Torches, Clocks, Mirrors and the Lorentz Transformation

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ABSTRACT

This article introduces the Lorentz transformations from elementary considerations, involving transmission of light signals between observers. The exposition is pedagogical and brings out connections with plane geometry.

The world around us consists of "events". It is not easy to define precisely what one means by an "event". (Try defining precisely what you mean by a "point" in geometry!) An example of an event is the explosion of a cracker on Deepavali. Events are characterised by a location in space and time. The collection of all events is called "spacetime". Events are to spacetime what points are to the plane in high school geometry. In coordinate geometry, one assigns Cartesian coordinates (x, y) to points. In the same way, an observer describes an event by giving four numbers, the time t at which it occurred and the three cartesian coordinates (x, y, z) of the place at which it occurred. We will simplify life by supposing that there is only one spatial dimension x. Events are then specified by giving *two* numbers — the time t and place x of occurrence.

If an observer on a moving train sees two events occur at the same place (say, in the same railway compartment) at different times, another observer who is standing on the railway platform will see them occur at different places. It was believed, till the last century, that if an observer saw two events occurring at the same time at different place, all observers would agree that these events were simultaneous. At the turn of the century, it was realised that this belief was in conflict with experiments. These experiments forced a revision of our ideas about space and time. These developments culminated in the Special Theory of Relativity, which was put forward by Einstein in 1905. Central to this theory is the Lorentz transformation. The purpose of this article is to present the Lorentz transformation from an elementary point of view.

Experiments show that

1. light travels at a finite speed*,
2. The speed of light is the same for all uniformly moving observers, regardless of their velocity.

We accept these as fundamental postulates. In order to derive the Lorentz transformation, all one needs to do is to *really believe* 1 and 2. 1 is easy but 2 will take some getting used to.

We consider observers who are in uniform relative motion. Each observer is equipped with a clock, a torch and a mirror. The clocks are all identical and of a

* The speed of light is usually denoted c and has been measured to be about 3×10^{10} cm/sec.

reputed Swiss make. But these observers are suspicious characters who trust only their own clocks. The clock reading of observer A will be denoted τ_A. The subscript indicates that this is a quantity measured by observer A. The torches will be used to flash light signals to other observers and the mirrors to bounce them back without delay. Let A be an observer and E an event. A flashes her torch at time τ_A^- (measured by her own clock!), and the light is bounced back to her by a mirror at E so that she receives the reflected signal at τ_A^+. Since the light took a time $\tau_A^+ - \tau_A^-$ to go "there and back", she will assign to the event E a space coordinate

$$x_A = c(\tau_A^+ - \tau_A^-)/2. \tag{1}$$

Since light takes as long to get there as to get back, she will assign to the event E a time which is midway between τ_A^+ and τ_A^-:

$$t_A = (\tau_A^- + \tau_A^+)/2. \tag{2}$$

Here again, the subscript A denotes that these are the coordinates of an event E assigned by observer A. This subscript is necessary to distinguish between different coordinates assigned by different observers to the same event E.

Suppose that B is another observer who is moving with respect to A. B will similarly assign to the event E space and time coordinates (t_B, x_B) which are related to τ_B^- and τ_B^+ by

$$x_B = c(\tau_B^+ - \tau_B^-)/2. \tag{3}$$

and

$$t_B = (\tau_B^- + \tau_B^+)/2. \tag{4}$$

We have used both postulates 1 and 2 above in arriving at Eqs. (1, 2, 3, 4). Postulate 1 is used when we choose to measure distances using light signals. Postulate 2 is needed when we use the same value ($c = c_A = c_B$) for the speed of light in Eqs. (1) and (3).

(t_B, x_B) will in general differ from (t_A, x_A). How are these assignments related? The "dictionary" relating (t_B, x_B) to (t_A, x_A) is called the *Lorentz transformation*. Our purpose here is to derive this transformation from elementary considerations involving clocks, torches and mirrors. This simple explanation of the Lorentz transformation is due to Bondi[1].

The first thing to understand is the Doppler effect: Suppose that A sees B receding* from her at a uniform rate. If A were to flash her torch on and off once a second, B would see these flashes separated by more than one second. (Between flashes, B would recede further and the next flash has more distance to travel.) If A and B set their clocks to zero when they crossed (let us call this event O), the time τ_A (according to A's clock, of course) at which A flashed her torch and time τ_B (according to B!) at which B sees the flash are related by the Doppler shift factor δ_{AB}:

$$\tau_A = \delta_{AB}\, \tau_B. \tag{5}$$

There is, of course, complete symmetry between A and B. B will see A receding from him and if B were to flash his torch at one second intervals, B would see these intervals stretched by the same factor, so that

$$\delta_{BA} = \delta_{AB}. \tag{6}$$

Suppose there were three observers, A, B and C, who crossed at the event O and set their clocks to zero at O. If A flashed her torch at one second intervals, C would see these flashes separated by δ_{AC} seconds. But this can be expressed as a product of the Doppler shift factors between AB and BC. For, one could arrange for B to flash signals to C in time with his seeing the flashes from A. Therefore $\delta_{AC} = \delta_{AB}\, \delta_{BC}$. Or, defining $\alpha = \ln\, \delta$,

$$\alpha_{AC} = \alpha_{AB} + \alpha_{BC}. \tag{7}$$

Now let E be an event. If A flashes a torch at time τ_A^- so that it passes B at time τ_B^- and reaches the event E, where it is bounced back so that it passes B at τ_B^+ and reaches A at τ_A^+. From the earlier discussion of the Doppler effect, we conclude (using the shorthand $\delta = \delta_{AB} = \delta_{BA} = e^{\alpha}$) that

$$\tau_B^- = \delta \tau_A^-, \tag{8}$$

since B is receiving signals from A and

* B could equally well be advancing. All the equations in this paper still hold.

$$\tau_A^+ = \delta \tau_B^+, \quad (9)$$

since A is receiving the returned signal from B.

From Eqs. (1)-(4) and (8), (9), we find (x_B, t_B) in terms of (x_A, t_A):

$$t_B = t_A \cosh \alpha - (x_A/c) \sinh \alpha, \quad (10)$$

$$x_B = -c\, t_A \sinh \alpha + x_A \cosh \alpha. \quad (11)$$

It is usual to write $\beta = \tanh \alpha$ and $\gamma = \cosh \alpha$, so that

$$t_B = \gamma t_A - \gamma \beta x_A / c, \quad (12)$$

$$x_B = -c\gamma\beta\, t_A + \gamma x_A. \quad (13)$$

This is the Lorentz transformation. Let us now express α, β, γ and δ more concretely in terms of the velocity between A and B. The events $x_B = 0$ (t_B anything) fall on a line, which is called the *World Line* of B. A will describe this line by the equation

$$x_A = c\beta t_A, \quad (14)$$

which describes an object moving with velocity $\beta = v/c = \tanh \alpha$. And

$$\gamma = 1/\text{sech}\, \alpha = 1/\sqrt{1 - \tanh^2 \alpha} = 1/\sqrt{(1 - v^2/c^2)}\,,$$

$$\delta = e^{\alpha} = \cosh \alpha + \sinh \alpha$$

$$= \cosh \alpha\,(1 + \tanh \alpha) = \gamma(1 + \beta)\,.$$

This gives the relativistic formula for the Doppler shift factor δ in terms of v:

$$\delta = \sqrt{\frac{1 + v/c}{1 - v/c}}\,. \quad (15)$$

If there were three observers A, B and C and the relative velocity between A and B was v_{AB} and the relative velocity between B and C was v_{BC}, what is the relative velocity between A and C? This is easily worked out from (7), the additivity of α. From the elementary identity

$$\tanh(\alpha_{AC}) = \frac{\tanh(\alpha_{AB}) + \tanh(\alpha_{BC})}{1 + \tanh(\alpha_{AB}) \tanh(\alpha_{BC})}, \quad (16)$$

for $\tanh(\alpha_{AB} + \alpha_{BC})$ follows the relativistic law for addition of velocities:

$$\beta_{AC} = \frac{\beta_{AB} + \beta_{BC}}{1 + \beta_{AB}\, \beta_{BC}} \quad (17)$$

One of the noteworthy features of the Lorentz transformation is the relativity of simultaneity. Events which appear simultaneous to one observer may not appear so to another. (12) shows us that events which occur at the same time at different places according to observer A (which take place at the same t_A) will occur at different times according to another observer B (do not occur at the same t_B). This may appear like a strange conclusion, but it is a clear consequence of postulates 1 and 2. The "strangeness" is all in the postulates!

Another point worth noting about the Lorentz transformation (10) and (11) is its similarity with rotations in the plane. Compare these with the formula:

$$x_B = x_A \cos\alpha + y_A \sin\alpha, \quad (18)$$

$$y_B = -x_A \sin\alpha + y_A \cos\alpha, \quad (19)$$

describing rotations through an angle α in the $x - y$ plane. The coordinates (x_A, y_A) and (x_B, y_B) represent the x and y coordinates of a point with respect to Cartesian coordinate systems A and B, which are rotated relative to each other by an angle α. Notice, also that points with the same x_A coordinate (but different y_A) have different x_B coordinates. This is in complete analogy to the discussion earlier concerning the relativity of simultaneity. The main difference between rotations and Lorentz transformations is that trigonometric functions appear in rotations, while Lorentz transformations involve hyperbolic functions. In some sense, a Lorentz transformation is a rotation in the $t - x$ plane through an imaginary angle! While rotations in the plane leave $x^2 + y^2$, the distance of any point from the origin unchanged, Lorentz transformations preserve $c^2t^2 - x^2$. This quantity is called the "interval". It plays the same role in the Special Theory of Relativity that distances do in plane geometry.

In Einstein's Special Theory of Relativity, the Lorentz transformation is applied to *all* the laws of

Physics. This theory is now firmly established experimentally and routinely used to describe particles whose speeds are comparable to that of light. For a more detailed elementary treatment of Special Relativity, see Bondi[1]. A more advanced treatment is given by Goldstein[2] or Jackson[3].

References

1. H. Bondi, *Relativity and Common Sense*, Dover, New York (1962).
2. H. Goldstein, *Classical Mechanics*, Addison-Wesley (1978).
3. J.D. Jackson, *Classical Electrodynamics*, Wiley Eastern, New Delhi (1978).

Do You Dare Firing a Gun

The environments in cities breed in criminals and underworld gangsters. So every big city has high risk areas wherein people do not dare to travel. However, Alliant Techn. Systems have come up with a significant device which will almost give no chance for anybody firing a gun within the city premises. The device so developed and named SECURES (System for Effective Control of Urban Environment Security) is basically a grid of street-corner acoustic sensors coupled to a network of radio transceivers. The noise due to a gunfire comes to the sensors which is difficult to be distinguished from a distance because of being mixed with car backfires and similar disturbances. However the sensors are coupled to an efficient signal processing circuitry which is so sharp that a confusion is always avoided. The culprit is located also, due to triangulation through sensors located at different spots. The information is quickly relayed to a command center via a digital packet radio and a local communications node. The whole mission is performed within less than a second which enables quick action.

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