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Generation of large-scale magnetic fields due to fluctuating α in shearing systems

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We explore the growth of large-scale magnetic fields in a shear flow, due to helicity fluctuations with a finite correlation time, through a study of the Kraichnan–Moffatt model of zero-mean stochastic fluctuations of the α parameter of dynamo theory. We derive a linear integro-differential equation for the evolution of the large-scale magnetic field, using the first-order smoothing approximation and the Galilean invariance of the α-statistics. This enables construction of a model that is non-perturbative in the shearing rate S and the α-correlation time τ_α. After a brief review of the salient features of the exactly solvable white-noise limit, we consider the case of small but non-zero τ_α. When the large-scale magnetic field varies slowly, the evolution is governed by a partial differential equation. We present modal solutions and conditions for the exponential growth rate of the large-scale magnetic field, whose drivers are the Kraichnan diffusivity, Moffatt drift, shear and a non-zero correlation time. Of particular interest is dynamo action when the α-fluctuations are weak; i.e. when the Kraichnan diffusivity is positive. We show that in the absence of Moffatt drift, shear does not give rise to growing solutions. But shear and Moffatt drift acting together can drive large-scale dynamo action with growth rate $\gamma \propto |S|$.

Key words: astrophysical plasmas

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1. Introduction

Magnetic fields are observed over a wide range of scales in various astrophysical objects (see, e.g. Han 2017). Their origins could be the result of turbulent dynamo processes which can lead to field generation on scales that are larger as well as smaller than the outer scale of the underlying turbulence (see, e.g. Moffatt 1978; Parker 1979; Krause & Rädler 1980; Zeldovich, Ruzmaikin & Sokolov 1983; Ruzmaikin, Shukurov & Sokoloff 1988; Brandenburg & Subramanian 2005). Of particular interest here is the subject of the large-scale dynamo (LSD), which may be studied in the framework of mean-field theory (Steenbeck, Krause & Rädler 1966; Moffatt 1978; Krause & Rädler 1980). The standard paradigm for LSD involves an α-effect which arises when

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the background turbulence possesses mean kinetic helicity, thus breaking the mirror symmetry of turbulence (see, e.g. Brandenburg & Subramanian 2005). The problem becomes more interesting and complicated when the usual α-effect is either absent or subcritical for dynamo growth. Mean velocity shear appears to play a vital role for LSD in such regimes of zero/subcritical α. As most astrophysical bodies also possess mean differential rotation, it is natural to ask if large-scale magnetic fields could grow in the presence of a background shear flow when α is a purely fluctuating quantity.

Early ideas of stochastically varying α with zero mean suggested that it causes a decrement in turbulent diffusion (Kraichnan 1976; Moffatt 1978). A number of subsequent studies then considered fluctuating α as an important ingredient for the evolution of magnetic fields in objects, such as, the Sun (Silant'ev 2000; Proctor 2007), accretion disks (Vishniac & Brandenburg 1997), galaxies (Sokolov 1997; Sur & Subramanian 2009). Numerical demonstration of the shear dynamo problem (Brandenburg *et al.* 2008; Yousef *et al.* 2008*a*,*b*; Singh & Jingade 2015) where large-scale magnetic fields were generated due to non-helically forced turbulence in shear flows, and failure to understand these in terms of simple ideas involving a shear–current effect (Kleeorin & Rogachevskii 2008; Rogachevskii & Kleeorin 2008; Sridhar & Subramanian 2009*a*,*b*; Sridhar & Singh 2010; Singh & Sridhar 2011; Kolekar, Subramanian & Sridhar 2012), brought the focus to a stochastic α which could potentially lead to the dynamo action generically in shearing systems (Heinemann, McWilliams & Schekochihin 2011; McWilliams 2012; Mitra & Brandenburg 2012; Proctor 2012; Richardson & Proctor 2012; Sridhar & Singh 2014). There is still a need to verify the model predictions for the growth of the first moment of the mean magnetic field in such systems by performing more simulations.

Squire & Bhattacharjee (2015*a*,*b*) recently proposed a new mechanism, called the magnetic shear current effect, which leads to the generation of a large-scale magnetic field due to the combined action of shear and small-scale magnetic fluctuations, if these are sufficiently strong and are near equipartition levels of turbulent motions. Such strong magnetic fluctuations are expected to be naturally present due to small-scale dynamo (SSD) action in astrophysical plasmas, which typically have large magnetic Reynolds number (Rm). This new effect thus raises the interesting possibility of the excitation of LSD due to SSD in presence of shear, and it challenges an understanding where SSD in high-Rm systems is thought to weaken the LSD, which could survive only when SSD is suppressed due to shear (Tobias & Cattaneo 2013; Pongkitiwanichakul *et al.* 2016; Nigro *et al.* 2017); but see also Kolokolov, Lebedev & Sizov (2011) and Singh, Rogachevskii & Brandenburg (2017) where it is found that the shear supports and even enhances the growth rate of SSD. However, we are here more concerned with the excitation of a large-scale shear dynamo, quite independent of any small-scale dynamo or strong magnetic fluctuations, which are both absent in most numerical simulations that are relevant. These simulations typically had Rm which were subcritical for SSD and the only source of magnetic fluctuations was due to the tangling of large-scale magnetic fields (Rogachevskii & Kleeorin 2007), and therefore these fluctuations could never be too strong in the kinematic regime of LSD.

In the present paper we explore the possibility of large-scale dynamo action in presence of background shear flow, due to an α that varies stochastically in space and time, with vanishing mean. Here we generalize the earlier work by Sridhar & Singh (2014), hereafter SS14, by including the full resistive term in determining the turbulent electromotive force (EMF). Such an extension in the absence of shear was done in Singh (2016). In § 2 we define our model by writing the dynamo equations in shearing coordinates. The integro-differential equation governing the evolution of the large-scale magnetic field is derived under a first-order smoothing approximation (FOSA) in § 3. This is non-perturbative in shearing rate S and the correlation time τ_α. Here we briefly review the exactly solvable limit of white-noise α fluctuations. In § 4 we reduce the evolution equation into a partial differential equation (PDE) for axisymmetric mean magnetic fields, by assuming small but non-zero τ_α. The dispersion relation giving the growth rate is then determined in § 5 where we present our results in different parameter regimes. We then discuss our findings and conclude in § 6.

2. The model

Let us begin with the standard dynamo equation in the presence of a background linear shear flow, $\boldsymbol{V}=SX_1\boldsymbol{e}_2$, where meso-scale magnetic field $\boldsymbol{B}$ evolves according to (see, Moffatt 1978; Krause & Rädler 1980; Brandenburg & Subramanian 2005; Sridhar & Singh 2014):

$$\left(\frac{\partial}{\partial\tau}+SX_1\frac{\partial}{\partial X_2}\right)\boldsymbol{B}-SB_1\boldsymbol{e}_2=\boldsymbol{\nabla}\times[\alpha(\boldsymbol{X},\tau)\boldsymbol{B}]+\eta_T\nabla^2\boldsymbol{B};\quad \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{B}=0. \tag{2.1}$$

Here we follow the same notation as in Sridhar & Singh (2014) where the position vector is denoted by $\boldsymbol{X}=(X_1, X_2, X_3)$ with components given in a fixed orthonormal frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$, and τ is the time variable. The shear rate, S, and total diffusivity, η_T, are treated as constant parameters, whereas $\alpha(\boldsymbol{X},\tau)$ provides a measure of meso-scale kinetic helicity of turbulence. We recall that (2.1) governing the dynamics of a meso-scale magnetic field is obtained by averaging over an ensemble of random velocity fields, $\{\boldsymbol{v}(\boldsymbol{X},\tau)\}$, which are assumed to have zero-mean isotropic fluctuations, uniform and constant kinetic energy density per unit mass and slow helicity fluctuations.

We employ here the double-averaging scheme (Kraichnan 1976; Moffatt 1983; Sokolov 1997) under which $\alpha(\boldsymbol{X},\tau)$ itself is a random variable of space and time, thus making (2.1) a stochastic partial differential equation. It is drawn from a superensemble with zero mean, $\overline{\alpha(\boldsymbol{X},\tau)}=0$. It's statistical properties are given below in (2.11). Next, we separate the meso-scale field, $\boldsymbol{B}=\overline{\boldsymbol{B}}+\boldsymbol{b}$, into large-scale, $\overline{\boldsymbol{B}}$, and fluctuating, $\boldsymbol{b}$, components, where the superensemble average of $\boldsymbol{b}$ vanishes, i.e. $\overline{\boldsymbol{b}}=\boldsymbol{0}$. The governing equation for the large-scale magnetic field $\overline{\boldsymbol{B}}$ can thus be obtained by Reynolds averaging the (2.1) over the superensemble:

$$\left(\frac{\partial}{\partial\tau}+SX_1\frac{\partial}{\partial X_2}\right)\overline{\boldsymbol{B}}-S\overline{B_1}\boldsymbol{e}_2=\boldsymbol{\nabla}\times\overline{\boldsymbol{\mathcal{E}}}+\eta_T\nabla^2\overline{\boldsymbol{B}},\quad \boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\boldsymbol{B}}=0, \tag{2.2}$$

$$\text{where }\overline{\boldsymbol{\mathcal{E}}}=\overline{\alpha(\boldsymbol{X},\tau)\boldsymbol{b}(\boldsymbol{X},\tau)}. \tag{2.3}$$

In order to determine the mean electromotive force (EMF), we must solve for the fluctuating field $\boldsymbol{b}$, which evolves as:

$$\left.\begin{aligned}\left(\frac{\partial}{\partial\tau}+SX_1\frac{\partial}{\partial X_2}\right)\boldsymbol{b}-Sb_1\boldsymbol{e}_2=\boldsymbol{\nabla}\times[\alpha\overline{\boldsymbol{B}}]+\boldsymbol{\nabla}\times[\alpha\boldsymbol{b}-\overline{\alpha\boldsymbol{b}}]+\eta_T\nabla^2\boldsymbol{b},\\ \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{b}=0,\quad \text{with initial condition } \boldsymbol{b}(\boldsymbol{X},0)=\boldsymbol{0}.\end{aligned}\right\} \tag{2.4}$$

As (2.2) and (2.4) involve inhomogeneous terms, it is convenient to solve these in the shearing coordinates $(\boldsymbol{x}, t)$ which are expressed in terms of the laboratory coordinates $(\boldsymbol{X},\tau)$ as (see, Sridhar & Subramanian 2009*a*; Sridhar & Singh 2010):

$$x_1=X_1;\quad x_2=X_2-S\,\tau\,X_1;\quad x_3=X_3;\quad t=\tau. \tag{2.5a–d}$$

The inverse transformation is:

$$X_1 = x_1; \quad X_2 = x_2 + S\,t\,x_1; \quad X_3 = x_3; \quad \tau = t. \tag{2.6a–d}$$

Now we can write equations (2.2)–(2.4) in terms of new fields that are functions of $\boldsymbol{x}$ and t: $\overline{\boldsymbol{H}}(\boldsymbol{x}, t) = \overline{\boldsymbol{B}}(\boldsymbol{X}, \tau)$; $\boldsymbol{h}(\boldsymbol{x}, t) = \boldsymbol{b}(\boldsymbol{X}, \tau)$; $a(\boldsymbol{x}, t) = \alpha(\boldsymbol{X}, \tau)$; and $\overline{\boldsymbol{E}}(\boldsymbol{x}, t) = \overline{\boldsymbol{\mathcal{E}}}(\boldsymbol{X}, \tau)$. Equations (2.2)–(2.4) then take the form (Sridhar & Singh 2014):

$$\frac{\partial \overline{\boldsymbol{H}}}{\partial t} - S\overline{H}_1\boldsymbol{e}_2 = \boldsymbol{\nabla}\times\overline{\boldsymbol{E}} + \eta_T\nabla^2\overline{\boldsymbol{H}}, \quad \boldsymbol{\nabla}\boldsymbol{\cdot}\overline{\boldsymbol{H}} = 0, \quad \overline{\boldsymbol{E}} = \overline{a\boldsymbol{h}}; \tag{2.7a–c}$$

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{h}}{\partial t} - Sh_1\boldsymbol{e}_2 = \boldsymbol{\nabla}\times[a\overline{\boldsymbol{H}}] + \boldsymbol{\nabla}\times[a\boldsymbol{h} - \overline{a\boldsymbol{h}}] + \eta_T\nabla^2\boldsymbol{h}, \\ &\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{h} = 0, \quad \text{with initial condition } \boldsymbol{h}(\boldsymbol{x}, 0) = \boldsymbol{0}; \end{aligned}\right\} \tag{2.8}$$

$$\text{where } \boldsymbol{\nabla} = \frac{\partial}{\partial \boldsymbol{x}} - \boldsymbol{e}_1 St\frac{\partial}{\partial x_2} \quad \text{is a time-dependent operator.} \tag{2.9}$$

We complete defining our model by specifying the statistics of the α fluctuations. We follow the exact same approach as given in detail in Sridhar & Singh (2014) and recall here only some key relevant points:

(i) Shear flows possess a natural symmetry known as Galilean invariance, relating the measurements of correlation functions made by comoving observers whose origins with respect to the laboratory frame translate with the same speed as that of the linear shear flow (Sridhar & Subramanian 2009*a*,*b*).

(ii) Here we are more interested in time-stationary Galilean-invariant α statistics, which can be expressed in the shearing frame as (see Sridhar & Singh (2014) for a derivation):

$$\overline{a(\boldsymbol{x}, t)a(\boldsymbol{x}', t')} = 2\mathcal{A}\left(\boldsymbol{x} - \boldsymbol{x}' + St'(x_1 - x_1')\boldsymbol{e}_2\right)\mathcal{D}(t - t'), \quad \text{with} \tag{2.10}$$

$$2\int_0^\infty \mathcal{D}(t)\,\mathrm{d}t = 1, \quad \mathcal{A}(0) = \eta_\alpha \geqslant 0. \tag{2.11}$$

The correlation time for the α fluctuations is defined as,

$$\tau_\alpha = 2\int_0^\infty \mathrm{d}t\, t\,\mathcal{D}(t). \tag{2.12}$$

The intrinsic anisotropy of the α fluctuations is measured by the Moffatt drift velocity,

$$\boldsymbol{V}_M = -\left(\frac{\partial\mathcal{A}(\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}\right)_{\boldsymbol{\xi}=\boldsymbol{0}} = \int_0^\infty \overline{\alpha(\boldsymbol{X}, \tau)\boldsymbol{\nabla}\alpha(\boldsymbol{X}, 0)}\,\mathrm{d}\tau. \tag{2.13}$$

In the above, we noted two properties of the spatial correlation function, $\mathcal{A}$, namely its value η_α and gradient $\boldsymbol{V}_M$ at zero separation. But we can associate one or more length scales relating to its variation in $\boldsymbol{\xi}$-space. In the estimates made below we use a single scale ℓ to denote this correlation length. The temporal correlation function, $\mathcal{D}$ is characterized by a single correlation time, τ_α. Hence the basic constant parameters of our model are $(\eta_\alpha, \boldsymbol{V}_M, \ell, \tau_\alpha)$.

3. Evolution equation for the large-scale magnetic field

Here we derive a closed equation for the large-scale magnetic field by exploiting the homogeneity of the problem in the sheared coordinates $\boldsymbol{x}$ by working with its conjugate Fourier variable $\boldsymbol{k}$. Let $\widetilde{Q}(\boldsymbol{k}, t) = \int \mathrm{d}^3x \exp(-\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x})Q(\boldsymbol{x}, t)$ be the Fourier transform of any quantity $Q(\boldsymbol{x}, t)$, with similar definition in terms of laboratory-frame coordinates, where $\boldsymbol{K}$ denotes the conjugate variable to $\boldsymbol{X}$. Note that the laboratory-frame wavevector $\boldsymbol{K}$ is time dependent and can be expressed in terms of sheared wavevectors $\boldsymbol{k}$ as, $\boldsymbol{K}(\boldsymbol{k}, t) = (k_1 - Stk_2, k_2, k_3)$; see (2.9). We need to first solve for $\widetilde{\boldsymbol{h}}(\boldsymbol{k}, t)$ as a functional of $\widetilde{a}(\boldsymbol{k}, t)$ and $\widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}, t)$. This is in general a complicated problem by itself, so for a first attempt we use the standard approach of the first-order smoothing approximation (FOSA) wherein the term, $\boldsymbol{\nabla}\times[a\boldsymbol{h} - \overline{a\boldsymbol{h}}]$, is dropped in (2.8). Analogous to (7.124) of Moffatt (1978), the condition for FOSA to be valid is:

$$\frac{\eta_\alpha \tau_\alpha}{\ell^2} \ll 1, \quad \text{OR} \quad \frac{\eta_\alpha}{\eta_T} \ll \frac{\eta_T \tau_\alpha}{\ell^2}, \tag{3.1a,b}$$

where we recall that ℓ is the correlation length of the α fluctuations. The first term of these condition comes from the short correlation assumption by comparing $\partial\boldsymbol{h}/\partial t$ with $\boldsymbol{\nabla}\times[a\boldsymbol{h} - \overline{a\boldsymbol{h}}]$; $\partial\boldsymbol{h}/\partial t$ is of the order $O(h_0/\tau_\alpha)$ and $\boldsymbol{\nabla}\times[a\boldsymbol{h} - \overline{a\boldsymbol{h}}]$ is of the order $O(\alpha_0 h_0/\ell)$. For FOSA to be valid, $\alpha_0 h_0/\ell \ll h_0/\tau_\alpha$. Using $\alpha_0^2 \sim \eta_\alpha/\tau_\alpha$ (from (2.11)), we can write the first condition in (3.1). Similarly, the second condition comes from comparing $\eta_T\nabla^2\boldsymbol{h}$ with $\boldsymbol{\nabla}\times[a\boldsymbol{h} - \overline{a\boldsymbol{h}}]$ in (2.8); $\eta_T\nabla^2\boldsymbol{h}$ is of the order $O(\eta_T h_0/\ell^2)$, and so, for FOSA to be valid, $\eta_T h_0/\ell^2 \gg \alpha_0 h_0/\ell$, which yields the second condition in (3.1) after rearranging and squaring the terms.

Then the fluctuating magnetic field evolves as:

$$\left(\frac{\partial}{\partial t} - \eta_T\nabla^2\right)\boldsymbol{h} - Sh_1\boldsymbol{e}_2 = \boldsymbol{\nabla}\times\boldsymbol{M}, \tag{3.2}$$

where $\boldsymbol{M}(\boldsymbol{x}, t) = a(\boldsymbol{x}, t)\overline{\boldsymbol{H}}(\boldsymbol{x}, t)$ is a source term for the fluctuating magnetic field, and $\boldsymbol{\nabla}$ is the time-dependent operator defined in (2.9). The FOSA solution for the fluctuating magnetic field in the Fourier space is given by (see appendix A for a derivation),

$$\widetilde{\boldsymbol{h}}(\boldsymbol{k}, t) = \int_0^t \mathrm{d}t'\, \widetilde{G}_{\eta_T}(\boldsymbol{k}, t, t')\left\{\mathrm{i}\boldsymbol{K}(\boldsymbol{k}, t') \times \widetilde{\boldsymbol{M}}(\boldsymbol{k}, t') + \boldsymbol{e}_2 S(t-t')[\mathrm{i}\boldsymbol{K}(\boldsymbol{k}, t') \times \widetilde{\boldsymbol{M}}(\boldsymbol{k}, t')]_1\right\}, \tag{3.3}$$

where the sheared Green's function in Fourier space:

$$\widetilde{G}_{\eta_T}(\boldsymbol{k}, t, t') = \exp\left[-\eta_T\left(k^2(t-t') - Sk_1k_2(t^2 - t'^2) + \frac{S^2}{3}k_2^2(t^3 - t'^3)\right)\right]. \tag{3.4}$$

This is derived in Sridhar & Singh (2010). It may be readily verified that (3.3) satisfies both constraints, $\boldsymbol{K}\boldsymbol{\cdot}\widetilde{\boldsymbol{h}} = 0$ and $\widetilde{\boldsymbol{h}}(\boldsymbol{k}, 0) = \boldsymbol{0}$. By making use of (3.3), and time-stationary Galilean-invariant statistics for the α fluctuations in Fourier space (see appendix B), we obtain the following expression for the mean EMF in Fourier space, after some straightforward algebra (see appendix C for a derivation):

$$\widetilde{\overline{\boldsymbol{E}}}(\boldsymbol{k}, t) = 2\int_0^t \mathrm{d}t'\mathcal{D}(t-t')\left\{\widetilde{\boldsymbol{U}}(\boldsymbol{k}, t, t') \times \widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}, t') + \boldsymbol{e}_2 S(t-t')[\widetilde{\boldsymbol{U}}(\boldsymbol{k}, t, t') \times \widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}, t')]_1\right\}, \tag{3.5}$$

where

$$\widetilde{\boldsymbol{U}}(\boldsymbol{k}, t, t') = \int \frac{\mathrm{d}^3k'}{(2\pi)^3} \widetilde{G}_{\eta_T}(\boldsymbol{k}-\boldsymbol{k}', t, t')\mathrm{i}\boldsymbol{K}(\boldsymbol{k}-\boldsymbol{k}', t')\widetilde{\mathcal{A}}(\boldsymbol{K}(\boldsymbol{k}', t')), \tag{3.6}$$

is a complex velocity field.

Fourier transforming (2.7), the equation governing the large-scale field is:

$$\frac{\partial \widetilde{\overline{\boldsymbol{H}}}}{\partial t} - S\widetilde{\overline{H}}_1\boldsymbol{e}_2 = \mathrm{i}\boldsymbol{K}(\boldsymbol{k}, t) \times \widetilde{\overline{\boldsymbol{E}}} - \eta_T K^2(\boldsymbol{k}, t)\widetilde{\overline{\boldsymbol{H}}}, \quad \boldsymbol{K}(\boldsymbol{k}, t) \boldsymbol{\cdot} \widetilde{\overline{\boldsymbol{H}}} = 0. \tag{3.7a,b}$$

Thus the set of (3.5)–(3.7) describes the evolution of the large-scale magnetic field, $\widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}, t)$ in terms of a closed, linear integro-differential equation, where both shear strength, S, and the α-correlation time, τ_α, are treated non-perturbatively. This is the principal general result of this paper, but solving these in full generality is beyond the scope of the present work, and we next pursue these equations analytically by making useful approximations.

3.1. *White-noise α fluctuations*

It is useful to recall basic properties of an exactly solvable limit of δ-correlated-in-time α fluctuations when the normalized correlation function $\mathcal{D}_{\mathrm{WN}}(t) = \delta(t)$ which is the Dirac delta function, giving $\tau_\alpha = 0$ from (2.12). Using this in (3.5) and (3.6), and noting that $\widetilde{G}_{\eta_T}(\boldsymbol{k}-\boldsymbol{k}', t, t) = 1$ from (3.4), we find the mean EMF:

$$\widetilde{\overline{\boldsymbol{E}}}_{\mathrm{WN}}(\boldsymbol{k}, t) = \widetilde{\boldsymbol{U}}_{\mathrm{WN}}(\boldsymbol{k}, t) \times \widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}, t) \quad \text{with } \widetilde{\boldsymbol{U}}_{\mathrm{WN}}(\boldsymbol{k}, t) = \mathrm{i}\boldsymbol{K}(\boldsymbol{k}, t)\eta_\alpha + \boldsymbol{V}_M, \tag{3.8}$$

where the α-diffusivity, $\eta_\alpha = \mathcal{A}(0)$, is given in (2.11) and the Moffatt drift velocity $\boldsymbol{V}_M$ is defined in (2.13). The Kraichnan diffusivity, η_K, is defined as, $\eta_K = \eta_T - \eta_\alpha$. Using these in (3.7) leads to the solution for the large-scale magnetic field (see Sridhar & Singh (2014) for more details):

$$\widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}, t) = \widetilde{\mathcal{G}}(\boldsymbol{k}, t)[\widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}, 0) + \boldsymbol{e}_2 St\, \widetilde{\overline{H}}_1(\boldsymbol{k}, 0)], \quad \boldsymbol{k} \boldsymbol{\cdot} \widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}, 0) = 0, \tag{3.9a,b}$$

where

$$\begin{aligned}\widetilde{\mathcal{G}}(\boldsymbol{k}, t) &= \exp\left\{-\int_0^t \mathrm{d}t'[\eta_K K^2(\boldsymbol{k}, t') + \mathrm{i}\boldsymbol{V}_M \boldsymbol{\cdot} \boldsymbol{K}(\boldsymbol{k}, t')]\right\} \\ &= \exp\{-\eta_K[k^2 t - Sk_1k_2t^2 + (S^2/3)k_2^2t^3] - \mathrm{i}[(\boldsymbol{V}_M \boldsymbol{\cdot} \boldsymbol{k})t - (S/2)V_{M1}k_2t^2]\}.\end{aligned} \tag{3.10}$$

This solution is identical to the one obtained in Sridhar & Singh (2014). Thus we find that the inclusion of the turbulent diffusion term in determining the mean EMF makes no difference for the dynamo solution in the white-noise limit. In agreement with earlier findings (Kraichnan 1976; Moffatt 1978; Sridhar & Singh 2014), we see from above that the α-diffusivity causes a reduction in the turbulent diffusion of the fields, and if it is sufficiently strong, i.e. when $\eta_K < 0$, this can lead to an instability giving growth of the large-scale magnetic field. Also, the Moffatt drift does not couple to the dynamo growth/decay and contributes only to the phase.

4. Axisymmetric large-scale dynamo equation with finite τ_α

We now turn to the principal aim of this work where we are more interested in exploring the possibility of a large-scale dynamo even when the α fluctuations are weak, i.e. when $\eta_K > 0$, by taking the memory effects into account. Assuming a small but finite correlation time for α fluctuations, $\tau_\alpha \neq 0$, we reduce the general set of (3.5)–(3.7) into a partial differential equation governing the dynamics of the large-scale magnetic field which evolves over times much larger than τ_α. In this case, the normalized time correlation function, $\mathcal{D}(t)$, is significant only for times $t \leqslant \tau_\alpha$ and it becomes negligible for larger times. The generalized mean EMF as given in (3.5) involves a time integral which can be solved under the small τ_α approximation.

Since the limit $\lim_{\tau_\alpha\to 0} \widetilde{\overline{\boldsymbol{E}}}(\boldsymbol{k}, t) = \widetilde{\overline{\boldsymbol{E}}}_{\mathrm{WN}}(\boldsymbol{k}, t)$, given by (3.8), is non-singular, we proceed by making the following ansatz where, for small τ_α, the mean EMF can be expanded in a power series in τ_α as:

$$\widetilde{\overline{\boldsymbol{E}}}(\boldsymbol{k}, t) = \widetilde{\overline{\boldsymbol{E}}}_{\mathrm{WN}}(\boldsymbol{k}, t) + \widetilde{\overline{\boldsymbol{E}}}^{(1)}(\boldsymbol{k}, t) + \widetilde{\overline{\boldsymbol{E}}}^{(2)}(\boldsymbol{k}, t) + \dots \tag{4.1}$$

where $\widetilde{\overline{\boldsymbol{E}}}_{\mathrm{WN}}(\boldsymbol{k}, t) \sim O(1)$ and $\widetilde{\overline{\boldsymbol{E}}}^{(n)}(\boldsymbol{k}, t) \sim O(\tau_\alpha^n)$ for $n \geqslant 1$. Below we verify this ansatz up to $n = 1$, for slowly varying magnetic fields. From (3.5) we determine $\widetilde{\overline{\boldsymbol{E}}}(\boldsymbol{k}, t)$ to first order in τ_α, for $t \gg \tau_\alpha$, by (i) changing the integration variable from t' to $s = t - t'$; (ii) setting the upper limit of the time integral to $+\infty$, since $\mathcal{D}(s)$ is significant only for times $s \leqslant \tau_\alpha$ as mentioned above, suggesting that only short times $0 \leqslant s < \tau_\alpha$ contribute appreciably to the integral in (3.5); and (iii) keeping the terms inside the { } in the integrand of (3.5) up to only first order in s. To be able to expand in s, we need to first express the (3.6) in the laboratory-frame wavevector $\boldsymbol{K} = \boldsymbol{K}(\boldsymbol{k}, t') = \boldsymbol{K}(\boldsymbol{k}, t - s)$, so that the Green's function in (3.4) and therefore the complex velocity field $\widetilde{\boldsymbol{U}}$ in (3.6) becomes time-translational symmetric.[1]

We first rewrite the mean EMF, given in (3.5), as

$$\widetilde{\overline{\boldsymbol{E}}}(\boldsymbol{k}, t) = 2\int_0^\infty \mathrm{d}s\, \mathcal{D}(s) \left\{ \widetilde{\boldsymbol{U}}(\boldsymbol{K}, s) \times \widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}, t - s) + \boldsymbol{e}_2 S s [\widetilde{\boldsymbol{U}}(\boldsymbol{K}, s) \times \widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}, t - s)]_1 \right\}, \tag{4.2}$$

where the complex velocity field, $\widetilde{\boldsymbol{U}}$, is

$$\widetilde{\boldsymbol{U}}(\boldsymbol{K}, s) = \int \frac{\mathrm{d}^3 K'}{(2\pi)^3} \widetilde{G}_{\eta_T}(\boldsymbol{K} - \boldsymbol{K}', s) \mathrm{i}(\boldsymbol{K} - \boldsymbol{K}') \widetilde{\mathcal{A}}(\boldsymbol{K}'). \tag{4.3}$$

Equation (4.3) is obtained by changing the integration variable in (3.6) to $\boldsymbol{K}' = \boldsymbol{K}(\boldsymbol{k}', t') = (k_1' - S(t - s)k_2', k_2', k_3')$ – which has unit Jacobian, giving $\mathrm{d}^3 k' = \mathrm{d}^3 K'$.

We make further simplification by considering only axisymmetric modes for which $k_2 = 0$. Note that for the non-axisymmetric modes, $\boldsymbol{K}(\boldsymbol{k}, t) = \boldsymbol{e}_1(k_1 - Stk_2) + \boldsymbol{e}_2 k_2 + \boldsymbol{e}_3 k_3$ increases monotonically with time, increasing the wavenumber, which would eventually decay by turbulent diffusivity. Therefore we focus our attention only on axisymmetric modes, for which $\boldsymbol{K}(\boldsymbol{k}, t - s) = \boldsymbol{K}(\boldsymbol{k}, t) = \boldsymbol{k} = (k_1, 0, k_3)$.

Let us first work out $\widetilde{\boldsymbol{U}}(\boldsymbol{k}, s)$ and $\widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}, t - s)$ correct up to $O(s)$.

[1]Green's function in (3.4) when expressed in the laboratory-frame wavevector becomes time-translational symmetric, i.e. $\widetilde{G}_{\eta_T}(\boldsymbol{k}, t, t') = \widetilde{G}_{\eta_T}(\boldsymbol{K}(\boldsymbol{k}, t'), t - t', 0) = \exp(-\eta_T\{K^2(t - t') - SK_1K_2(t - t')^2 + (S^2/3)K_2^2(t - t')^3\})$.

(1) $\underline{\widetilde{\boldsymbol{U}}(\boldsymbol{k},s) \text{ to } O(s):}$ Taylor expanding $\widetilde{\boldsymbol{U}}(\boldsymbol{k},s)$ gives,

$$\widetilde{\boldsymbol{U}}(\boldsymbol{k},s) = \widetilde{\boldsymbol{U}}(\boldsymbol{k},0) + s\left.\frac{\partial\widetilde{\boldsymbol{U}}}{\partial s}\right|_{s=0} + O(s^2). \tag{4.4}$$

$$\text{where } \widetilde{\boldsymbol{U}}(\boldsymbol{k},0) = \mathrm{i}\boldsymbol{k}\eta_\alpha + \boldsymbol{V}_M \quad \text{(from (3.8))}, \tag{4.5}$$

$$\text{and} \quad \left.\frac{\partial\widetilde{\boldsymbol{U}}}{\partial s}\right|_{s=0} = -\mathrm{i}\eta_T\int\frac{\mathrm{d}^3K'}{(2\pi)^3}(\boldsymbol{K}-\boldsymbol{K}')^2(\boldsymbol{K}-\boldsymbol{K}')\widetilde{\mathcal{A}}(\boldsymbol{K}'). \tag{4.6}$$

Equation (4.6) is obtained by differentiating equation (4.3) with respect to s and taking the limit $s\to 0$; note that $\boldsymbol{K}=\boldsymbol{k}$, since $k_2=0$. Using the Fourier transform for $\widetilde{\mathcal{A}}$, together with the properties of the δ-function, we get

$$\begin{aligned}\left.\frac{\partial\widetilde{\boldsymbol{U}}}{\partial s}\right|_{s=0} = {} & -\mathrm{i}\eta_T\big\{k^2(\boldsymbol{k}\mathcal{A}(\boldsymbol{\xi}) + \mathrm{i}[\nabla\mathcal{A}(\boldsymbol{\xi})]) + 2\mathrm{i}\boldsymbol{k}(\boldsymbol{k}\boldsymbol{\cdot}[\nabla\mathcal{A}(\boldsymbol{\xi})]) \\ & - \boldsymbol{k}[\nabla^2\mathcal{A}(\boldsymbol{\xi})] - \mathrm{i}[\nabla^2\{\nabla\mathcal{A}(\boldsymbol{\xi})\}] - 2(\boldsymbol{k}\boldsymbol{\cdot}\nabla)\{\nabla\mathcal{A}(\boldsymbol{\xi})\}\big\}_{\boldsymbol{\xi}=0}.\end{aligned} \tag{4.7}$$

Equation (4.7) can be evaluated once we know the functional form for spatial correlator $\mathcal{A}(\boldsymbol{\xi})$. Neglecting derivatives of $\mathcal{A}$ that are higher than the first order – see Singh (2016) for detail – we have:

$$\left.\frac{\partial\widetilde{\boldsymbol{U}}}{\partial s}\right|_{s=0} = -\eta_T k^2(\mathrm{i}\boldsymbol{k}\eta_\alpha + \boldsymbol{V}_M) - 2\eta_T(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{V}_M)\boldsymbol{k}. \tag{4.8}$$

Equation (4.5) and (4.8) together thus provide the function $\widetilde{\boldsymbol{U}}(\boldsymbol{k},s)$ correct up to $O(s)$.

(2) $\underline{\widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k},t-s) \text{ to } O(s):}$ We write as,

$$\widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k},t-s) = \widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k},t) - s\frac{\partial\widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k},t)}{\partial t} + \cdots. \tag{4.9}$$

where it is assumed that $|\widetilde{\overline{\boldsymbol{H}}}| \gg s|\partial\widetilde{\overline{\boldsymbol{H}}}/\partial t| \gg s^2|\partial^2\widetilde{\overline{\boldsymbol{H}}}/\partial t^2|$, etc. In (4.9), we need $\partial\widetilde{\overline{\boldsymbol{H}}}/\partial t$ only up to $O(1)$ to find $\widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k},t-s)$ up to $O(s)$. We write this by substituting (3.8) in (3.7) and using $\widetilde{\boldsymbol{U}}_{\mathrm{WN}}(\boldsymbol{k},t) = \widetilde{\boldsymbol{U}}_{\mathrm{WN}}(\boldsymbol{k}) = \mathrm{i}\boldsymbol{k}\eta_\alpha + \boldsymbol{V}_M$:

$$\left.\frac{\partial\widetilde{\overline{\boldsymbol{H}}}}{\partial t}\right|_{O(1)} = S\widetilde{\overline{H}_1}\boldsymbol{e}_2 + \mathrm{i}\boldsymbol{k}\times\widetilde{\overline{\boldsymbol{E}}}_{\mathrm{WN}} - \eta_T k^2\widetilde{\overline{\boldsymbol{H}}} = S\widetilde{\overline{H}_1}\boldsymbol{e}_2 - (\eta_K k^2 + \mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{V}_{\mathrm{M}})\widetilde{\overline{\boldsymbol{H}}}. \tag{4.10}$$

The time integral in (4.2) is then solved by using the definitions provided in (2.11) and (2.12) when we substitute the expressions derived above for the terms in { } in (4.2). We get, after straightforward algebra, the following expression for the mean EMF, which is correct up to $O(\tau_\alpha)$:

$$\begin{aligned}\widetilde{\overline{\boldsymbol{E}}}(\boldsymbol{k},t) = {} & \widetilde{\overline{\boldsymbol{E}}}_{\mathrm{WN}}(\boldsymbol{k},t) + \tau_\alpha\{(\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\widetilde{\boldsymbol{U}}_{\mathrm{WN}})\widetilde{\overline{\boldsymbol{E}}}_{\mathrm{WN}} - 2\eta_{\mathrm{T}}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{V}_{\mathrm{M}})\boldsymbol{k}\times\widetilde{\overline{\boldsymbol{H}}}\} \\ & + S\tau_\alpha\{\widetilde{\overline{H}_1}\boldsymbol{e}_2\times\widetilde{\boldsymbol{U}}_{\mathrm{WN}} + \boldsymbol{e}_2[\widetilde{\boldsymbol{U}}_{\mathrm{WN}}\times\widetilde{\overline{\boldsymbol{H}}}]_1\}.\end{aligned} \tag{4.11}$$

This verifies the ansatz of (4.1) up to $n = 1$, as claimed. It is important to note that the (4.11) is valid only for slowly varying large-scale magnetic fields. To lowest order this condition can be explicitly stated as: $|\tilde{\tilde{\boldsymbol{H}}}| \gg \tau_\alpha |\partial \tilde{\tilde{\boldsymbol{H}}}/\partial t|$. To obtain the sufficient condition for the validity of (4.11), use (4.10) for $\partial \tilde{\tilde{\boldsymbol{H}}}/\partial t$ to get the following conditions for three dimensionless quantities which need to be small:

$$|S\tau_\alpha| \ll 1, \quad |\eta_K k^2 \tau_\alpha| \ll 1, \quad |kV_M\tau_\alpha| \ll 1. \tag{4.12a–c}$$

Since we have expanded EMF in small τ_α, it is only the first of the two FOSA conditions in (3.1) that becomes relevant. This must be added to the above three conditions for (4.11) to be valid. Using (4.11) in (3.7) we obtain:

$$\begin{aligned} \frac{\partial \tilde{\tilde{\boldsymbol{H}}}}{\partial t} &= [S\tilde{\tilde{H}}_1\boldsymbol{e}_2 + \eta_\alpha k^2 \tilde{\tilde{\boldsymbol{H}}} - \mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{V}_M)\tilde{\tilde{\boldsymbol{H}}}][1 + \mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{V}_M)\tau_\alpha - \eta_\alpha k^2\tau_\alpha] - \eta_T k^2 \tilde{\tilde{\boldsymbol{H}}} \\ &\quad + 2\mathrm{i}\eta_T k^2 \tau_\alpha (\boldsymbol{k}\cdot\boldsymbol{V}_M)\tilde{\tilde{\boldsymbol{H}}} + S\tau_\alpha[V_{M2}\tilde{\tilde{H}}_3 - V_{M3}\tilde{\tilde{H}}_2 - \mathrm{i}\eta_\alpha k_3 \tilde{\tilde{H}}_2][-\mathrm{i}k_3\boldsymbol{e}_1 + \mathrm{i}k_1\boldsymbol{e}_3], \\ &\quad \text{with } \boldsymbol{k} = (k_1, 0, k_3), \text{ and } k_1\tilde{\tilde{H}}_1 + k_3\tilde{\tilde{H}}_3 = 0. \end{aligned} \tag{4.13}$$

Equation (4.13) is the linear partial differential equation obtained by reducing the linear integro-differential equation (see (3.5)–(3.7)) under the condition of (4.12). It describes the evolution of an axisymmetric, large-scale magnetic field over times that are much larger than τ_α. It depends on (i) the diffusivity η_T; (ii) properties of the α correlation in terms of η_α, $\boldsymbol{V}_M$ and τ_α; (iii) shear S. These must satisfy the three conditions given in (4.12) and the first condition in (3.1) for the validity of (4.13). We note here again that the set of (3.5)–(3.7) is non-perturbative in both S and τ_α, whereas (4.13) is valid only when $|S\tau_\alpha| \ll 1$.

5. Growth rate of modes when τ_α is non-zero

As usual in numerical works on the related subject (see, e.g. Brandenburg *et al.* 2008; Singh & Jingade 2015) where ‘horizontal’ (plane of shear; in this case the $X_1 - X_2$ plane) averages are performed to define the large-scale magnetic fields, it is useful to consider one-dimensional propagating modes. This is equivalent to setting K_1 and K_2 equal to zero. Here we only need to set $k_1 = 0$ in (4.13). In this case the wavevector $\boldsymbol{k} = (0, 0, k)$ points along the ‘vertical’ ($\pm\boldsymbol{e}_3$) direction, thus resulting in a uniform $\tilde{\tilde{H}}_3$ which is of no interest for dynamo action. Hence we set $\tilde{\tilde{H}}_3 = 0$, and take $\tilde{\tilde{\boldsymbol{H}}}(k, t) = \tilde{\tilde{H}}_1(k, t)\boldsymbol{e}_1 + \tilde{\tilde{H}}_2(k, t)\boldsymbol{e}_2$. Making these substitutions in (4.13) we find:

$$\begin{aligned} \frac{\partial \tilde{\tilde{\boldsymbol{H}}}}{\partial t} &= [S\tilde{\tilde{H}}_1\boldsymbol{e}_2 + \eta_\alpha k^2 \tilde{\tilde{\boldsymbol{H}}} - \mathrm{i}kV_{M3}\tilde{\tilde{\boldsymbol{H}}}][1 + \mathrm{i}kV_{M3}\tau_\alpha - \eta_\alpha k^2\tau_\alpha] - \eta_T k^2 \tilde{\tilde{\boldsymbol{H}}} \\ &\quad + 2\mathrm{i}\eta_T \tau_\alpha (k^3 V_{M3})\tilde{\tilde{\boldsymbol{H}}} + S[\mathrm{i}kV_{M3}\tau_\alpha - \eta_\alpha k^2\tau_\alpha]\tilde{\tilde{H}}_2\boldsymbol{e}_1. \end{aligned} \tag{5.1}$$

Seeking modal solutions of the form,

$$\tilde{\tilde{\boldsymbol{H}}}(k, t) = [\tilde{\tilde{H}}_{01}(k)\boldsymbol{e}_1 + \tilde{\tilde{H}}_{02}(k)\boldsymbol{e}_2]\exp(\lambda t), \tag{5.2}$$

and substituting this in (5.1) we get the following dispersion relation:

$$\begin{aligned} \lambda_\pm &= -\eta_K k^2 - \eta_\alpha^2 k^4 \tau_\alpha + (kV_{M3})^2\tau_\alpha + \mathrm{i}kV_{M3}[2(\eta_\alpha + \eta_T)k^2\tau_\alpha - 1] \\ &\quad \pm |S|\sqrt{[\mathrm{i}kV_{M3}\tau_\alpha - \eta_\alpha k^2\tau_\alpha][1 + \mathrm{i}kV_{M3}\tau_\alpha - \eta_\alpha k^2\tau_\alpha]}. \end{aligned} \tag{5.3}$$

We are more interested in the growth rate $\gamma = \mathrm{Re}\{\lambda\}$, as the dynamo action corresponds to the case when $\gamma > 0$. From the dispersion relation (5.3) we have:

$$\left.\begin{aligned}
&\gamma_{\pm} = \mathrm{Re}\{\lambda_{\pm}\} = -\eta_K k^2 - \eta_\alpha^2 k^4 \tau_\alpha + (kV_{M3})^2 \tau_\alpha \pm |S| [\chi_R^2 + \chi_I^2]^{1/4} \cos(\psi/2), \\
&\quad \text{where} \ \cos(\psi) = \frac{\chi_R}{(\chi_R^2 + \chi_I^2)^{1/2}}, \quad \cos(\psi/2) = \frac{\sqrt{\chi_R + \sqrt{\chi_R^2 + \chi_I^2}}}{\sqrt{2}(\chi_R^2 + \chi_I^2)^{1/4}}, \\
&\chi_R = \eta_\alpha k^2 \tau_\alpha (\eta_\alpha k^2 \tau_\alpha - 1) - (kV_{M3}\tau_\alpha)^2, \quad \chi_I = -kV_{M3}\tau_\alpha (2\eta_\alpha k^2 \tau_\alpha - 1).
\end{aligned}\right\} \tag{5.4}$$

Below, we make some comments about the growth rate derived above.

(i) The growth rate γ of the large-scale dynamo is linear in the shear rate $|S|$, assuming that the parameters $(\eta_K, \eta_\alpha, V_{M3}, \tau_\alpha)$ are all independent of S. This linear scaling is observed in earlier numerical works (Brandenburg *et al.* 2008; Yousef *et al.* 2008*a*,*b*; Singh & Jingade 2015).

(ii) For zero shear, the growth rate as given from (5.4) becomes identical to the one derived in Singh (2016), where the generalization to the Kraichnan problem as well as the possibility of Moffatt drift driven dynamos were explored in detail.

(iii) The last term involving shear in (5.4) is identical to the corresponding term in the expression for the growth rate derived in Sridhar & Singh (2014), with an important difference being that there the angle ψ was defined using the tangent function, which introduces an error when either of the two, χ_R and χ_I, take negative values. Here we correct this by explicitly writing $\cos(\psi/2)$ in terms of χ_R and χ_I.

5.1. *Dimensionless growth rate function*

The growth rate function $\varGamma$ is defined using dimensionless quantities,

$$\varGamma_{\pm} = \gamma_{\pm}\tau_\alpha; \quad \beta = \eta_\alpha k^2 \tau_\alpha; \quad \varepsilon_S = S\tau_\alpha; \quad \varepsilon_K = \eta_K k^2 \tau_\alpha; \quad \varepsilon_M = kV_{M3}\tau_\alpha, \tag{5.5a–e}$$

where β and ε_K measure the wavenumber of modal mean magnetic field in terms of η_α and η_K, respectively. With the first condition of (3.1), $\beta/k^2 = \eta_\alpha \tau_\alpha \ll \ell^2$. These parameters can vary as,

$$0 \leqslant \beta \ll (k\ell)^2; \quad \beta + \varepsilon_K > 0; \quad |\varepsilon_S| \ll 1; \quad |\varepsilon_K| \ll 1; \quad |\varepsilon_M| \ll 1. \tag{5.6a–e}$$

The parameter β can be larger or smaller than unity depending on whether the mean-field varies over scales smaller or larger than ℓ, respectively. The second condition comes from $\beta + \varepsilon_K = \eta_T k^2 \tau_\alpha > 0$, and last three constraints come from (4.12). Multiplying the expression for $\gamma_\pm$ in (5.4) by τ_α, and denoting by $\varGamma_>$ ($\varGamma_<$) the larger (smaller) of $\varGamma_+$ and $\varGamma_-$, we get

$$\varGamma_{\gtrless} = -\varepsilon_K - \beta^2 + \varepsilon_M^2 \pm \frac{|\varepsilon_S|}{\sqrt{2}} \sqrt{\beta(\beta - 1) - \varepsilon_M^2 + \sqrt{[\beta(\beta - 1) + \varepsilon_M^2]^2 + \varepsilon_M^2}}. \tag{5.7}$$

Note that the radicand in (5.7) is greater than zero. In figure 1 we show the behaviour of $\varGamma$ as a function of β by keeping other parameters fixed. Below we list some properties of the growth rate function as defined in (5.7).

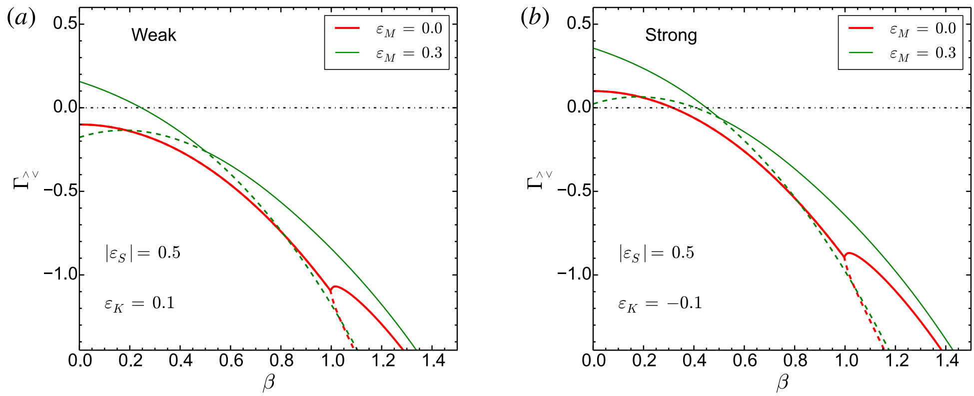

FIGURE 1. The two roots, $\Gamma_>$ (solid) and $\Gamma_<$ (dashed), of the growth rate function defined in (5.7) are shown as a function of β for $\varepsilon_M = 0$ (red; thick) and 0.3 (green; thin) with $|\varepsilon_S| = 0.5$, where (*a*) and (*b*) correspond to weak ($\varepsilon_K = 0.1$) and strong ($\varepsilon_K = -0.1$)α fluctuations, respectively.

(i) For fixed ε_K, β and ε_M, $\Gamma_>$ ($\Gamma_<$) increases (decreases) monotonically with shear.

(ii) When ε_M is non-zero, then the radicand in (5.7) vanishes at $\beta = 1/2$, where the two roots coincide; see green solid and dashed curves in figure 1. Both roots are identical for $0 \leqslant \beta \leqslant 1$ when $\varepsilon_M = 0$ and branch out for $\beta > 1$; see red solid and dashed curves.

(iii) In the absence of the Moffatt drift, the necessary condition for dynamo action is that the α fluctuations must be strong, i.e. $\varepsilon_K < 0$, regardless of the strength of the shear parameter $|\varepsilon_S|$ which should be kept smaller than unity in the present model. The dynamo is then driven through the $-\varepsilon_K$ term in (5.7) by the process of negative diffusion first suggested by Kraichnan (1976).

(iv) Moffatt drift always contributes positively to the dynamo growth. Considering the case of zero shear, we see from (5.7) that $\varepsilon_M > \varepsilon_M^{crit}$, with $\varepsilon_M^{crit} = \sqrt{\varepsilon_K + \beta^2}$, can always facilitate LSD in both the weak and strong α fluctuation regimes, for sufficiently low values of β such that $\varepsilon_M^{crit} \ll 1$.

(v) The growth rate is always negative for $\beta \gg 1$ due to the $-\beta^2$ term in (5.7), as is also shown in figure 1.

(vi) The growth rate for $\beta \approx 0$ i.e. largest scale possible is given for small values of $\varepsilon_M \ll 1$ as,

$$\Gamma_> \simeq -\varepsilon_K + \frac{|\varepsilon_S|}{\sqrt{2}}|\varepsilon_M|^{1/2}, \tag{5.8}$$

which implies Moffatt drift couples strongly with shear and growth is possible for weak α-fluctuations i.e. $\varepsilon_K > 0$ when $|\varepsilon_S||\varepsilon_M|^{1/2} > \sqrt{2}\varepsilon_K$.

5.2. *Growth rates as functions of the wavenumber*

We henceforth consider only the dominant root $\Gamma_>$ and study its wavenumber dependence. Following SS14, we first identify natural length and time scales whose corresponding wavenumber and frequency are defined as,

$$k_\alpha = (\eta_\alpha \tau_\alpha)^{-1/2} > 0; \quad \sigma = |\eta_K| k_\alpha^2, \tag{5.9a,b}$$

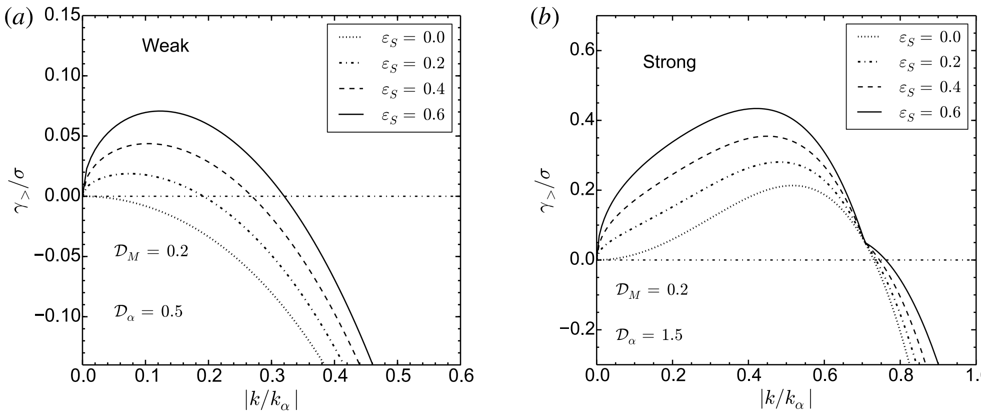

FIGURE 2. Normalized growth rate $\gamma_>/\sigma$ as a function of $|k/k_\alpha|$ for $\mathcal{D}_M = 0.2$. (*a*) and (*b*) correspond to weak ($\mathcal{D}_\alpha = 0.5$) and strong ($\mathcal{D}_\alpha = 1.5$)α fluctuations respectively. Solid, dashed, dash-dotted and dotted curves correspond to $\varepsilon_S = 0.6$, 0.4, 0.2 and 0, respectively.

where k_α can be recognized as an inverse diffusion length due to α-diffusivity η_α. Here $|k| > k_\alpha$ and $|k| < k_\alpha$ are called high and low wavenumbers, respectively. From (5.5) and (5.9) we see that $\beta = (k/k_\alpha)^2$. Since the parameters ε_K and ε_M involve the wavenumber k in their definitions, we find it better to rewrite an expression for $\varGamma_>$ using new dimensionless dynamo numbers, which are defined in terms of known constants:

$$\mathcal{D}_\alpha = \frac{\eta_\alpha}{\eta_T}, \quad \mathcal{D}_M = \frac{V_{M3}^2 \tau_\alpha}{\eta_\alpha}. \tag{5.10a,b}$$

We first make use of (5.4)–(5.10) to express the growth rates as a function of wavenumber and constant dynamo parameters in the weak and strong regimes of α-fluctuations.

Weak α fluctuations: Here, $\eta_\alpha < \eta_T$, i.e. $\mathcal{D}_\alpha < 1$ and $|\eta_K| = +\eta_K = \eta_T - \eta_\alpha$, giving

$$\frac{\gamma_>}{\sigma} = -\left(\frac{k}{k_\alpha}\right)^2 + \frac{\mathcal{D}_\alpha}{1-\mathcal{D}_\alpha}\left[-\left(\frac{k}{k_\alpha}\right)^4 + \mathcal{D}_M\left(\frac{k}{k_\alpha}\right)^2 + \frac{|\varepsilon_S|}{\sqrt{2}}\sqrt{\mathcal{R}_a(k)+\mathcal{R}_b(k)}\right] \tag{5.11}$$

$$\text{with } \mathcal{R}_a(k) = \left(\frac{k}{k_\alpha}\right)^2\left[\left(\frac{k}{k_\alpha}\right)^2 - 1\right] - \mathcal{D}_M\left(\frac{k}{k_\alpha}\right)^2 \tag{5.12}$$

$$\text{and } \mathcal{R}_b(k) = \left(\left\{\left[\left(\frac{k}{k_\alpha}\right)^2 - 1\right] + \mathcal{D}_M\left(\frac{k}{k_\alpha}\right)^2\right\}^2 + \mathcal{D}_M\left(\frac{k}{k_\alpha}\right)^2\right)^{1/2}. \tag{5.13}$$

In figure 2(*a*) we show the wavenumber dependence of the normalized growth rate $\gamma_>/\sigma$ for different choices of the shear parameter, at fixed $\mathcal{D}_\alpha$ and $\mathcal{D}_M$, when the α fluctuations are weak. Interestingly, the growth rate is positive for fairly small wavenumbers, thus facilitating a truly large-scale dynamo, with a wavenumber cutoff beyond which the growth rate turns negative. At much larger wavenumbers, The growth rate varies as $\gamma \propto -k^4$ due to the η_T-correction in the present model. Shear boosts the growth rates at all wavenumbers, and thus it can support the dynamo action for sufficiently strong Moffatt drift.

Strong α fluctuations: In this case, $\eta_\alpha > \eta_T$, i.e. $\mathcal{D}_\alpha > 1$ and $|\eta_K| = -\eta_K$, giving

$$\frac{\gamma_>}{\sigma} = + \left(\frac{k}{k_\alpha}\right)^2 + \frac{\mathcal{D}_\alpha}{\mathcal{D}_\alpha - 1}\left[- \left(\frac{k}{k_\alpha}\right)^4 + \mathcal{D}_M \left(\frac{k}{k_\alpha}\right)^2 + \frac{|\varepsilon_S|}{\sqrt{2}}\sqrt{\mathcal{R}_a(k) + \mathcal{R}_b(k)}\right]. \quad (5.14)$$

Here the small wavenumbers grow as all the effects, Kraichnan diffusivity, Moffatt drift and shear, contribute positively to the dynamo action; see figure 2(*b*). Similar to the case of weak α fluctuations, the growth rate here too is a non-monotonic function of k and it becomes negative for sufficiently large wavenumbers.

5.2.1. *Dynamo action for zero Moffatt drift*

This corresponds to the Kraichnan problem, extended to include a non-zero τ_α. There are two cases to consider, the one in the absence of shear and the other when shear is present.

1. Shear absent (only η_α and τ_α non-zero)

Using (5.11)–(5.14) by setting $\mathcal{D}_M = 0$ and $|\varepsilon_S| = 0$ the normalized growth rate can be expressed as, for

Weak α fluctuations: when $0 < \eta_\alpha < \eta_T$, i.e. $\mathcal{D}_\alpha < 1$,

$$\frac{\gamma}{\sigma} = - \left(\frac{k}{k_\alpha}\right)^2 - \frac{\mathcal{D}_\alpha}{1 - \mathcal{D}_\alpha}\left(\frac{k}{k_\alpha}\right)^4. \quad (5.15)$$

Here, the growth is negative definite for all values of k and a monotonically decreasing function of k. At large wavenumbers, it varies as $\gamma \propto -k^4$, a correction due to finite τ_α and inclusion of a finite resistive term in the fluctuating field equation. The first term in the (5.15) is due to Kraichnan diffusivity (compare it with (3.10) by setting $S = 0$ and $\boldsymbol{V}_M = 0$).

Strong α fluctuations: when $0 < \eta_T < \eta_\alpha$, i.e. $\mathcal{D}_\alpha > 1$,

$$\frac{\gamma}{\sigma} = \left(\frac{k}{k_\alpha}\right)^2 - \frac{\mathcal{D}_\alpha}{\mathcal{D}_\alpha - 1}\left(\frac{k}{k_\alpha}\right)^4. \quad (5.16)$$

In this regime, the growth rate is positive for a certain range of wavenumbers and it becomes negative for large wavenumbers as mentioned above. In figure 3 we compare our model (which has non-zero τ_α) with the original Kraichnan model – we see that a non-zero τ_α introduces a high wavenumber cutoff in the case of strong α-fluctuations, which agrees with the conclusions of Singh (2016).

2. The effect of shear

Using (5.7) we rewrite the growth rate function more explicitly as:

$$\Gamma_> = -\varepsilon_K - \beta^2, \quad \text{when } 0 \leqslant \beta \leqslant 1 \quad (5.17)$$

$$\Gamma_> = -\varepsilon_K - \beta^2 + |\varepsilon_S|\sqrt{\beta(\beta - 1)}, \quad \text{when } \beta > 1. \quad (5.18)$$

We note that the shear does not couple to the dynamo growth rate when β is smaller than or equal to unity, or in other words, when $|k| < k_\alpha$.[2] Since $|\varepsilon_S| \ll 1$, the dominant term in (5.18) is $-\beta^2$ for $k > k_\alpha$, which makes the growth rate negative definite. Thus,

[2]Compare equations (5.17) and (5.18) with equation (85) in SS14, where the authors had obtained the contribution of shear for $|k| < k_\alpha$, due to the error in the angle evaluation in (5.4) which is corrected here.

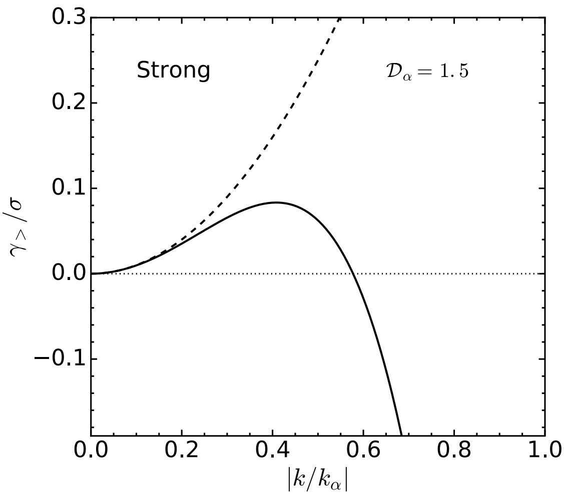

FIGURE 3. Normalized growth rate when Moffatt drift and shear are zero. Plotted as a function of $|k/k_\alpha|$. Solid curve shows finite τ_α correction and dashed-dotted curve is for the white-noise case.

for weak α fluctuations which have $\varepsilon_K > 0$, shear alone cannot drive a large-scale dynamo at any wavenumber. Therefore, the necessary condition for dynamo action in this case is that the α fluctuations must be strong. We now look at the properties of growth rate as a function of wavenumber.

Weak α fluctuations: Here, $\eta_\alpha < \eta_T$, i.e. $\mathcal{D}_\alpha < 1$ and $|\eta_K| = +\eta_K = \eta_T - \eta_\alpha$, giving

$$\frac{\gamma_>}{\sigma} = -\left(\frac{k}{k_\alpha}\right)^2 - \frac{\mathcal{D}_\alpha}{1-\mathcal{D}_\alpha}\left(\frac{k}{k_\alpha}\right)^4, \quad \text{when } 0 < |k| < k_\alpha \tag{5.19}$$

$$= -\left(\frac{k}{k_\alpha}\right)^2 + \frac{\mathcal{D}_\alpha}{1-\mathcal{D}_\alpha}\left[-\left(\frac{k}{k_\alpha}\right)^4 + |\varepsilon_S|\sqrt{\left(\frac{k}{k_\alpha}\right)^2\left[\left(\frac{k}{k_\alpha}\right)^2 - 1\right]}\right], \quad \text{when } |k| > k_\alpha. \tag{5.20}$$

We can see from (5.19) that the growth rate is negative definite in the range $0 < |k| < k_\alpha$, as inferred above. Dynamo action is not possible for $|k| > k_\alpha$ for the following reason. When $|k| > k_\alpha$, shear contributes to the growth rate (see (5.20)). Since the model is valid for $|\varepsilon_S| \ll 1$, in order to increase the strength of that term we can increase $\mathcal{D}_\alpha$ (while keeping it less than unity), but this will also strengthen the second term, which is $\propto -k^4$, due to the finite η_T correction in the fluctuating field equation, thereby killing dynamo action.

Strong α fluctuations: Here, $\eta_T < \eta_\alpha$, i.e. $\mathcal{D}_\alpha > 1$ and $|\eta_K| = -\eta_K = \eta_\alpha - \eta_T$, giving

$$\frac{\gamma_>}{\sigma} = \left(\frac{k}{k_\alpha}\right)^2 - \frac{\mathcal{D}_\alpha}{\mathcal{D}_\alpha - 1}\left(\frac{k}{k_\alpha}\right)^4, \quad \text{when } 0 < |k| < k_\alpha \tag{5.21}$$

$$= \left(\frac{k}{k_\alpha}\right)^2 + \frac{\mathcal{D}_\alpha}{\mathcal{D}_\alpha - 1}\left[-\left(\frac{k}{k_\alpha}\right)^4 + |\varepsilon_S|\sqrt{\left(\frac{k}{k_\alpha}\right)^2\left[\left(\frac{k}{k_\alpha}\right)^2 - 1\right]}\right], \quad \text{when } |k| > |k_\alpha|. \tag{5.22}$$

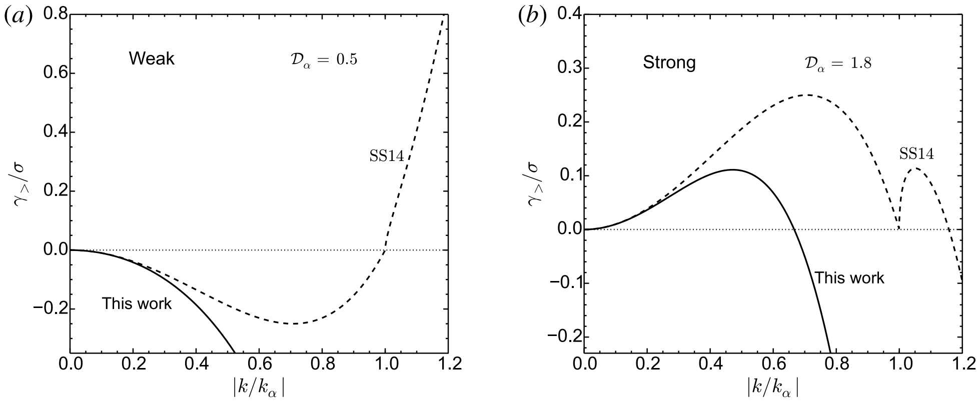

FIGURE 4. Normalized growth rate $\gamma_>/\sigma$ as a function of $|k/k_\alpha|$ for $|\varepsilon_S| = 0.3$. (*a,b*) Correspond to weak ($\mathcal{D}_\alpha = 0.5$) and strong ($\mathcal{D}_\alpha = 1.8$)α fluctuations, respectively. Solid and dashed curves correspond to this work and SS14, respectively.

The growth rate γ becomes positive for a certain range of wavenumber depending upon the strength of the α-fluctuations, eventually becoming negative at large wavenumbers due to the $-k^4$ term arising due to the finite η_T correction. This behaviour is compared in figure 4 with Sridhar & Singh (2014); we can see that there is good agreement at low wavenumbers whereas at large wavenumbers there is a difference. The derivation of Sridhar & Singh (2014) had neglected the effect of turbulent resistivity on the fluctuating component of the magnetic field, and they had noted that this would lead to an overestimation of growth rates at large wavenumbers. This is what we find in the present work: retaining this term makes the growth rate negative at large wavenumbers and, for weak α-fluctuations, the behaviour is indeed qualitatively different. Therefore, including the η_T term gives a bonafide large-scale dynamo action by predicting the high wavenumber cutoff. Thus, in the absence of Moffatt drift, the necessary condition for the large-scale dynamo when shear is present is the same as the case when it is absent.

6. Conclusions

We have studied the effect of α fluctuations on the growth of large-scale magnetic fields in a shearing background. Our derivation of the mean electromotive force is based on the first-order smoothing approximation (FOSA), whose range of validity is given in (3.1). These are such that FOSA is, in general, valid for all 'weak' α fluctuations ($\eta_\alpha < \eta_T$), which is the case of primary interest for dynamo action. We have extended the analysis of Sridhar & Singh (2014) by including the effect of the turbulent resistivity, η_T, on the fluctuating component of the magnetic field. We derived the integro-differential equation for the large-scale magnetic field, which is non-perturbative in shear strength, S, and the α-correlation time, τ_α, similar to Sridhar & Singh (2014). For the exactly solvable case of white-noise α-fluctuations, dynamo action is possible only when the α-fluctuations are strong; this is also similar to Sridhar & Singh (2014). In order to explore dynamo action in the regime of weak α-fluctuations it is necessary to consider a non-zero τ_α. Considering a small but non-zero τ_α and a slowly varying large-scale magnetic field, we reduced the integro-differential equation to a partial differential equation. We present an expression for the mean EMF, correct up to first order in τ_α. We also corrected an error in Sridhar & Singh (2014) in the expression for the growth rate, γ. Our salient conclusions are listed below:

(i) In the absence of Moffatt drift (i.e. $\boldsymbol{V}_M = 0$) the growth rate is independent of shear when $0 < |k| < k_\alpha$, and there is no dynamo action for weak α-fluctuations even when $|k| > k_\alpha$ for moderately small shear (i.e. $|\varepsilon_S| \ll 1$) – see figure 4(*a*).

(ii) For dynamo action with weak α-fluctuations, it is necessary that $\boldsymbol{V}_M \neq 0$: Moffatt drift couples strongly to shear and excites dynamo modes for $|k| < k_\alpha$ – see item (iv) in § 5.1 and figure 2.

We briefly comment on different approaches adopted in some related earlier works involving α fluctuations in a shearing background. Heinemann *et al.* (2011) considered tensorial $\hat{\boldsymbol{\alpha}}$-fluctuations due to a quasi-two-dimensional velocity field, whose dynamics is governed by the Navier–Stokes equation at low Reynolds number, where the stochastic motions occur due to a Gaussian random forcing which is δ-correlated in time. A double-averaging scheme was employed, first over the 'horizontal' (or xy) coordinates, and second over the statistics of the forcing function. They found that the first moment of the magnetic field does not grow, while there is a growth of the mean-squared magnetic field. Note that the spatial fluctuations in $\hat{\boldsymbol{\alpha}}$ were ignored there, and the correlation time of only temporally fluctuating $\hat{\boldsymbol{\alpha}}$ was assumed to be the same as that of the velocity field. Mitra & Brandenburg (2012) also studied a model with tensorial $\hat{\boldsymbol{\alpha}}$ and allowed only temporal fluctuations which were further restricted to be δ-correlated in time. When cross-correlations between different $\hat{\boldsymbol{\alpha}}$ components were assumed to be zero, they found growing solutions for the second moment of the mean magnetic field, but not for the first moment. However, when cross-correlations were allowed, large enough shear promoted the growth of even the mean magnetic field. Ignoring spatial structures and memory effects of the stochastic α appear to be a serious limitation. We remedy this in the present investigation where an essential generalization is made to explore new physical mechanisms driving large-scale dynamos, but by focussing here on the scalar α fluctuations to keep the analysis simple.

Thus, our model is a minimal extension of Kraichnan (1976) and Moffatt (1978), where α is assumed to be a fluctuating pseudo-scalar field, and η_T is constant. We have constructed a model of large-scale dynamo action with essential roles played by the Moffatt drift and a non-zero correlation time. Hence our focus has been to keep the tensorial structure of α as simple as possible, while exploring the effect of spatio-temporal variations that are natural to turbulent flows. We note that our work is almost completely complementary to Mitra & Brandenburg (2012), wherein α fluctuations are tensorial but have very restrictive space–time properties: no spatial variation at all and with a zero correlation time. Indeed non-zero correlation times and non-trivial spatial statistics appear essential for dynamo action, as emphasized in item (2) above. We note here that there seems to be some numerical evidence for pseudo-tensorial α and tensorial η_T fluctuations (Brandenburg *et al.* 2008; Rheinhardt *et al.* 2014; Singh & Jingade 2015). Our results, obtained for pseudo-scalar α, can be readily extended to tensorial fields.

Our analytical results for the growth rates of modes relies on a perturbative expansion in τ_α, which could also be generalized. Another important assumption is the role of the shear in the statistics of the α fluctuations: these fluctuations have been specified by a Galilean-invariant two-point correlation function in factored form $\mathcal{A}(\boldsymbol{R})\mathcal{D}(t)$, where $\boldsymbol{R} = \boldsymbol{x} - \boldsymbol{x}' + St'(x_1 - x_1')\boldsymbol{e}_2$. Even though the functional form of $\mathcal{A}$

has dependence on shear through the argument $\boldsymbol{R}$, to the first-order expansion in τ_α, neither $\mathcal{A}(0) = \eta_\alpha$ nor $\boldsymbol{V}_M$ depends on shear explicitly. This is a limitation, since we can expect a background shear flow to introduce anisotropy in the turbulent flow which is the source of the fluctuations. Future modelling must seek to be guided by numerical simulations that are designed to measure the statistics of α fluctuations.

Appendix A. Derivation of (3.3)

Fourier transforming (3.2), we obtain:

$$\left.\begin{aligned} \left(\frac{\partial}{\partial t} + \eta_T K^2\right)\widetilde{\boldsymbol{h}} - S\widetilde{h}_1\boldsymbol{e}_2 = \mathrm{i}\boldsymbol{K}\times\widetilde{\boldsymbol{M}}, \quad \boldsymbol{K}\boldsymbol{\cdot}\widetilde{\boldsymbol{h}} = 0, \quad \widetilde{\boldsymbol{h}}(\boldsymbol{k}, 0) = \boldsymbol{0}, \\ \text{with } \boldsymbol{K}(\boldsymbol{k}, t) = \boldsymbol{e}_1(k_1 - Stk_2) + \boldsymbol{e}_2k_2 + \boldsymbol{e}_3k_3, \\ \text{and} \quad \widetilde{\boldsymbol{M}}(\boldsymbol{k}, t) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k'\widetilde{a}^*(\boldsymbol{k}', t)\,\widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}+\boldsymbol{k}', t). \end{aligned}\right\} \tag{A 1}$$

We integrate (A 1) component-wise to write the following solution, satisfying both constraints, $\boldsymbol{K}\boldsymbol{\cdot}\widetilde{\boldsymbol{h}} = 0$ and $\widetilde{\boldsymbol{h}}(\boldsymbol{k}, 0) = \boldsymbol{0}$:

$$\begin{aligned} \widetilde{\boldsymbol{h}}(\boldsymbol{k}, t) \;=\; & \int_0^t \mathrm{d}t'\,\widetilde{G}_{\eta_T}(\boldsymbol{k}, t, t')[\mathrm{i}\boldsymbol{K}(\boldsymbol{k}, t')\times\widetilde{\boldsymbol{M}}(\boldsymbol{k}, t')] \\ & + \boldsymbol{e}_2 S\int_0^t \mathrm{d}t'\int_0^{t'} \mathrm{d}t''\,\widetilde{G}_{\eta_T}(\boldsymbol{k}, t, t'')[\mathrm{i}\boldsymbol{K}(\boldsymbol{k}, t'')\times\widetilde{\boldsymbol{M}}(\boldsymbol{k}, t'')]_1. \end{aligned} \tag{A 2}$$

Green's function $\widetilde{G}_{\eta_T}(\boldsymbol{k}, t, t')$ is given in (3.4), from where we can see a property that $\widetilde{G}_{\eta_T}(\boldsymbol{k}, t, t')\times\widetilde{G}_{\eta_T}(\boldsymbol{k}, t, t'') = \widetilde{G}_{\eta_T}(\boldsymbol{k}, t, t'')$, which is used in getting (A 2). Reducing the double time integral in (A 2) to a single time integral by using,

$$\int_0^t \mathrm{d}t'\int_0^{t'} \mathrm{d}t'' f(t'') = \int_0^t \mathrm{d}t'(t-t')f(t'), \tag{A 3}$$

we obtain the FOSA solution for the fluctuating magnetic field as given in (3.3).

Appendix B. Two-point α-correlator in Fourier space

Here we derive a general expression for time-stationary Galilean-invariant two-point α-correlator in Fourier space, where (2.11) transforms to:

$$\begin{aligned} \overline{\widetilde{a}(\boldsymbol{k}_1, t)\widetilde{a}^*(\boldsymbol{k}_3, t')} &= \int \mathrm{d}^3x_1\mathrm{d}^3x_3 \exp(-\mathrm{i}\boldsymbol{k}_1\boldsymbol{\cdot}\boldsymbol{x}_1 + \mathrm{i}\boldsymbol{k}_3\boldsymbol{\cdot}\boldsymbol{x}_3)\overline{a(\boldsymbol{x}_1, t)a(\boldsymbol{x}_3, t')} \\ &= 2\mathcal{D}(t-t')\int \mathrm{d}^3x_1\,\mathrm{d}^3x_3 \exp[-\mathrm{i}(\boldsymbol{k}_1\boldsymbol{\cdot}\boldsymbol{x}_1 - \boldsymbol{k}_3\boldsymbol{\cdot}\boldsymbol{x}_3)]\mathcal{A}(\boldsymbol{x}_1 - \boldsymbol{x}_3 + St'(x_{11} - x_{31})\boldsymbol{e}_2). \end{aligned} \tag{B 1}$$

Using new integration variables, $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_3$ and $\boldsymbol{r}' = (\boldsymbol{x}_1 + \boldsymbol{x}_3)/2$, we get

$$\begin{aligned} \overline{\widetilde{a}(\boldsymbol{k}_1, t)\widetilde{a}^*(\boldsymbol{k}_3, t')} \;=\; & 2\mathcal{D}(t-t')\int \mathrm{d}^3r\,\mathrm{d}^3r' \exp\left[-\mathrm{i}(\boldsymbol{k}_1 - \boldsymbol{k}_3)\boldsymbol{\cdot}\boldsymbol{r}' - \frac{\mathrm{i}}{2}(\boldsymbol{k}_1 + \boldsymbol{k}_3)\boldsymbol{\cdot}\boldsymbol{r}\right] \\ & \times\mathcal{A}(\boldsymbol{r} + St'r_1\boldsymbol{e}_2) = 2\mathcal{D}(t-t')(2\pi)^3\delta(\boldsymbol{k}_1 - \boldsymbol{k}_3) \\ & \times\int \mathrm{d}^3r \exp(-\mathrm{i}\boldsymbol{k}_1\boldsymbol{\cdot}\boldsymbol{r})\mathcal{A}(\boldsymbol{r} + St'r_1\boldsymbol{e}_2). \end{aligned} \tag{B 2}$$

Another change of the integration variable to $\boldsymbol{R}=\boldsymbol{r}+St'r_1\boldsymbol{e}_2$ gives us a compact form for the two-point correlator:

$$\left.\begin{aligned}\overline{\widetilde{a}(\boldsymbol{k}_1,t)\widetilde{a}^*(\boldsymbol{k}_3,t')}&=2\mathcal{D}(t-t')(2\pi)^3\delta(\boldsymbol{k}_1-\boldsymbol{k}_3)\,\widetilde{\mathcal{A}}(\boldsymbol{K}(\boldsymbol{k}_1,t')),\\ \text{where }\ \widetilde{\mathcal{A}}(\boldsymbol{K})&=\int \mathrm{d}^3R\exp(-\mathrm{i}\boldsymbol{K}\boldsymbol{\cdot}\boldsymbol{R})\mathcal{A}(\boldsymbol{R}).\end{aligned}\right\}\tag{B 3}$$

Note that $\widetilde{\mathcal{A}}(-\boldsymbol{K})=\widetilde{\mathcal{A}}^*(\boldsymbol{K})$ because $\mathcal{A}(\boldsymbol{R})$ is a real function and that the argument of the complex spatial power spectrum $\widetilde{\mathcal{A}}(\boldsymbol{K})$ is a time-dependent wavevector.

Appendix C. Derivation of (3.5) and (3.6)

We derive an expression for the mean EMF in Fourier space by using (3.3) as:

$$\begin{aligned}\widetilde{\overline{\boldsymbol{E}}}(\boldsymbol{k},t)&=\int \mathrm{d}^3x\exp(-\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x})\,\overline{\boldsymbol{E}}(\boldsymbol{x},t)=\int \mathrm{d}^3x\exp(-\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x})\,\overline{a(\boldsymbol{x},t)\boldsymbol{h}(\boldsymbol{x},t)}\\ &=\frac{1}{(2\pi)^3}\int \mathrm{d}^3k'\mathrm{d}^3k''\delta(\boldsymbol{k}'+\boldsymbol{k}''-\boldsymbol{k})\,\overline{\widetilde{a}(\boldsymbol{k}',t)\,\widetilde{\boldsymbol{h}}(\boldsymbol{k}'',t)}\\ &=\frac{1}{(2\pi)^3}\int \mathrm{d}^3k'\,\mathrm{d}^3k''\delta(\boldsymbol{k}'+\boldsymbol{k}''-\boldsymbol{k})\int_0^t \mathrm{d}t'\,\widetilde{G}_{\eta_T}(\boldsymbol{k}'',t,t')\\ &\quad\times\{[\mathrm{i}\boldsymbol{K}(\boldsymbol{k}'',t')\times\overline{\widetilde{a}(\boldsymbol{k}',t)\widetilde{\boldsymbol{M}}(\boldsymbol{k}'',t')}]+\boldsymbol{e}_2S(t-t')[\mathrm{i}\boldsymbol{K}(\boldsymbol{k}'',t')\times\overline{\widetilde{a}(\boldsymbol{k}',t)\widetilde{\boldsymbol{M}}(\boldsymbol{k}'',t')}]_1\}.\end{aligned}\tag{C 1}$$

This is given in terms of the quantity $\overline{\widetilde{a}(\boldsymbol{k}',t)\widetilde{\boldsymbol{M}}(\boldsymbol{k}'',t')}$, which can be determined by using the definition of $\widetilde{\boldsymbol{M}}$ from (A 1) and then using the time-stationary Galilean-invariant expression for the two-point $\widetilde{a}$-correlator as given by (B 3). We get

$$\begin{aligned}\overline{\widetilde{a}(\boldsymbol{k}',t)\widetilde{\boldsymbol{M}}(\boldsymbol{k}'',t')}&=\frac{1}{(2\pi)^3}\int \mathrm{d}^3k'''\,\overline{\widetilde{a}(\boldsymbol{k}',t)\widetilde{a}^*(\boldsymbol{k}''',t')}\,\widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}''+\boldsymbol{k}''',t')\\ &=2\mathcal{D}(t-t')\,\widetilde{\mathcal{A}}(\boldsymbol{K}(\boldsymbol{k}',t'))\,\widetilde{\overline{\boldsymbol{H}}}(\boldsymbol{k}'+\boldsymbol{k}'',t').\end{aligned}\tag{C 2}$$

Substituting (C 2) in (C 1) and solving the k''-integral using the property of the δ-function, we immediately find the expression for mean EMF as given in (3.5) in terms of a generalized complex velocity vector $\widetilde{\boldsymbol{U}}$ defined by (3.6).

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