta)}\|_p \coloneqq \sqrt{h_{ab}(p)\hat{V}_{I_1}^{(\delta)a}(p)\hat{V}_{I_1}^{(\delta)b}(p)} \tag{9.65}$$

with $\hat{V}_{I_1}^{(\delta)a}(p)$ being the constant extension of $\hat{V}_{I_1}^{(\delta)a}$ at $v_{[i,I,\delta]_1}$. As in Sec. IX. A. 2, the partial derivative ∂_a can be taken with respect to any coordinates as its tangent space index a is contracted with that of $\hat{V}_{I_1}^{(\delta)a}$; if we take it to be the coordinate derivative with respect to $\{x^\delta\}$ then it passes through $\hat{V}_{I_1}^{(\delta)a}(p)$ and only acts on h_{ab}, f. Using (9.64) in (9.55) and taking the limit $\bar{\delta} \to 0$, we have that

$$\lim_{\bar{\delta}\to 0}(\Psi_{f,h_{ab},P_0}|\hat{D}[\vec{M}_i]_{\bar{\delta}}c_{[i,I,\hat{J},\hat{K},\delta]_1}\rangle = \frac{3\hbar N}{4\pi \mathrm{i}} M(v_{[i,I,\delta]_1},\{x^\delta\})\nu_{v_{[i,I,\delta]_1}}^{-2/3} g_{c_{[i,I,\hat{J},\hat{K},\delta]_1}}\sum_{I_1} h_{I_1} q_{I_1}^{i_1=i}\hat{V}_{I_1}^{(\delta)a}(\partial_a\|\vec{\hat{V}}_{I_1}^{(\delta)}\|_p f(p,\{x^\delta\}))|_{p=v_{[i,I,\delta]_1}}. \tag{9.66}$$

In the notation of Appendix F, we may write this as

$$\lim_{\bar{\delta}\to 0}(\Psi_{f,h_{ab},P_0}|\hat{D}[\vec{M}_i]_{\bar{\delta}}c_{[i,I,\hat{J},\hat{K},\delta]_1}\rangle = \frac{3\hbar N}{4\pi \mathrm{i}}\nu_{v_{[i,I,\delta]_1}}^{-2/3} g_{c_{[i,I,\hat{J},\hat{K},\delta]_1}}\sum_{I_1} h_{I_1} q_{I_1}^{i_1=i}(H_{I_1}^1(p=v_{[i,I,\delta]_1})). \tag{9.67}$$

Next consider the term $\hat{D}[\vec{M}_i]_{\bar{\delta}}c$ in (9.53). From (8.6),

$$\hat{D}[\vec{M}_i]_{\bar{\delta}}c = \frac{3\hbar}{4\pi \mathrm{i}} M(v,\{x\})\nu_v^{-2/3}\sum_{J,\hat{R}_1,\hat{S}_1}\frac{c_{(i,J,\hat{R},\hat{S},\bar{\delta})} - c}{(N-1)(N-2)\bar{\delta}}, \tag{9.68}$$

As can be seen from (9.53) and (9.68), the term $\hat{D}[\vec{M}_i]_{\bar{\delta}}c$ does *not* contribute to the commutator because it is multiplied by a product of lapse functions evaluated at the *same point* v. Nevertheless it is instructive to evaluate it for reasons which will become clear towards the end of this section. A similar analysis to that involved in obtaining (9.67) yields

$$\lim_{\bar{\delta}\to 0}(\Psi_{f,h_{ab},P_0}|\hat{D}[\vec{M}_i]_{\bar{\delta}}c\rangle = \frac{3\hbar N}{4\pi \mathrm{i}} M(v,\{x\})\nu_v^{-2/3} g_c \sum_J h_J q_J^i \hat{V}_J^a(\partial_a\|\vec{\hat{V}}_J\|_p f(p,\{x\}))|_{p=v} \tag{9.69}$$

which can be written in the notation of Appendix F as

$$\lim_{\bar{\delta}\to 0}(\Psi_{f,h_{ab},P_0}|\hat{D}[\vec{M}_i]_{\bar{\delta}}c\rangle = \frac{3\hbar N}{4\pi \mathrm{i}}\nu_v^{-2/3} g_c \sum_J h_J q_J^i(H_J^1(p=v)). \tag{9.70}$$

In the above calculation the Q factor is the same as that in (9.60):

$$Q(c_{0(i,J,\delta_0)}, c_0, S_1) \coloneqq \frac{N(N-1)(N-2)}{[(N-1)(N-2)(2+\cos^2\theta+(N-3)|\cos\theta|)]}, \tag{9.71}$$

and we have used (9.43) and (9.49) to set $j = 1$ in Eq. (G12). Note that the sequence label S_1 is identical for (9.60) and (9.71). The charge independence of the Q factor (9.60) together with the discussion in Sec. VIII. D implies that the Q factor for (9.71) *must necessarily* be the same as that for (9.60).

Next we compute the contraction limit $\delta \to 0$ of (9.67). From Appendix G. 2, and using $q \gg 1$, we have, as $\delta \to 0$,

$$\sum_{\hat{J}_1,\hat{K}_1} g_{c_{[i,I,\hat{J},\hat{K},\delta]_1}} = g_c(\delta)^{\frac{4}{3}(q-1)} Q(c_{0[i,I,\delta_0]_1^{0,1}}, S_2) h_I (1+O(\delta^2)) \tag{9.72}$$

where we have used (9.43) and (9.48) to set $j = 2$ in Eq. (G12). From Eqs. (F14), (F27) and (F16), as $\delta \to 0$ we have

$$\begin{aligned}
&\sum_{I_1} h_{I_1} q_{I_1}^{i_1=i}(H_{I_1}^1(p = v_{[i,I,\delta]_1}))\\
&= \delta^{-\frac{4}{3}(q-1)}\Bigg\{\left(M(v_{[i,I,\delta]_1},\{x\})\hat{V}_I^a\partial_a\left(f(v_{[i,I,\delta]_1},\{x\})\sqrt{h_{ab}(v_{[i,I,\delta]_1})\hat{V}_I^a\hat{V}_I^b}\right)\right.\\
&\quad\times[(N-1)(N-2)q_{I_1=I}^{i_1=i} + \left(\sum_{I_1\neq I} q_{I_1}^{i_1=i}\right)(N-2)(1+\cos\theta(1+\cos^2\theta+(N-3)|\cos\theta|)]) + O(\delta^2)\Bigg\}\\
&= \delta^{-\frac{4}{3}(q-1)}\{(M(v_{[i,I,\delta]_1},\{x\})q_{I_1=I}^{i_1=i}\hat{V}_I^a\partial_a\left(f(v_{[i,I,\delta]_1},\{x\})\sqrt{h_{ab}(v_{[i,I,\delta]_1})\hat{V}_I^a\hat{V}_I^b}\right)\\
&\quad\times[(N-1)(N-2)-(N-2)(\cos\theta)(1+\cos^2\theta+(N-3)|\cos\theta|)]) + O(\delta^2)\}
\end{aligned} \tag{9.73}$$

where we have used gauge invariance to go from the first equality to the second.

Next, we choose the Q factor in (9.72) to be

$$Q(c_{[i,I,\delta_0]_1^{0,1}}, S_2) = \frac{\nu_v^{-2/3}}{\nu_{v_{[i,I,\delta]_1}}^{-2/3}} \frac{AN(N-1)(N-2)}{[(N-1)(N-2)-(N-2)(\cos\theta)(1+\cos^2\theta+(N-3)|\cos\theta|)]} \tag{9.74}$$

where we shall specify the positive constant A shortly and where, as in (9.31), $N > 3$, $|\cos\theta| < 1$ implies that Q is positive. Note that in (9.74), we have $S_2 = (d_i, d_i)$ whereas in (9.31), $S_2 = (h, h)$ so that the Q factors for these two equations can be (and are) chosen to be distinct from each other.

Using (9.72)–(9.74) in (9.67) yields

$$\begin{aligned}
\sum_{\hat{J}_1,\hat{K}_1} \lim_{\bar{\delta}\to 0}(\Psi_{f,h_{ab},P_0}|\hat{D}[\vec{M}_i]_{\bar{\delta}} c_{[i,I,\hat{J},\hat{K},\delta]_1}\rangle &= A(N)(N-1)(N-2)\frac{3\hbar N}{4\pi \mathrm{i}}\nu_v^{-2/3}\\
&\quad\times g_c h_I q_{I_1=I}^{i_1=i}\left(M(v_{[i,I,\delta]_1},\{x\})\hat{V}_I^a\partial_a\left(f(v_{[i,I,\delta]_1},\{x\})\sqrt{h_{ab}(v_{[i,I,\delta]_1})\hat{V}_I^a\hat{V}_I^b}\right)\right) + O(\delta^2).
\end{aligned} \tag{9.75}$$

In the above equation note that $q_{I_1=I}^{i_1}$ refers to the charge on the $(I_1 = I)$th edge of $c_{[i,I,\hat{J},\hat{K},\delta]_1}$. Since the transition involved is of electric diffeomorphism type, there is no charge flipping so that this charge is equal to the charge on the Ith edge of c so that we have

$$q_{I_1=I}^{i_1=i} = q_I^i. \tag{9.76}$$

Using (9.76) in (9.75) we have

$$\begin{aligned}
\sum_{\hat{J}_1,\hat{K}_1} \lim_{\bar{\delta}\to 0}(\Psi_{f,h_{ab},P_0}|\hat{D}[\vec{M}_i]_{\bar{\delta}} c_{[i,I,\hat{J},\hat{K},\delta]_1}\rangle &= A(N)(N-1)(N-2)\frac{3\hbar N}{4\pi \mathrm{i}}\nu_v^{-2/3} g_c\\
&\quad\times h_I q_I^i\left(M(v_{[i,I,\delta]_1},\{x\})\hat{V}_I^a\partial_a\left(f(v_{[i,I,\delta]_1},\{x\})\sqrt{h_{ab}(v_{[i,I,\delta]_1})\hat{V}_I^a\hat{V}_I^b}\right)\right) + O(\delta^2).
\end{aligned} \tag{9.77}$$

Expanding the second line of (9.77) in a Taylor approximation and summing over I, we obtain

$$\begin{aligned}&\sum_I h_I q_I^i \Big(M(v_{[i,I,\delta]_1}, \{x\}) \hat{V}_I^a \partial_a \Big(f(v_{[i,I,\delta]_1}, \{x\}) \sqrt{h_{ab}(v_{[i,I,\delta]_1}) \hat{V}_I^a \hat{V}_I^b} \Big) \Big) \\ &= \sum_I h_I \Big\{ q_I^i \Big(M(v, \{x\}) \hat{V}_I^a \partial_a \Big(f(v, \{x\}) \sqrt{h_{ab}(v) \hat{V}_I^a \hat{V}_I^b} \Big) \Big) \\ &\quad + \delta (q_I^i)^2 \hat{V}_I^b \partial_b \Big(M(p, \{x\}) \Big(\hat{V}_I^a \partial_a \Big(f(p, \{x\}) \sqrt{h_{ab}(p) \hat{V}_I^a \hat{V}_I^b} \Big) \Big) \Big) \Big|_{p=v} \Big\} + O(\delta^2). \end{aligned} \tag{9.78}$$

Next, consider the dual action of (9.53) on the anomaly free state in the limit $\bar{\delta} \to 0$:

$$\begin{aligned}\lim_{\bar{\delta}\to 0} (\Psi_{f,h_{ab},P_0} | \hat{D}[\vec{M}_i]_{\bar{\delta}} \hat{D}[\vec{N}_i]_\delta c\rangle &= \frac{3\hbar}{4\pi \mathrm{i}} N(v, \{x\}) \nu_v^{-2/3} \frac{1}{(N-1)(N-2)\delta} \Bigg(\sum_{I,\hat{J}_1,\hat{K}_1} \lim_{\bar{\delta}\to 0} (\Psi_{f,h_{ab},P_0} | \hat{D}[\vec{M}_i]_{\bar{\delta}} c_{[i,I,\hat{J},\hat{K},\delta]_1}\rangle \Bigg) \\ &\quad - \sum_{I,\hat{J}_1,\hat{K}_1} (\Psi_{f,h_{ab},P_0} | \hat{D}[\vec{M}_i]_{\bar{\delta}} c\rangle). \end{aligned} \tag{9.79}$$

The second line of (9.79) can be evaluated using (9.78). The zeroth order term in δ in this expansion is $A(N)(N-1)(N-2)$ times the right-hand side of (9.69). In the term on the third line of (9.79), the amplitude is exactly that of (9.69) and the indices $I, \hat{J}_1, \hat{K}_1$ are dummy indices for this amplitude so that the amplitude is simply multiplied by a factor of N (coming from the sum over I) and $(N-1)(N-2)$ (from the sum over the hatted indices). Hence the zeroth order term of the second line cancels the contribution from the third line *only if we set* $A = 1$.

On the other hand, as mentioned above the term on the third line of (9.79) does not contribute to the commutator. Hence we are not restricted to the choice $A = 1$ if we are only interested in the commutator. This commutator is

$$\begin{aligned}\lim_{\delta\to} \lim_{\bar{\delta}\to 0} (\Psi_{f,h_{ab},P_0} | (\hat{D}[\vec{M}_i]_{\bar{\delta}} \hat{D}[\vec{N}_i]_\delta - N \leftrightarrow M) | c\rangle &= \frac{3\hbar}{4\pi \mathrm{i}} \nu_v^{-2/3} \lim_{\delta\to 0} \frac{1}{(N-1)(N-2)\delta} \\ &\quad \times \Bigg\{ \Bigg(\sum_{I,\hat{J}_1,\hat{K}_1} \lim_{\bar{\delta}\to 0} (\Psi_{f,h_{ab},P_0} | N(v, \{x\}) \hat{D}[\vec{M}_i]_{\bar{\delta}} c_{[i,I,\hat{J},\hat{K},\delta]_1}\rangle \Bigg) \\ &\quad - \Bigg(\sum_{I,\hat{J}_1,\hat{K}_1} \lim_{\bar{\delta}\to 0} (\Psi_{f,h_{ab},P_0} | M(v, \{x\}) \hat{D}[\vec{N}_i]_{\bar{\delta}} c_{[i,I,\hat{J},\hat{K},\delta]_1}\rangle \Bigg) \Bigg\}. \end{aligned} \tag{9.80}$$

Using (9.78) in (9.80), taking the $\delta \to 0$ limit and leaving A undetermined (and in particular, not necessarily equal to unity), we obtain $\hat{V}_I^a$

$$\begin{aligned}(\Psi_{f,h_{ab},P_0} | \sum_{i=1}^3 [\hat{D}[\vec{M}_i], \hat{D}[\vec{N}_i]] c\rangle &= A \left(\frac{3\hbar N}{4\pi \mathrm{i}}\right)^2 \nu_v^{-4/3} g_c \Bigg\{ \sum_{i,I} h_I (q_I^i)^2 [N(v, \{x\}) \hat{V}_I^b (\partial_b M(p, \{x\})) \\ &\quad - M(v, \{x\}) \hat{V}_I^b (\partial_b N(p, \{x\}))] [\hat{V}_I^a \partial_a (f(p, \{x\}) \sqrt{h_{ab}(p) \hat{V}_I^a \hat{V}_I^b})] |_{p=v} \Bigg\}. \end{aligned} \tag{9.81}$$

Making contact through (9.43) with (9.40) this can be written succinctly as

$$(\Psi_{f,h_{ab},P_0} | \sum_{i=1}^3 [\hat{D}[\vec{M}_i], \hat{D}[\vec{N}_i] c\rangle = A \left(\frac{3\hbar N}{4\pi \mathrm{i}}\right)^2 (\nu_v^{-4/3}) g_c \sum_i \sum_I h_I (q_I^i)^2 (H_I^2(M, N; p = v) - H_I^2(N, M; p = v)). \tag{9.82}$$

Comparing this with (9.42), we obtain

$$(\Psi_{f,h_{ab},P_0} | \sum_{i=1}^3 [\hat{D}[\vec{M}_i], \hat{D}[\vec{N}_i] c\rangle = \frac{4A}{3\beta_M \beta_N} (\Psi_{f,h_{ab},P_0} | [\hat{C}[M], \hat{C}[N] c\rangle. \tag{9.83}$$

Comparing this with (2.11), we obtain an anomaly free commutator if:

(a) we choose successive actions of the Hamiltonian constraint to have opposite flips so that $\beta_M = -\beta_N$ so that $\beta_M \beta_N = -1$;
(b) we choose $A = \frac{1}{4}$.

Making these choices we obtain the desired anomaly free result. Note that because we have been obliged to choose $A \neq 1$, the continuum limit product of two electric diffeomorphism constraints is *not* defined; only their commutator is well defined. However, the electric diffeomorphism constraint operator is not one of the constraint operators used to generate the constraint algebra; its role is restricted to the demonstration of an anomaly free commutator between a pair of Hamiltonian constraints in accord with (2.11). Hence the ill defined-ness of the product of two electric diffeomorphism constraint operators is not an obstruction to our treatment of the constraint algebra.

X. MULTIPLE PRODUCTS OF HAMILTONIAN CONSTRAINTS

In Sec. X. A we derive, through an inductive proof, the expression for the action of multiple products of Hamiltonian constraint operators on an anomaly free state. In Sec. X. B we show that the action derived in X. A yields anomaly free single commutators (see Sec. I for our usage of the term "anomaly free single commutator"). Since the detailed calculations below are similar to those of Sec. IX, we shall only highlight the main steps of these calculations in our exposition.

A. Multiple products of Hamiltonian constraints: Derivation

1. Introductory remarks

We note the following:

(1) The discrete action of a product of n Hamiltonian constraints in (8.9) requires a choice of β (which characterizes the charge flips) for each constraint action. In the rest of this work, we choose $\beta = \beta_{\rm i}$ for the ith Hamiltonian constraint in (8.9) to be $+1$ if i is *odd* and -1 if i is *even*. This is consistent with the choice made in Sec. IX. A for the case of $n = 2$.
(2) Note that Eq. (8.9) is evaluated through the two steps outlined in Sec. VIII. C so that the coordinate patch with respect to which the amplitude of a deformed child generated by the operator product is evaluated is the appropriate *contraction* coordinate patch. Only for the undeformed state c, the amplitude is evaluated with respect to the *reference* coordinates associated with c.
(3) Recall that the cone angle is acute or obtuse depending on whether the deformations are upward or downward. Hereon we will tailor our choice of cone angle to the choice of bra set so that $|\cos\theta|$ is fixed and the same for all deformations of ket correspondents of members of the bra set and is chosen such that

$$|\cos\theta|(3Nq_{\max}^{\rm primordial}) < 1 \qquad (10.1)$$

with $q_{\max}^{\rm primordial}$ defined as in (6.9), where the set of edge charges in that equation can be taken to be those of any primordial charge net in the bra set.[41] Note that this condition is equivalent to the condition

$$|\cos\theta|(3Nq_{\max}^{\rm net}) < 1 \qquad (10.2)$$

where

$$q_{\max}^{\rm net} = \max_{(i=1,2,3),(I=1,..,N)} |q_I^i| \qquad (10.3)$$

where the charges q_I^i are the *net* edge charges[42] at the nondegenerate vertex of any element of the bra set. It is easy to check that this equivalence follows immediately from Appendix C together with the definition of the bra set in Sec. VII. A.
(4) Equations (6.7) and (6.8)are defined as conditions on primordial charges. Appendix C shows that the net charges and primordial charges on corresponding edges are identical or flipped images of each other. Hence the Eqs. (6.7) and (6.8)) also hold for net charges on multiply deformed children of primordial charge nets and we shall so interpret them when we refer to them hereon.

2. Summary of choices

It is useful to note that from Secs. VI. B, VI. C, VI. E and (3) of Sec. X. A. 1, that the action (8.9) is fixed once the following choices have been made:

(a) The set of primordial states $S_{\rm primordial}$.
(b) A primary coordinate patch $\{x_0\}$ around a point p_0.
(c) A set of primordial reference states, one for each diffeomorphism class of states in $S_{\rm primordial}$, the nondegenerate vertex of each such reference state being located at p_0 and linear with respect to $\{x_0\}$. These primordial reference states are divided into exhaustive and mutually exclusive classes, each class defining a bra set so that members of each class have the same set of unordered edge charges. For each class we choose a cone angle θ which satisfies (10.1).

[41] The value of $q_{\max}^{\rm primordial}$ is independent of the choice of primordial charge net in the bra set since any such primordial has the same set of unordered edge charges (see Sec. VII. A).

[42] This notation is consistent with the Note at the beginning of Sec. IX; note that this equation is in general distinct from (4.1) because the charges on the right-hand side of that equation refer to the actual edge charges not the net edge charges.

(d) A reference state for each distinct diffeomorphism class of elements of $S_{\rm primary}$, each such reference state itself being an element of $S_{\rm primary}$.[43]
(e) A choice of reference diffeomorphism, one for each element c of the ket set, which maps the reference state c_0 for this element c, to c.
(f) A choice of deformation such that any (single or multiple) deformation of any element of the complement of the ket set is also in this complement.

Once the choices (a)–(f) are made, the formalism is rigid in that the choice of upward/downward conical deformations is fixed as in Sec. V, the contraction procedure is fixed as in Sec. VI. C, the discrete action of operator products is fixed as in Sec. VI. D with the sign flips chosen in accord with (1) above, the anomaly free basis states are chosen as in Sec. VII and the continuum limit is defined as in Eq. (8.10).

3. Notation

Recall that the Cauchy manifold is a C^k-semianalytic manifold for some $k \gg 1$. We compute the continuum limit (8.10) for arbitrary $n < k$ with $\hat{O}_{\rm i}(N_{\rm i}), {\rm i} = 1, .., n$ being Hamiltonian constraint operators. We restrict our attention to the case that c in (8.10) is in the bra set because, as mentioned in Sec. VIII. A, for c not in the bra set, the dual action vanishes.

We denote the nondegenerate vertex of c by v and its associated reference coordinate patch by $\{x\}$. We shall be interested in a proof by mathematical induction. In the course of that proof, it will suffice to develop notation only for singly deformed states. The single deformations of interest will be denoted as

$$[i, I, \hat{J}, \hat{K}, \beta, \delta]_1 = (i, I, \hat{J}_1, \hat{K}_1, \beta, \delta). \tag{10.4}$$

The vertex of the singly deformed state $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}$ is denoted by $v_{[i,I,\delta]_1}$ and its associated *contraction* coordinate patch by $\{x^\delta\}$. In the induction proof it will turn out that the single deformation of (10.4) will play the role of the first of $m+1$ deformations applied to c. Accordingly, in relation to the notation of Sec. VI. C, in this section we have set

$$\{x_\alpha\} \equiv \{x\}, \quad j_1 = m+1, \quad \epsilon_{{\rm j}_1} \equiv \delta, \quad \{x_\alpha^{\epsilon_{{\rm j}_1}}\} \equiv \{x^\delta\}. \tag{10.5}$$

[43] Recall that $S_{\rm primary}$ is the set of primaries generated from reference primordials through repeated conical deformations of the type constructed in Appendix B and Sec. V with respect to $\{x_0\}$ so that $S_{\rm primary}$ is determined once (a)–(b) above are fixed. In particular the cone angles characterizing the conical deformations are fixed by (b). Recall also that the ket set $S_{\rm ket}$ comprises of all diffeomorphic images of elements in $S_{\rm primary}$ and hence is also determined once (a)–(b) are fixed.

As usual, wherever required explicitly, we denote the density weighted object B evaluated at point p in the coordinate system $\{y\}$ by $B(p, \{y\})$. We shall also make extensive use of the notation developed in Appendix F.

4. Proof by induction

Let n be a positive integer with $n \leq k-1$ where k is the differentiability class of the semianalytic Cauchy slice. Define k_n as

$$k_n = \frac{n-1}{2} \quad \text{if } n \text{ is odd}, \qquad k_n = \frac{n}{2} \quad \text{if } n \text{ is even}. \tag{10.6}$$

Let c be in the bra set and let the ith net charge at the Ith edge at its nondegenerate vertex v be q_I^i. Define

$$|\vec{q}_I| = \sqrt{\sum_{i=1}^{3} (q_I^i)^2}. \tag{10.7}$$

Claim.—The continuum limit of the dual action of a product of n Hamiltonian constraints when n is even is

$$(\Psi_{f,h_{ab},P_0}| \left(\prod_{\rm i=1}^{n} \hat{C}(N_{\rm i})\right) |c\rangle = (-3)^{k_n} \left(\frac{3\hbar N}{8\pi i}\right)^n (\nu^{-\frac{2}{3}})^n g_c \sum_I |\vec{q}_I|^n h_I H_I^n(N_1, .., N_n; v), \tag{10.8}$$

and the continuum limit of the dual action of a product of n Hamiltonian constraints when n is odd is

$$(\Psi_{f,h_{ab},P_0}| \left(\prod_{\rm i=1}^{n} \hat{C}(N_{\rm i})\right) |c\rangle = (-3)^{k_n} \left(\frac{3\hbar N}{8\pi i}\right)^n (\nu^{-\frac{2}{3}})^n g_c \sum_I |\vec{q}_I|^{n-1} \left(\sum_{i=1}^{3} q_I^i\right) \times h_I H_I^n(N_1, .., N_n; v), \tag{10.9}$$

and where we have used Eq. (9.40) to define $H_I^n(N_1, .., N_n; p)$ so that

$$H_I^n(N_1, .., N_n; v) = \left(\prod_{\rm i=1}^{n} N_{\rm n-i+1}(p, \{x\}) \hat{V}_I^{a_{\rm n-i+1}}(p) \partial_{a_{\rm n-i+1}}\right) \times (f(p, \{x\}) \sqrt{h_{ab}(p) \hat{V}_I^a(p) \hat{V}_I^b(p)})\Big|_{p=v} \tag{10.10}$$

and the product is ordered from left to right in increasing i.

Proof by induction.—Step 1: In Sec. IX. A we have shown that (10.9) holds for $n = 1$ [see (9.27)] and that (10.8) holds for $n = 2$ [see (9.41)]. More in detail, clearly, we have shown (10.9) holds for $n = 1$ and that (10.8) holds for $n = 2$ *for any choice of* (a)–(f), Sec. X. A. 2 with c being in a bra set resulting from these choices.

Step 2: Assume that (10.8) holds for $n = m$, m even, *for any choice of* (a)–(f), Sec. X. A. 2. Then we show below that (10.9) holds for $n = m + 1$ for any choice of (a)–(f), Sec. X. A. 2. We have that

$$(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m+1}\hat{C}(N_{\mathrm{i}})\right)|c\rangle = \lim_{\delta\to 0}(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)\hat{C}_\delta(N_{\mathrm{m+1}})|c\rangle. \tag{10.11}$$

Using (8.5) we obtain

$$(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)\hat{C}_\delta(N_{\mathrm{m+1}})|c\rangle = (-1)^m\frac{3\hbar}{8\pi\mathrm{i}}N(x(v))\nu_v^{-2/3}\frac{1}{(N-1)(N-2)\delta}\sum_{i,I\hat{J}_1,\hat{K}_1}((\Psi_{f,h_{ab},P_0}| \\ \times\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)|c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}\rangle - (\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)|c\rangle). \tag{10.12}$$

Here the $(-1)^m$ factor is unity for m even in accord with (1), Sec. X. A. 1.[44] The second amplitude in the last line of (10.12) is given by (10.8) with $n = m$. The first amplitude (within the summation symbol) looks as if it could be evaluated through a direct application of (10.8). However from (2), Sec. X. A. 1, the coordinates associated with $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}$ are the contraction coordinates whereas (10.8) is applicable only if these coordinates were reference coordinates. Recall however that we have assumed (10.12) for *any* choice of (a)–(f). Our strategy is then to make choices for (a)–(f) such that (10.8) is directly applicable to the first term in the context of such choices.

We proceed as follows. Consider some fixed choice of (a)–(f) in Sec. X. A. 2, for which (10.8) is used to evaluate the second term in the last line of (10.12). In this fixed choice, as in Sec. VI, let c_0 be the reference state for c, let the reference diffeomorphism which maps c_0 to c be α, let the deformation of c_0 with respect to the primary coordinates $\{x_0\}$ at parameter δ_0 be $c_{0(i,I,\beta,\delta_0)}$ and let the contraction image of $c_{0(i,I,\beta,\delta_0)}$ by the appropriate contraction diffeomorphism be $c_{0[i,I,\hat{J},\hat{K},\beta,\delta]_1}$ so that $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}$ is the image by α of $c_{0[i,I,\hat{J},\hat{K},\beta,\delta]_1}$. Using (10.5) this contraction diffeomorphism from (6.33) is

$$\Phi^{\epsilon_{\mathrm{j}_1=m+1}=\delta,\{x_0\},\hat{J}_1,\hat{K}_1}_{c_0,(i,I,\beta,\delta_0),S_{\mathrm{j}_1=m+1}} \equiv \phi \tag{10.13}$$

where for notational simplicity in this step (i.e. Step 2), we have denoted the contraction diffeomorphism on the left-hand side of (10.13) by ϕ. Now consider the choices (a')–(c') below which are images of the fixed choice made above by the diffeomorphism ϕ. These ϕ choices are then as follows:

(a') The set of primordials $S_{\phi,\mathrm{primordial}}$ is chosen to be the image of the set $S_{\mathrm{primordial}}$ of primordials chosen in accordance with choice (a); since $S_{\mathrm{primordial}}$ is closed under diffeomorphisms we have that $S_{\phi,\mathrm{primordial}}$ is equal to $S_{\mathrm{primordial}}$.

(b') The primary coordinate patch is $\phi^*\{x_0\}$ around the point $\phi(p_0)$.

(c') The set of primordial reference states is just the set of images by ϕ of the fixed choice (c) of reference primordials. The cone angles, as measured by the primary coordinates in (b'), for deformations of primordial reference states are chosen to be identical to the choices in (c) for their diffeomorphically related counterparts.

Next, note that the set of primaries, $S_{\phi,\mathrm{primary}}$, are now generated from the reference primordials of (c') through conical deformations with respect to $\phi^*\{x_0\}$; it follows that $S_{\phi,\mathrm{primary}}$ consists of the images of the elements of S_{primary} by ϕ. The ket set generated from $S_{\phi,\mathrm{primary}}$ is then identical to the ket set S_{Ket} generated from S_{primary} because the ket set is closed under the action of diffeomorphisms. Next, consider the diffeomorphism class $[c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}]$ of $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}$. Clearly we have that $[c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}] = [c_{0(i,I,\beta,\delta_0)}]$. Further, since $c_{0(i,I\beta,\delta_0)} \in S_{\mathrm{primary}}$, Eq. (10.13) and (c') above imply that $c_{0[i,I,\hat{J},\hat{K},\beta,\delta]_1} \in S_{\phi,\mathrm{primary}}$. Hence may choose $c_{0[i,I,\hat{J},\hat{K},\beta,\delta]_1}$ to be a reference state for $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}$. Recall that $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}$ is the image by α of $c_{0[i,I,\hat{J},\hat{K},\beta,\delta]_1}$ Accordingly we choose (d') and (e') as follows:

(d') We choose the reference state for $[c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}]$ to be $c_{0[i,I,\hat{J},\hat{K},\beta,\delta]_1}$ and choose reference states for other diffeomorphism classes of elements of the ket set arbitrarily.

(e') We choose the reference diffeomorphism for $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}$ to be α and choose the remaining reference diffeomorphisms arbitrarily.

Finally, since the ket set is unaltered we retain the choice of (f) i.e. we set (f') to be the same as the fixed choice (f) above. It is then easy to see that the bra set B_{P_0} chosen with

[44]The equation as it is written would also be valid for m odd where from (1), Sec. X. A. 1 we require an overall $-1 = (-1)^m$ factor coming from our choice of the $m + 1$th β flip when m is odd.

respect to (a)–(f) is also a valid bra set with respect to (a')–(f').

Accordingly we consider the same bra set B_{P0} as before and choose h_{ab}, f also as before and obtain a state Ψ^{ϕ}_{f,h,P_0} based on its amplitude evaluations in the context of the choices (a')–(f') above. Our choices (a')–(f') [especially (d'), (e')] ensure that the *reference coordinates* for $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}$ in the context of these choices is the same as the *contraction coordinates* for this state in the context of the fixed choices for (a)–(f) above. It is then straightforward to check that the contraction coordinates for any deformed state generated by the action of $(\prod_{\mathrm{i}=1}^{m}\hat{C}_{\epsilon_{\mathrm{i}}}(N_{\mathrm{i}}))$ on $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}$ in the context of choices (a')–(f') coincides with the contraction coordinates for the same deformed state when it is generated by the action of $(\prod_{\mathrm{i}=1}^{m}\hat{C}_{\epsilon_{\mathrm{i}}}(N_{\mathrm{i}}))\hat{C}_{\delta}(N_{\mathrm{m}+1})$ on c. It follows that the evaluation of (10.8) in accordance with Sec. X. A. 2 in the context of the choices (a')–(f'), and, with c replaced by $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}$ and with Ψ_{f,h,P_0} replaced by Ψ^{ϕ}_{f,h,P_0}, coincides precisely with the first term (within the summation symbol) in the last line of Eq. (10.12). It then follows from (10.8) that

$$(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)|c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}\rangle = (-3)^{k_m}\left(\frac{3\hbar N}{8\pi i}\right)^m (\nu_{v_{[i,I\delta]_1}}^{-\frac{2}{3}})^m g_{c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}}\sum_{L_1}|\vec{q}_{L_1}|^m h_{L_1}H^m_{L_1}(N_1,..,N_m;v_{[i,I,\delta]_1}), \quad (10.14)$$

where, using the notation (10.5) and(9.40), the coordinate dependent parts of $H^m_{L_1}(N_1,..,N_m;v_{[i,I,\delta]_1}$ are evaluated with respect to the coordinates $\{x^{\delta}\}$ which serve both as the reference coordinates for the state $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}$ in the choice scheme (a')–(f') or as the contraction coordinates in the fixed choice scheme of (a)–(f) above. We now revert back to the latter interpretation of these coordinates.

Using the contraction behavior of $h_{L_1}, H^m_{L_1}, g_{c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}}$ derived in Appendixes F, G. 2 together with (10.5), we obtain, as $\delta \to 0$

$$\begin{aligned}&\sum_{\hat{J}_1,\hat{K}_1}(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)|c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}\rangle\\ &= (-3)^{k_m}\left(\frac{3\hbar N}{8\pi i}\right)^m(\nu_{v_{[i,I\delta]_1}}^{-\frac{2}{3}})^m g_c Q(c_{0[i,I,\beta,\delta_0]_1^{1,0}},S_{m+1})h_I\\ &\times\left\{|\vec{q}_{L_1=I}|^m(N-1)(N-2)+\cos^m(\theta)\left(\sum_{L_1\neq I}|\vec{q}_{L_1\neq I}|^m\right)(N-2)(1+\cos^2\theta+(N-3)|\cos\theta|\right\}\\ &\times\left(\prod_{\mathrm{i}=1}^{m}N^{a_{\mathrm{m-i+1}}}_{\mathrm{m-i+1}}(p,\{x\})\hat{V}_I^{a_{\mathrm{m-i+1}}}(p)\partial_{a_{\mathrm{m-i+1}}}(f(p,\{x\})\sqrt{h_{ab}(p)\hat{V}_I^a(p)\hat{V}_I^b(p)})|_{p=v_{[i,I,\delta]_1}}\right)+O(\delta^2). \end{aligned}\quad (10.15)$$

We set

$$Q(c_{0[i,I,\beta,\delta_0]_1^{1,0}},S_{m+1}) = \frac{(\nu_v^{-\frac{2}{3}})^m}{(\nu_{v_{[i,I\delta]_1}}^{-\frac{2}{3}})^m}\left(\frac{(N)(N-1)(N-2)|\vec{q}_I|^m}{\{|\vec{q}_{L_1=I}|^m(N-1)(N-2)+\cos^m(\theta)(\sum_{L_1\neq I}|\vec{q}_{L_1\neq I}|^m)(N-2)(1+\cos^2\theta+(N-3)|\cos\theta|\}}\right). \quad (10.16)$$

Here $\vec{q}_I$ refers to the charge on the Ith edge at v in c and $\vec{q}_{L_1}$ to the charge on the L_1th edge at $v_{[i,I,\delta]_1}$ in $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}$. Since m is even, the Q factor above is manifestly positive as required. With this choice of Q we obtain

$$\begin{aligned}&\sum_{\hat{J}_1,\hat{K}_1}(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)|c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}\rangle\\ &= (-3)^{k_m}\left(\frac{3\hbar N}{8\pi i}\right)^m(\nu_v^{-\frac{2}{3}})^m g_c N(N-1)(N-2)h_I|\vec{q}_I|^m\\ &\times\prod_{\mathrm{i}=1}^{m}N^{a_{\mathrm{m-i+1}}}_{\mathrm{m-i+1}}(p,\{x\})\hat{V}_I^{a_{\mathrm{m-i+1}}}(p)\partial_{a_{\mathrm{m-i+1}}}(f(p,\{x\})\sqrt{h_{ab}(p)\hat{V}_I^a(p)\hat{V}_I^b(p)}|_{p=v_{[i,I,\delta]_1}}+O(\delta^2). \end{aligned}\quad (10.17)$$

As in our treatment of similar terms in Sec. IX, we expand the right-hand side of the above equation in a Taylor approximation in powers of δ.[45] It is easy to see that the zeroth order contribution exactly cancels the contribution from the second term in (10.12). The first order term provides the only contribution to (10.12) which survives in the $\delta \to 0$ limit. It is straightforward to check that taking this limit of (10.12), we obtain

$$
\begin{aligned}
&(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)\hat{C}_\delta(N_{\mathrm{m}+1})|c\rangle \\
&= (-3)^{k_m}\left(\frac{3\hbar N}{8\pi i}\right)^{m+1}(\nu_v^{-\frac{2}{3}})^{m+1}g_c\sum_{i,I}\Bigg\{|\vec{q}_I|^m N_{\mathrm{m}+1}(p,\{x\})h_I q_I^i \\
&\times \hat{V}_I^a(p)\partial_a\left(\prod_{\mathrm{i}=1}^{m}N_{\mathrm{m-i+1}}^{a_{\mathrm{m-i+1}}}(p,\{x\})\hat{V}_I^{a_{\mathrm{m-i+1}}}(p)\partial_{a_{\mathrm{m-i+1}}}(f(p,\{x\})\sqrt{h_{ab}(p)\hat{V}_I^a(p)\hat{V}_I^b(p)})\right)_{p=v}\Bigg\}. \qquad (10.18)
\end{aligned}
$$

From (10.6) it follows that when m is even $k_m = k_{m+1}$. Using the notation (9.40) together with this fact in (10.18) yields

$$
(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m+1}\hat{C}(N_{\mathrm{i}})\right)|c\rangle = (-3)^{k_{m+1}}\left(\frac{3\hbar N}{8\pi i}\right)^{m+1}(\nu_v^{-\frac{2}{3}})^{m+1}g_c\sum_{i,I}\{|\vec{q}_I|^{m-1}q_I^i h_I H_I^{m+1}(N_1,..,N_{m+1};v) \qquad (10.19)
$$

which is the desired result (10.9) with $n = m+1$.

Since the fixed choice (a)–(f) underlying this derivation is arbitrary and since the assumed form for $n = m$ holds for any such choice, the result (10.19) also holds for any choice of (a)–(f).

Step 3: Assume that (10.9) holds for $n = m$, m odd, *for any choice of* (a)–(f). Then we show below that (10.8) holds for $n = m+1$ for any choice of (a)–(g). The first part of our analysis is identical to the first part of the analysis in Step 2. Note that in Step 2, Eqs. (10.11)–(10.13) hold regardless of whether m is odd or even [see footnote 44 with regard to the validity of (10.11) when m is odd]. The second amplitude in the last line of (10.12) is now given by (10.9) with $n = m$. To apply (10.9) to the first amplitude in the last line of (10.12) we repeat the analysis subsequent to (10.12) till (but not inclusive of) (10.14). The choices (a')–(f') allow us to apply (10.9) also to the first amplitude in the last line of (10.12). Accordingly, this term evaluates to

$$
(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)|c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}\rangle = (-3)^{k_m}\left(\frac{3\hbar N}{8\pi i}\right)^m(\nu_{v_{[i,I\delta]_1}}^{-\frac{2}{3}})^m g_{c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}}\sum_{L_1}|\vec{q}_{L_1}|^{m-1}\left(\sum_{i_1}q_{L_1}^{i_1}\right)h_{L_1}H_{L_1}^m(N_1,..,N_m;v_{[i,I\delta]_1}), \qquad (10.20)
$$

where, similar to (10.14), using the notation (10.5) and (9.40), the coordinate dependent parts of $H_{L_1}^m(N_1,..,N_m;v_{[i,I,\delta]_1})$ are evaluated with respect to the coordinates $\{x^\delta\}$. Once again, using the contraction behavior of $h_{L_1}, H_{L_1}^m, g_{c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}}$ derived in Appendixes F, G. 2 together with (10.5), we obtain, as $\delta \to 0$

$$
\begin{aligned}
&\sum_{\hat{J}_1,\hat{K}_1}(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)|c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}\rangle \\
&= (-3)^{k_m}\left(\frac{3\hbar N}{8\pi i}\right)^m(\nu_{v_{[i,I\delta]_1}}^{-\frac{2}{3}})^m g_c Q(c_{0[i,I,\beta,\delta_0]_1^{1,0}},S_{m+1})h_I\Bigg\{|\vec{q}_{L_1=I}|^{m-1}\left(\sum_{i_1}q_{L_1=I}^{i_1}\right)(N-1)(N-2) \\
&+\cos^m(\theta)\left(\sum_{L_1\neq I}|\vec{q}_{L_1\neq I}|^{m-1}\left(\sum_{i_1}q_{L_1}^{i_1}\right)\right)(N-2)(1+\cos^2\theta+(N-3)|\cos\theta|\Bigg\} \\
&\times\left(\prod_{\mathrm{i}=1}^{m}N_{\mathrm{m-i+1}}^{a_{\mathrm{m-i+1}}}(p,\{x\})\hat{V}_I^{a_{\mathrm{m-i+1}}}(p)\partial_{a_{\mathrm{m-i+1}}}\left(f(p,\{x\})\sqrt{h_{ab}(p)\hat{V}_I^a(p)\hat{V}_I^b(p)}\right)|_{p=v_{[i,I,\delta]_1}}\right)+O(\delta^2). \qquad (10.21)
\end{aligned}
$$

[45] This Taylor expansion is valid provided $m+1 \leq k-1$.

We set

$$Q(c_{0[i,I,\beta,\delta_0]_1^{1,0}}, S_{m+1}) = \frac{(\nu_v^{-\frac{2}{3}})^m}{(\nu_{v_{[i,I\delta]_1}}^{-\frac{2}{3}})^m} \times \frac{3(N)(N-1)(N-2)|\vec{q}_I|^{m-1}(\sum_{i_1} q_{L_1=I}^{i_1})}{[|\vec{q}_{L_1=I}|^{m-1}(\sum_{i_1} q_{L_1=I}^{i_1})(N-1)(N-2)] + [\cos^m\theta(\sum_{L_1\neq I}|\vec{q}_{L_1}|^{m-1}(\sum_{i_1} q_{L_1}^{i_1}))(N-2)(1+\cos^2\theta+(N-3)|\cos\theta|]}. \tag{10.22}$$

For θ constrained by Eqs. (10.1) and (10.2), it is straightforward to check that the sign of the denominator is that of its first term $|\vec{q}_{L_1=I}|^{m-1}(\sum_{i_1} q_{L_1=I}^{i_1})(N-1)(N-2)$ which is then the same as the sign of the numerator so that Q is positive. Note that Q in (10.22) is different from that in (10.16); this is not a problem because their associated constraint strings S_{m+1} are different in that in one case m is even and in the other m is odd. Note that because the transition $[i, I, \hat{J}, \hat{K}, \beta, \delta]_1$ is a Hamiltonian constraint generated one [with $\beta = -1$ in accord with (1), Sec. X. A. 1], we have that

$$q_{I_1=I}^{i_1} = {}^{(i)}q_I^{i_1} = \delta^{ii_1} q_I^{i_1} + \sum_k \epsilon^{ii_1k} q_I^k \tag{10.23}$$

where the left-hand side refers to the Ith edge charge in the child $c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}$ and the right-hand side to the Ith edge charge in the parent c. Using (10.22) and (10.23) in (10.21), together with the fact that the norm of the charge vector is flip independent, we obtain

$$\sum_{\hat{J}_1,\hat{K}_1} (\Psi_{f,h_{ab},P_0}| \left(\prod_{\mathrm{i}=1}^m \hat{C}(N_\mathrm{i})\right) |c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}\rangle = (-3)^{k_m} \left(\frac{3\hbar N}{8\pi i}\right)^m (\nu_v^{-\frac{2}{3}})^m g_c 3N(N-1)(N-2) \times h_I |\vec{q}_I|^{m-1} \left(\sum_{i_1} {}^{(i)}q_I^{i_1}\right) \prod_{\mathrm{i}=1}^m N_{\mathrm{m-i+1}}^{a_{\mathrm{m-i+1}}}(p,\{x\}) \hat{V}_I^{a_{\mathrm{m-i+1}}}(p) \partial_{a_{\mathrm{m-i+1}}} \left(f(p,\{x\})\sqrt{h_{ab}(p)\hat{V}_I^a(p)\hat{V}_I^b(p)}\right)\Big|_{p=v_{[i,I,\delta]_1}} + O(\delta^2). \tag{10.24}$$

Expanding in a Taylor approximation subject to footnote 45 we obtain

$$\sum_I \sum_i \sum_{\hat{J}_1,\hat{K}_1} (\Psi_{f,h_{ab},P_0}| \left(\prod_{\mathrm{i}=1}^m \hat{C}(N_\mathrm{i})\right) |c_{[i,I,\hat{J},\hat{K},\beta,\delta]_1}\rangle = (-3)^{k_m} \left(\frac{3\hbar N}{8\pi i}\right)^m (\nu_v^{-\frac{2}{3}})^m g_c 3N(N-1)(N-2) \sum_{i,I} h_I |\vec{q}_I|^{m-1} \left(\sum_{i_1} {}^{(i)}q_I^{i_1}\right) \times \left\{ \prod_{\mathrm{i}=1}^m N_{\mathrm{m-i+1}}^{a_{\mathrm{m-i+1}}}(p,\{x\}) \hat{V}_I^{a_{\mathrm{m-i+1}}}(p) \partial_{a_{\mathrm{m-i+1}}} \left(f(p,\{x\})\sqrt{h_{ab}(p)\hat{V}_I^a(p)\hat{V}_I^b(p)}\right)\Big|_{p=v} + \delta q_I^i V_I^a \partial_a \left(\prod_{\mathrm{i}=1}^m N_{\mathrm{m-i+1}}^{a_{\mathrm{m-i+1}}}(p,\{x\}) \hat{V}_I^{a_{\mathrm{m-i+1}}}(p) \partial_{a_{\mathrm{m-i+1}}} \left(f(p,\{x\})\sqrt{h_{ab}(p)\hat{V}_I^a(p)\hat{V}_I^b(p))}\right)\right)\Big|_{p=v} \right\} + O(\delta^2). \tag{10.25}$$

Using (9.34), it is straightforward to see that the contribution, to (10.12) [with m odd in (10.12)], of the zeroth order term in δ in (10.25) cancels with the contribution, to (10.12) of the second term in the last line of (10.12). Hence only the first order term in δ in (10.25) contributes to (10.12). Using (9.35) and (10.25) in (10.12) yields

$$(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)\hat{C}_{\delta}(N_{\mathrm{m}+1})|c\rangle$$
$$= (-3)^{k_m}\left(\frac{3\hbar N}{8\pi i}\right)^{m+1}(\nu_v^{-\frac{2}{3}})^{m+1}(-3)g_c\sum_{i,I}\Big\{|\vec{q}_I|^{m-1}(q_I^i)^2N_{\mathrm{m}+1}(p,\{x\})h_I$$
$$\times\hat{V}_I^a(p)\partial_a\left(\prod_{\mathrm{i}=1}^{m}N_{\mathrm{m-i+1}}^{a_{\mathrm{m-i+1}}}(p,\{x\})\hat{V}_I^{a_{\mathrm{m-i+1}}}(p)\partial_{a_{\mathrm{m-i+1}}}\left(f(p,\{x\})\sqrt{h_{ab}(p)\hat{V}_I^a(p)\hat{V}_I^b(p)}\right)\right)_{p=v}\Big\} \tag{10.26}$$

where the overall (-3) factor comes from the 3 in the numerator of (10.22) and the $(-1)^m = -1$ factor in (10.12). From (10.6) it follows that when m is odd $k_{m+1} = k_m + 1$. Using this fact together with the definition of the charge vector norm (10.7) and the notation (9.40), in (10.26) yields

$$(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m+1}\hat{C}(N_{\mathrm{i}})\right)|c\rangle = (-3)^{k_{m+1}}\left(\frac{3\hbar N}{8\pi i}\right)^{m+1}(\nu_v^{-\frac{2}{3}})^{m+1}g_c\sum_{i,I}|\vec{q}_I|^{m+1}h_IH_I^{m+1}(N_1,..,N_{m+1};v) \tag{10.27}$$

which is the desired result (10.8) with $n = m + 1$.

Once again, since the fixed choice (a)–(f) underlying this derivation is arbitrary and since the assumed form for $n = m$ holds for any such choice, the result (10.27) also holds for any choice of (a)–(f).

Steps 1, 2 and 3 above complete the proof of the Claim. The caveat in footnote 45 restricts the validity of the proof to the case that $n \leq k - 1$, consistent with the Claim.

B. Anomaly free single commutators

In Sec. X. B. 1 we summarize our notation. In Sec. X. B. 2 we compute the action of a multiple product of Hamiltonian constraints multiplied by a single electric diffeomorphism constraint. We use this in Sec. X. B. 3 to compute the action of a multiple product of Hamiltonian constraints multiplied by a single commutator between a pair of electric diffeomorphism constraints. We show that the result is the same as that of the action of this product of Hamiltonian constraints multiplied by the appropriate commutator between a pair of Hamiltonian constraints. Hence this single commutator between a pair of Hamiltonian constraints is anomaly free in the sense that it can be replaced, within the particular string of operators under consideration, by the commutator between a pair of electric diffeomorphism constraints in line with (2.11). In Sec. X. C we use this result to show that each of the commutators in (1.1) is anomaly free in the sense that each of them can be replaced by a corresponding appropriate electric diffeomorphism commutator.

1. Notation

We denote the nondegenerate vertex of c by v and its associated reference coordinate patch by $\{x\}$. As in Sec. X. A it will suffice to develop notation only for singly deformed states. The single deformations of interest will be denoted as

$$[i,I,\hat{J},\hat{K},\beta=0,\delta]_1 = (i,I,\hat{J}_1,\hat{K}_1,\delta). \tag{10.28}$$

The vertex of the singly deformed state $c_{[i,I,\hat{J},\hat{K},\delta]_1}$ is denoted by $v_{[i,I,\delta]_1}$ and its associated *contraction* coordinate patch by $\{x^\delta\}$. The single deformation of (10.28) will play the role of the first of $m + 1$ deformations applied to c in Sec. X. B. 2 and the role of the first of $m + 2$ deformations in Sec. X. B. 3. Accordingly, in relation to the notation of Sec. VI. C, in Sec. X. B. 2 we set

$$\{x_\alpha\} \equiv \{x\}, \quad j_1 = m+1, \quad \epsilon_{\mathrm{j}_1} \equiv \delta, \quad \{x_\alpha^{\epsilon_{\mathrm{j}_1}}\} \equiv \{x^\delta\}, \tag{10.29}$$

and in Sec. X. B. 3 we set

$$\{x_\alpha\} \equiv \{x\}, \quad j_1 = m+2, \quad \epsilon_{\mathrm{j}_1} \equiv \delta, \quad \{x_\alpha^{\epsilon_{\mathrm{j}_1}}\} \equiv \{x^\delta\}. \tag{10.30}$$

2. Single electric diffeomorphism

In this Sec. we evaluate the action of $(\prod_{\mathrm{i}=1}^m \hat{C}(N_{\mathrm{i}}))\times \hat{D}(\vec{N}_{m+1i})$. We have that

$$(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)\hat{D}(\vec{N}_{m+1i})|c\rangle = \lim_{\delta\to 0}(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)\hat{D}_\delta(\vec{N}_{m+1i})|c\rangle. \tag{10.31}$$

Using (8.6) we obtain

$$(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)\hat{D}_\delta(\vec{N}_{m+1i})|c\rangle$$
$$=\frac{3\hbar}{4\pi \mathrm{i}}N(x(v))\nu_v^{-2/3}\frac{1}{(N-1)(N-2)\delta}\sum_{I\hat{J}_1,\hat{K}_1}\left((\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)|c_{[i,I,\hat{J},\hat{K},\delta]_1}\rangle-(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)|c\rangle\right). \tag{10.32}$$

We use (10.8) and (10.9) to evaluate the amplitudes in the last line. The second amplitude admits a direct application of these equations with some fixed choice for (a)–(f), Sec. X. A. 2. These equations are applied to the evaluation of the first amplitude with the choices (a')–(f') outlined in Step 2, Sec. X. A. 4 except that we set $\beta = 0$ there. The calculational details differ slightly for m even and m odd.

Case A: m even.—Using the contraction behavior of various quantities in Appendixes F, G. 2 and the notation (10.29) and (9.40), we obtain

$$\sum_{\hat{J}_1,\hat{K}_1}(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)|c_{[i,I,\hat{J},\hat{K},\delta]_1}\rangle$$
$$= (-3)^{k_m}\left(\frac{3\hbar N}{8\pi i}\right)^m(\nu_{v_{[i,I\delta]_1}}^{-\frac{2}{3}})^m g_c Q(c_{0[i,I,\delta_0]_1^{1,0}}, S_{m+1})h_I\Big\{|\vec{q}_{L_1=I}|^m(N-1)(N-2)+\cos^m(\theta)\left(\sum_{L_1\neq I}|\vec{q}_{L_1\neq I}|^m\right)$$
$$\times(N-2)(1+\cos^2\theta+(N-3)|\cos\theta|)\Big\}\prod_{\mathrm{i}=1}^{m}N_{\mathrm{m-i+1}}^{a_{\mathrm{m-i+1}}}(p,\{x\})\hat{V}_I^{a_{\mathrm{m-i+1}}}(p)\partial_{a_{\mathrm{m-i+1}}}(f(p,\{x\})\sqrt{h_{ab}(p)\hat{V}_I^a(p)\hat{V}_I^b(p)})|_{p=v_{[i,I,\delta]_1}}$$
$$+O(\delta^2). \tag{10.33}$$

We set

$$Q(c_{0[i,I,\delta_0]_1^{1,0}}, S_{m+1})$$
$$=\frac{(\nu_v^{-\frac{2}{3}})^m}{(\nu_{v_{[i,I\delta]_1}}^{-\frac{2}{3}})^m}\left(\frac{(N)(N-1)(N-2)|\vec{q}_I|^m}{\{|\vec{q}_{L_1=I}|^m(N-1)(N-2)+\cos^m(\theta)(\sum_{L_1\neq I}|\vec{q}_{L_1\neq I}|^m)(N-2)(1+\cos^2\theta+(N-3)|\cos\theta|)\}}\right). \tag{10.34}$$

For m even, clearly Q is positive.[46]

Expanding the contribution of (10.33) to (10.32) in a Taylor approximation in powers of δ subject to footnote 45, the zeroth order contribution cancels the contribution from the second term in the last line of (10.32). Only the first order contribution remains in the $\delta \to 0$ limit in (10.32) and we obtain

$$(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)\hat{D}(\vec{N}_{m+1i})|c\rangle = \lim_{\delta\to 0}(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)\hat{D}_\delta(\vec{N}_{m+1i}|c\rangle$$
$$= (-3)^{k_m}2\left(\frac{3\hbar N}{8\pi i}\right)^{m+1}(\nu_v^{-\frac{2}{3}})^{m+1}g_c\sum_I|\vec{q}_I|^m q_I^i h_I H_I^{m+1}(N_1,..,N_{m+1};v) \tag{10.35}$$

where we have used the notation (9.40) so that

$$H_I^{m+1}(N_1,..,N_{m+1};v) \coloneqq \prod_{i=1}^{l}N_{l-i+1}(p,\{x\})\hat{V}_{L_m}^{a_{l-i+1}}(p)\partial_{a_{l-i+1}}(f(p,\{x\})\sqrt{h_{ab}(p)\hat{V}_{L_m}^a(p)\hat{V}^b{}_{L_m}(p)})\Big|_{p=v}. \tag{10.36}$$

The reader may skip to Sec. X. B. 3 wherein we continue on to the electric diffeomorphism commutator calculation for m even in Sec. X. B. 3.

Case B: m odd.—Using Appendixes F, G. 2 and the notation (10.29) and (9.40), we obtain

$$\sum_{\hat{J}_1,\hat{K}_1}\left(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)|c_{[i,I,\hat{J},\hat{K},\delta]_1}\rangle\right.$$
$$= (-3)^{k_m}\left(\frac{3\hbar N}{8\pi i}\right)^m(\nu_{v_{[i,I\delta]_1}}^{-\frac{2}{3}})^m g_c Q(c_{0[i,I,\delta_0]_1^{1,0}}, S_{m+1})h_I\Big\{|\vec{q}_{L_1=I}|^{m-1}\left(\sum_{i_1}q_{L_1=I}^{i_1}\right)(N-1)(N-2)$$
$$+\cos^m(\theta)\left(\sum_{L_1\neq I}|\vec{q}_{L_1}|^{m-1}\left(\sum_{i_1}q_{L_1}^{i_1}\right)\right)(N-2)(1+\cos^2\theta+(N-3)|\cos\theta|)\Big\}$$
$$\times\left(\prod_{\mathrm{i}=1}^{m}N_{\mathrm{m-i+1}}^{a_{\mathrm{m-i+1}}}(p,\{x\})\hat{V}_I^{a_{\mathrm{m-i+1}}}(p)\partial_{a_{\mathrm{m-i+1}}}(f(p,\{x\})\sqrt{h_{ab}(p)\hat{V}_I^a(p)\hat{V}_I^b(p)})\right)\Bigg)_{p=v_{[i,I,\delta]_1}}+O(\delta^2). \tag{10.37}$$

[46]Note that the Q factors in (10.34) and (10.22) are identical functions of their associated child-parent charges. In principle they could have been chosen to differ from each other because their sequence labels differ in that the $m+1$th operator type is h for (10.22) and d_i for (10.34).

We set

$$Q(c_{0[i,I,\delta_0]_1^{1,0}},S_{m+1}) = \frac{(\nu_v^{-\frac{2}{3}})^m}{(\nu_{v_{[i,I\delta]_1}}^{-\frac{2}{3}})^m} \frac{(N)(N-1)(N-2)|\vec{q}_I|^{m-1}(\sum_j q_I^j))}{|\vec{q}_{L_1=I}|^m(\sum_{i_1} q_{L_1=I}^{i_1})(N-1)(N-2)+\cos^m(\theta)(\sum_{L_1\neq I}|\vec{q}_{L_1}|^{m-1}(\sum_{i_1} q_{L_1}^{i_1}))(N-2)(1+\cos^2\theta+(N-3)|\cos\theta|)}. \tag{10.38}$$

Since the transition $[i,I,\hat{J},\hat{K},\delta]_1$ is an electric diffeomorphism type, we have that

$$q^i_{L_1=I} = q^i_I. \tag{10.39}$$

Using this with (10.1) and (10.2) implies that $Q>0$. Expanding the contribution of (10.37) to (10.32) in a Taylor approximation in powers of δ subject to footnote 45, the zeroth order contribution cancels the contribution from second term in the last line of (10.32). Only the first order contribution remains in the $\delta\to 0$ limit in (10.32) and we obtain

$$\begin{aligned}(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^m \hat{C}(N_\mathrm{i})\right)\hat{D}(\vec{N}_{m+1i})|c\rangle &= \lim_{\delta\to 0}(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^m \hat{C}(N_\mathrm{i})\right)\hat{D}_\delta\left(\vec{N}_{m+1i}|c\rangle\right.\\ &= (-3)^{k_m}2\left(\frac{3\hbar N}{8\pi i}\right)^{m+1}(\nu_v^{-\frac{2}{3}})^{m+1}g_c\sum_I|\vec{q}_I|^{m-1}\left(\sum_j q_I^j\right)\Big)q_I^i h_I H_I^{m+1}(N_1,..,N_{m+1};v)\end{aligned} \tag{10.40}$$

where, as in (10.35), we have used the notation (9.40).

3. Electric diffeomorphism commutator

In this section we evaluate the action of $(\prod_{\mathrm{i}=1}^m \hat{C}(N_\mathrm{i}))[\hat{D}(\vec{N}_{m+1i}),\hat{D}(\vec{N}_{m+2i})]$. We have that

$$\begin{aligned}&(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^m \hat{C}(N_\mathrm{i})\right)[\hat{D}(\vec{N}_{m+1i}),\hat{D}(\vec{N}_{m+2i})]|c\rangle\\ &\quad= \lim_{\delta\to 0}(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^m \hat{C}(N_\mathrm{i})\right)(\hat{D}(\vec{N}_{m+1i}),\hat{D}_\delta(\vec{N}_{m+2i})-\hat{D}(\vec{N}_{m+2i}),\hat{D}_\delta(\vec{N}_{m+1i}))|c\rangle.\end{aligned} \tag{10.41}$$

Using (8.6) we obtain

$$\begin{aligned}&(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^m \hat{C}(N_\mathrm{i})\right)\hat{D}(\vec{N}_{m+1i})\hat{D}_\delta(\vec{N}_{m+2i}|c\rangle\\ &\quad= \frac{3\hbar}{4\pi\mathrm{i}}N_{m+2}(v,\{x\})\nu_v^{-2/3}\frac{1}{(N-1)(N-2)\delta}\\ &\qquad\times\sum_{I\hat{J}_1,\hat{K}_1}\left((\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^m \hat{C}(N_\mathrm{i})\right)\hat{D}(\vec{N}_{m+1i})|c_{[i,I,\hat{J},\hat{K},\delta]_1}\rangle-(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^m \hat{C}(N_\mathrm{i})\right)\hat{D}(\vec{N}_{m+1i})|c\rangle\right).\end{aligned} \tag{10.42}$$

We may use (10.35) and (10.40) to evaluate the amplitudes in the last two lines. The second amplitude admits a direct application of these equations with some fixed choice for (a)–(f), Sec. X. A. 2. It is straightforward to check that the application of (10.35) and (10.40) to the evaluation of the second amplitude results in an expression with an overall factor $N_{m+2}(v,\{x\})N_{m+1}(v,\{x\})$. It then follows from (10.41) that this term does not contribute to the commutator and, hence, we disregard it.

Equations (10.35) and (10.40) may be applied to the evaluation of the first amplitude with the choices (a')–(f') outlined in Step 2, Sec. X. A. 4 except that, once again, we set $\beta=0$ there. The calculational details for this contribution differ slightly for m even and m odd.

Case A: m even.—Using (10.35) as indicated above to evaluate the first amplitude in the last line of (10.42), we obtain its contraction limit using the Appendixes F, G. 2 together with (10.30) to be

$$\sum_{\hat{J}_1,\hat{K}_1} (\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)\hat{D}(\vec{N}_{m+1i})|c_{[i,I,\hat{J},\hat{K},\delta]_1}\rangle$$
$$= (-3)^{k_m}2\left(\frac{3\hbar N}{8\pi i}\right)^{m+1}(\nu_{v_{[i,I\delta]_1}}^{-\frac{2}{3}})^{m+1}g_c Q(c_{0[i,I,\delta_0]_1^{1,0}},S_{m+2})h_I\{|\vec{q}_{L_1=I}|^m q^i_{L-1=I}(N-1)(N-2)$$
$$+\cos^{m+1}(\theta)\left(\sum_{L_1\neq I}|\vec{q}_{L_1}|^m q^i_{L_1}\right)(N-2)(1+\cos^2\theta+(N-3)|\cos\theta|)\}$$
$$\times\left(\prod_{\mathrm{i}=1}^{m+1}N^{a_{\mathrm{m-i+1}}}_{\mathrm{m-i+1}}(p,\{x\})\hat{V}_I^{a_{\mathrm{m-i+1}}}(p)\partial_{a_{\mathrm{m-i+1}}}(f(p,\{x\})\sqrt{h_{ab}(p)\hat{V}_I^a(p)\hat{V}_I^b(p)})\Big|_{p=v_{[i,I,\delta]_1}}\right)+O(\delta^2). \tag{10.43}$$

Note that the transition $[i,I,\hat{J},\hat{K},\delta]_1$ is an electric diffeomorphism type deformation so that

$$q^i_{L_1=I}=q^i_I. \tag{10.44}$$

We set for some $A>0$:

$$Q(c_{0[i,I,\delta_0]_1^{1,0}},S_{m+2})$$
$$=\frac{(\nu_v^{-\frac{2}{3}})^{m+1}}{(\nu_{v_{[i,I\delta]_1}}^{-\frac{2}{3}})^{m+1}}\frac{A(N)(N-1)(N-2)|\vec{q}_I|^m q^i_I}{\{|\vec{q}_{L_1=I}|^m q^i_{L_1=I}(N-1)(N-2)+\cos^{m+1}(\theta)(\sum_{L_1\neq I}|\vec{q}_{L_1}|^m q^i_{L_1})(N-2)(1+\cos^2\theta+(N-3)|\cos\theta|\}}. \tag{10.45}$$

Equation (10.44) together with (10.1) and (10.2) once again implies that $Q>0$. Next, we expand (10.43) in a Taylor expansion in powers of δ. It is easy to check that the zeroth order term does not contribute to the commutator (10.42). Only the first order term contributes. Using this first order term in (10.42) and taking the contraction limit, we obtain

$$(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)[\hat{D}(\vec{N}_{m+1i}),\hat{D}(\vec{N}_{m+2i})]|c\rangle$$
$$=(-3)^{k_m}A4\left(\frac{3\hbar N}{8\pi i}\right)^{m+2}(\nu_v^{-\frac{2}{3}})^{m+2}g_c\sum_I|\vec{q}_I|^m(q^i_I)^2h_I\Big\{N_{\mathrm{m+2}}(p,\{x\})\hat{V}_I^a(p)\partial_a$$
$$\times\left(\prod_{\mathrm{i}=1}^{m+1}N^{a_{\mathrm{m-i+1}}}_{\mathrm{m-i+1}}(p,\{x\})\hat{V}_I^{a_{\mathrm{m-i+1}}}(p)\partial_{a_{\mathrm{m-i+1}}}(f(p,\{x\})\sqrt{h_{ab}(p)\hat{V}_I^a(p)\hat{V}_I^b(p)}\right)-N_{\mathrm{m+2}}(p,\{x\})\leftrightarrow N_{\mathrm{m+1}}(p,\{x\})\Big\}_{p=v}. \tag{10.46}$$

Summing over i in (10.46) and using the notation (9.40) and the definition of the charge norm (10.7), we obtain

$$(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)[\hat{D}(\vec{N}_{m+1i}),\hat{D}(\vec{N}_{m+2i})]|c\rangle$$
$$=(4A)(-3)^{k_m}\left(\frac{3\hbar N}{8\pi i}\right)^{m+2}(\nu_v^{-\frac{2}{3}})^{m+2}g_c\sum_I\{|\vec{q}_I|^{m+2}h_I(H_I^{m+2}(N_1,..,N_{m+1},N_{m+2};v)-H_I^{m+1}(N_1,..,N_{m+2},N_{m+1};v). \tag{10.47}$$

On the other hand, replacing the commutator $\sum_i[\hat{D}(\vec{N}_{m+1i}),\hat{D}(\vec{N}_{m+2i})]$ with $[\hat{C}(N_{m+1}),\hat{C}(N_{m+2})]$ and noting that $m+2$ is even, we obtain, from (10.8)

$$(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)[\hat{C}(N_{m+1}),\hat{C}(N_{m+2})]|c\rangle$$
$$=(-3)^{k_{m+2}}\left(\frac{3\hbar N}{8\pi i}\right)^{m+2}(\nu^{-\frac{2}{3}})^{m+2}g_c\sum_I|\vec{q}_I|^{m+2}h_I(H_I^{m+2}(N_1,..,N_{m+1},N_{m+2};v)-H_I^{m+1}(N_1,..,N_{m+2},N_{m+1};v)). \quad (10.48)$$

From (10.6) we have that $k_{m+2}=k_m+1$. It is then easy to see that an anomaly free commutator results if we choose $A=1/4$ in (10.47).

Case B: m odd.—Using (10.40) as indicated above to evaluate the first amplitude in the last line of (10.42), we obtain its contraction limit using the Appendixes F, G. 2 and Eq. (10.30) as

$$\sum_{\hat{J}_1,\hat{K}_1}(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)\hat{D}(\vec{N}_{m+1i})|c_{[i,I,\hat{J},\hat{K},\delta]_1}\rangle$$
$$=(-3)^{k_m}2\left(\frac{3\hbar N}{8\pi i}\right)^{m+1}(\nu_{v_{[i,I\delta]_1}}^{-\frac{2}{3}})^{m+1}g_cQ(c_{0[i,I,\delta_0]_1^{1,0}},S_{m+2})h_I\Bigg\{|\vec{q}_{L_1=I}|^{m-1}\left(\sum_{j_1}q_{L_1=I}^{j_1}\right)q_{L-1=I}^{i_1=i}(N-1)(N-2)$$
$$+\cos^{m+1}(\theta)\left(\sum_{L_1\neq I}|\vec{q}_{L_1}|^{m-1}\left(\sum_{i_1}q_{L_1}^{i_1}\right)q_{L_1}^{i_1=i}\right)(N-2)(1+\cos^2\theta+(N-3)|\cos\theta|)\Bigg\}$$
$$\times\left(\prod_{\mathrm{i}=1}^{m+1}N_{\mathrm{m-i+1}}^{a_{\mathrm{m-i+1}}}(p,\{x\})\hat{V}_I^{a_{\mathrm{m-i+1}}}(p)\partial_{a_{\mathrm{m-i+1}}}(f(p,\{x\})\sqrt{h_{ab}(p)\hat{V}_I^a(p)\hat{V}_I^b(p)})\Big|_{p=v_{[i,I,\delta]_1}}\right)+O(\delta^2). \quad (10.49)$$

Note that the transition $[i,I,\hat{J},\hat{K},\delta]_1$ is an electric diffeomorphism type deformation so that

$$q_{L_1=I}^{i_1=i}=q_I^i. \quad (10.50)$$

We set for some $A>0$

$$Q(c_{0[i,I,\delta_0]_1^{1,0}},S_{m+2})$$
$$=\frac{(\nu_v^{-\frac{2}{3}})^{m+1}}{(\nu_{v_{[i,I\delta]_1}}^{-\frac{2}{3}})^{m+1}}$$
$$\times\frac{A(N)(N-1)(N-2)|\vec{q}_I|^{m-1}(\sum_jq_I^j)q_I^i}{|\vec{q}_{L_1=I}|^{m-1}(\sum_{i_1}q_{L_1=I}^{i_1})q_{L_1=I}^i(N-1)(N-2)+\cos^{m+1}(\theta)(\sum_{L_1\neq I}|\vec{q}_{L_1}|^{m-1}(\sum_{i_1}q_{L_1}^{i_1})q_{L_1}^i)(N-2)(1+\cos^2\theta+(N-3)|\cos\theta|)}. \quad (10.51)$$

Equation (10.50) together with (10.1) and (10.2) once again implies that $Q>0$. Expanding (10.43) in a Taylor expansion in powers of δ, it is easy to check that the zeroth order term does not contribute to the commutator (10.42). Only the first order term contributes. Using this first order term in (10.42) and taking the contraction limit, we obtain

$$(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)[\hat{D}(\vec{N}_{m+1i}),\hat{D}(\vec{N}_{m+2i})]|c\rangle$$
$$=(-3)^{k_m}A4\left(\frac{3\hbar N}{8\pi i}\right)^{m+2}(\nu_v^{-\frac{2}{3}})^{m+2}g_c\sum_I|\vec{q}_I|^{m-1}\left(\sum_jq_I^j\right)(q_I^i)^2h_I\Bigg\{N_{\mathrm{m+2}}(p,\{x\})\hat{V}_I^a(p)\partial_a$$
$$\times\left(\prod_{\mathrm{i}=1}^{m+1}N_{\mathrm{m-i+1}}^{a_{\mathrm{m-i+1}}}(p,\{x\})\hat{V}_I^{a_{\mathrm{m-i+1}}}(p)\partial_{a_{\mathrm{m-i+1}}}\left(f(p,\{x\})\sqrt{h_{ab}(p)\hat{V}_I^a(p)\hat{V}_I^b(p)}\right)-N_{\mathrm{m+2}}(p,\{x\})\leftrightarrow N_{\mathrm{m+1}}(p,\{x\})\right\}_{p=v}. \quad (10.52)$$

Summing over i in (10.52) and using the notation (9.40) and the definition of the charge norm (10.7), we obtain

$$(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)[\hat{D}(\vec{N}_{m+1i}),\hat{D}(\vec{N}_{m+2i})]|c\rangle$$
$$=(4A)(-3)^{k_m}\left(\frac{3\hbar N}{8\pi i}\right)^{m+2}(\nu_v^{-\frac{2}{3}})^{m+2}g_c\sum_I\{|\vec{q}_I|^{m+1}\left(\sum_j q_I^j\right)h_I(H_I^{m+2}(N_1,..,N_{m+1},N_{m+2};v)-H_I^{m+1}(N_1,..,N_{m+2},N_{m+1};v). \tag{10.53}$$

On the other hand, replacing the commutator $\sum_i[\hat{D}(\vec{N}_{m+1i}),\hat{D}(\vec{N}_{m+2i})]$ with $[\hat{C}(N_{m+1}),\hat{C}(N_{m+2})]$ and noting that $m+2$ is odd, we obtain, from (10.9)

$$(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)[\hat{C}(N_{m+1}),\hat{C}(N_{m+2})]|c\rangle$$
$$=(-3)^{k_{m+2}}\left(\frac{3\hbar N}{8\pi i}\right)^{m+2}(\nu^{-\frac{2}{3}})^{m+2}g_c$$
$$\times\sum_I|\vec{q}_I|^{m+1}\left(\sum_j q_I^j\right)h_I(H_I^{m+2}(N_1,..,N_{m+1},N_{m+2};v)-H_I^{m+1}(N_1,..,N_{m+2},N_{m+1};v). \tag{10.54}$$

From (10.6) we have that $k_{m+2}=k_m+1$. It is then easy to see that, once again, an anomaly free commutator results if we choose $A=1/4$ in (10.53).

C. Multiple single anomaly free commutators

Consider the action of the operator in Eq. (1.1) on the anomaly free state Ψ_{f,h_{ab},P_0}. In this section we show this action is invariant under the replacement of each Hamiltonian constraint commutator in (1.1) by a corresponding electric diffeomorphism commutator, this replacement being a quantum implementation of the anomaly free condition (2.11). We proceed as follows.

First we further develop the notation for operator sequence labels of Q factors developed in Sec. VIII. D as follows. Consider the sequence

$$(\underbrace{h,..,h}_{m_1},t_1,t_1,\underbrace{h,..,h}_{m_2},t_2,t_2,\underbrace{h,..,h}_{m_n},t_n,t_n,\underbrace{h,..,h}_{p_n}) \tag{10.55}$$

where each t_i is either h or d_k, $k\in 1,2,3$ and m_i, p_n are whole numbers. The Q factors which we define below only depend on whether or not a t_i is Hamiltonian or electric diffeomorphism; the particular component of the electric diffeomorphism does not matter. Since the β factors for the Hamiltonian constraint are such that $\beta^2=1$ and since $\beta=0$ for an electric diffeomorphism, we denote the essential part of the sequence above through the symbol σ_n as follows:

$$\sigma_n(m_1,m_2,..,m_n;\beta_1^2,..,\beta_n^2;p_n). \tag{10.56}$$

Thus, the specification of the arguments of σ_n allow us to reconstruct the sequence (10.55) up to irrelevant (for the Q factors of interest) ambiguities regarding the specific components of electric diffeomorphism operators in such a sequence. Next, consider any operator product of the type $\hat{O}_{\epsilon_1,..,\epsilon_{q_n}}(M_1,..,M_{q_n})$ corresponding to a discrete approximant for the operator product $\hat{O}(M_1,..,M_{q_n})$ where each of the operators in the operator product are either Hamiltonian or electric diffeomporphism operators similar to the operator product in (6.18). Let the sequence associated with this operator product be $\mathcal{S}_{q_n}$ and any subsequence of this sequence of operators from the first to the $\mathrm{j_k}$th, $\mathrm{j_k}\leq q_n$ be $\mathcal{S}_{\mathrm{j_k}}$.

Next consider a "big" operator product consisting of the sequence of operators of type (10.56) followed by the operator product $\hat{O}_{\epsilon_1,..,\epsilon_{q_n}}(M_1,..,M_{q_n})$ where the latter occurs to the right of the former and so acts first on any charge net c. We denote the operator sequence for such a big product by

$$S(\sigma(m_1,m_2,..,m_n;\beta_1^2,..,\beta_n^2;p_n),\mathcal{S}_{q_n})$$
$$\equiv\sigma(m_1,m_2,..,m_n;\beta_1^2,..,\beta_n^2;p_n),\mathcal{S}_{q_n}. \tag{10.57}$$

If, in this big operator product, we replace $\hat{O}_{\epsilon_1,..,\epsilon_{q_n}}(M_1,..,M_{q_n})$ by an operator consisting of the product of the first $\mathrm{j_k}$ operators in $\hat{O}_{\epsilon_1,..,\epsilon_{q_n}}(M_1,..,M_{q_n})$, then we denote the sequence corresponding to the new big operator product by

$$S(\sigma_n(m_1,m_2,..,m_n;\beta_1^2,..,\beta_n^2;p_n),\mathcal{S}_{\mathrm{j_k}})$$
$$\equiv\sigma_n(m_1,m_2,..,m_n;\beta_1^2,..,\beta_n^2;p_n),\mathcal{S}_{\mathrm{j_k}}. \tag{10.58}$$

We shall be interested in Q factors for child-parent contractions $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_k^{k-1,k}}$ where the child and parent states are generated by the action of $\hat{O}_{\epsilon_1,..,\epsilon_{q_n}}(M_1,..,M_{q_n})$ on some charge net c and when the operator product sequence label for Q is of the type (10.56) or (10.57). The Q factor for such a situation in the case wherein all the operators in (10.55) are Hamiltonian [so that all the β_i^2 are unity in (10.56)] is, using the above notation in conjunction with that of Sec. VIII. D, then

$$Q(c_{0[i,I,\beta,\delta_0]_k^{k-1,k}}, S(\sigma_n(m_1,m_2,..,m_n;1,1,..1;p_n),\mathcal{S}_{\mathrm{j_k}})) \\ \equiv Q(c_{0[i,I,\beta,\delta_0]_k^{k-1,k}};\sigma_n(m_1,m_2,..,m_n;1,1,..1;p_n),\mathcal{S}_{\mathrm{j_k}}). \tag{10.59}$$

We define the Q factors labeled by the sequence (10.58) for the transition $c_{[i,I,\beta,\delta_0]_k^{k-1,k}}$ to be such that these Q factors for all choices of σ_n in (10.58) are the same as that in (10.59):

$$Q(c_{0[i,I,\beta,\delta_0]_k^{k-1,k}};\sigma_n(m_1,m_2,..,m_n;\beta_1^2,..,\beta_n^2;p_n),\mathcal{S}_{\mathrm{j_k}}) \\ \coloneqq Q(c_{0[i,I,\beta,\delta_0]_k^{k-1,k}};\sigma_n(m_1,m_2,..,m_n;1,1,..1;p_n),\mathcal{S}_{\mathrm{j_k}}), \\ \forall \mathrm{j_k}\in\{1,..,q_n\}, q_n>0, \quad \forall\ \beta_i^2\in\{0,1\}, \\ \forall\ p_n, m_i, i=1,..,n \tag{10.60}$$

where p_n and m_i, $i=1,..,n$ range over the set of whole numbers.

It is straightforward to see that the kinds of operator products implicated in a demonstration that operator strings of the type (1.1) have anomaly free single commutators are exactly those for which the sequence labels are of the type (10.56). We now construct such a demonstration through an inductive proof on the index n which occurs in (10.56). Note that the index n corresponds to the number of single commutators involved.

First consider the case $n=1$. Let $\hat{O}_{\epsilon_1,..,\epsilon_{r_1}}(M_1,..,M_{r_1})$, $r_1>0$ be a product of r_1 Hamiltonian constraints. Then using Sec. X. B together with Eq. (10.60) with $q_1\coloneqq r_1$ it follows that for any $r_1>0$ we have that

$$(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right)[\hat{C}(N_{m+1}),\hat{C}(N_{m+2})]\hat{O}_{\epsilon_1,..,\epsilon_{r_1}} \\ \times(M_1,..,M_{r_1})|c\rangle \\ =(-3)\sum_i(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{m}\hat{C}(N_{\mathrm{i}})\right) \\ \times[\hat{D}(\vec{N}_{m+1i}),\hat{D}(\vec{N}_{m+2i})]\hat{O}_{\epsilon_1,..,\epsilon_{r_1}}(M_1,..,M_{r_1})|c\rangle. \tag{10.61}$$

From Sec. X. B, the above equation also holds if we replace $\hat{O}_{\epsilon_1,..,\epsilon_{r_1}}(M_1,..,M_{r_1})$ by the identity operator. Taking the continuum limit of (10.61) we see that the result on anomaly free single commutators holds for the case that $n=1$ with $p_1=r_1$ in (10.56). Similarly taking the continuum limit of the equation obtained by replacing $\hat{O}_{\epsilon_1,..,\epsilon_{r_1}}(M_1,..,M_{r_1})$ by the identity operator in (10.61), this result also holds for the case that $n=1$ and $p_1=0$. Thus we have established the desired result for the case that $n=1$ which corresponds to the case of a single commutator.

Next let us assume that the anomaly free single commutator property holds for all operator strings with sequences (10.55) for some $n=s$. More in detail consider an operator product consisting of m_1 Hamiltonian constraints and a Hamiltonian constraint commutator followed by m_2 Hamiltonian constraints and a Hamiltonian constraint commutator, all the way up to m_s Hamiltonian constraints and the sth Hamiltonian constraint commutator, followed by a product of p_s Hamiltonian constraints. Then the assumption is that in any such product any subset of these Hamiltonian commutators can be replaced by sums over electric diffeomorphism commutators in accordance with (2.11). From this assumption we now show that the same statement hold for $n=s+1$.

First define

$$\hat{O}(\sigma_n(m_1,m_2,..,m_n;1,1..,1;p_n)) \\ \coloneqq\prod_{j=1}^{n}\left(\left(\prod_{\mathrm{i}=1}^{m_j}\hat{C}(N_{\mathrm{i}})\right)[\hat{C}(A_j),\hat{C}(B_j)]\right)\left(\prod_{\mathrm{k}=1}^{p_n}\hat{C}(F_{\mathrm{k}})\right) \tag{10.62}$$

where the sequence on the left-hand side has all its β_i^2 as unity. Next, define the operator

$$\hat{O}(\sigma_n(m_1,m_2,..,m_n;\beta_1^2,..,\beta_n^2;p_n)) \tag{10.63}$$

as follows. For each i for which $\beta_i^2=0$ in $\sigma_n(m_1,m_2,..,m_n;\beta_1^2,..,\beta_n^2;p_n)$, replace the ith Hamiltonian constraint commutator in (10.62) by an appropriate sum over electric diffeomorphism commutators consistent with (2.11). Note that this $\hat{O}(\sigma_n)$ notation is consistent with (10.56) in that each of the constraint operator products obtained by expanding out the commutators in (10.63) correspond to the (same) sequence $\sigma_n(m_1,m_2,..,m_n;\beta_1^2,..,\beta_n^2;p_n)$. In this notation our assumption for $n=s$ may be written as

$$(\Psi_{f,h_{ab},P_0}|\hat{O}(\sigma_s(m_1,m_2,..,m_s;1..,1;p_s))|c\rangle \\ =(\Psi_{f,h_{ab},P_0}|\hat{O}(\sigma_s(m_1,m_2,..,m_s;\beta_1^2,..,\beta_s^2;p_s))|c\rangle \\ \forall\ \{\beta_i^2,m_i,i=1,..,s\}\quad\text{and}\quad\forall p_s. \tag{10.64}$$

Next let $\hat{O}^1_{\epsilon_1,\epsilon_2}(G_1,G_2)$ be the discrete approximant to the Hamiltonian constraint commutator $[\hat{C}(G_1),\hat{C}(G_2)]$ and let $\hat{O}^2_{\epsilon_1,\epsilon_2}(G_1,G_2)$ be the discrete approximant to the appropriate sum over electric diffeomorphism commutators through (2.11). Since the action of these discrete approximants is a finite linear combination of charge nets with Q factors as defined in (10.60), it follows for $\alpha = 1, 2$ that

$$(\Psi_{f,h_{ab},P_0}|\hat{O}(\sigma_s(m_1, m_2, .., m_s; 1.., 1; p_s))\hat{O}^{\alpha}_{\epsilon_1,\epsilon_2}(G_1, G_2)|c\rangle$$
$$= (\Psi_{f,h_{ab},P_0}|\hat{O}(\sigma_s(m_1, m_2, .., m_s; \beta_1^2, .., \beta_s^2; p_s))$$
$$\hat{O}^{\alpha}_{\epsilon_1,\epsilon_2}(G_1, G_2)|c\rangle \quad \forall \{\beta_i^2, i = 1, .., s\}. \quad (10.65)$$

It is then straightforward to see that taking the continuum limit of the left-hand and right-hand sides and applying the anomaly free single commutator result of Sec. X. B yields the result

$$(\Psi_{f,h_{ab},P_0}|\hat{O}(\sigma_{s+1}(m_1, m_2, .., m_{s+1}; 1.., 1; p_{s+1} = 0))|c\rangle$$
$$= (\Psi_{f,h_{ab},P_0}|\hat{O}(\sigma_{s+1}(m_1, m_2, .., m_{s+1}; \beta_1^2, .., \beta_{s+1}^2;$$
$$p_{s+1} = 0))|c\rangle \quad \forall \{\beta_i^2, i = 1, .., s+1\} \quad (10.66)$$

where we have set $p_s \coloneqq m_{s+1}$. Next consider the approximant $\hat{O}_{\vec{\epsilon}} \coloneqq \prod_{t=1}^{p_{s+1}} \hat{C}_{\epsilon_t}(P_t)$ to a product of $p_{s+1} \neq 0$ Hamiltonian constraint operators. Again using (10.60) we may substitute $|c\rangle$ in (10.66) by the finite linear combination of charge nets $\hat{O}_{\vec{\epsilon}}|c\rangle$. Taking the continuum limits of the resulting equations, we obtain

$$(\Psi_{f,h_{ab},P_0}|\hat{O}(\sigma_{s+1}(m_1, m_2, .., m_{s+1}; 1.., 1; p_{s+1}))|c\rangle$$
$$= (\Psi_{f,h_{ab},P_0}|\hat{O}(\sigma_{s+1}(m_1, m_2, .., m_{s+1}; \beta_1^2, .., \beta_{s+1}^2;$$
$$p_{s+1}))|c\rangle \quad \forall \{\beta_i^2, i = 1, .., s+1\}, \quad (10.67)$$

which is the desired result for $n = s + 1$. This completes our inductive proof of an anomaly free single commutator implementation of the algebra of Hamiltonian constraints. It only remains to show that this implementation is diffeomorphism covariant. We show this in the next section.

XI. DIFFEOMORPHISM COVARIANCE

We implement diffeomorphism covariance of the continuum limit action of products of constraint operators on any anomaly free basis state by tailoring the underlying discrete action to the metric label of the basis state being acted upon. This idea, of tailoring the action of a discrete approximant to an operator to the state it acts upon, is a familiar one in the case that the states lie in the kinematic Hilbert space of LQG (see for example [3,13,24], and also Sec. II of this paper). Here we apply this idea to the space of kinematically non-normalizable anomaly free states.

As a prelude to the detailed technical description in Secs. XI. A and XI. B below, we now describe the broad idea behind this implementation. Recall from Sec. VII. B that an anomaly free basis state is labeled by a density $-1/3$ function f, a metric with no conformal symmetries h_{ab} and a choice of bra set B_{P0}. Given this state Ψ_{f,h_{ab},P_0} we can construct all its amplitudes i.e. all the complex numbers $(\Psi_{f,h_{ab},P_0}|c\rangle$ for any charge net c. It then turns out that we can construct enough information about the metric h_{ab} from these amplitudes so as to distinguish this metric from any of its diffeomorphic images. If we restrict the space of permissible metric labels for anomaly free basis states to be the space of all diffeomorphic images of h_{ab}, this means that the metric label of any anomaly free basis state can be uniquely identified from the state itself through its amplitudes. In this sense the state "knows" about its metric label. Hence it is meaningful to define the discrete action of constraint operators on this state in such a way that this discrete action depends on the metric label of the state.

It turns out that the dual action of the unitary operator corresponding to a diffeomorphism ϕ maps a state with metric label h_{ab} to one with metric label h^{ϕ}_{ab}, where h^{ϕ}_{ab} is the image of h_{ab} by ϕ. The idea is then to use the metric label h^{ϕ}_{ab} to identify the diffeomorphism ϕ, since, due to the lack of (conformal) isometries, any permissible metric label is uniquely associated with the diffeomorphism which maps h_{ab} to this label. Then for this metric label h^{ϕ}_{ab} we choose the primary coordinates and reference diffeomorphisms to be appropriate images, by ϕ of the primary coordinates and reference diffeomorphisms chosen for the state labeled by h_{ab}. These "image" structures are then used to regulate and define constraint operator products along the lines of Sec. VI and V. It can then be shown that this diffeomorphism covariant choice of regulating structures leads to a diffeomorphism covariant continuum limit action of products of constraints.

In Sec. XI. A we formulate and prove a precise statement which shows that anomaly free basis states have the requisite sensitivity to their metric labels. In Sec. XI. B we use this sensitivity to define a covariant choice of reference structures and express the action of finite diffeomorphisms on anomaly free states in the context of this covariant choice. In Sec. XI. C we demonstrate that this covariant choice results in an implementation of diffeomorphism covariance of the continuum limit action of products of constraints. For the remainder of this section we shall restrict attention to $-\frac{1}{3}$ density scalars f which vanish at most at a finite number of points in Σ.[47]

A. Metric label sensitivity of an anomaly free state

Let h_{0ab} be a metric which has no conformal symmetries. Let $\mathcal{H}_{h_0}$ be the space of all diffeomorphic images of h_{0ab} by all C^k semianalytic diffeomorphisms. Let h_{1ab}, h_{2ab} be two distinct elements of $\mathcal{H}_{h_0}$. Note that the two metrics cannot be conformally related everywhere because they are

[47] This is for technical simplicity; it seems plausible to us that our considerations can be generalized for the case where f is not restricted in this manner. We leave such a generalization for future work.

distinct, diffeomorphic to each other and have no conformal symmetries. Hence there exists a point $a \in \Sigma$, and, from the fact that the metrics are C^{k-1}, a neighborhood $U(a,\delta)$ of a for some small enough $\delta > 0$ such that in a fixed coordinate patch $\{y\}$ in this neighborhood, we have that

$$\left| \frac{h_{1ij}}{h_1^{\frac{1}{3}}} - \frac{h_{2ij}}{h_2^{\frac{1}{3}}} \right| > C. \tag{11.1}$$

The above inequality holds for every point in $U(a,\epsilon)$, for at least one fixed pair $i, j \in \{1,2,3\}$ and for some fixed positive constant C. Here our notation is such that s_{ij} denotes the coordinate components of the metric s_{ab} in the chart $\{y\}$ and s denotes its determinant in this chart. Thus Eq. (11.1) indicates that at least one component, in any fixed semianalytic chart, of the (conformally invariant) metric densities in this equation differ by some minimum non zero amount in a small enough neighborhood of at least one point of the Cauchy slice.[48]

Next, consider an anomaly free state which is labeled by some element s_{ab} of $\mathcal{H}_{h_0}$. We now show that at any point $p \in \Sigma$ this metric can be reconstructed, up to an overall scaling at p, to arbitrary accuracy from the amplitudes of the anomaly free state. Accordingly, fix a point p on the Cauchy slice Σ and consider some choice set (a)–(f) of Sec. X. A. With this choice set consider a primordial state c such that $c \in B_{P0}$ and such that c has its nondegenerate vertex at p with reference coordinate patch $\{x\}$. The action of a single electric diffeomorphism on c generates the child, $c_1 \equiv c_{(i,I,\hat{J},\hat{K},\beta=0,\delta)}$ where δ is measured by $\{x\}$. From Sec. VII, the amplitude $(\Psi_{f,s_{ab},P_0}|c_1\rangle$ of the anomaly free state Ψ_{f,s_{ab},P_0} for the state c_1 is evaluated using $\{x_1\}$ where $\{x_1\}$ denotes the reference coordinate patch for c_1 around v_1. In view of the fact that f vanishes only at a finite number of points, it follows that we can choose δ such that f is nonvanishing at v_1 and we so choose δ.

Next, consider any diffeomorphism χ which is identity in some neighborhood of v_1 and consider the state $c_{1\chi}$ which is the image of c_1 by χ. Let the reference coordinate patch for $c_{1\chi}$ be denoted by $\{x_{1\chi}\}$. It is straightforward to see that the Lemma in P2 implies that we can use the coordinates $\chi^*\{x_1\}$ to evaluate the amplitude $(\Psi_{f,s_{ab},P_0}|c_{1,\chi}\rangle$ instead of the reference coordinates $\{x_{1\chi}\}$. Note however that since χ is identity in a vicinity of v_1, this is the same as evaluating the amplitude with respect to $\{x_1\}$. Since the coordinate dependent part of the amplitude is $f\sum h_I H_I$ (see Sec. VII. B) and since this part only depends on the vertex structure of $c_{1\chi}$ at v_1 it follows that

$$(\Psi_{f,s_{ab},P_0}|c_{1,\chi}\rangle = Bg_{c_{1\chi}}, \tag{11.2}$$

$$(\Psi_{f,s_{ab},P_0}|c_1\rangle = Bg_{c_1} \tag{11.3}$$

where $B = f\sum_{L_1} h_{L_1} H_{L_1}$ and the coordinate dependent evaluation of the function f at v_1 and the coordinate dependent normalization of the edge tangent vectors $\vec{e}_{J_1}$ at v_1 are with respect to $\{x_1\}$ as argued above, both for c_1 and for $c_{1\chi}$.

Next, we construct diffeomorphisms which are identity in a neighborhood of v_1 but which move the C^0 kinks of c_1 to certain desired positions. Since these diffeomorphisms are of the type χ above, the amplitudes for the diffeomorphic images of c_1 by these diffeomorphisms satisfy (11.3) and, therefore, serve to evaluate the function g [see (G1), Appendix G] when its arguments have been placed at these desired positions. By placing these arguments at positions close enough to p, we may use the contraction behavior of g (see Appendix G. 2) to extract the information about the metric label s_{ab} in the vicinity of the point p. Accordingly we proceed as follows.

First, note that in the state c_1 the C^0 kink $\tilde{v}_{\hat{J}_1}$ lies at a distance δ^{p_1} from p along the Lth edge of c with $L = \hat{J}_1$, the C^0 kink $\tilde{v}_{\hat{K}_1}$ lies at a distance $Q\delta^{p_2}$ from p along Mth edge of c with $M = \hat{K}_1$. The values of p_2, p_1 are given in the Appendix G. 2. The exact specification and value of Q is not needed here. The remaining C^0 kinks lie within a distance δ^{p_3} of p where $p_3 > p_2$ (see Appendix G. 2), all these distances being measured by $\{x\}$. Next, consider any $\epsilon \ll \delta$. Clearly we can apply diffeomorphisms of the type constructed in (iii), Sec. VI. C to move kinks at coordinate distances $\delta^{p_1}, Q\delta^{p_2}, \delta^{p_3}$ to coordinate distances $\epsilon^{p_1}, Q\epsilon^{p_2}, \epsilon^{p_3}$. Further, these diffeomorphisms can be constructed in such a way that they are identity in a neighborhood of v_1. Let us apply these diffeomorphisms to c_1.

Next consider the region $R_{\epsilon,\tau}$ bounded by two spherical shells of radius $Q\epsilon^{p_2} \pm \tau$ around the vertex p of c, with $\tau \ll \epsilon^{p_3}$. Let $\vec{\xi}_\epsilon$ be any semianalytic vector field which is tangent to the sphere of radius $Q\epsilon^{p_2}$ around p and let $F_{\epsilon,\tau}$ be a semianalytic function which is 1 on this sphere and which vanishes outside R_τ. By choosing ξ_ϵ appropriately we can use an appropriate finite diffeomorphism generated by the vector field $F_{\epsilon,\tau}\vec{\xi}_\epsilon$ to move the point $\tilde{v}_{\hat{K}_1}$ to any desired location on the sphere of radius $Q\epsilon^{p_2}$ while leaving the positions of the remaining kinks unaltered. More in detail by moving this kink by such a diffeomorphism $\phi_{u,\epsilon}$ to a position on this sphere such that the straight line from the origin of the sphere at p to this position has unit tangent $\vec{\hat{u}}$, we obtain, from Appendix G. 2 that

$$g_{c_{1\phi_{u,\epsilon}}} = \epsilon^{p_2-p_1} Q \frac{\|\vec{\hat{u}}\|}{\|\vec{\hat{e}}_{\hat{J}_1}\|}(1 + O(\epsilon^{p_2-p_1}))g_c \tag{11.4}$$

where the metric norms are calculated at the point p. Clearly $\phi_{u,\epsilon}$ is of the type χ in (11.3). It follows that as $\epsilon \to 0$, we have that

[48]It is straightforward to see that transiting from one fixed coordinate chart to another only affects the value of the constant C and the choice of i, j in (11.1).

$$(\Psi_{f,s_{ab},P_0}|c_{1\phi_{u,\epsilon}}\rangle = B\epsilon^{p_2-p_1}Q\frac{\|\vec{\hat{u}}\|}{\|\vec{\hat{e}}_{\hat{j}}\|}(1+O(\epsilon^{p_2-p_1}))g_c \tag{11.5}$$

$$\Rightarrow B_1 s_{ab}\hat{u}^a\hat{u}^b(1+O(\epsilon^{(p_2-p_1)}) = \frac{((\Psi_{f,s_{ab},P_0}|c_{1\phi_{u,\epsilon}}\rangle)^2}{\epsilon^{2(p_2-p_1)}} \tag{11.6}$$

where $B_1 \coloneqq \frac{Q^2B^2}{s_{ab}\hat{e}^a_{\hat{j}}\hat{e}^b_{\hat{j}}}$ and $g_c = 1$ because c is primordial and has no C^0 kinks. Note that B_1 is independent of the position $\vec{\hat{u}}$. By varying the position $\vec{\hat{u}}$ and by choosing ϵ as small as we wish, clearly we can reconstruct the metric at p up to an overall factor to any desired accuracy. More in detail, let us suppress the labels on the right-hand side of (11.6) which do not vary with $\vec{\hat{u}}, \epsilon$ and set

$$\frac{((\Psi_{f,s_{ab},P_0}|c_{1\phi_{u,\epsilon}}\rangle)^2}{\epsilon^{2(p_2-p_1)}} \coloneqq F(\vec{\hat{u}},\epsilon). \tag{11.7}$$

Clearly, from appropriate linear combinations of evaluations of F for six appropriately chosen values $\vec{\hat{u}}_\alpha, \alpha = 1,..,6$ we can reconstruct the six coordinate components of the metric s_{ab} upto an overall factor to any desired accuracy:

$$B_1 s_{\mu\nu} + O(\epsilon^{(p_2-p_1)}) = \sum_\alpha \lambda^\alpha_{\mu\nu}F(\vec{\hat{u}}_\alpha,\epsilon). \tag{11.8}$$

Since $\vec{\hat{u}}_\alpha, \lambda^\alpha_{\mu\nu}, \alpha = 1,..,6, \mu,\nu = 1,\ 2,\ 3$ are fixed and independent of ϵ, we retain only the ϵ dependence of the right-hand side of the above equation and set

$$\sum_\alpha \lambda^\alpha_{\mu\nu}F(\vec{\hat{u}}_\alpha,\epsilon) \eqqcolon s^\epsilon_{\mu\nu} \tag{11.9}$$

so that we have that

$$B_1 s_{\mu\nu} + O(\epsilon^{(p_2-p_1)}) = s^\epsilon_{\mu\nu} \tag{11.10}$$

$$\Rightarrow (B_1)^3 s + O(\epsilon^{(p_2-p_1)}) = s^\epsilon \tag{11.11}$$

$$\Rightarrow (B_1)^{-1}s^{-\frac{1}{3}} + O(\epsilon^{(p_2-p_1)}) = (s^\epsilon)^{-\frac{1}{3}} \tag{11.12}$$

where we have used s, s^ϵ to denote the determinants of $s_{\mu\nu}, s^\epsilon_{\mu\nu}$. Multiplying the left- and right-hand sides of Eqs. (11.12), (11.10) we get

$$\frac{s_{\mu\nu}}{s^{\frac{1}{3}}} = \frac{s^\epsilon_{\mu\nu}}{(s^\epsilon)^{\frac{1}{3}}} + O(\epsilon^{(p_2-p_1)}). \tag{11.13}$$

Since $\{x\}$ is an admissible semianalytic chart on Σ, we can transit to any other fixed ϵ independent semianalytic chart. Since the Jacobian factors are independent of ϵ, Eq. (11.13) holds in *any* such chart in obvious notation. Next, taking the limit as $\epsilon \to 0$ of (11.13) and letting p vary over Σ, it follows that the conformally invariant metric density $\frac{s_{\mu\nu}}{s^{\frac{1}{3}}}$ can be reconstructed on all of Σ from the set of amplitudes defined by any anomaly free state Ψ_{f,s_{ab},P_0} with metric label s_{ab}. Note that this result is *independent* of the choice scheme used to define Ψ_{f,s_{ab},P_0} (recall that a choice of reference coordinates is needed to evaluate the amplitudes which define Ψ_{f,s_{ab},P_0}).

Next, we use the machinery developed above to prove the following statement:

Statement.—Consider a choice scheme S_1 and anomaly free basis state Ψ_{f_1,h_{1ab},P^1_0} defined in this choice scheme for the scalar density, metric and bra set labels $f_1,\ h_{1ab}, B_{P^1_0}$ where f_1 vanishes at most at a finite number of points, $h_{1ab} \in \mathcal{H}_{h_0}$ and $B_{P^1_0}$ is an admissible bra set in the scheme S_1. Likewise consider a second choice scheme S_2 and anomaly free basis state Ψ_{f_2,h_{2ab},P^2_0} with f_2 vanishing at most at a finite number of points, $h_{2ab} \in \mathcal{H}_{h_0}$ and $B_{P^2_0}$ admissible in S_2. Let $h_{1ab} \neq h_{2ab}$. Then $\Psi_{f_1,h_{1ab},P^1_0} \neq \Psi_{f_2,h_{2ab},P^2_0}$ where the inequality indicates that the two states are distinct in the sense of distributions.

Proof.—First suppose $B_{P^1_0} \neq B_{P^2_0}$. Let c be such that $c \in B_{P^1_0}, c \notin B_{P^2_0}$. Let $f_1 \neq 0$ at the nondegenerate vertex of c (if it vanishes replace c by some diffeomorphic image of c such that $f_1 \neq 0$ at the nondegenerate vertex of this image and rename this state as c). Then $(\Psi_{f_1,h_{1ab},P^1_0}|c\rangle \neq 0$ but $(\Psi_{f_2,h_{2ab},P^2_0}|c\rangle = 0$ so the 2 states are different.

Next consider the case $B_{P^1_0} = B_{P^2_0}$ and denote $B_{P^1_0} = B_{P^2_0} \equiv B_{P_0}$. In what follows we shall frequently refer to the argumentation (11.2)–(11.13) in the first part of this section. Consider a primordial state c in B_{P_0}, and the electric diffeomorphism deformation of c in scheme S_1. Call the deformed ket c_1. Proceed as in the first part of this section replacing f, s_{ab} by $f_1,\ h_{1ab}$ so as to obtain (11.13) with s_{ab} replaced by h_{1ab}.

Next, consider the amplitude $(\Psi_{f_2,h_{2ab},P_0}|c_1\rangle$ evaluated in the scheme S_2 (we emphasize that c_1 is still the deformed child produced in scheme S_1 from its parent c). Let the reference coordinates for the evaluation be $\{x_2\}$. Now if f_2 vanishes at v_1 we have $(\Psi_{f_2,h_{2ab},P_0}|c_1\rangle = 0$ whereas $(\Psi_{f_1,h_{1ab},P_0}|c_1\rangle \neq 0$ so that the two anomaly free states are again distinct. Next, let $f_2 \neq 0$ at v_1. Consider again the action of a diffeomorphism χ which is identity in the vicinity of v_1 on c_1. Once again the Lemma of P2 implies that we may continue to use $\{x_2\}$ for amplitude evaluations $(\Psi_{f_2,h_{2ab},P^2_0}|c_{1\chi}\rangle$. It is easy to check that the subsequent analysis also holds so that we have (11.2)–(11.13) with the replacements $f_2,\ h_{2ab}$ for f, s_{ab} in those equations. Thus we have derived the equations:

$$\frac{h_{1\mu\nu}}{h_1^{\frac{1}{3}}} + O(\epsilon^{(p_2-p_1)}) = \frac{h^{\epsilon}_{1\mu\nu}}{(h^{\epsilon}_1)^{\frac{1}{3}}}, \tag{11.14}$$

$$\frac{h_{2\mu\nu}}{h_2^{\frac{1}{3}}} + O(\epsilon^{(p_2-p_1)}) = \frac{h^{\epsilon}_{2\mu\nu}}{(h^{\epsilon}_2)^{\frac{1}{3}}} \tag{11.15}$$

where $\frac{h^{\epsilon}_{1\mu\nu}}{(h^{\epsilon}_1)^{\frac{1}{3}}}$ is *exactly* the same function of the amplitudes $(\Psi_{f_1,h_{1ab},P_0}|c_{1\phi_{u,\epsilon}}\rangle$ as $\frac{h^{\epsilon}_{2\mu\nu}}{(h^{\epsilon}_2)^{\frac{1}{3}}}$ is of the amplitudes $(\Psi_{f_2,h_{2ab},P_0}|c_{1\phi_{u,\epsilon}}\rangle$ as can be seen from (11.7).

Next, recall that the left-hand sides of (11.14) and (11.15) are both evaluated at the vertex p of c. Choose c to be such that its vertex p lies in $U(a,\delta)$ so that (11.1) holds at p. It directly follows that by choosing ϵ small enough in (11.14) and (11.15), the right-hand sides of these equations differ. This implies that there must be at least one value of $\alpha \in \{1,2,..,6\}$ such that the amplitudes $(\Psi_{f_1,h_{1ab},P_0}|c_{1\phi_{u_\alpha,\epsilon}}\rangle$, $(\Psi_{f_2,h_{2ab},P_0}|c_{1\phi_{u_\alpha,\epsilon}}\rangle$ of the two anomaly free states on the *same* charge net $c_{1\phi_{u_\alpha,\epsilon}}$ differ. Hence the two states are distinct and this concludes the proof.

The statement which we have proved above implies that given two distinct metric labels $h_{ab}, h'_{ab} \in \mathcal{H}_{h_0}$, we are free to choose two different choice schemes, one for the definition of anomaly free states with metric label h_{ab} and for the definition of the discrete action of constraints on these states, and a second for anomaly free states with metric label h'_{ab} and for the definition of the discrete action of constraints on *these* states. This freedom of choice leads to no inconsistency because we are guaranteed that two states with distinct metric labels are distinct. We shall use this freedom in the next section.

B. Diffeomorphism covariant regulating choices

In what follows we refer to a particular implementation of the choice scheme (a)–(f) summarized in (2), Sec. X. A. 1 by the letter S or by addending suitable symbols/subscripts to S; for example S_1 or S' etc. In Sec. XI. B. 1 we define a covariant choice of such schemes by tying each such choice scheme to the metric label of the anomaly free basis state under consideration. In Sec. XI. B. 2 we derive the action of a diffeomorphism on an anomaly free basis state.

1. Covariant choice schemes

Consider, as in Sec. XI. A the space $\mathcal{H}_{h_0}$ of metrics diffeomorphic to h_{0ab}. Let h_{0ab} be associated with some choice scheme S_0. We shall use the notation of Secs. VI. B and VI. C for the reference structures associated with this choice. Accordingly, the metric and the associated primary coordinate patch, the reference state for the diffeomorphism class of states of c, the reference diffeomorphism mapping this reference state to c and the reference coordinate patch for c are

$$h_{0ab}, \quad \{x_0\}, \quad c_0, \quad \alpha, \quad \alpha^*\{x_0\}. \tag{11.16}$$

Let $h_{ab} \in \mathcal{H}_{h_0}$. Since h_{0ab} has no (conformal) symmetries there exists a unique diffeomorphism ϕ such that $h_{ab} = \phi^* h_{0ab}$. We define the choice scheme S_h associated with this metric to be the images by ϕ of the choice scheme S_0[49] so that the metric, the associated primary coordinate patch, the reference state for the diffeomorphism class of states of c_ϕ, the reference diffeomorphism mapping this reference state to c_ϕ and the reference coordinate patch for c_ϕ are

$$h_{ab} = \phi^* h_{0ab}, \quad \phi^*\{x_0\}, \quad \phi \circ c_0, \quad \phi\circ\alpha\circ\phi^{-1}, \quad \phi^*\alpha^*\{x_0\} \tag{11.17}$$

where we have denoted the image of c by the diffeomorphism ϕ by $c_\phi \equiv \phi\circ c$ so that $\hat{U}(\phi)|c\rangle = |c_\phi\rangle = |\phi\circ c\rangle$. Here, we have chosen the cone angle as measured by $\{x_0\}$ for conical deformations $c_{0[i,I,\beta,\delta_0}$ of any c_0 in the scheme S_0 to be the same as that measured by $\phi^*\{x_0\}$ for conical deformations $\phi\circ c_{0[i,I,\beta,\delta_0}$ of $\phi\circ c_0$ in the scheme S_h.

Note that the choice schemes $\{S_h, h_{ab} \in \mathcal{H}_{h_0}\}$ yield the same set of primordials and the same ket set. Further if we choose the bra set B_{P0} in scheme S_0 then this same bra set is admitted as a bra set in the choice scheme S_h for any $h_{ab} \in \mathcal{H}_{h_0}$.

Accordingly consider the anomaly free state $\Psi_{f,h_{ab},B_{P0}}$. We shall adopt a *covariant regulator scheme* for the definition of the constraint operator products of Secs. IX and X by which we mean that the discrete action of any such operator on $\Psi_{f,h_{ab},B_{P_0}}$ is defined with respect to the choice scheme S_h. More in detail, let

$$\hat{O}(\{N_{\rm i},\epsilon_{\rm i}, i=1,..m\}) \equiv \hat{O}(\{N_{\rm i},\epsilon_{\rm i}\}) \coloneqq \left(\prod_{\rm i=1}^{m}\hat{O}_{{\rm i},\epsilon_{\rm i}}(N_{\rm i})\right),$$
$$\epsilon_{\rm i} < \epsilon_{\rm j} \quad \text{iff} \quad {\rm i}<{\rm j}, \tag{11.18}$$

where the product is ordered from left to right in increasing i and each $\hat{O}_{{\rm i},\epsilon_{\rm i}}(N_{\rm i})$ is chosen to be the discrete approximant to a Hamiltonian or electric diffeomorphism constraint operator, so that the resulting operator product $\hat{O}(\{N_{\rm i},\epsilon_{\rm i}\})$ is of the type encountered in Secs. IX and X. Then the action of this operator on any state $\Psi_{f,h_{ab},B_{P_0}}$ with $h_{ab} \in \mathcal{H}_{h_0}$, evaluated on any charge net c yields the amplitude:

$$(\Psi_{f,h_{ab},P_0}|\hat{O}(\{N_{\rm i},\epsilon_{\rm i}\})|c\rangle \tag{11.19}$$

where this amplitude is evaluated as in Secs. IX and X with respect to choice scheme S_h. Denoting the continuum limit operator defined through the discrete approximant

[49]The choice (f) in Sec. X. A. 2 will be assumed to be the same for S_h and S_0.

$\hat{O}(\{N_{\rm i}, \epsilon_{\rm i}\})$ by $\hat{O}(\{N_{\rm i}\})$ we have that

$$(\Psi_{f,h_{ab},P_0}|\hat{O}(\{N_{\rm i}\})|c\rangle \coloneqq (\lim_{\epsilon_m\to 0}(\lim_{\epsilon_{n-1}\to 0}\ldots(\lim_{\epsilon_1\to 0}(\Psi_{f,h_{ab},P_0}|\hat{O}(\{N_{\rm i},\epsilon_{\rm i}\})|c\rangle \tag{11.20}$$

where the discrete action amplitude on the right-hand side is defined with respect to the scheme S_h.

It is straightforward to see that the following property holds in this covariant regulator scheme. Consider the metric label $h_{ab} \in \mathcal{H}_{h_0}$ and its image $\phi^* h_{ab}$ by the diffeomorphism ϕ. Given a state c, let its reference coordinates, reference state and reference diffeomorphism mapping this reference state to c for the choice scheme associated with h_{ab} be

$$\{x\}^h_c, \qquad c^h_0, \qquad \alpha^h_c. \tag{11.21}$$

Then the reference coordinates, reference state and reference diffeomorphism for the state c_ϕ for the choice scheme associated with $\phi^* h_{ab}$ is

$$\{x\}^{\phi^* h}_{c_\phi} = \phi^*\{x\}^h_c, \quad (c^{\phi^* h}_\phi)_0 = \phi\circ c^h_0, \quad \alpha^{\phi^* h}_{c_\phi} = \phi\circ\alpha^h_c\circ\phi^{-1}. \tag{11.22}$$

2. *Action of finite diffeomorphisms on anomaly free states*

Hereon, we need to keep track of metric labels and associated reference structures. Accordingly, we use (11.21) to rewrite (7.5) so that the evaluation of the amplitude $(\Psi_{f,h_{ab},P_0}|c>$ in the choice scheme S_h is

$$(\Psi_{f,h_{ab},P_0}|c\rangle = g_c(h_{ab},\{\tilde{v}\})\left(\sum_I h_I H_I\right)|_{h_{ab},v,\{x\}^h_c} f(v,\{x\}^h_c). \tag{11.23}$$

Here I indexes the edges at the nondegenerate vertex v of c. The notation $g_c(h_{ab},\{\tilde{v}\})$ tells us the function g of Sec. G is evaluated at the C^0 kinks of c and the geodesic distances between these kinks are determined by the metric h_{ab}. The subscript h_{ab}, c to the sum over I indicates that the edge tangents which go into the definition of H_I, h_I are unit with respect to the $\{x\}^h_c$ coordinates and are evaluated at v with respect to the metric h_{ab}.

In this notation we have, once again in the S_h scheme that

$$(\Psi_{f,h_{ab},P_0}|\hat{U}^\dagger(\phi)|c\rangle = g_{c_{\phi^{-1}}}(h_{ab},\{\phi^{-1}(\tilde{v})\}), \left(\sum_I h_I H_I\right)|_{h_{ab},\phi^{-1}(v),\{x\}^h_{c_{\phi^{-1}}}} f(\phi^{-1}(v),\{x\}^h_{c_{\phi^{-1}}}). \tag{11.24}$$

Next, from (11.21) and (11.22), note that in the $C_{\phi^* h}$ scheme the reference coordinates for $c = \phi\circ c_{\phi^{-1}}$ are $\phi^*\{x\}^h_{c_{\phi^{-1}}}$. This implies that

$$(\Psi_{\phi^* f,\phi^* h_{ab},P_0}|c\rangle = g_c(\phi^* h_{ab},\{\tilde{v}\}), \left(\sum_I h_I H_I\right)|_{\phi^* h_{ab},v,\phi^*\{x\}^h_{c_{\phi^{-1}}}} f(v,\phi^*\{x\}^h_{c_{\phi^{-1}}}). \tag{11.25}$$

Using the properties of pushforwards by diffeomorphisms and that fact that $\phi\circ c_{\phi^{-1}} = c$, we have that

$$g_c(\phi^* h_{ab},\{\tilde{v}\}) = g_{c_{\phi^{-1}}}(h_{ab},\{\phi^{-1}(\tilde{v})\}), \tag{11.26}$$

$$(\phi^* f)(v,\phi^*\{x\}^h_{c_{\phi^{-1}}}) = f(\phi^{-1}(v),\{x\}^h_{c_{\phi^{-1}}}). \tag{11.27}$$

It is also straightforward to see, from the properties of pushforwards and the definition of h_I, H_I, that

$$\left(\sum_I h_I H_I\right)|_{\phi^* h_{ab},v,\phi^*\{x\}^h_{c_{\phi^{-1}}}} = \left(\sum_I h_I H_I\right)|_{h_{ab},\phi^{-1}(v),\{x\}_{c^h_{\phi^{-1}}}}. \tag{11.28}$$

From (11.26), (11.27) and (11.28) together with (11.24) and (11.25), it follows that

$$(\Psi_{f,h_{ab},P_0}|\hat{U}^\dagger(\phi)|c\rangle = (\Psi_{\phi^* f,\phi^* h_{ab},P_0}|c\rangle. \tag{11.29}$$

This equality holds for every c in the bra set B_{P0}. Further both sides vanish for any $c \notin B_{P0}$. Hence we have the following equality of anomaly free basis states:

$$\hat{U}(\phi)\Psi_{f,h_{ab},P_0} = \Psi_{\phi^* f,\phi^* h_{ab},P_0}. \tag{11.30}$$

C. Action of products of constraints and finite diffeomorphisms

As explained in the Introduction, the Poisson bracket relation (2.8) between a pair of Hamiltonian constraints is replaced by (2.11) and this relation is implemented in quantum theory in Secs. IX and X. Here we are interested in the remaining Poisson bracket relations (2.6) and (2.7) between the diffeomorphism constraints and between the diffeomorphism and Hamiltonian constraints. In LQG the primary operators related to diffeomorphisms are the unitary operators which implement finite diffeomorphisms generated by the diffeomorphism constraints rather than the diffeomorphism constraints themselves. Hence in quantum theory we replace (2.6) and (2.7) by the relations

$$\hat{U}(\phi_1)\hat{U}(\phi_2) = \hat{U}(\phi_1\circ\phi_2), \tag{11.31}$$

$$\hat{U}^\dagger(\phi)\hat{C}[N]\hat{U}(\phi) = \hat{C}[\phi_* N]. \tag{11.32}$$

These relations are to be imposed on the algebra generated by arbitrary products of finite diffeomorphism unitaries and Hamiltonian constraint operators. Hence we are interested in the imposition of these relations within operator products of the form

$$\left(\prod_{i_1=1}^{m_1}\hat{U}(\psi_{i_1})\right)\hat{O}^{(1)}\left(\prod_{i_2=m_1+1}^{m_2}\hat{U}(\psi_{i_2})\right)\hat{O}^{(2)}\ldots \times\left(\prod_{i_n=m_{n-1}+1}^{m_n}\hat{U}(\psi_{i_n})\right)\hat{O}^{(n)}\left(\prod_{i_1=1}^{m_1}\hat{U}(\psi_{i_{n+1}})\right) \tag{11.33}$$

where the ψ's are semianalytic diffeomorphisms and the $\hat{O}$'s are products of Hamiltonian constraint operators.

Note also that we would like to show that the relation (2.11) is also valid within each such product of Hamiltonian constraint operators. More in detail, by considering appropriate linear combinations of products of the type (11.33), we may define a product where, now, each $\hat{O}$ in (11.33) contains products of single commutators between pairs of Hamiltonian constraints i.e. we may consider multiple products of single commutators of the type in (1.1) and Sec. X. C. We may then obtain a new operator product by replacing each of these commutators by appropriate electric diffeomorphism commutators as indicated by (2.11) and we would like to show that the first operator products with Hamiltonian constraint commutators equals this new product obtained by these replacements. Hence we are interested in computing the action of operator products of the form (11.33) where each $\hat{O}$ can be (a) a product of Hamiltonian constraints, (b) a product of the type (1.1) or (c) the product in (b) with the replacement of Hamiltonian commutators by appropriate electric diffeomorphism ones.

Since LQG provides a representation of the relation (11.31) on its kinematic Hilbert space, it immediately follows that this relation is automatically implemented on the space of distributions through dual action. Since the anomaly free states are distributions, it follows that this relation is already imposed. Given that this relation is imposed it is easy to see that operator products of the form (11.33) are equivalent to products of the form

$$\hat{U}(\phi_1)^\dagger\hat{O}^{(1)}\hat{U}(\phi_1)\hat{U}(\phi_2)^\dagger\hat{O}^{(2)}\hat{U}(\phi_2)\ldots \times\hat{U}(\phi_n)^\dagger\hat{O}^{(n)}\hat{U}(\phi_n)\hat{U}(\phi_{n+1}) \tag{11.34}$$

where the ϕ's are semianalytic diffeomorphisms. The imposition of (11.32) and (2.11) on such operator products yields the relation

$$\begin{aligned}&(\hat{U}(\phi_1)^\dagger\hat{O}^{(1)}(\{N_{\mathrm{i}_1},\mathrm{i}_1=1,..,m_1\})\hat{U}(\phi_1))(\hat{U}(\phi_2)^\dagger\hat{O}^{(2)}(\{N_{\mathrm{i}_2},\mathrm{i}_2=m_1+1,..,m_2\})\hat{U}(\phi_2))\\&\quad\ldots(\hat{U}(\phi_n)^\dagger\hat{O}^{(n)}(\{N_{\mathrm{i}_n},\mathrm{i_n}=m_{n-1}+1,..,m_n\})\hat{U}(\phi_n))\hat{U}(\phi_{n+1})\\&=\hat{O}^{(1)}(\{\phi_{1*}N_{\mathrm{i}_1},\mathrm{i}_1=1,..,m_1\})\hat{O}^{(2)}(\{\phi_{2*}N_{\mathrm{i}_2},\mathrm{i}_2=m_1+1,..,m_2\})\\&\quad..\hat{O}^{(n)}(\{\phi_{n*}N_{\mathrm{i}_n},\mathrm{i_n}=m_{n-1}+1,..,m_n\})\hat{U}(\phi_{n+1}).\end{aligned} \tag{11.35}$$

Note that each $\hat{O}^{(i)}$ operator is the continuum limit of some discrete approximant of the type (11.18), each such product being defined in some choice scheme S^i. We show below that this choice scheme, and hence the (continuum limit) action of the operator product (11.34) is *uniquely* fixed from the following two inputs: Input (A): Any such choice scheme S^i must be consistent with the covariant choice scheme defined in Sec. XI. B. 1. By this we mean that any amplitude $(\Psi_{\bar{f},\bar{h}_{ab},\bar{P}_0}|\hat{O}(\{N_{\mathrm{i}},\epsilon_{\mathrm{i}}\})|c\rangle$ with $\hat{O}(\{N_{\mathrm{i}},\epsilon_{\mathrm{i}}\})$ defined as in (11.18) must be evaluated in the choice scheme $S_{\bar{h}}$ [see the discussion around (11.19)]. Input (B): The discrete action of any such *discrete* approximant of the type (11.18) in any choice scheme on a charge net c yields a *finite* linear combination of charge nets [see (a)–(f) of Sec. X. A. 2].

Accordingly, in what follows we shall restrict our attention to the covariant choice scheme defined in Sec. XI. B. 1. In Sec. XI. C. 1 we prove a key identity. In Sec. XI. C. 2 we derive the action of the operator product (11.34) from the inputs (A) and (B) above together with the identity proved in Sec. XI. C. 1. The resulting action will be seen to implement the relation (11.35) on the domain of anomaly free states.

1. A key identity

Claim.—Let $\hat{O}(\{N_{\mathrm{i}},\epsilon_{\mathrm{i}}\})$ be defined as in (11.18). Then the following identity holds for all, $f,\{N_{\mathrm{i}}\},c$ and all $h_{ab}\in\mathcal{H}_{h_0}$:

$$(\Psi_{f,h_{ab},P_0}|\hat{U}^\dagger(\phi)\hat{O}(\{N_{\mathrm{i}},\epsilon_{\mathrm{i}}\})\hat{U}(\phi)|c\rangle = (\Psi_{f,h_{ab},P_0}|\hat{O}(\{(\phi_*N_{\mathrm{i}}),\epsilon_{\mathrm{i}}\})|c\rangle + O(\vec{\epsilon}) \tag{11.36}$$

where $O(\vec{\epsilon})$ indicates a quantity which vanishes in the continuum limit:

$$\lim_{\epsilon_m\to 0}\lim_{\epsilon_{n-1}\to 0}\ldots\lim_{\epsilon_1\to 0}O(\vec{\epsilon})=0. \tag{11.37}$$

Proof.—From (11.29), (11.30) and (B) above, it follows that

$$(\Psi_{f,h_{ab},P_0}|\hat{U}^\dagger(\phi)\hat{O}(\{N_{\rm i},\epsilon_{\rm i}\})\hat{U}(\phi)|c\rangle = (\Psi_{\phi^* f,\phi^* h_{ab},P_0}|\hat{O}(\{N_{\rm i},\epsilon_{\rm i}\})\hat{U}(\phi)|c\rangle. \quad (11.38)$$

Introduce the following notation for the action, on the anomaly free state $\Psi_{\bar{f},\bar{h}_{ab},P_0}$, of the continuum limit operator $\hat{O}(\{N_{\rm i})$ obtained from its discrete approximant $\hat{O}(\{N_{\rm i},\epsilon_{\rm i}\})$:

$$(\Psi_{\bar{f},\bar{h}_{ab},P_0}|\hat{O}(\{N_{\rm i},{\rm i}=1,..,m\})|s\rangle \coloneqq \mathcal{A}_m(\bar{f},\bar{h},\{N_{\rm i}\},s,\{x\}^{\bar{h}}_s). \quad (11.39)$$

Here the left-hand side is defined as a continuum limit of its discrete approximant as in (11.20). The right-hand side is the appropriate explicitly calculated amplitude[50] from (10.8) and (10.9) with the substitutions $\bar{f},\bar{h},s,\{x\}^{\bar{h}}_s$ for $f,h,c,\{x\}$ in those expressions. The resulting expression depends on the arguments $\bar{f},\bar{h},\{N_{\rm i}\},s,\{x\}^{\bar{h}}_s$. Since we work within the covariant choice scheme, the coordinates $\{x\}^{\bar{h}}_s$ with respect to which the explicit expression is defined are determined by $\bar{h}_{ab}$ and s; nevertheless despite this redundancy, it is useful for pedagogical purposes to retain this argument in $\mathcal{A}_m$.

From the definition of the continuum limit it follows that the corresponding discrete action can be written as

$$(\Psi_{\bar{f},\bar{h}_{ab},P_0}|\hat{O}(\{N_{\rm i},\epsilon_{\rm i}\})|s\rangle == \mathcal{A}_m(\bar{f},\bar{h},\{N_{\rm i}\},s,\{x\}^{\bar{h}}_s) + O(\vec{\epsilon}). \quad (11.40)$$

Setting $|c_\phi\rangle \coloneqq \hat{U}(\phi)|c\rangle$, Eqs. (11.38) and (11.40) imply that

$$(\Psi_{f,h_{ab},P_0}|\hat{U}^\dagger(\phi)\hat{O}(\{N_{\rm i},\epsilon_{\rm i}\})\hat{U}(\phi)|c\rangle = \mathcal{A}_m(\phi^* f,\phi^* h,\{N_{\rm i}\},c_\phi,\{x\}^{\phi^* h}_{c_\phi}) + O(\vec{\epsilon}). \quad (11.41)$$

From (11.22) it follows that

$$\mathcal{A}_m(\phi^* f,\phi^* h,\{N_{\rm i}\},c_\phi,\{x\}^{\phi^* h}_{c_\phi}) = \mathcal{A}_m(\phi^* f,\phi^* h,\{N_{\rm i}\},c_\phi,\phi^*\{x\}^h_c). \quad (11.42)$$

Using the properties of pullbacks by diffeomorphisms together with definitions of the various quantities which figure in the explicit expressions (10.8) and (10.9), it is straightforward to see that

$$\mathcal{A}_m(\phi^* f,\phi^* h,\{N_{\rm i}\},c_\phi,\phi^*\{x\}^h_c) = \mathcal{A}_m(f,h,\{\phi_* N_{\rm i}\},c,\{x\}^h_c). \quad (11.43)$$

Using the appropriate substitutions in (11.40) we have that

$$\mathcal{A}_m(f,h,\{\phi_* N_{\rm i}\},c,\{x\}^h_c) = (\Psi_{f,h_{ab},P_0}|\hat{O}(\{(\phi_* N_{\rm i}),\epsilon_{\rm i}\})|c\rangle + O(\vec{\epsilon}). \quad (11.44)$$

The claimed identity (11.36) immediately follows from Eqs. (11.41), (11.42), (11.43) and (11.44).

Key identity: As a corollary, we have the following key identity which we shall use repeatedly in the next section:

$$(\Psi_{f,h_{ab},P_0}|\hat{O}(\{N_{\rm i},\epsilon_{\rm i}\})\hat{U}(\phi)|c\rangle = (\Psi_{\phi_* f,\phi_* h_{ab},P_0}|\hat{O}(\{(\phi_* N_{\rm i}),\epsilon_{\rm i}\})|c\rangle + O(\vec{\epsilon}). \quad (11.45)$$

To see this, substitute f, h by $\phi_* f,\phi_* h$ in (11.36) to obtain

$$(\Psi_{\phi_* f,\phi_* h_{ab},P_0}|\hat{U}^\dagger(\phi)\hat{O}(\{N_{\rm i},\epsilon_{\rm i}\})\hat{U}(\phi)|c\rangle = (\Psi_{\phi_* f,\phi_* h_{ab},P_0}|\hat{O}(\{(\phi_* N_{\rm i}),\epsilon_{\rm i}\})|c\rangle + O(\vec{\epsilon}). \quad (11.46)$$

From Input (B), $\hat{O}(\{N_{\rm i},\epsilon_{\rm i}\})\hat{U}(\phi)|c\rangle$ is a finite linear combination of charge nets so that we may apply (11.29) to the left-hand side of (11.46) and obtain

$$(\Psi_{\phi_* f,\phi_* h_{ab},P_0}|\hat{U}^\dagger(\phi)\hat{O}(\{N_{\rm i},\epsilon_{\rm i}\})\hat{U}(\phi)|c\rangle = (\Psi_{f,h_{ab},P_0}|\hat{O}(\{N_{\rm i},\epsilon_{\rm i}\})\hat{U}(\phi)|c\rangle. \quad (11.47)$$

Equation (11.45) immediately follows from (11.47) and (11.46).

An alternative way to state the Claim is to dispense with Inputs (A) and (B) and instead state that if (a), (b) below hold then Eqs. (11.36) hold where (a), (b) are as follows:

(a) We define the amplitude evaluation of any anomaly free basis state labeled by any metric $\bar{h}_{ab} \in \mathcal{H}_{h_0}$ on any state $\bar{c}$ to be with respect to the scheme $S_{\bar{h}}$.

(b) We choose the discrete action of the operator approximant $\hat{O}(\{N_{\rm i},\epsilon_{\rm i}\})$ on the left-hand side (lhs) of (11.36) to be evaluated in the $S_{\phi^* h}$ scheme and that of the operator approximant $\hat{O}(\{(\phi_* N_{\rm i}),\epsilon_{\rm i}\})$ on the right-hand side (rhs) in the S_h scheme.

It is straightforward to repeat the steps of the proof with inputs (a) and (b) and thereby prove the claim. Similarly, the corollary can be restated as follows. Let (a) hold and let the discrete action of $\hat{O}(\{N_{\rm i},\epsilon_{\rm i}\})$ on the lhs of (11.45) be in the S_h scheme and that of $\hat{O}(\{(\phi_* N_{\rm i}),\epsilon_{\rm i}\})$ on the rhs in the $S_{\phi^* h}$ scheme. Then Eq. (11.45) holds. Once again the proof is basically a straightforward repetition of the proof of the corollary sketched above.

[50]Recall that we have shown in Sec. X. C that this amplitude is consistent with (2.11).

2. Action of the operator product in Eq. (11.34)

In this section we evaluate the action of the operator (11.34) on the anomaly free state $\Psi_{f,h_{ab},B_{P0}}$. This action is obtained from that of the left-hand side of (11.35) on this state:

$$\begin{aligned}&(\Psi_{f,h_{ab},B_{P0}}|(\hat{U}(\phi_1)^\dagger\hat{O}^{(1)}(\{N_{\mathrm{i}_1},\epsilon_{\mathrm{i}_1}\mathrm{i}_1=1,..,m_1\})\\ &\times\hat{U}(\phi_1))(\hat{U}(\phi_2)^\dagger(\hat{O}^{(2)}(\{N_{\mathrm{i}_2},\epsilon_{\mathrm{i}_2},\mathrm{i}_2=m_1+1,..,m_2\})\hat{U}(\phi_2))\\ &\ldots(\hat{U}(\phi_n)^\dagger\hat{O}^{(n)}(\{N_{\mathrm{i}_n},\epsilon_{\mathrm{i}_n},\mathrm{i}_n=m_{n-1}+1,..,m_n\})\hat{U}(\phi_n))\hat{U}(\phi_{n+1})|c\rangle.\end{aligned} \tag{11.48}$$

In order to evaluate this discrete action we use Input (A), (B) above iteratively as follows. For any charge net s we have that

$$\begin{aligned}&(\Psi_{f,h_{ab},B_{P0}}|(\hat{U}(\phi_1)^\dagger\hat{O}^{(1)}(\{N_{\mathrm{i}_1},\epsilon_{\mathrm{i}_1},\mathrm{i}_1=1,..,m_1\})\hat{U}(\phi_1))|s\rangle\\ &=(\Psi_{f,h_{ab},B_{P0}}|\hat{O}^{(1)}(\{\phi_{1*}N_{\mathrm{i}_1},\epsilon_{\mathrm{i}_1},\mathrm{i}_1=1,..,m_1\})\hat{U}(\phi_2)^\dagger|s_1\rangle+O_1(\vec{\epsilon}^{(1)})\\ &=(\Psi_{\phi_2^*f,\phi_2^*h_{ab},B_{P0}}|\hat{O}^{(1)}(\{\phi_2^*\phi_{1^*}N_{\mathrm{i}_1},\epsilon_{\mathrm{i}_1},\mathrm{i}_1=1,..,m_1\})|s_1\rangle+O_1(\vec{\epsilon}^{(1)})\\ &=(\Psi_{\phi_2^*f,\phi_2^*h_{ab},B_{P0}}|\hat{O}^{(1)}(\{\phi_2^*\phi_{1*}N_{\mathrm{i}_1},\epsilon_{\mathrm{i}_1},\mathrm{i}_1=1,..,m_1\})\hat{O}^{(2)}(\{N_{\mathrm{i}_2},\epsilon_{\mathrm{i}_2}\mathrm{i}_2=m_1+1,..,m_2\})\hat{U}(\phi_2)|s_2\rangle+O_1(\vec{\epsilon}^{(1)})\\ &=(\Psi_{f,h_{ab},B_{P0}}|\hat{O}^{(1)}(\{\phi_{1*}N_{\mathrm{i}_1},\epsilon_{\mathrm{i}_1},\mathrm{i}_1=1,..,m_1\})\hat{O}^{(2)}(\{\phi_{2*}N_{\mathrm{i}_2},\epsilon_{\mathrm{i}_2},\mathrm{i}_2=m_1+1,..,m_2\})|s_2\rangle+O_2(\vec{\epsilon}^{(2)})+O_1(\vec{\epsilon}^{(1)})\end{aligned} \tag{11.49}$$

where we have used (11.36) in the second line, (11.45) in the third and fifth lines and where we have defined s_1, s_2, by

$$|s\rangle=\hat{U}(\phi_2)^\dagger|s_1\rangle,|s_1\rangle \coloneqq \hat{O}^{(2)}(\{N_{\mathrm{i}_2},\epsilon\mathrm{i}_2,\mathrm{i}_2=m_1+1,..,m_2\})\hat{U}(\phi_2)|s_2\rangle. \tag{11.50}$$

The symbol $O_1(\vec{\epsilon}^{(1)})$ denotes a term which vanishes in the partial continuum limit which sends the parameters $\epsilon_1,\epsilon_2,..\epsilon_{m_1}$ to zero (in that order) while keeping $\epsilon_{\mathrm{j}},\mathrm{j}>m_1$ fixed. Similarly the term $O_2(\vec{\epsilon}^{(2)})$ vanishes in the partial continuum limit over $\{\epsilon_{\mathrm{i}},\mathrm{i}=1,..,m_2\}$ while keeping $\epsilon_{\mathrm{j}},\mathrm{j}>m_2$ fixed. Clearly this procedure may be iterated to obtain an expression for (11.48). It is easy to check that the continuum limit of this expression yields the evaluation of the right-hand side of Eq. (11.35) on the anomaly free state.

Note that the application of Inputs (A), (B) to the calculation above fixes the choice scheme for the definition of each of $\hat{O}^{(j)}(N_{\mathrm{i}_j},\epsilon_{\mathrm{i}_j})$ in (11.48) to be $S_{\phi_j^*h}$. We may also restate the result by dispensing with Inputs (A) and (B) and instead state that if (a), (b) below hold, then (11.35) holds. Here (a) is identical to (a), Sec. XI. C. 1 and (b) is as follows: (b) In Eq. (11.48) let the choice scheme for the definition of the discrete action of $\hat{O}^{(j)}(N_{\mathrm{i}_j},\epsilon_{\mathrm{i}_j})$ be chosen to be $S_{\phi_j^*h}$. It is straightforward to see that a proof may be constructed by basically repeating the steps which lead us from (11.48) to (11.49), iterating, and then taking the continuum limit. Viewed in this way, defining anomaly free states through (a) above, we have shown that there exist discrete actions of operator approximants whose continuum limit action lead to the relationship (11.35).

The considerations in this section show that the operator actions of Secs. IX and X are consistent with Eqs. (11.31), (11.32) so that we have a diffeomorphism covariant anomaly free single commutator implementation of the constraint algebra. More in detail given any two operator strings of the type (11.33) related by the substitution of commutators between Hamiltonian constraints by the appropriate combination of electric diffeormorphism commutators, we (a) first convert the strings to the form (11.34) through (11.31), (b) use (11.32) to remove all the $\hat{U}(\phi_i)$, $\hat{U}^\dagger(\phi_i)$, $i=1,..,n-1$ operators from the string and (c) appeal to the anomaly free single commutator results of Secs. IX and X. The steps (a)–(c) show that we have a diffeomorphism covariant anomaly free single commutator implementation of the constraint algebra.

XII. BRIEF SUMMARY OF RESULTS

Consider an operator product of the form

$$\hat{O}=\left(\prod_{i_1=1}^{m_1}\hat{U}(\psi_{i_1})\right)\hat{O}^{(1)}\left(\prod_{i_2=m_1+1}^{m_2}\hat{U}(\psi_{i_2})\right)\hat{O}^{(2)}\ldots\times\left(\prod_{i_n=m_{n-1}+1}^{m_n}\hat{U}(\psi_{i_n})\right)\hat{O}^{(n)}\left(\prod_{i_1=1}^{m_1}\hat{U}(\psi_{i_{n+1}})\right) \tag{12.1}$$

where (a) the ψ's are semianalytic diffeomorphisms, (b) the $\hat{O}^{(i)}$'s are products of Hamiltonian constraint operators of each of density weight $4/3$ and each smeared by a lapse of density weight $-1/3$, and (c) the total number of Hamiltonian constraints in the operator product (12.1) is less than k, the Cauchy slice Σ being a C^k semianalytic manifold. Such an operator product has a well-defined dual action on any anomaly free basis state Ψ_{f,h_{ab},P_0}, this dual action being inferred from the amplitudes

$$(\Psi_{f,h_{ab},P_0}|\hat{O}|c\rangle, \quad \forall\ c \tag{12.2}$$

where c ranges over the set of all charge net states. These amplitudes are such that the relations

$$\hat{U}(\phi_1)\hat{U}(\phi_2) = \hat{U}(\phi_1 \circ \phi_2), \tag{12.3}$$

$$\hat{U}^\dagger(\phi)\hat{C}[N]\hat{U}(\phi) = \hat{C}[\phi_* N] \tag{12.4}$$

hold so that any individual operator string, *within* the big operator product (12.1), which is of the form of the left-hand side of (12.3) or (12.4) can be replaced, in the amplitude evaluation (12.2), by the corresponding right-hand side and *vice versa*.

An anomaly free basis state Ψ_{f,h_{ab},P_0} is a linear combination of charge net bras, these bras comprising the bra set B_{P_0}. The coefficients of these bras in this linear combination are determined by a scalar density f of weight $-1/3$ which vanishes at most at a finite number of points, and a metric h_{ab} with no conformal symmetries. The explicit action of a product of $n \leq k-1$ Hamiltonian constraints on the anomaly free basis state Ψ_{f,h_{ab},P_0} is given, for n even and $c \in B_{P0}$ by

$$(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{n}\hat{C}(N_{\mathrm{i}})\right)|c\rangle = (-3)^{\frac{n}{2}}\left(\frac{3\hbar N}{8\pi i}\right)^n (\nu^{-\frac{2}{3}})^n g_c \sum_I |\vec{q}_I|^n h_I H_I^n(N_1,..,N_n;v), \tag{12.5}$$

and for n odd and $c \in B_{P0}$ by

$$(\Psi_{f,h_{ab},P_0}|\left(\prod_{\mathrm{i}=1}^{n}\hat{C}(N_{\mathrm{i}})\right)|c\rangle = (-3)^{\frac{n-1}{2}}\left(\frac{3\hbar N}{8\pi i}\right)^n (\nu^{-\frac{2}{3}})^n g_c \sum_I |\vec{q}_I|^{n-1} \times \left(\sum_{i=1}^{3} q_I^i\right) h_I H_I^n(N_1,..,N_n;v), \tag{12.6}$$

where

$$H_I^n(N_1,..,N_n;v) := \left(\prod_{\mathrm{i}=1}^{n} N_{\mathrm{n-i+1}}^{a_{\mathrm{n-i+1}}}(p,\{x\})\hat{V}_I^{a_{\mathrm{n-i+1}}}(p)\partial_{a_{\mathrm{n-i+1}}}\right) \times (f(p,\{x\})\sqrt{h_{ab}(p)\hat{V}_I^a(p)\hat{V}_I^b(p)})\Big|_{p=v} \tag{12.7}$$

and where the products above are ordered from left to right in increasing i. Here the reference coordinate patch (around the nondegenerate vertex v of c) associated with the metric h_{ab} is $\{x\}$. The vertex structure is such that the edges of the charge net c in a small vicinity of v are straight lines in the $\{x\}$ coordinates. The Ith such edge has unit coordinate edge tangents $\hat{V}_I^a$ with $\hat{V}_I^a$ pointing outward or inward from v depending on the kink structure of c in the vicinity of v. These edge tangents are extended to constant (with respect to $\{x\}$) vector fields at any point p in the vicinity of v in Eq. (12.7). The ith edge charge on the edge I is denoted by q_I^i and ν is the "volume" eigenvalue at v in c. The function g_c depends on the network of geodesic distances, as measured by h_{ab}, between all pairs of C^0 kinks in c; a C^0 kink is a point at the intersection of two edges such that the edge tangents of the two edges at this point are not proportional to each other. For $c \notin B_{P0}$ the right-hand sides of (12.5) and (12.6) vanish.

Equations (12.6) and (12.7) are consistent with anomaly free single commutators. By this we mean that (a) these equations can be used to compute the action, on an anomaly free basis state, of any operator string of the form (1.1), and, (b) each of the commutators in the resulting expression can be replaced by the appropriate electric diffeomorphism commutators in accordance with (the quantum correspondent of) Eq. (2.11).

The action of a diffeomorphism ϕ on the anomaly free basis state Ψ_{f,h_{ab},P_0} yields the state $\Psi_{\phi^* f,\phi^* h_{ab},P_0}$:

$$(\Psi_{f,h_{ab},P_0}|\hat{U}^\dagger(\phi)|c\rangle = (\Psi_{\phi^* f,\phi^* h_{ab},P_0}|c\rangle \tag{12.8}$$

where ϕ^* is the pushforward action of ϕ.

The explicit action of any operator product of the form (12.1) can be obtained from (12.5), (12.6), (12.8) through a judicious use of the identities (12.3) and (12.4) and the fact that the reference coordinate patch $\{y\}$ associated with the metric $\phi^* h_{ab}$ for the state $\hat{U}(\phi)|c\rangle$ is the pushforward of the reference coordinate patch $\{x\}$ associated with the metric h_{ab} for the state c so that $\{y\} = \phi^*\{x\}$.

XIII. DISCUSSION

A. Characterization of anomaly free domain

In the previous section we showed that the finite span of anomaly free basis states constitute an arena wherein the constraint algebra admits a diffeomorphism covariant and anomaly free implementation. We refer to this finite span as the anomaly free domain $\mathcal{D}_{AF}$. We know very little about this domain. For example given all the amplitudes $\langle\Psi|c\rangle$ of a state $\Psi \in \mathcal{D}_{AF}$, we do not know of any operational way of using these amplitudes to reconstruct the expansion of Ψ in terms of anomaly free basis states. We do not even know if this expansion is unique. On the other hand, the action of the constraint operators depends on the basis expansion by virtue of the covariant choice scheme wherein the regulation (and hence continuum limit action) of constraint operators depends on the metric label of the basis state being acted upon. Hence if the expansion in basis states is not unique neither is the definition of the action of the constraints. Nevertheless *given any such expansion*, the Hamiltonian constraint commutators can be replaced by appropriate electric diffeomorphism constraint commutators and the action of the constraint operator products in (11.34) is diffeomorphism covariant *within the context of this particular basis state expansion*. If there are several such expansions then defining all operators of interest with respect to any one fixed expansion ensures that the *relations* between these operators are consistent with anomaly free commutators and diffeomorphism covariance. It is in this sense that (11.31), (11.32) and (2.11) hold.

B. Physical states and their off shell deformations

The anomaly free states introduced in Sec. VII and used in Secs. VIII–XI do not satisfy the Hamiltonian constraint as can be seen from Eq. (9.27). They also do not satisfy the diffeomorphism constraint, as can be seen from Eq. (11.30). Hence they are *off shell* states. We would like to see them as off shell deformations of on shell states. The simplest way to do this is to define the distribution Ψ_{sol,P_0}:

$$(\Psi_{\mathrm{sol},P_0}| = \sum_{\langle\bar{c}|\in B_{P0}} \langle\bar{c}|. \qquad (13.1)$$

It is then easy to check that the action of the distribution Ψ_{sol,P_0} on (9.25) vanishes *independent* of which $h_{ab} \in \mathcal{H}_{h_0}$ is used to regulate the constraint in that equation. More in detail, if we fix any $h_{ab} \in \mathcal{H}_{h_0}$ and use the choice scheme S_h, we have that the continuum limit of the resulting Hamiltonian constraint on Ψ_{sol,P_0} vanishes. Further, by inspection, Ψ_{sol,P_0} is invariant under the (dual) action of operators which implement finite diffeomorphisms. Hence Ψ_{sol,P_0} is a solution to all the constraints and constitutes a physical state.

Next, consider the following one parameter family of states based on the bra set B_{P0}:

$$\Psi_{f,h_{ab},P_0,\tau} = \Psi_{\mathrm{sol},P_0} + \tau\Psi_{f,h_{ab},B_{P0}}, \qquad \tau > 0. \qquad (13.2)$$

Clearly, $\Psi_{f,h_{ab},P_0,\tau}$ is an off shell state such that its action on operator products of the type (11.34) is diffeomorphism covariant and implements anomaly free single commuators. Further, $\Psi_{f,h_{ab},P_0,\tau}$ can be deformed into the physical state Ψ_{sol,P_0} by allowing τ to vanish. Thus the one parameter set of states $\{\Psi_{f,h_{ab},P_0,\tau}, \tau > 0\}$ constitute an off shell deformation of the physical state Ψ_{sol,P_0} such that on these states the implementation of the constraint algebra is diffeomorphism covariant and displays anomaly free single commutators. More generally, we may consider any state Ψ in $\mathcal{D}_{AF}$ and construct $\Psi_\tau = \Psi_{\mathrm{sol},P_0} + \tau\Psi$ as off shell deformations of Ψ_{sol,P_0}. The comments of Sec. XIII. A above then apply to the manner in which the implementation of the constraint algebra on such states is consistent with (11.31), (11.32) and (2.11).

C. Contrast with the conventional notion of anomaly free constraint algebras

As mentioned in the Introduction, the conventional notion of anomaly free constraint algebras also includes multiple (as opposed to single) anomaly free commutators. In the absence of structure functions, this conventional notion is powerful and appropriate as it (a) typically incorporates a representation of some underlying Lie group of gauge tranformations and (b) ensures that there is a sufficiently large space of physical states.

In contrast, in the case of gravity, as is well known, the 4d diffeomorphism group (and its Lie algbera of vector fields) is not represented through the constraint algebra because a spatial slice with respect to one spacetime metric is generically not spatial with respect to the image of this metric by a diffeomorphism. Further, due to the presence of structure functions, the multiple Poisson brackets between constraints, while weakly vanishing, yield constraints with more and more complicated phase space dependent lapses and shifts rather than simple Lie algebra like structures. Thus property (a) of the Lie group case seems absent so that the motivation for anomaly free multiple commutators stems in this context mainly from (b). However, if we drop the requirement of anomaly free multiple commutators, we may nevertheless directly check (b) i.e. the conventional notion with regard to (b) may be viewed only as a sufficient rather than necessary condition for a nontrivial physical state space. Another reason to question the need for anomaly free multiple commutators is that they represent properties which are higher that leading order in $\hbar$ and hence their implementation seems to be unnecessary from a naive view of obtaining the correct classical limit.

Our view point is then as follows. While the constraints do not offer a representation of 4d diffeomorphisms, there exists a subset of constraints whose algebra is that of 3d diffeomorphisms. Accordingly we seek quantum representation of the group of 3d diffeomorphisms and LQG provides this. Next, even though the 4d diffeomorphism Lie algebra is unavailable, one *can* nevertheless interpret the *single* Poisson bracket (2.8) as the representation of 4d deformations in *spacetime* of the 3d Cauchy slice [8]. More in detail, in Ref. [8] it is shown that commutator of a pair of such infinitesimal geometric deformations normal to the

Cauchy slice to *leading order* exactly mirrors the single Poisson bracket (2.8).[51] Hence we seek to represent these single Poisson brackets in an anomaly free manner through (2.11). After doing this we may then check (b) i.e. we may check if we have a large enough solution space. While the work in this paper suggests that the constraint action is compatible with a large enough solution space, a confirmation of this suggestion rests on Sec. XIII. G below.

D. Dependence of solution space on regulating choices

The key regulating choice is that of the primary coordinates when the metric label is h_0, other choices being fixed through our covariant regulator choice requirement. This preferred choice seems to lead to the existence of preferred structural properties of physical states in that such states are combinations of charge net bras which are multiple deformations of primordial bras, these multiple deformations being defined with respect to this choice of primary coordinates. Hence it would be advisable to see if we could build on the work here so that our considerations yield physical states which are combinations of multiply deformed charge net bras these multiple deformations arising from *all* possible choices of linear coordinates for primordial states.

E. Structural inputs in our demonstration of anomaly free commutators

1. Interventions

The interventions by judiciously chosen holonomies in order to define deformations in Secs. IV and V play an *essential* role in our demonstration of the existence of anomaly free commutators. These interventions are far from obvious and could not have been arrived at without guidance from the requirement of anomaly freedom. Thus the requirement of anomaly free commutators plays a key role in homing in on the (hopefully) correct choice of (discrete approximants to) the Hamiltonian constraint.

2. Gauge invariance

$U(1)^3$ gauge invariance plays a key role in our considerations; without it, the results of Appendix C which related net charges to primordial ones would not hold. As a result of the interventions in Sec. XIII. E. 1, it is the properties of the *net* charges (as opposed to the charges themselves) which become important (see the Note and related discussion at the beginning of Sec. IX). $U(1)^3$ gauge invariance then plays a key role in our considerations; without it, the results of Appendix C which relate net charges to primordial ones would not hold and there would no longer be a correlation between properties of primordial charges and those of net charges. This would negatively impact many important structures/concepts such as the definition of nondegeneracy of CGR vertices, the properties of the bra set discussed Sec. VI. B, the invariance of the inequalities (6.7) and (6.8) under the replacement of primordial charges by net charges, and the equivalence of (10.1) with (10.2).

3. Linearity

Linearity of charge net vertices plays a key role in our constructions. It allows for unambiguous extensions of graphs, such extensions being required for the construction of certain conical deformations (see Sec. V). It also allows for a natural definition of "along edge" vertex displacements (see Sec. III. B. 2). This definition together with the linearity of the scrunching diffeomorphism G (6.17) leads to constant Jacobian factors which can be pulled out when analyzing the contraction behavior of the function H_m^l in (F26) and (F27). This contraction behavior neatly dovetails with that of the function g_c. All this would be impacted if we did not have linearity. Our proof of the validity of the replacement of reference coordinates by contraction coordinates for amplitude evaluations relies on the invariance of the regular conicality of deformations under rigid translations; this, too, relies crucially on linearity.

It is not clear to us if our constructions can be generalized if we drop the requirement of linear vertices; however, any such putative construction would be incredibly baroque. The linearity property implies that higher order moduli vanish [25]; since no physically interesting operators in LQG to date involve higher order moduli, linearity does not seem to signify a strong physical restriction. Linearity also plays a role in interpretations of kinematic states [26] and in the application of the Minkowski theorem to our considerations in P1 [27].

4. Restrictions on charge labels

As mentioned earlier the "eternal nondegeneracy" restriction on all members of a primary family is a key property without which it would be difficult to proceed. However, it may be worthwhile to think about how this restriction may be weakened or removed. The restriction (6.6) seems to be an overkill and likely can be removed without damaging our final results; this should be confirmed. The restrictions (6.6) and (6.5) seem to play a role only for the products involving more than two constraints (see Sec. X). In this regard, see Sec. XIII. F below.

5. Restriction on valence

We have restricted our attention to the case that the valence N of any primordial at its nondegenerate vertex is even. The reason is that any regular conical deformation of a

[51]It would be of interest to see if this correspondence also holds between multiple Poisson brackets and higher order contributions to the commutator between infinitesimal geometric deformations; if the correspondence breaks down due to "embedding dependence" [8], this would provide added justification for dropping the requirement of anomaly free multiple commutators.

GR vertex with N odd results in a vertex which is not GR. This follows from the fact that the projections perpendicular to the cone axis of the edges pointing along the cone for a regular conical configuration separate into pairs which point opposite to each other. This in turn is due to the fact that π is an integer multiple of the azimuthal angle $\phi = \frac{2\pi}{N-1}$ between successive projections i.e. $\pi = \frac{N-1}{2}\phi$.

The GR property is used crucially in the proof of the Lemma in Sec. III of P2, this Lemma being used in the arguments of Sec. VIII. C involving the replacement of reference coordinates by contraction coordinates. If N is odd, let us assume that there do exist charge nets which satisfy eternal nondgeneracy under repeated conical deformations. It should still be possible to replace reference by contraction coordinates by restricting attention to charge nets in the ket set which have no vertex symmetries. By this we mean that the only diffeomorphism which maps any such charge net to itself is necessarily identity in the vicinity of the nondegenerate vertex of the charge net. This can be achieved, for example, by arranging for the primordial charges on distinct edges to be unequal to each other, provided the various charge restrictions in Sec. XIII. E. 4 can be show to hold. A careful check must also be made that all other considerations in this work go through for the N odd case. While these issues need careful investigation, we believe that with suitable genericity assumptions which lead to the absence of vertex symmetries, it should be possible to generalize our work to the case of N odd.

6. Role of the C^1, C^2 kinks

The reader is urged to peruse the last paragraph of Sec. V. E wherein the necessity of correlation of the upward direction between members of a lineage is emphasized. The upward directions at any parent vertex are inferred from the positioning of the C^1, C^2 and C^0 kinks about the parent vertex and the placement of kinks around the child vertex is correlated with the set upward directions at the parent vertex. While the C^0 kinks occur naturally from our picture of the deformations generated by the constraints as the ‘abrupt pulling of edges along some particular edge, we have introduced the C^1, C^2 kinks purely as diffeomorphism invariant markers for the reconstruction of consistent upward directions. Their presence stems from our desire to exercise adequate control on the calculations in this work. However we feel that they constitute an inessential technical overkill and that it should be possible to do away with them.

F. Products of more than two constraints

It seems unlikely to us that the treatment of products of multiple constraint products in Sec. X will go through for the $SU(2)$ case of Euclidean gravity. This is because the analogs of an ith charge component is the ith component of a left or right invariant vector field on $SU(2)$ i.e. the analogs of these charges are gauge *variant* operators. Hence it seems difficult to define the Q factors in Eqs. (10.22) and (10.38). On the other hand all the ingredients in our treatment are fixed already by the requirements of an anomaly free commutator for the case of two constraints (see Sec. IX). Given these ingredients, the $\mathbf{-1}$ structure of the constraints ensures that any solution to the constraints is of the type discussed in Sec. XIII. B above. Hence even if we manage to generalize only the considerations of Sec. IX (and Sec. XIII) to the $SU(2)$ case, it would constitute significant progress.

G. Multivertex states

The extension of our results to the multivertex case is a key open problem. It is only in the context of such an extension that we can analyze propagation in the sense of Smolin [6,9]. In [10] we make reasonable assumptions on the solution space emerging from such a putative extension and analyze the issue of propagation.

H. Semianalytic assumption

We have assumed that semianalytic vector fields generate semianalytic diffeomorphisms and used this assumption in many of our constructions. An important open technical problem is to construct a proof of the validity of this assumption.

I. Speculations on role of the metric label

Anomaly free basis states have a metric label which plays a key role in our implementation of diffeomorphism covariance. We have restricted metric labels to have no conformal symmetries; is it possible to allow for metric labels with (asymptotic) symmetries such as (asympotitically) flat metrics? Can these metric labels have any other fundamental role to play (for example in coupling to matter or in considerations of Lorentz invariance or semiclassicality)?

ACKNOWLEDGMENTS

I thank Fernando Barbero for his comments on a draft version of this work and for his kind help with the figures. I thank Thomas Thiemann for detailed discussions on moduli. I thank Jerzy Lewandowski for help with the semianalytic category.

APPENDIX A: DEFINITION OF C^0, C^1, C^2 KINKS

C^0 kink: Let 2 semianalytic C^k edges e and f intersect at a point p. Let the edge tangent at p, in some parametrization t of e, be $\dot{e}^a$. Let the edge tangent at p in some parametrization s of f be $\dot{f}^a$. Then p is called a C^0 kink if $\dot{e}^a, \dot{f}^a$ are linearly independent. Clearly this property is invariant under diffeomorphisms.

Next, consider e, f as above. Let the intersection point p be the end point of e and the beginning point of f. Consider a semianalytic coordinate patch in an open neighborhood of p. Dots will refer to derivatives of coordinate components of points of e, f with respect to their respective parameters t, s at the point p.

p is a C^1 kink iff:

(a1) There exists $\lambda_1 > 0$ such that $\dot{f}^a = \lambda_1 \dot{e}^a$.

(b1) There exists no λ_2 such that $\ddot{f}^a - (\lambda_1)^2 \ddot{e}^a = \lambda_2 \dot{e}^a$.

p is a C^2 kink iff:

(a2) There exists $\lambda_1 > 0$ such that $\dot{f}^a = \lambda_1 \dot{e}^a$.

(b2) There exists λ_2 such that $\ddot{f}^a - (\lambda_1)^2 \ddot{e}^a = \lambda_2 \dot{e}^a$.

(c2) There exists no λ_3 such that $\dddot{f}^a - (\lambda_1)^3 \dddot{e}^a - 3\lambda_1 \lambda_2 \ddot{e}^a = \lambda_3 \dot{e}^a$.

Here the conditions (a1), (a2) ensure that there exists a reparametrization of e such that first order parameter derivatives of e, f coincide at p. The condition (b1) implies that no reparametrization of e, for which the first order parameter derivatives of e, f coincide at p, is such that the second order parameter derivatives coincide. The conditions (b2), (c2) imply that such a reparametrization exists but no such reparametrization can also make the third order parameter derivatives of e, f coincide at p. It is straightforward to verify that these conditions are invariant under change of semianalytic coordinate patch around p (assuming, that the differentiability degree k of the semianalytic manifold is greater than 3) as well as under change of parametrizations of e, f. A straightforward consequence is that the defining properties of C^1, C^2 kinks are diffeomorphism invariant.

APPENDIX B: REGULAR DOWNWARD CONICAL DEFORMATIONS OF LINEAR GR VERTEX STRUCTURE

The deformation is constructed in two steps. The first, described in Sec. B. 1, endows the deformation with a regular cone structure with the nonconducting edges in the vicinity of the displaced vertex lying along a downward regular cone with axis along the conducting edge. Here by regular we mean that if we take the outward pointing upper conducting edge as the z axis then the nonconducting edges are at equispaced azimuthal angles around this axis along the cone. In the second step described in Sec. B. 2 we introduce a C^m kink $m \in \{1, 2\}$ on the upper conducting edge. The same techniques used below can be adapted to (a) construct regular downward conical deformations of linear CGR vertex structures as discussed in Sec. IV, (b) use (a) as in Sec. V. A. 1 to construct regular upward conical deformations of linear GR vertex structures, and, (c) use (b) to construct regular upward linear CGR vertex structures as in Sec. V. A. 2.

1. Step 1: Obtaining a regular cone about the conducting line

Let c be a charge net with a single linear nondegenerate GR vertex v with N edges, N being even. We are interested in deforming this charge net along its Ith edge to obtain the deformed charge net $\bar{c}$. If the deformation is generated by the Hamiltonian constraint this deformed charge net $\bar{c}$ is obtained as the product of three charge net holonomies; the first holonomy is based on the deformed graph depicted in Fig. 1(b) and the second and third on the undeformed graph of Fig. 1(a) underlying c. If the deformation is generated by an electric diffeomorphism, the charge net is based only on the deformed graph depicted in Fig. 1(b) but is colored differently from the first holonomy for the Hamiltonian constraint alluded to above. The displaced vertex of $\bar{c}$ is CGR if the deformation is generated by the Hamiltonian constraint and GR if the deformation is generated by an electric diffeomorphism. Hence only in the former case do we have a conducting line and an upper conducting edge. Nevertheless, in this section we abuse this terminology slightly and refer to the edge in $\bar{c}$ along which the deformation has taken place, variously, as the conducting line, conducting edge or upper conducting edge.

In this section we construct the precise deformation which leads to the deformed graph structure of Fig. 1(b). Since we are exclusively concerned with graph structure near the deformed vertex, we shall not be interested in the colorings of c, $\bar{c}$ in this section. We shall use the language "deformation of c" to mean "deformation of the graph underlying c so as to yield the deformed part of the graph near the deformed vertex in the graph underlying $\bar{c}$."

Let $\{x\}$ be the chosen (linear) coordinate patch around v so that there is a small enough coordinate ball, $B_{2\delta}(v)$ of radius 2δ around v whose surface intersects each edge emanating from v only once and such that these edges within this ball are straight lines. In what follows all our considerations will be restricted to this ball and we shall freely use coordinate structures with respect to $\{x\}$ such as (the restriction to this ball of) planes, lines, rigid rotations etc. In what follows we shall also use the notation $B_\tau(p)$ to denote a coordinate ball of radius τ around the point p.

Let e_I be the edge of c at v along which the deformation is to be constructed. We use hatted indices to denote the edges of c at v other than the Ith so that such an edge is denoted $e_{\hat{J}}, \hat{J} \neq I$. Let $B_\delta(v)$ intersect each $e_{\hat{J}}$ at the point $\tilde{v}_{\hat{J}}$ and e_I at the point v_I. Join v_I to each $\tilde{v}_{\hat{J}}$ by the straight line $l_{I\hat{J}}$.

Each $l_{I\hat{J}}$ is in the coordinate plane $P_{I\hat{J}}$ spanned by the tangent vectors $\vec{e}_I(v), \vec{e}_{\hat{J}}(v)$ at v (these vectors are in the direction of the straight line edges $e_{\hat{J}}$ and e_I). Since v is GR, these N-1 planes (one for each $\hat{J}$) only intersect along the straight line along e_I. Consider any such plane $P_{I\hat{J}}$ and the rotation vector field $\vec{\xi}_{I\hat{J}}$ about the axis passing through v_I normal to this plane. Consider $B_\epsilon(v_I)$, $\epsilon < \delta$ so that $B_\epsilon(v_I) \subset B_{2\delta}(v)$. Let f be a semianalytic function of compact support which is unity in $B_{\frac{\epsilon}{2}}(v_I)$ and vanishes outside $B_\epsilon(v_I)$. Let $\phi(f\xi_{I\hat{J}}, t)$ be the one parameter family of diffeomorphisms generated by $f\vec{\xi}_{I\hat{J}}$. For an appropriate value of $t = t(\theta, \hat{J})$ apply this diffeomorphism *only* to the line $l_{I\hat{J}}$ so as to rotate it rigidly within $B_{\frac{\epsilon}{2}}(v_I)$ to the required cone angle θ while retaining its semianalytic character. Performing this "rotation" for each line $l_{I\hat{J}}$ yields a downward cone structure in the vicinity of v_I. With a slight abuse of notation we shall continue to refer to $\{l_{I\hat{J}}\}$ so deformed by the same symbol. The above structure defines a deformation of c wherein $\{\tilde{v}_{\hat{J}}\}$ are the C^0 kinks, v_I is the displaced vertex and the edges $\{e_{\hat{J}}\}$ have been "abruptly dragged" at the C^0 kinks so as to form the curves $l_{\hat{J},I}$ which are straight lines in the vicinity of v_I and point along the downward cone there with cone angle θ. Further, due to the use of semianalytic diffeomorphisms the graph so obtained remains semianalytic and, due to the details of the procedure no unwanted intersections have been created.

Our considerations hereon are restricted to $B_{\frac{\epsilon}{2}}(v_I)$. Recall that the edges at v_I in $B_{\frac{\epsilon}{2}}(v_I) \subset B_\epsilon(v)$ are all straight lines. Since in this section we have occasion to refer to both the undeformed edges of c as well as their deformed counterparts in $\bar{c}$, we denote these deformed counterparts through "bars." Accordingly, using the same numbering for the deformed edges at v_I in the deformed charge net as for their undeformed counterparts at v in c, we denote the deformed counterpart of e_J at v_I in $\bar{c}$ by $\bar{e}_J$. Using the hatted index notation, the edges $\{\bar{e}_{\hat{J}}\}$ are nonconducting at v_I and the conducting edge is $\bar{e}_I$.

Consider the projections of each of the nonconducting edges transverse to the conducting edge at v_I in $\bar{c}$. These projections take the form of radially directed rays in a two-dimensional disk $D_\perp$ emanating outward from its center. As shown in P1, the *angular order* of these projections around this center is a coordinate invariant property. We shall now further deform the structure around v_I so that these transverse projections are at equal angles $\phi = \frac{2\pi}{N-1}$ with respect to each other while maintaining the downward cone angle θ of their unprojected nonconducting edge correspondents. For this purpose it is useful to change our notation slightly and denote the nonconducting edges by $\bar{e}_0, \bar{e}_1, ..\bar{e}_{N-2}$ where we have numbered the edges in the angular order of the transverse projections of their tangents at v_I with respect to $\bar{e}_I$ and we have arbitrarily picked $\bar{e}_0$ to be some particular nonconducting edge. Let the edges $\bar{e}_1, ..\bar{e}_{N-2}$ be such that their transverse projections onto $D_\perp$ make angles $\phi_1, .., \phi_{N-2}$ with respect to $\bar{e}_0$. Starting at $\bar{e}_0$ and moving anticlockwise around the cone axis in order of increasing ϕ, let $\bar{e}_i$ be the first edge encountered such that

$$\phi \geq i\frac{2\pi}{N-1}, \qquad \phi_j < j\frac{2\pi}{N-1} \quad \forall\ j < i. \tag{B1}$$

Assume $i > 1$. Let $\vec{\xi}_\phi$ be the rotational vector field about the axis $\bar{e}_I$. Consider a semianalytic function of compact support f_{i-1} such that $f_{i-1} = 1$ on $B_{\frac{\epsilon}{4}}(v_I)$ and $f_i = 0$ outside $B_{\frac{\epsilon}{2}}(v_I)$ with f_i decreasing from 1 to 0 in the region between the boundaries of these 2 spheres. Let $\Phi(f_{i-1}\xi_\phi, t)$ be the one parameter family of semianalytic diffeomorphisms generated by $f_{i-1}\vec{\xi}_\phi$. Then for an appropriate value of $t = t_{i-1}$ apply the diffeomorphism $\Phi(f_i\xi_\phi, t_{i-1})$ *only* to $\bar{e}_{i-1}$ so that its transverse angle with respect to the $\bar{e}_0$ in the vicinity of v_I is increased to $(i-1)\frac{2\pi}{N-1}$.[52] It can be checked that this deformation does not create any additional intersections between any edges. Next repeat this procedure for the edge $\bar{e}_{i-2}$ replacing f_{i-1} by f_{i-2} which is unity in $B_{\frac{\epsilon}{8}}(v_I)$, vanishes outside $B_{\frac{\epsilon}{4}}(v_I)$ and is between 1 and 0 in the region between the boundaries of these 2 spheres. This brings $\bar{e}_{i-2}$ "forward" to its desired angular position. Repeat this procedure for all the $i-1$ edges between $\bar{e}_i$ and $\bar{e}_0$. This leads to a situation in which we have straight line edges in $B_{\frac{\epsilon}{2^i}}(v_I)$ and we proceed to the next step.[53]

The next step is to check if $\bar{e}_i$ is already at its correct position. If so we skip this paragraph and move on to the step outlined in the next paragraph. If $\bar{e}_i$ is not at its correct position, we apply a similar procedure to $\bar{e}_i$ with f_i unity in $B_{\frac{\epsilon}{2^{i+1}}}(v_I)$ and vanishing outside $B_{\frac{\epsilon}{2^i}}(v_I)$ so as to rotate $\bar{e}_i$ *clockwise* about $\bar{e}_I$ in the vicinity of v_I to its desired position. At the end of all this, we have created no additional intersections, and we have edges $\bar{e}_0, \bar{e}_1, .., \bar{e}_i$ at their desired positions with these edges being straight lines in $B_{\frac{\epsilon}{2^{i+1}}}(v_I)$.

Next, if $i < N-1$, repeat the considerations above for the edges $\bar{e}_j, j > i$ by replacing the role of $\bar{e}_0$ in the above procedure by $\bar{e}_i$. Clearly the procedure terminates in a finite number of steps at the end of which the deformed graph remains semianalytic without any additional intersections between its edges, and, the vertex structure in a small neighborhood of v_I is exactly of the "regular conical" type required.

Let us now revert to our old notation which numbered the deformed and undeformed edge counterparts identically. It is important to note that the following property holds for the graph deformation we have just defined. Consider the projections of the edge tangents for $\hat{J} \neq I$ transverse to the Ith edge tangent at the vertex v of the undeformed charge net c. Call this set of projections as $\{\vec{\hat{e}}_{\hat{J},\perp}\}$. The elements of this set can be ordered in order of increasing transverse angle ϕ. Let this ordering be $(\vec{\hat{e}}_{\hat{J}_1,\perp}, \vec{\hat{e}}_{\hat{J}_2,\perp}, .., \vec{\hat{e}}_{\hat{J}_{N-1},\perp})$. Recall again that each undeformed edge $e_{\hat{J}}$ has a unique deformed

[52] If $\bar{e}_{i-1}$ is already at its desired position set $t_{i-1} = 0$.

[53] In the case that $i = 1$ satisfies (B1), there are no "in between" edges and we can directly proceed to the next step. Also note that if there is no edge i which satisfies (B1), then all edges are either at their correct positions or need to be rotated anticlockwise to their positions. In such a case we start our procedure as above by setting $i-1 = N-2$ so as to first bring the $N-2$th edge to its correct position, then $N-3$th edge all the way up to the second edge (the second edge is e_1 since our numbering starts from 0) so that all edges are now at their correct positions.

nonconducting counterpart $\bar{e}_{\hat{J}}$ which departs from the undeformed edge $e_{\hat{J}}$ at the C^0 kink $\tilde{v}_{\hat{J}}$ and terminates at v_I. Call the set of projections of these nonconducting edge tangents transverse to the conducting edge tangent at the vertex v_I as $\{\vec{\hat{\bar{e}}}_{\hat{J},\perp}\}$. These also may be ordered in a similar manner. Then the property which holds for the deformation is that these two orderings are identical i.e. the collection $(\vec{\hat{\bar{e}}}_{\hat{J}_1,\perp}, \vec{\hat{\bar{e}}}\hat{J}_2, \perp, .., \vec{\hat{\bar{e}}}_{\hat{J}_{N-1},\perp})$ is also ordered in order of increasing ϕ.

2. Step 2: Introducing a C^2 or C^1 kink on the conducting line

The step above leaves us with a regular cone in some small $\eta \ll \delta$ sized neighborhood $B_\eta(v_I)$ of v_I in the deformed charge net $\bar{c}$. Recall that $\bar{c}$ is the deformed charge net obtained in Sec. B. 1 above and, for a Hamiltonian constraint deformation is based on the graph depicted in Fig. 1(c) and for an electric diffeomorphism deformation on Fig. 1(b). We now show how to place a C^2 kink on the upper conducting edge of $\bar{c}$.

Since the deformation of c is along its Ith edge, the upper conducting edge in $\bar{c}$ is a subset of e_I and we confine our considerations to this subset below. Let $\tau \ll \eta$ and let $v_{\frac{\eta}{2},I}$, $v_{\frac{\eta+\tau}{2},I}$ be at distances $\frac{\eta}{2}, \frac{\eta+\tau}{2}$ from v_I on e_I so that the outward pointing upper conducting edge runs from v_I on to $v_{\frac{\eta}{2},I}$ and then to $v_{\frac{\eta+\tau}{2},I}$. We seek to deform this edge so that $v_{\frac{\eta}{2},I}$ becomes a C^2 kink. We require that in doing this (a) the segment of the edge between v_I and $v_{\frac{\eta}{2},I}$ remains undeformed (and hence a straight line), and (b) the deformation be confined to within a distance $\tau \ll \frac{\eta}{2}$ of $v_{\frac{\eta}{2},I}$ in a small enough cylindrical (with axis along e_I) neighborhood U_{cyl} of the edge e_I that no intersections of the deformed edge with any other edge of the graph underlying $\bar{c}$ ensue.

Clearly the desired deformation can be generated through the action of a small loop holonomy h_{l_τ} with charge $q^i_{l_\tau}$ where the loop consists of two semianalytic segments $l_{1\tau}, l_{2\tau}$ where (i) $l_{1\tau}$ runs from $v_{\frac{\eta}{2}I}$ along e_I for a distance $\frac{\tau}{2}$ to the point $v_{\frac{\eta+\tau}{2},I}$ on e_I, (ii) $l_{2\tau}$ runs from $v_{\frac{\eta+\tau}{2},I}$ to $v_{\frac{\eta}{2}I}$ within a small enough cylindrical neighborhood U_{cyl} of e_I such that it does not intersect the deformed graph of Step 1 except at the two points $v_{\frac{\eta}{2}I}$, $v_{\frac{\eta+\tau}{2},I}$ and such that it joins $v_{\frac{\eta}{2}I}$ at a C^2 kink but leaves $v_{\tau,I}$ as a semianalytic C^k extension, and (iii) $q^i_{l_\tau}$ is chosen to be the negative of the charge which colors the (outgoing) upper conducting edge at v_I in $\bar{c}$ (recall that the part of e_I between $v_{\frac{\eta}{2}I}$ and $v_{\frac{\eta+\tau}{2},I}$ is a subset of this upper conducting edge).

Recall that the deformed charge net obtained at the end of Step 1 in the case of Hamiltonian constraint type deformation can be thought of as being generated by the product of appropriately defined holonomies [see, for example, the line preceding Eq. (3.2) and the discussion in Sec. II after Eq. (2.24)]. This deformation is further modified through multiplication by the holonomy h_{l_τ}. By choosing U_{cyl} and τ small enough, the area of the loop can be made as small as desired so that the corresponding classical holonomy is unity to $O(\eta^m)$ for any desired m. This implies that the above C^2 kink modification of the discrete approximant constraint action still yields an acceptable approximant to the constraint action. In a similar fashion multiplication by this holonomy of the electric diffeomorphism constraint approximant which generates the deformation of Step 1 also yields an acceptable approximant which generates the C^2 kink modified deformation.

To summarize: The end result of our constructions is that in a small enough neighborhood of the deformed vertex v_I the nonconducting edges are straight lines which form a regular downward pointing cone around an axis in the direction of the conducting edge which is also a straight line in this neighborhood. The nonconducting edges emanating from v_I meet their undeformed counterparts in C^0 kinks whereas the conducting edge emanating from v_I is distinguished by its having a C^2 kink. The area of the C^2 kink (i.e. the area of a holonomy which can create this kink) can be made as small as desired. In particular, given some $\alpha_0 \ll \delta_0$ the departure of the edge from linearity can be confined to a small sphere of radius $2\alpha_0$ around the kink.

It is straightforward to see that similar constructions enable the placement of C^2 or C^1 kinks at desired locations on the conducting line of the deformed charge nets encountered in the main text.

APPENDIX C: COLORING OF MULTIPLY DEFORMED STATES

The concept of net charge plays a key role in this section. Equation (5.1) defines the *net charge* on a conducting edge to be the sum of its *outgoing* upper and lower conducting charges. Here we extend this definition to the case that the edge is nonconducting; in such a case we define the net charge $q^i_{\mathrm{net}I}$ on such an edge to be its outgoing charge.

Next, let c be a state with a single nondegenerate GR or CGR vertex v which is linear with respect to the coordinate patch $\{x\}$. Let c be deformed as described in Sec. V by the deformation (i, I, β, δ). The detailed locations of kinks associated with this deformation are not important here. We have the following cases:

Case 1: The parent vertex is GR and there is no intervention. From Secs. III, IV and V it follows that the displaced vertex in the child is either GR or CGR. We have two subcases:

Case 1a: Parent vertex is GR, Child vertex is displaced along the edge e_I of the parent in the outgoing direction $\vec{\hat{e}}_I$: In this case, for any $J \neq I$, the Jth edge at the displaced vertex of the child is colored with (i,β) flipped images (i.e. unflipped charges if $\beta = 0$ and β times the i flipped charges if $\beta = \pm 1$) of the charges on its undeformed parental counterpart. By gauge invariance the net charge on the Ith edge at the displaced vertex in the child is the (i,β) flipped image of the charge on the Ith edge of c.

Case 1b: Parent vertex is GR, child vertex is displaced along the *linear extension* of the edge e_I of the parent in the incoming direction $-\vec{\hat{e}}_I$ opposite to the outgoing direction: In this case there is no conducting line in the child at the displaced vertex and, by construction, all edges at the displaced vertex of the child have (i,β) flipped charges of their undeformed parental counterparts.

Case 2: Parent vertex is CGR: Let the conducting line through the parent vertex v in c be the Kth one. Due to the intervention by h_l (see Secs. IV and V), the parent vertex v in c becomes a GR vertex in c_l. The edges at v in c_l are equipped with the net charges of their counterparts at v in c. We have two subcases:

Case 2a: The child vertex is displaced along the edge e_I of c_l in the outgoing direction $\vec{\hat{e}}_I$. There are two further subcases:

Case 2a.1: $I \neq K$: The displaced vertex is not located on the conducting line in c. Hence the inverse intervention h_l^{-1} leaves this vertex untouched. From Case 1 above, the displaced vertex in the child $c_{l(i,I,\beta,\delta)}$ is either CGR or GR. Since this vertex is untouched by the inverse intervention, this vertex remains CGR or GR in $c_{(i,I,\beta,\delta)}$. Note also from Case 1a that the net charges at the displaced vertex in the deformed child $c_{l(i,I,\beta,\delta)}$ are the (i,β) flipped images of the corresponding net charges at v in c_l. Since the inverse intervention does not touch this vertex, the net charges at the displaced vertex in $c_{(i,I,\beta,\delta)}$ are also the (i,β) flipped images of the charges on c_l, these charges on c_l being the same as the net charges at v in c.

Case 2a.2: $I = K$: The displaced vertex is located on the conducting line passing through v in c. From Case 1 it follows that the displaced vertex is CGR or GR in $c_{l(i,K,\beta,\delta)}$. Since the displacement is along the Kth edge in c_l, in the case that the displaced vertex is CGR the upper and lower conducting edges at this vertex in $c_{l(i,K,\beta,\delta)}$ are also the Kth ones. Since the inverse intervention can only affect the vertex structure at the displaced vertex in $c_{l(i,K,\beta,\delta)}$ along this Kth conducting line, it follows that the displaced vertex in $c_{(i,K,\beta,\delta)}$ is also either GR or CGR. Moreover the inverse intervention cannot change the net charges at the displaced vertex so that the net charges at the displaced vertex in $c_{(i,K,\beta,\delta)}$ are the same as those at this vertex in $c_{l(i,K,\beta,\delta)}$. The latter, from Case 1a, are the (i,β) flipped images of their counterparts at v in c_l, these charges in c_l being the same as the net charges at v in c.

Case 2b: The child vertex is displaced along the *linear extension* of the edge e_I of the c_l in the incoming direction $-\vec{\hat{e}}_I$ opposite to the outgoing direction. There are two subcases:

Case 2b.1: $I \neq K$: The displaced vertex is not located on the conducting line in c so that the inverse intervention h_l^{-1} leaves this vertex untouched. From Case 1b above it follows that (i) the displaced vertex in $c_{l(i,I,\beta,\delta)}$, and hence in $c_{(i,I,\beta,\delta)}$, is GR (ii) the net charges at the displaced vertex in $c_{l(i,I,\beta,\delta)}$, and hence in $c_{(i,I,\beta,\delta)}$, are the (i,β) flipped images of the charges on c_l. The charges at v in c_l are the same as the net charges at c because the inverse intervention cannot change net charges.

Case 2b.2: $I = K$: The displaced vertex is located on the conducting line in c. From Case 1b, it follows that the displaced vertex is GR in $c_{l(i,K,\beta,\delta)}$. The inverse intervention can only affect the vertex structure at the displaced vertex in $c_{l(i,K,\beta,\delta)}$ along the Kth edge of $c_{l(i,K,\beta,\delta)}$ at this vertex. It follows that the displaced vertex in $c_{(i,K,\beta,\delta)}$ can only be GR or CGR. Moreover the inverse intervention cannot change the net charges at the displaced vertex so that the net charges at the displaced vertex in $c_{(i,K,\beta,\delta)}$ are the same as those in $c_{l(i,K,\beta,\delta)}$. The latter, from Case 1b, are the (i,β) flipped images of their counterparts at v in c_l and the charges at v in c_l are the same as the net charges at v in c.

Case 3: Parent vertex is GR but an intervention is required: This case is that of (2), Sec. V. B. 1. It is easy to check that this is identical to the case of a CGR vertex with vanishing upper conducting charge and no new structures beyond those already encountered ensue. Since our arguments for the CGR case did not depend on the specific values of the edge charges, we still have that Conclusion 1 below holds.

Thus we have Conclusion 1: The displaced vertex in the child is either GR or CGR. The net charges on the edges of the child at its displaced vertex are the (i,β) flipped images of the net charges on their counterparts in the parent.

Also note the following:

(1a) In Case 1a above the undeformed parts of the edges $e_{J\neq I}$ emanating from v in $c_{(i,I,\beta,\delta)}$ have vanishing ith charge when $\beta \neq 0$. By gauge invariance, the Ith edge at v in $c_{(i,I,\beta,\delta)}$ also has vanishing ith charge so that v is degenerate in $c_{(i,I,\beta,\delta)}$. Note that v remains GR in $c_{(i,I,\beta,\delta)}$. If $\beta = 0$ (i.e. for electric diffeomorphism type deformations), v is absent in $c_{(i,I,\beta,\delta)}$.

(1b) In Case 1b above, similar to (1a) the undeformed parts of the edges $e_{J\neq I}$ emanating from v in $c_{(i,I,\beta,\delta)}$ have vanishing ith charge when $\beta \neq 0$. By gauge invariance, net charge along the Ith edge at v in

$c_{(i,I,\beta,\delta)}$ has vanishing ith component so that v is degenerate in $c_{(i,I,\beta,\delta)}$. Note, however, that because of the necessity of a graph extension, the vertex v in $c_{(i,I,\beta,\delta)}$ can be either CGR or GR. If $\beta = 0$, then because of the graph extension v is present (and bivalent) in $c_{(i,I,\beta,\delta)}$.

(2) The deformed charge nets $c_{l(i,I,\beta,\delta)}$ are obtained by actions of the type in Cases 1a, 1b at the GR vertex v of c_l. Accordingly we have that (2a) The transition from c_l to $c_{l(i,I,\beta,\delta)}$ is of type Case 1a: For $\beta \neq 0$, (1a) implies that the edges at the vertex v in $c_{l(i,I,\beta,\delta)}$ each have net charge with vanishing ith component. The action of the inverse intervention does not change these net charges so that the net charges at v in $c_{(i,I,\beta,\delta)}$ also have vanishing ith component. If v in $c_{(i,I,\beta,\delta)}$ remains CGR, it is then easy to see that independent of which edge at v in $c_{(i,I,\beta,\delta)}$ we assign as upper, due to the fact that the corresponding intervention which removes the lower conducting edge at v does not change the net charge, we have that v is unambiguously degenerate in $c_{(i,I,\beta,\delta)}$ (see Definition 3, Sec. V. E).

If $\beta = 0$ then from (1a) v is absent in $c_{l(i,I,\beta,\delta)}$ so that it is bivalent in $c_{(i,I,\beta,\delta}$.

(2b) The transition from c_l to $c_{l(i,I,\beta,\delta)}$ is of type Case 1b. Here it is important to delineate the two subcases, (2b.1) with $I \neq K$ and (2b.2) with $I = K$:

(2b.1) From (1b), if $\beta \neq 0$, the edges at the vertex v in $c_{l(i,I,\beta,\delta)}$ each have net charge with vanishing ith component. Note however that from (1b) the vertex v in $c_{l(i,I,\beta,\delta)}$ can be GR or CGR. If it is CGR, the conducting line at v in $c_{l(i,I,\beta,\delta)}$ is along the Ith edge of c and its extension. Since the inverse intervention only affects the vertex structure at v along the Kth edge in $c_{l(i,I,\beta,\delta)}$, this Ith conducting line is also present at v in $c_{(i,I,\beta,\delta)}$. In addition the inverse intervention restores the lower part of the Kth conducting edge so that there are now *two* conducting lines through v in $c_{(i,I,\beta,\delta)}$. Note however that the inverse intervention cannot change net charges so that the net charges on these lines still have vanishing ith component. Definition 5, Sec. V then implies that this "doubly CGR" vertex is degenerate.

If $\beta = 0$, then from (1b) v is bivalent in $c_{l(i,I,\beta,\delta)}$ and the inverse intervention renders this vertex 4 valent but with only two linearly independent edge tangents. Hence the vertex is planar (and hence neither GR nor CGR) and hence, degnerate, in $c_{(i,I,\beta,\delta)}$.

(2b.2) From (1b), if $\beta \neq 0$, the edges at the vertex v in $c_{l(i,K,\beta,\delta)}$ each have net charge with vanishing ith component. If the vertex v is CGR in $c_{l(i,K,\beta,\delta)}$ as a result of the graph extension, then the conducting line through v is along the Kth edge in c_l and its extension. Since the inverse intervention is also along the Kth edge at v in c, the vertex v in $c_{(i,K,\beta,\delta)}$ is either GR or CGR but not doubly CGR. Since the inverse intervention cannot change net charges, the net charges at v in $c_{(i,K,\beta,\delta)}$ have vanishing ith component so that v is unambiguously degenerate in $c_{(i,K,\beta,\delta)}$. If $\beta = 0$, only the Kth line passes through v in $c_{l(i,K,\beta,\delta)}$ so that v is bivalent in $c_{l(i,K,\beta,\delta)}$. The inverse intervention near v is also along this line and the vertex v remains bivalent (and hence degenerate) in $c_{(i,K,\beta,\delta)}$.

(3) As mentioned in Case 3 above, this is identical to the case of a CGR vertex with vanishing upper conducting charge and no new structures beyond those already encountered ensue. Since our arguments for the CGR case did not depend on the specific values of the edge charges, we still have that Conclusion 2 below holds.

It is straightforward to check that in all cases, leaving aside the vertex v and its displaced image in the child, the only other vertices created by the deformation are of valence at most 3 and hence degenerate. Hence the only possibly nondegenerate vertices of $c_{(i,I,\beta,\delta)}$ are v (which we have shown is degenerate) and its displaced image.

Thus we have Conclusion 2: The vertex v (if present) in $c_{(i,I,\beta,\delta)}$ is degenerate.

It then follows, if (as is assumed in the main text) the displaced vertex is nondegenerate, then the deformed child of a parent with a single linear, nondegenerate GR or CGR vertex also has a single GR or CGR vertex with net charges which are (i, β) flipped images of their parental correspondents. Applying this to any c in the ket set, it follows, by virtue of the fact that any such Ket arises as a multiple deformation of some primordial ket, that (a) any $c \in S_{\mathrm{Ket}}$ has a single nondegenerate vertex and (b) the net charges at the nondegenerate vertex of $c \in S_{\mathrm{Ket}}$ are identical to, or the flipped images of, the charges on such a primordial ancestor.

APPENDIX D: EXAMPLES OF P RIMORDIAL STATES

Consider a 4 valent gauge invariant linear vertex which is linear with respect to some chart $\{x, y, z\}$. Let its outward pointing edges be in either upward or downward conical conformation with respect to $\{x, y, z\}$ (by which we mean that one edge points along the cone axis and the remaining three are arranged in a regular upward or downward configuration about this axis). We assume without loss of generality that the 4th edge e_4 points along the z (or $-z$) axis and that the remaining three edges e_1, e_2, e_3 point along a cone with angle $0 < \theta < \pi$ about the $-z$ axis. Let the projections of the outgoing tangents to e_1, e_2, e_3 be $\vec{e}_{1\perp}, \vec{e}_{2\perp}, \vec{e}_{3\perp}$. Let the edges be placed such that these projections are ordered anticlockwise about the z axis as $\{\vec{e}_{1\perp}, \vec{e}_{2\perp}, \vec{e}_{3\perp}\}$ and let the angle between successive projections be $\frac{2\pi}{3}$ so that the configuration is regular conical.

Let the triplet of $U(1)$ charges on the Ith edge be $\vec{q}_I = (q_I^1, q_I^2, q_I^3)$. Choose $\vec{q}_I, I = 1, 2, 3$ to be linearly independent vectors and set $\vec{q}_4 = -(\sum_{I=1}^3 \vec{q}_I)$ so that the vertex is gauge invariant. It is straightforward to verify that if the cone is in downward conformation (so that e_4 points along the $+z$ axis), the volume eigenvalue ν (see (2.16)) is

$$\nu = 4|q|, \qquad q \coloneqq \frac{1}{48}\epsilon^{ijk} q_1^i q_2^j q_3^k \tag{D1}$$

and if the cone is in upward conformation (so that e_4 points along the $-z$ axis) then the volume eigenvalue is

$$\nu = 2|q|. \tag{D2}$$

Let us denote a primordial charge net with a single 4 valent vertex of the type which results in (D1) by c_D and that with a single 4 valent vertex of the type which results in (D2) by c_U. We note the following:

(1) For both these choices of primordials, the linear dependence of 3 of the 4 charge triplets together with gauge invariance implies that these four charge triplets define a GR set of charge vectors i.e. any three of these vectors are linearly independent. Since we have only used linear independence of three charge vectors, gauge invariance and the conical conformation of the edge tangents, it follows that we could have chosen *any* of the four edges to be along the cone axis and still obtained nondegeneracy.

(2) Let us subject the charges $\vec{q}_I, I = 1, 2, 3, 4$ to a (β, i) flip. It is easy to see that flipped charges also form a gauge invariant set and that the volume eigenvalue is invariant under the replacement of the charges $\vec{q}_I, I = 1, 2, 3, 4$ by their flipped images.

(3) From Appendix C it follows that the net charges which color the edges at the vertex of any charge net obtained through multiple deformations of a primordial charge net are just multiply flipped images of the charges on the primordial.

From (1)–(3) above in conjunction with Conclusion 2 of Appendix D, the deformations constructed in Sec. V. A–V. C and Definition 3 of the nondegeneracy of CGR vertices in Sec. V. E, it follows that any multiple deformation of either c_D or c_U results in a deformed charge net which has a single nondegenerate GR or CGR vertex.

Finally, it is straightforward to see that we can easily arrange for conditions (6.5), (6.6) to be satisfied, for example by setting the charges on c_D, C_U to be $q_I^i = M\delta^{i_I} + 1$, $I = 1, 2, 3, M \gg 1$.

APPENDIX E: JACOBIAN BETWEEN REFERENCE AND CONTRACTION COORDINATES

To avoid notational clutter we adopt the following change in notation in this section relative to Step 1, Sec. VIII. C. We set $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m} = s$, $(c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m})_0 = s^{\mathrm{ref}}$, $\alpha_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m} = \phi_{\mathrm{ref}}$, $c_{0[i,I,\beta,\delta_0]_m} = s^{\mathrm{con}}$, and denote the diffeomorphism which maps s^{con} to s by ϕ_{con} [here the action of ϕ_{con} is obtained by the action of an appropriate (composite) contraction diffeomorphism followed by the diffeomorphism α of Step 1, Sec. VIII. C]. The reference coordinates for s are $\phi^*_{\mathrm{ref}}\{x_0\}$ where ϕ^*_{ref} is the *pushforward* action of ϕ_{ref} and the contraction coordinates for s are $\phi^*_{\mathrm{con}}\{x_0\}$. We are interested in evaluating the Jacobian between these two coordinate systems at the nondegenerate vertex v of s.

First note the following. The states s^{con} and s^{ref} are diffeomorphic to s and hence to each other. Hence there exists a diffeomorphism ϕ which maps s^{ref} to s^{con}. This diffeomorphism must map the nondegenerate vertex v_0^{ref} of s^{ref} to the nondegenerate vertex v_0^{con} of s^{con} and also map the set of nearest kinks at these vertices to each other. In particular the nearest C^1 kink if present in s^{ref} must be also be present in s^{con} and be mapped to it and similarly for a nearest C^2 kink if present. Since at least one of these kinks must be present and since the upward direction inferred from the location of either or both of these kinks, if present, is uniquely defined (see Sec. V. B), the upward direction $\vec{V}$ at v^{ref} is mapped to that at v^{con}. Since both s^{ref} and s^{con} are primaries, the vertex structure in a small vicinity of their vertices must be either upward or downward conical. Note however that if the structure is upward conical in s^{ref} at v^{ref} then it must be upward conical in s^{con} at v^{con}, and similarly for downward conical structures. This follows from the fact that no diffeomorphism connected to identity can map an upward conical structure to a downward one.

To prove this, proceed as follows. Consider an upward or downward cone with respect to $\vec{V}$:

(i) (i) From P2, it follows that the anticlockwise ordering of the projections, transverse to $\vec{V}$ in the coordinates $\{x_0\}$, of the *outward pointing* edge tangents which do not point along $\vec{V}$ is invariant under orientation preserving changes of coordinates. A quick way to see this is as follows. Clearly, the two-dimensional vector space of these transverse projections is isomorphic to the vector space of equivalence classes of vectors where two vectors are defined to be equivalent if they differ by a vector proportional to $\vec{V}$. Let us denote the transverse projection of an edge tangent $\vec{v}$ (or equivalently its equivalence class as defined in the previous sentence) by $\vec{v}_\perp$. Since in the regular conical conformation with respect to $\{x_0\}$, the angle between two successive projections in this anticlockwise ordering is less that π, the condition that two projections $\vec{v}_{1\perp}, \vec{v}_{2\perp}$, with $\vec{v}_{2\perp}$ occurring immediately after $\vec{v}_{1\perp}$ in this ordering are successive is equivalent to the conditions that (a) no other edge tangent projection can be written as a linear combination of $\vec{v}_{1\perp}, \vec{v}_{2\perp}$ with *positive* coefficients i.e. there exist no $\alpha, \beta > 0$ for which there exists an edge tangent $\vec{v}_3$ such that $\alpha\vec{v}_{1\perp} + \beta\vec{v}_{2\perp} = \vec{v}_{3\perp}$, and (b) the vectors $\vec{v}_1, \vec{v}_2, \vec{V}$ form a right-handed triple i.e. with respect to the alternating Levi-Civita tensor η_{abc} we have that $\eta_{abc}v_1^a v_2^b V^c > 0$ (this condition encodes the fact that $\vec{v}_{2\perp}$ is encountered *after* $\vec{v}_{1\perp}$ in the anticlockwise ordering; we have implicitly assumed that $\{x_0\}$ is right handed). The conditions (a) and (b) are invariant under positive rescalings of $\vec{v}_1, \vec{v}_2, \vec{V}$. Clearly, any coordinate transformation can only rescale these vectors with positive rescalings since they refer to coordinate invariant directions at the tangent space of the vertex in question. This proves that the ordering of these projections is defined in a coordinate invariant way.

(ii) The $N-1$ edges not pointing along $\vec{V}$ are arranged in a regular cone of angle θ with respect to $\vec{V}$ when viewed in the $\{x_0\}$ coordinates. Let $\hat{\vec{V}}$ be the unit vector along $\vec{V}$ unit with respect to the $\{x_0\}$ coordinate norm and consider three successive units (with respect to $\{x_0\}$) outward pointing edge tangents $\hat{\vec{v}}_i, i = 1, 2, 3$ arranged in anticlockwise order as discussed in (a) so that the angle between two successive pairs in these coordinates is $\phi = \frac{2\pi}{N-1}$. It is straightforward to show that

$$\hat{\vec{v}}_1 + \hat{\vec{v}}_3 - 4\cos\theta \sin^2\frac{\phi}{2}\hat{\vec{V}} = 2\cos\phi\hat{\vec{v}}_2. \tag{E1}$$

This implies the relation

$$a\vec{v}_1 + b\vec{v}_3 - c\cos\theta\vec{V} = d\cos\phi\vec{v}_2, \quad \text{for some} \quad a, b, c, d > 0. \tag{E2}$$

This implies that if $N = 4$, for some $\alpha, \beta, \gamma > 0$, we have that

$$-(\alpha\vec{v}_1 + \beta\vec{v}_3 - \gamma\vec{V}) = \vec{v}_2, \quad \text{for a upward cone} \tag{E3}$$

$$-(\alpha\vec{v}_1 + \beta\vec{v}_3 + \gamma\vec{V}) = \vec{v}_2, \quad \text{for an downward cone;} \tag{E4}$$

and that if $N > 4$, for some $\alpha, \beta, \gamma > 0$, we have that

$$\alpha\vec{v}_1 + \beta\vec{v}_3 - \gamma\vec{V} = \vec{v}_2, \quad \text{for an upward cone;} \tag{E5}$$

$$\alpha\vec{v}_1 + \beta\vec{v}_3 + \gamma\vec{V} = \vec{v}_2, \quad \text{for a downward cone.} \tag{E6}$$

The above equations retain their form (as well as the positivity properties of α, β, γ) irrespective of the choice of coordinates because a change of coordinates only provides a positive rescaling to the vectors in these equations.

(iii) Clearly, any diffeomorphism ϕ connected to identity which maps s^{ref} to s^{con} is such that (a) it maps outward pointing edge tangents at v_{ref} in s^{ref} to outward pointing edge tangents at v_{con} in s^{con}; (b) it maps, as noted in the second paragraph of this section, the upward direction at v_{ref} to that at v_{con}; (c) it retains the anticlockwise ordering of the $(N-1)$ edge tangents (which are not along the cone axis) around the upward direction; this immediately follows from the fact that ϕ (which is orientation preserving by virtue of its being connected to identity) preserves conditions (i) (a),(b); and (d) from (a)–(c), it follows that ϕ preserves conditions (E3)–(E6) so that if any one of these conditions hold at v_{ref} in s^{ref}, the same condition holds at v_{con} in s^{con}.

The statement (d) implies that an upward conical deformation cannot be mapped into a downward conical deformation (and vice versa) by ϕ. Next, recall that the multiple deformation which generates any primary from any reference primordial is confined to a coordinate ball $B_{\Delta_0}(p_0)$ with respect to the Primary Coordinates $\{x_0\}$. We show the following Lemma.

Lemma L1: Let the vertex structures in a small vicinity of the nondegenerate vertices of $s^{\mathrm{ref}}, s^{\mathrm{con}}$ be downward conical. Denote the coordinate ball of size η (in $\{x_0\}$ coordinates) around a point p by $B_\eta(p)$. Then there exist small enough open balls $B_\tau(v_0^{\mathrm{ref}}), B_\tau(v_0^{\mathrm{con}})$, together with a rigid rotation R and a rigid translation T (with R, T defined with respect to the $\{x_0\}$ chart) such that $B_\tau(v_0^{\mathrm{con}}) = RTB_\tau(v_0^{\mathrm{ref}})$ and such that $RT(s^{\mathrm{ref}}|_{B_\tau(v_0^{\mathrm{ref}})}) = s^{\mathrm{con}}|_{B_\tau(v_0^{\mathrm{con}})}$ where $c|_U$ refers to the restriction of the colored graph defining the charge net c to the set U.

Proof.—Clearly, there exists small enough τ such that $B_\tau(v_0^{\mathrm{ref}}), B_\tau(v_0^{\mathrm{con}})$ intersect $s^{\mathrm{ref}}, s^{\mathrm{con}}$ only in their conical vertex structures. Thus $s^{\mathrm{ref}}|_{B_\tau(v_0^{\mathrm{ref}})}, s^{\mathrm{con}}|_{B_\tau(v_0^{\mathrm{con}})}$ both comprise of regular conical structures with respect to $\{x_0\}$ i.e. each of these restrictions comprise of a cone vertex with downward pointing nonconducting edges around the upward pointing cone axis. Further both cones have the same angle θ. Since $s^{\mathrm{ref}}, s^{\mathrm{con}}$ are diffeomorphic (to s and hence) to each other, there exists a diffeomorphism ϕ which maps the preferred upward pointing axes to each other and the set of downward pointing conical edges to each other. Such a diffeomorphism induces a map between the sets of downward pointing unit edge tangents so that

$$\{\phi^*(\vec{e}_{\hat{J}}), \hat{J} = 1, .., N-1\} = \{\beta_{\hat{I}}\vec{e}_{\hat{I}}, \beta_{\hat{I}} > 0, \hat{I} = 1, .., N-1\} \tag{E7}$$

where the edge tangents $\vec{e}_{\hat{J}}, \vec{e}_{\hat{I}}$ on the left- and right-hand sides are unit with respect to the coordinates $\{x_0\}$ at

$v_0^{\rm ref}$, $v_0^{\rm con}$ respectively. From (iii)(c) above it follows that the sets of projections of these edge tangents transverse to the cone axis have the same ordering. Thus we may use ϕ to identify each downward pointing edge of the cone in $s^{\rm ref}$ with a downward pointing edge in $s^{\rm con}$.

Let T be the rigid translation which maps $v_0^{\rm ref}$ to $v_0^{\rm con}$. Clearly $T(B_\tau(v_0^{\rm ref})) = B_\tau(v_0^{\rm con})$. Next, rotate $(Ts^{\rm ref})|_{B_\tau(v_0^{\rm con})}$ by R_1 so that its distinguished "up" direction $\vec{V}$ aligns with that of $s^{\rm con}|_{U^{\rm con}}$. Next, rotate around this preferred direction by R_2 so that one of the downward pointing edges of $(R_1Ts^{\rm ref})|_{B_\tau(v_0^{\rm con})}$ aligns with its counterpart in $s^{\rm con}|_{U^{\rm con}}$ as identified by ϕ. Since the transverse ordering is preserved, this automatically aligns all the remaing downward pointing edges of $(R_1Ts^{\rm ref})|_{B_\tau(v_0^{\rm con}}$ with their counterparts in $s^{\rm con}|_{U^{\rm con}}$. Since R_1, R_2 preserve (the coordinate ball) $B_\tau(v_0^{\rm con})$ we set $R = R_2R_1$ to obtain the desired transformation RT.

Next note that there exists a small enough neighborhood V of the nondegenerate vertex v of s such that V is covered by the reference as well as the contraction coordinates so that $(\phi_{\rm ref})^{-1}V, (\phi_{\rm con})^{-1}V$ are in the domain of the coordinate patch $\{x_0\}$, and such that $s|_V$ only contains the nondegenerate vertex v and segments of the edges emanating therefrom. Also note that there exists small enough τ such that the open balls of the Lemma above are such that $B_\tau(v_0^{\rm con}) \subset (\phi_{\rm con})^{-1}V$, $B_\tau(v_0^{\rm ref}) \subset (\phi_{\rm ref})^{-1}V$. It follows from the Lemma above that $\phi_{\rm con}RTB_\tau(v_0^{\rm ref}) \subset V$ and that this set is an open neighborhood of v. Hence $(\phi_{\rm ref})^{-1}\phi_{\rm con}RTB_\tau(v_0^{\rm ref})$ is covered by $\{x_0\}$ and is an open neighborhood of $v_0^{\rm ref}$. Setting $(\phi_{\rm ref})^{-1}\phi_{\rm con}RT \equiv \psi$, and $\psi B_\tau(v_0^{\rm ref}) = B_\tau^\psi(v_0^{\rm ref})$, we have that both $B_\tau(v_0^{\rm ref})$ and $B_\tau^\psi(v_0^{\rm ref})$ are open neighborhoods of $v_0^{\rm ref}$ and we can compute the Jacobian

$$J^\mu_\nu = \frac{\partial(\psi^* x_0)^\mu}{\partial x_0^\nu}\bigg|_{v_0^{\rm ref}}. \tag{E8}$$

From the proof of the Lemma in P2 and the discussion above it follows that (a) ψ maps the set of edge tangents at $s^{\rm ref}$ at its vertex $v_0^{\rm ref}$ to itself modulo overall scaling (b) the upward direction is mapped to itself and the anticlockwise arrangement of the transverse projections of the downward pointing edge directions are mapped to themselves. From the fact that the reference deformations of Appendix B are regular conical, the results of (a) and (b) can be implemented through the action of linear transformation on the tangent space at $v_0^{\rm ref}$ which takes the form of a constant times an $SO(3)$ matrix in the $\{x_0\}$ coordinates. Then it follows from the last part of the proof of the Lemma in P2 that the action of ψ on the tangent space as expressed through the Jacobian in the above equation must be that of constant times as rotation i.e. we have that

$$J^\mu_\nu = C\bar{R}^\mu_\nu \tag{E9}$$

for some $C > 0$ and some $SO(3)$ matrix $\bar{R}$.

Next note that with $p \in \phi_{\rm con}RTB_\tau(v_0^{\rm ref})$ and $\bar{p} \coloneqq (RT)^{-1}(\phi_{\rm con})^{-1}p \in B_\tau(v_0^{\rm ref})$ we have that

$$\begin{aligned}\frac{\partial(\phi^*_{\rm ref}x_0)^\mu(p)}{\partial((\phi_{\rm con})^*RT^*x_0)^\nu(p)} &= \frac{\partial x_0^\mu((\phi_{\rm ref})^{-1}p)}{\partial x_0^\nu((RT)^{-1}(\phi_{\rm con})^{-1}p)}\\ &= \frac{\partial(\psi^*x_0)^\mu(\bar{p})}{\partial x_0^\nu(\bar{p})}.\end{aligned} \tag{E10}$$

Setting $p = v$ we have from (E8) and (E9) that

$$\frac{\partial((\phi_{\rm ref})^*x_0)^\mu(p)}{\partial((\phi_{\rm con})^*(RT)^*x_0)^\nu(p)}\bigg|_{p=v} = C\bar{R}^\mu{}_\nu. \tag{E11}$$

Next note that with $p \in \phi_{\rm con}RTB_\tau(v_0^{\rm ref})$,

$$\begin{aligned}\frac{\partial((\phi_{\rm ref})^*x_0)^\mu(p)}{\partial((\phi_{\rm con})^*x_0)^\alpha(p)} &= \frac{\partial((\phi_{\rm ref})^*x_0)^\mu(p)}{\partial((\phi_{\rm con})^*(RT)^*x_0)^\alpha(p)}\\ &\times \frac{\partial((\phi_{\rm con})^*(RT)^*x_0)^\alpha(p)}{\partial((\phi_{\rm con})^*x_0)^\nu(p)},\end{aligned} \tag{E12}$$

so that it remains to evaluate the second Jacobian in this equation. Setting $\bar{p} = (\phi_{\rm con})^{-1}p \in B_\tau(v_0^{\rm con})$ we have that

$$\frac{\partial((\phi_{\rm con})^*(RT)^*x_0)^\mu(p)}{\partial((\phi_{\rm con})^*x_0)^\nu(p)} = \frac{\partial x_0^\mu((RT)^{-1}\bar{p})}{\partial x_0^\nu(\bar{p})}. \tag{E13}$$

Since R, T are rigid rotations and translations in $\{x_0\}$, we have that $x_0^\mu((RT)^{-1}\bar{p}) = x_0^\mu(T^{-1}R^{-1}\bar{p}) = (R^{-1})^\mu_\nu x_0^\nu(\bar{p}) - t^\mu$ where t^μ is a constant vector corresponding to the translation T from which it follows that the Jacobian in the above equation is the $(R^{-1})^\mu_\nu$. Together with (E11) and (E12), this implies that the Jacobian between the reference and the contraction coordinates is a constant times a rotation.

It is straightforward to check that an appropriate version of Lemma 1 and the following argumentation leads to the same conclusion for the upward conical case.

APPENDIX F: CONTRACTION BEHAVIOR OF VARIOUS QUANTITIES OF INTEREST

1. Notation

To avoid notational clutter we adopt the following change in notation in this section relative to Step 2, Sec. VIII. C. We set, similar to Eq. (8.20),

$$\begin{aligned} &x_\alpha^{\epsilon_{j_1}..\epsilon_{j_m}} \equiv x^{(\delta)} \qquad x_\alpha^{\epsilon_{j_1}..\epsilon_{j_{m-1}}} \equiv x \qquad \epsilon_{j_m} \equiv \delta\\ &c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m} \equiv s_\delta \qquad c_{[i,I,\beta,\epsilon,\hat{J},\hat{K}]_m^{m-1}} \equiv s \qquad \beta_m \equiv \beta. \end{aligned} \tag{F1}$$

We are interested in the transition from the immediate parent $s \equiv c_{[i,I,\beta,\epsilon,\hat{J},\hat{K}]_m^{m-1}}$ to its child $s_\delta \equiv c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$. In this transition the parent s is deformed conically along (or opposite to) its I_{m-1}th edge at its nondegenerate vertex v so as to displace this vertex to the point v^δ which is the nondegenerate vertex of the child s_δ. In our notation (F1), this transition is $(i_{m-1}, I_{m-1}, \beta_m, \epsilon_{\mathrm{j_m}}) \equiv (i_{m-1}, I_{m-1}, \beta, \delta)$. We denote the upward direction at v in s by $\vec{V}_{I_{m-1}}$ so that $\vec{V}_{I_{m-1}}$ is parallel or antiparallel to the edge tangent $\hat{\vec{e}}_{I_{m-1}}$ at v (see Sec. V. B).

Finally, all the edge charges referred to below will be *net* charges (see the first paragraph of Appendix C for the definition of net charge).

2. Contraction coordinates and choices of upward direction

From (F1), the child s_δ of its immediate parent s has contraction coordinates $\{x^{(\delta)}\}$. Step 1 of Sec. VIII. C allows the evaluation of the coefficients which multiply the bra correspondent of s_δ in these contraction coordinates. To extract their contraction behavior we need to transit to the parental coordinates $\{x\}$ in terms of which (see Sec. VI. D. 2) δ is measured. Following Eqs. (6.15) and (6.16), it is useful to rotate the $\{x\}, \{x^{(\delta)}\}$ coordinates by R_s^{-1} so as to obtain $\{y\}, \{y^{(\delta)}\}$ coordinates with y^3 pointing along the straight line passing through v and v^δ. From the last remark of Sec. VI. C it follows that $y^{(\delta)3}$ also points along this direction. Note that this direction is parallel (antiparallel) to $\vec{V}_{I_{m-1}}$ if the deformation is downward (upward). Note also that $\vec{V}_{I_{m-1}}$ is defined, strictly speaking, only at v in s. However due to the fact that v in s is linear with respect to $\{x\}$ (and, hence, $\{y\}$) we can naturally define $\vec{V}_{I_{m-1}}$ at every point in the domain of these coordinates. It is in this sense that we refer to the direction defined by $\vec{V}_{I_{m-1}}$ in this section. Note that the direction so defined is consistent with C^2, C^1 kink placements generated by the transformation $(i_{m-1}, I_{m-1}, \beta, \delta)$.

More in detail, any edge that emanates from v_δ in c_δ which bears such a (C^1 or C^2) kink is, in the vicinity of v^δ, a straight line pointing along the I_{m-1}th edge emanating from v in c (or along the linear extension (with respect to $\{y\}$) of this edge. Hence its outward pointing edge tangent is along or opposite to $\vec{V}_{I_{m-1}}$. The placement of its nearest kink (see Secs. V. B and V. C) then defines the upward direction $\vec{V}^{(\delta)}_{I_m}$ for such an edge to be equal to $\vec{V}_{I_{m-1}}$.

3. Contraction behavior of H_{L_m}, h_{L_m}, f

Note that the structure in the immediate vicinity of the nondegenerate vertex of s_δ is regular conical in terms of $\{x^{(\delta)}\}$ (or $\{y^{(\delta)}\}$) because this structure and this coordinate system are images *by the same diffeomorphism* (see Sec. VI, especially Sec. VI. D. 2) of reference deformations and reference coordinates in which the reference deformations are regular conical. Recall, from Sec. F. 2 above, that the upward direction at v_δ in c_δ is $\vec{V}_{I_{m-1}}$. Let the cone angle as measured by the $\{y^{(\delta)}\}$ coordinates, with respect to this upward direction be θ so that $\theta < \frac{\pi}{2}$ defines an upward conical deformation and $\theta > \frac{\pi}{2}$ defines a downward conical deformation. Consider any J_mth edge of s_δ with $J_m \neq I_{m-1}$ in the immediate vicinity of v_δ.[54] Such an edge points along the cone. Let its azimuthal angle in the $\{y^{(\delta)}\}$ coordinates be ϕ_{J_m}. Thus we have that the unit (with respect to the $\{y^{(\delta)}\}$ coordinates) outward pointing edge tangent $\hat{\vec{e}}^{(\delta)}_{J_m}$ along this edge has coordinates (in the $\{y^{(\delta)}\}$ chart)[55]

$$\hat{e}^{(\delta)\bar{\mu}}_{J_m} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \pm\cos\theta) \tag{F2}$$

where the $+$ sign in front of $\cos\theta$ corresponds to downward deformations (for which $y^{(\delta)3}, y^3$ run upward) and the $-$ sign to upward deformations (for which $y^{(\delta)3}, y^3$ run downward). Using Eq. (6.15), the components of $\hat{\vec{e}}^{(\delta)}_{J_m}$ in the $\{y\}$ coordinates are

$$\hat{e}^{(\delta)\mu}_{J_m} = (\delta^{q-1}\sin\theta\cos\phi, \delta^{q-1}\sin\theta\sin\phi, \pm\cos\theta). \tag{F3}$$

Since the y^3 direction is along $\vec{V}_{I_{m-1}}$ for downward deformations and opposite to $\vec{V}_{I_{m-1}}$ for upward deformations, the above equation takes the form

$$\hat{e}^{(\delta)a}_{J_m} = \cos\theta \hat{V}^a_{I_{m-1}} + \delta^{q-1} w^a_{J_m} \tag{F4}$$

where $w^a_{J_m}$ is a δ independent vector in the $\{y\}$ (and hence in the $\{x\}$) coordinates and $\hat{V}^a_{I_{m-1}}$ is the normalized (in the $\{x\}$ or, equivalently, $\{y\}$ coordinates) vector parallel to $V^a_{I_{m-1}}$. Note that these edges are such that the nearest kink is a C^0 kink so that the upward direction $V^{(\delta)a}_{J_m}$ is along the outward pointing J_mth edge tangent (see Sec. V). Denoting its normalized (with respect to the $\{y^\delta\}$ coordinates associated with v_δ) by $\hat{V}^{(\delta)a}_{J_m}$, we may write (F4) as

$$\hat{V}^{(\delta)a}_{J_m} = \hat{e}^{(\delta)a}_{J_m} = \cos\theta \hat{V}^a_{I_{m-1}} + \delta^{q-1} w^a_{J_m}. \tag{F5}$$

Next consider the upper conducting edge $e_{I_m = I_{m-1},u}$ (if present) at v^δ in s_δ. A similar analysis shows that the unit (with respect to $\{x^\delta\}$) edge tangent along this edge is also unit with respect to the $\{x\}$ coordinates so that we have

[54] Recall that we use the edge enumeration convention described in Sec. VI. A

[55] Since $\{y^{(\delta)}\}$ and $\{x^{(\delta)}\}$ are related by the rotation R_s, normalization in both these systems is identical.

$$\hat{e}^{(\delta)a}_{I_m=I_{m-1},u} = \hat{V}^a_{I_{m-1}}. \tag{F6}$$

Similarly the lower conducting edge (if present[56]) at v_δ in s_δ has unit (with respect to $\{x^\delta\}$) tangent

$$\hat{e}^{(\delta)a}_{I_m=I_{m-1},d} = -\hat{V}^a_{I_{m-1}}. \tag{F7}$$

Note that by definition the outward upper conducting edge tangent is along the upward direction $V^{(\delta)a}_{J_m}$ at v_δ, and the lower one opposite to it so that we may write (F6) and (F7) as the single equation[57]

$$\hat{e}^{(\delta)a}_{I_m=I_{m-1},u} = \hat{V}^a_{I_{m-1}} = -\hat{e}^{(\delta)a}_{I_m=I_{m-1},d} = \hat{V}^{(\delta)a}_{I_m=I_{m-1}}. \tag{F8}$$

It is then straightforward to obtain the following estimates for the metric norms of the unit (with respect to $\{x^{(\delta)}\}$) edge tangent vectors at v^δ in s_δ:

$$\left\|\hat{e}^{(\delta)a}_{J_m\neq I_{m-1}}\right\|_{v^\delta} = |\cos\theta|\sqrt{h_{ab}(v^\delta)\hat{V}^a_{I_{m-1}}\hat{V}^b_{I_{m-1}}} \times (1 + C_{1,J_m}(v^\delta)\delta^{q-1} + C_{2,J_m}(v^\delta)(\delta^{q-1})^2)^{\frac{1}{2}} \tag{F9}$$

$$C_{1,J_m}(v^\delta) \coloneqq 2\frac{h_{ab}(v^\delta)w^a_{J_m}\hat{V}^b_{I_{m-1}}}{\cos\theta h_{ab}(v^\delta)\hat{V}^a_{I_{m-1}}\hat{V}^b_{I_{m-1}}} \tag{F10}$$

$$C_{2,J_m}(v^\delta) \coloneqq \frac{h_{ab}(v^\delta)w^a_{J_m}w^b_{J_m}}{\cos^2\theta h_{ab}(v^\delta)\hat{V}^a_{I_{m-1}}\hat{V}^b_{I_{m-1}}} \tag{F11}$$

$$\left\|\hat{e}^{(\delta)a}_{I_m=I_{m-1},u}\right\|_{v^\delta} = \left\|\hat{e}^{(\delta)a}_{I_m=I_{m-1},d}\right\|_{v^\delta} = \sqrt{h_{ab}(v^\delta)\hat{V}^a_{I_{m-1}}\hat{V}^b_{I_{m-1}}}. \tag{F12}$$

From (F6)–(F12), we obtain the following behavior for the quantities in (7.3) and (7.4):

$$H_{I_m=I_{m-1}} = \sqrt{h_{ab}(v_\delta)\hat{V}^a_{I_{m-1}}\hat{V}^b_{I_{m-1}}},$$
$$H_{J_m\neq I_{m-1}} = |\cos\theta| H_{I_m=I_{m-1}} + O(\delta^{q-1}), \tag{F13}$$

$$h_{I_m=I_{m-1}} = (N-1)(N-2) + O(\delta^{q-1}),$$
$$h_{J_m\neq I_{m-1}} = \frac{N-2}{|\cos\theta|}(1+\cos^2\theta + (N-3)|\cos\theta|) + O(\delta^{q-1}). \tag{F14}$$

Next, consider any density weight $-\frac{1}{3}$ scalar density S evaluated at v_δ. From (6.16) it follows that

$$S(v_\delta,\{x^\delta\}) = \delta^{-\frac{2}{3}(q-1)}S(v_\delta,\{x\}) \tag{F15}$$

where we have used the notation $S(p,\{z\})$ to signify the evaluation of S at the point p in the coordinate system $\{z\}$. Setting $S \eqqcolon f$, Eq. (F15) yields the contraction behavior of f. Note also that if $\{y\}$ is related to $\{x\}$ by a rotation [say, as in Eqs. (6.15) and (6.16)] we have by virtue of the fact that the determinant of a rotation matrix is unity, that

$$f(p,\{x\}) = f(p,\{y\}). \tag{F16}$$

Next consider the quantity $H^l_{L_m}$ defined by

$$H^l_{L_m} \coloneqq \prod_{i=1}^{l} N_i(v_\delta,\{y^{(\delta)}\})\hat{V}^{(\delta)a_i}_{L_m}\partial_{a_i}(f(v_\delta,\{y^{(\delta)}\})\|\vec{e}^{(\delta)}{}_{L_m}\|). \tag{F17}$$

Here N_i, f are density weight $-1/3$ scalars. The unit (with respect to the $\{y^{(\delta)}\}$ coordinates) upward direction the L_mth edge (or line) at v^δ is denoted by $\hat{V}^{(\delta)a_i}_{L_m}$, where the upward direction $V^{(\delta)a_i}_{L_m}$ is chosen in accord with the criteria of Sec. V. The product in (F17) is ordered from left to right in decreasing value of i so that the factor $(N_1\hat{V}^{(\delta)a_1}_{L_m}\partial_{a_1})$ is rightmost. The vector $\vec{\hat{V}}^{(\delta)}_{L_m}$ and its norm with respect to the metric h_{ab} are defined only at the vertex v^δ of c_δ. In order to render (F17) well defined, we extend the domain of definition of $\vec{\hat{V}}^{(\delta)}_{L_m}$ from v_δ to a small neighborhood $U(v^\delta)$ thereof so that the vector field on this extended domain is constant in the $\{y^{(\delta)}\}$ chart. This neighborhood is small enough that Eqs. (6.11) hold so that the vector field is also constant in the $\{y\}$ coordinates. Thus, for any point p in this neighborhood, we define this "constant extension" $\hat{V}^{(\delta)a_i}_{L_m}$ from which we define

$$H^l_{L_m}(N_1,N_2,..N_l;p) \coloneqq \prod_{i=1}^{l} N_i(p,\{y^{(\delta)}\})\hat{V}^{(\delta)a_i}_{L_m}(p)\partial_{a_i}(f(p,\{y^{(\delta)}\}) \times \sqrt{h_{ab}(p)\hat{V}^{(\delta)a}{}_{L_m}(p)\hat{V}^{(\delta)b}{}_{L_m}(p)}). \tag{F18}$$

Then we render (F17) well defined by setting

[56]Recall (see for example footnote 13) that while the upper or lower conducting edge may be absent in s_δ because the upper or lower conducting charge happens to vanish, from Appendix C it must be the case that at least one of these edges is present in the child due to the relation of the net conducting charge with the primordial charge.

[57]Note that (F8) is consistent with last part of the discussion in Sec. F. 1.

$$H^l_{L_m} := H^l_{L_m}(N_1, N_2, ..N_l; p = v^\delta). \quad \text{(F19)}$$

We are interested in the contraction behavior of $H^l_{L_m}$ as defined above. Note that since the transformation between $\{y\}$ and $\{y^{(\delta)}\}$ is linear in the domain of interest, constant vector fields in one system are also constant in the other. It then immediately follows that with such extensions of vectors $\hat{e}^{(\delta)a_i}_{L_m}\hat{V}^{(\delta)a_i}_{L_m}, \vec{\hat{V}}_{I_{m-1}}, \vec{w}_{J_m}$ in Eqs. (F4)–(F8), these equations continue to hold in $U(v^\delta)$. It then follows that replacing these vectors by their constant extensions in (F9) and (F12) and replacing $h_{ab}(v^\delta)$ by $h_{ab}(p)$ in these equations, we obtain equations which hold in $U(v^\delta)$. These equations can then be used to write (F18) in terms of quantities natural to the $\{y\}$ coordinates. Evaluating this form of the equations at v^δ then allows us to derive the contraction behavior of $H^l_{L_m}$ as defined by (F19). Accordingly (F9) and (F12) are extended to $U(v^\delta)$ as

$$\left\|\hat{e}^{(\delta)a}_{J_m\neq I_{m-1}}\right\|_p = |\cos\theta|\sqrt{h_{ab}(p)\hat{V}^a_{I_{m-1}}(p)\hat{V}^b_{I_{m-1}}(p)} \times (1 + C_{1,J_m}(p)\delta^{q-1} + C_{2,J_m}(p)(\delta^{q-1})^2)^{\frac{1}{2}} \quad \text{(F20)}$$

$$C_{1,J_m}(p) := 2\frac{h_{ab}(p)w^a_{J_m}(p)\hat{V}^b_{I_{m-1}}(p)}{\cos\theta h_{ab}(p)\hat{V}^a_{I_{m-1}}(p)\hat{V}^b_{I_{m-1}}(p)} \quad \text{(F21)}$$

$$C_{2,J_m}(p) := \frac{h_{ab}(p)w^a_{J_m}(p)w^b_{J_m}(p)}{\cos^2\theta h_{ab}(p)\hat{V}^a_{I_{m-1}}(p)\hat{V}^b_{I_{m-1}}(p)} \quad \text{(F22)}$$

$$\left\|\hat{e}^{(\delta)a}_{I_m=I_{m-1},u}\right\|_p = \left\|\hat{e}^{(\delta)a}_{I_m=I_{m-1},d}\right\|_p = \sqrt{h_{ab}(p)\hat{V}^a_{I_{m-1}}(p)\hat{V}^b_{I_{m-1}}(p)}. \quad \text{(F23)}$$

Using (F20)–(F23), (F15), (F5), (F8) in Eq. (F18), and noting that the only objects in these equations with a nontrivial p dependence are h_{ab}, N_i, f the other quantities being constant in $\{y\}$ coordinates, it is straightforward to obtain

$$H^l_{L_m=I_{m-1}}(N_1, N_2, ..N_l; p) = \delta^{-(l+1)\frac{2}{3}(q-1)}\left\{\prod_{i=1}^{l} N_i(p,\{y\})\hat{V}^{a_i}_{I_{m-1}}(p)\partial_{a_i}\left(f(p,\{y\})\sqrt{h_{ab}(p)\hat{V}^a_{I_{m-1}}(p)\hat{V}^b_{I_{m-1}}(p)}\right)\right\}, \quad \text{(F24)}$$

$$H^l_{L_m\neq I_{m-1}}(N_1, N_2, ..N_l; p) = \delta^{-(l+1)\frac{2}{3}(q-1)}\left\{|\cos\theta|(\cos^l\theta)\prod_{i=1}^{l} N_i(p,\{y\})\hat{V}^{a_i}_{I_{m-1}}(p)\partial_{a_i}\left(f(p,\{y\})\sqrt{h_{ab}(p)\hat{V}^a_{I_{m-1}}(p)\hat{V}^b_{I_{m-1}}(p)}\right) + O(\delta^{q-1})\right\}. \quad \text{(F25)}$$

As emphasized above, the derivatives in (F24) and (F25) are along constant coordinate directions in the $\{y\}$ coordinates. The only objects with nontrivial p dependence are $h_{ab}(p), N_i, f$ and so the above expressions only involve coordinate derivatives of components of the metric and of the evaluations of N_i, f in the $\{y\}$ coordinates. Setting $p = v^\delta$ after evaluating these derivatives, we write the contraction behavior of $H^l_{L_m}$ in a notation similar to that used in (F17) as

$$H^l_{L_m=I_{m-1}} := H^l_{L_m}(N_1, N_2, ..N_l; p = v^\delta) = \delta^{-(l+1)\frac{2}{3}(q-1)}\left\{\prod_{i=1}^{l} N_i(v_\delta,\{y\})\hat{V}^{a_i}_{I_{m-1}}\partial_{a_i}\left(f(v_\delta,\{y\})\sqrt{h_{ab}(v^\delta)\hat{V}^a_{I_{m-1}}\hat{V}^b_{I_{m-1}}}\right)\right\}, \quad \text{(F26)}$$

$$H^l_{L_m\neq I_{m-1}} := H^l_{L_m}(N_1, N_2, ..N_l; p = v^\delta) = \delta^{-(l+1)\frac{2}{3}(q-1)}\left\{|\cos\theta|(\cos^l\theta)\prod_{i=1}^{l} N_i(v_\delta,\{y\})\hat{V}^{a_i}_{I_{m-1}}\partial_{a_i}\left(f(v_\delta,\{y\})\sqrt{h_{ab}(v^\delta)\hat{V}^a_{I_{m-1}}\hat{V}^b_{I_{m-1}}}\right) + O(\delta^{q-1})\right\}. \quad \text{(F27)}$$

APPENDIX G: THE FUNCTION g_c

1. Specification of the function g_c

Recall that $g: \Sigma^{m(N-1)} \to \mathbf{R}$ with no smoothness restrictions and that we are interested in the specification of g only when none of its arguments are coincident.

First, define the function $d(a_1, a_2)$ between any two distinct points $a_1, a_2 \in \Sigma$ as follows: If there exists a unique geodesic with length $l, l < 1$ which joins a_1 to a_2 then we define $d = l$ else we set $d = 1$. We shall refer to d as a "distance" function.

Let U_m be the set of $m(N-1)$ (noncoincident) points in Σ which serve as arguments of g. Consider the case in which the elements of S can be uniquely segregated into $m-k+2$ sets of points S_i, $i = k-1, k, \ldots, m-1, m$ with each S_i, $i \geq k$ containing $(N-1)$ points as follows. Let S_m be such that the distance between any two elements of S_m is less than the distance between any element of S_m and any element of U_m not in S_m, as well as between any two elements of U_m not in S_m. This means that the $\binom{N-1}{2}$ distances between points in S_m are the shortest distances among the $\binom{m(N-1)}{2}$ distances between points in U_m.

To define S_{m-1} we remove the points belonging to S_m from U. Call the resulting set of $(m-1)(N-1)$ points as U_{m-1}. Let S_{m-1} be such that the distance between any two elements of S_{m-1} is less than the distance between any element of S_{m-1} and any element of U_{m-1} not in S_{m-1}, as well as between any two elements of U_{m-1} not in S_{m-1}. This means that the $\binom{N-1}{2}$ distances between points in S_{m-1} are the shortest distances among the $\binom{(m-1)(N-1)}{2}$ distances between points in U_{m-1}. We assume that the structure of points in U_m is such that this procedure can be iterated so as to define $S_{m-2}, S_{m-3}\ldots, S_k$ and that the procedure cannot be iterated beyond this so that the remaining $(k-1)(N-1)$ points are contained in S_{k-1}, where if $k = 1$, S_0 is the empty set.

Next, in each set S_i, $i > k-1$, consider the $\binom{N-1}{2}$ distances between pairs of points. Order these distances in decreasing value and denote this ordered set by $(d_1^{(i)}, d_2^{(i)}, \ldots, d_{\binom{N-1}{2}}^{(i)})$, where $d_r^{(i)} \leq d_s^{(i)}$ iff $r > s$. Then we define

$$g \coloneqq \prod_{i=k}^{m} \frac{d_{N-1}^{(i)}}{d_1^{(i)}}. \tag{G1}$$

If $m > 1$ and U_m is such that there exists no $k \geq 1$ for which the above segregation exists, we set $g = 1$. If $m = 1$ then we define $k = 1$ so that all the points are in the set S_1 and interpret (G1) as

$$g \coloneqq \frac{d_{N-1}^{(1)}}{d_1^{(1)}}. \tag{G2}$$

2. Contraction behavior of g_c

Consider the set of C^0 kinks of the child $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ which contract to the parent vertices v of the parent $c_{[i,I,\beta,\epsilon,\hat{J},\hat{K}]_m^{m-1}}$. For the purposes of this section, we denote the contraction parameter ϵ_{j_m} by ϵ, the child $c_{[i,I,\hat{J},\hat{K},\beta,\epsilon]_m}$ by c and the parent $c_{[i,I,\beta,\epsilon,\hat{J},\hat{K}]_m^{m-1}}$ by c_{par} and the *contraction* coordinates associated with v_{par} in c_{par} by $\{x\}$.

Then for small enough ϵ the function g_c as defined in Appendix G. 1 above separates as $g_c = g_1 g_{c_{\mathrm{par}}}$ where g_1 is a function only of the $N-1$ C^0 kinks which contract to the parent vertex with, from (G1),

$$g_1 = \frac{d_{N-1}}{d_1} \tag{G3}$$

where the $\binom{N-1}{2}$ distances between pairs of these C^0 kinks are ordered in decreasing value and denoted by $(d_1, d_2, \ldots, d_{\binom{N-1}{2}})$, where $d_r \leq d_s$ iff $r > s$.

For small enough ϵ these distances are the geodesic distances between pairs of C^0 kinks. Using the fact that geodesic normal coordinates are (at least) C^3 functions (recall that Σ is a semianalytic manifold of differentiability class much larger than unity) of coordinate charts on Σ, it is straightforward to show that the geodesic distance d between points separated by a coordinate distance δ is estimated as

$$d(a_1, a_2) = \delta \|\vec{\hat{e}}_{a_1,a_2}\| + O(\delta^2) \tag{G4}$$

where $\vec{\hat{e}}_{a_1,a_2}$ is the unit (with respect to the coordinate norm) coordinate vector along the coordinate straight line connecting a_1 to a_2 and $\|\vec{\hat{e}}_{a_1,a_2}\|$ is the metric norm (with respect to h_{bc} at either of the points a_1 or a_2):

$$\|\vec{\hat{e}}_{a_1,a_2}\| = \sqrt{h_{bc}(a)\hat{e}^b_{a_1,a_2}\hat{e}^c_{a_1,a_2}}, \qquad a = a_1 \quad \text{or} \quad a = a_2 \tag{G5}$$

where the choice of $a = a_1$ or a_2 only affects the expression (G4) at $O(\delta^2)$. We may now use (G4) to estimate the required geodesic distances between the contracting C^0 kinks.

We shall use the notation in (iii), (iv) Sec. VI. C. Note that for the contraction of C^0 kinks we have $p_1 < p_2 < p_3$ [see (i)–(iii), Step 2, Sec. VI. D. 1. b]. The C^0 kinks are situated such that (a) one of them lies along the $\hat{J}$th edge at the nondegenerate vertex v_{par} in c_{par} at a coordinate distance ϵ^{p_1} from v_{par}, (b) a second lies along the $\hat{K}$th edge at the nondegenerate vertex v_{par} in c_{par} at a coordinate distance $Q\epsilon^{p_2}$ from v_{par}, and (c) the remaining $N-3$ kinks vertices lie at coordinate distances of size ϵ^{p_3} from v_{par}.

Clearly the largest distances among the pairs of these kinks will be those between the kink in (a) and the others. There are $N-2$ such distances. Clearly the $N-1$th distance in the prescribed decreasing order will be one of the distances between the kink in (b) and those in (c). These distances can be readily estimated using (G4) and elementary plane geometry. We obtain

$$d_{N-1} = Q\epsilon^{p_2}\|\vec{\hat{e}}_{\hat{K}}\|(1+O(\epsilon^{p_2-p_1})),$$
$$d_1 = \epsilon^{p_1}\|\vec{\hat{e}}_{\hat{J}}\|(1+O(\epsilon^{p_2-p_1})) \tag{G6}$$

where $\|\vec{\hat{e}}_{\hat{L}}\|$ denotes the metric norm of the unit coordinate vector $\vec{\hat{e}}_{\hat{L}}$ at the point v_{par} in the coordinate system $\{x\}$ associated with v_{par} in c_{par},

$$\|\vec{\hat{e}}_{\hat{L}}\| = \sqrt{h_{ab}(v_{\mathrm{par}})\hat{e}^a_{\hat{L}}\hat{e}^b_{\hat{L}}}, \tag{G7}$$

and where we have used the following inequalities which follow from (i)–(iii), Step 2, Sec. VI. D. 1. b [see (G11) and (G12) below]:

$$p_3 > p_2 > p_1, \quad p_3 - p_2 > p_2 - p_1, \quad p_1 > p_2 - p_1. \tag{G8}$$

From (G3) and (G6), we have that

$$g_1 = \epsilon^{p_2-p_1} Q\frac{\|\vec{\hat{e}}_{\hat{K}}\|}{\|\vec{\hat{e}}_{\hat{J}}\|}(1+O(\epsilon^{p_2-p_1})). \tag{G9}$$

Note also that during the contraction o/f the $N-1$ kinks created in the transition from c_{par} to c, the position of any preexisting kinks in c_{par} are left unchanged by virtue of (vi), Step 2, Sec. VI. D. 1. b. Hence the contraction behavior of g_c is

$$g_c = \epsilon^{p_2-p_1} Q\frac{\|\vec{\hat{e}}_{\hat{K}}\|}{\|\vec{\hat{e}}_{\hat{J}}\|}(1+O(\epsilon^{p_2-p_1}))g_{c_{\mathrm{par}}}. \tag{G10}$$

Note that from (i)–(iii), Step 2, Sec. VI. D. 1. b we have, for some $j \geq 1, p, q \gg 1$ that

$$p_1 = jp\frac{2}{3}(q-1), \qquad p_2 = j(p+1)\frac{2}{3}(q-1),$$
$$p_3 = j(p+1)\frac{2}{3}(q-1) + j\frac{4}{3}(q-1) \tag{G11}$$

so that

$$p_2 - p_1 = j\frac{2}{3}(q-1), \qquad p_3 - p_2 = j\frac{4}{3}(q-1). \tag{G12}$$

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