Journal of Physics: Conference Series **360** (2012) 012024 doi:10.1088/1742-6596/360/1/012024

SU(2) gauge theory of gravity with topological invariants

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Abstract. The most general gravity Lagrangian in four dimensions contains three topological densities, namely Nieh-Yan, Pontryagin and Euler, in addition to the Hilbert-Palatini term. We set up a Hamiltonian formulation based on this Lagrangian. The resulting canonical theory depends on three parameters which are coefficients of these terms and is shown to admit a real $SU(2)$ gauge theoretic interpretation with a set of seven first-class constraints. Thus, in addition to the Newton's constant, the theory of gravity contains three (topological) coupling constants, which might have non-trivial imports in the quantum theory, e.g. in quantum geometry.

1. Introduction

The classical dynamics of a system is not affected by the addition of topological densities in the Lagrangian. This is so because such densities can always be locally written as total divergences. However, quantum dynamics might depend on them. The cases of the Sine-Gordon quantum mechanical model or QCD provide perfect examples of such a phenomenon where topological terms leave their imprints on the quantum theory[1].

In gravity theory in 3+1 dimensions, there are three possible topological terms, namely, Nieh-Yan, Pontryagin and Euler, which can be added to the Lagrangian. In terms of tetrads and spin-connections, these can be written as[1]:

$$\begin{aligned} I_{NY} &= e\Sigma^{\mu\nu}_{IJ}\tilde{R}_{\mu\nu}{}^{IJ}(\omega) \ + \ \epsilon^{\mu\nu\alpha\beta}D_\mu(\omega)e_{I\nu}D_\alpha(\omega)e^I_\beta \ = \ \partial_\mu\left[\epsilon^{\mu\nu\alpha\beta}\ e^I_\nu\ D_\alpha(\omega)e_{I\beta}\right] \\ I_P &= \epsilon^{\mu\nu\alpha\beta}R_{\mu\nu IJ}(\omega)R_{\alpha\beta}{}^{IJ}(\omega) \ = \ 4\partial_\mu\left[\epsilon^{\mu\nu\alpha\beta}\omega_\nu{}^{IJ}\left(\partial_\alpha\omega_{\beta IJ}+\frac{2}{3}\omega_{\alpha I}{}^K\omega_{\beta KJ}\right)\right] \\ I_E &= \epsilon^{\mu\nu\alpha\beta}R_{\mu\nu IJ}(\omega)\tilde{R}_{\alpha\beta}{}^{IJ}(\omega) \ = \ 4\partial_\mu\left[\epsilon^{\mu\nu\alpha\beta}\tilde{\omega}_\nu{}^{IJ}\left(\partial_\alpha\omega_{\beta IJ}+\frac{2}{3}\omega_{\alpha I}{}^K\omega_{\beta KJ}\right)\right] \end{aligned} \qquad (1)$$

where $R_{\mu\nu}^{IJ}(\omega)=\partial_{[\mu}\omega_{\nu]}^{\ IJ}+\omega_{[\mu}^{\ IK}\omega_{\nu]K}^{\ \ J}$ and $D_\mu(\omega)e^I_\nu=\partial_\mu e^I_\nu+\omega_\mu{}^{IJ}e_{\nu J}$. Although these topological densities are functions of local geometric quantities, they encode only the global properties of the manifold. The Nieh-Yan density depends on torsion and in Euclidean theory, its integral over a compact manifold is a sum of three integers associated with the homotopy maps $\pi_3(SO(5))=Z$ and $\pi_3(SO(4))=Z+Z$. Among the other two which depend on the curvature, the Pontryagin-class characterises the integers corresponding to the map $\pi_3(SO(4))=Z+Z$ and the Euler-class

[1] The quantity $\tilde{X}^{IJ}$ the dual of X_{IJ} in the internal space: $\tilde{X}^{IJ}=\frac{1}{2}\ \epsilon^{IJKL}X_{KL}$

characterises the combination of Betti numbers. While the first two densities are P and T odd, the third is P and T even (see [2] and the references within).

In order to understand their possible import in the quantum theory, it is important to set up a classical Hamiltonian formulation of the theory containing all these terms in the action. In ref.[3], such an analysis has been presented for a theory based on Lagrangian density containing the standard Hilbert-Palatini term and the Nieh-Yan density. The resulting theory, in time gauge, has been shown to correspond to the well-known canonical gauge theoretic formulation of gravity based on Sen-Ashtekar-Barbero-Immirzi *real* $SU(2)$ gauge fields [4]. Here η^{-1}, the inverse of the coefficient of Nieh-Yan term, is identified with the Barbero-Immirzi parameter γ. The framework in [3] supersedes the earlier formulation of Holst [5] in the sense that unlike the Holst term, the Nieh-Yan density
(a) does not need any further modifications for the inclusion of matter couplings and the equations of motion continue to be independent of η for all couplings;
(b) provides a topological interpretation for Barbero-Immirzi parameter, leading to a complete analogy between η and the θ-parameter of non-abelian gauge theories (from the classical perspective).
As an elucidation of these facts, the method has been applied to spin-$\frac{1}{2}$ fermions[3] and supergravity theories[6].

Here we include all three topological terms in the Hilbert-Palatini Lagrangian[2]:

$$\mathcal{L}(e,\omega) ~=~ \frac{1}{2}\, e\, \Sigma^{\mu\nu}_{IJ}\, R_{\mu\nu}{}^{IJ}(\omega) ~+~ \frac{\eta}{2}\, I_{NY} ~+~ \frac{\theta}{4}\, I_P ~+~ \frac{\phi}{4}\, I_E \tag{2}$$

where, $\Sigma^{\mu\nu}_{IJ} = \frac{1}{2}\, e^{\mu}_{[I} e^{\nu}_{J]}$. In order to understand how the canonical theory of gravity gets affected by such additions, a Hamiltonian analysis based on this Lagrangian is presented below, demonstrating how we obtain a real SU(2) formulation of gravity with all three topological densities.

2. Hamiltonian formulation

To set up a Hamiltonian description, we assume that the spacetime is of the form $\Sigma X R$ where Σ is a compact manifold. We decompose the 16 (spacetime) tetrad fields e^I_μ into the fields V^I_a, M_I, N^a and N (16=9+3+3+1) (see [2] for further details):

$$\begin{aligned} e^I_t &= N M^I + N^a V^I_a, \qquad e^I_a = V^I_a ~; \\ e^t_I &= -\frac{M_I}{N}~, \qquad e^a_I = V^a_I + \frac{N^a M_I}{N}; \\ M_I V^I_a &= 0 ~~, \qquad M_I M^I = -1 ~; \\ V^I_a V^b_I &= \delta^b_a ~, \qquad V^I_a V^a_J = \delta^I_J + M^I M_J ~. \end{aligned} \tag{3}$$

The internal space metric is Lorentzian, i.e., $\eta_{IJ} := dia(-1,1,1,1)$. Next, instead of the variables V^a_I and M^I, we define a new set of 12 variables as:

$$E^a_i ~=~ 2e\Sigma^{ta}_{0i} ~\equiv~ e\left(e^t_0 e^a_i - e^t_i e^a_0\right) = -\sqrt{q}\, M_{[0}V^a_{i]}, \qquad \chi_i = -M_i/M^0 \tag{4}$$

Before writing the full Lagrangian, we note that with the help of Bianchi identities $\epsilon^{abc} D_a(\omega) R_{bcIJ} = 0$ and $\epsilon^{abc} D_a(\omega)\tilde{R}_{bcIJ} = 0$, the last two terms in (2) can be written as[2]:

$$\frac{\theta}{4}\, I_P + \frac{\phi}{4}\, I_E = e^a_{IJ}\, \partial_t \omega^{(\eta)IJ}_a \tag{5}$$

[2] The quantity $X^{(\eta)IJ}$ is defined as: $X^{(\eta)IJ} = X^{IJ} + \eta\, \tilde{X}^{IJ}$

with $(1+\eta^2)\, e^a_{IJ} \;=\; \epsilon^{abc}\left\{(\theta+\eta\phi)\, R_{bcIJ}(\omega) + (\phi-\eta\theta)\, \tilde{R}_{bcIJ}(\omega)\right\}$. Using (3), (4) and (5), the Lagrangian in (2) can be written as:

$$\mathcal{L} \;=\; \pi^a_{IJ}\partial_t\omega_a^{(\eta)IJ} \;+\; t^a_I\partial_t V^I_a \;-\; NH \;-\; N^a H_a \;-\; \frac{1}{2}\omega_t^{IJ} G_{IJ} \tag{6}$$

where $\pi^a_{IJ} \;=\; e\;\Sigma^{ta}_{IJ} \;+\; e^a_{IJ}$ and

$$\begin{aligned} G_{IJ} &= -\,2D_a(\omega)\pi^{a(\eta)}_{IJ} - t^a_{[I}V_{J]a} \quad , \\ H_a &= \pi^b_{IJ}R^{(\eta)IJ}_{ab}(\omega) \;-\; V^I_a D_b(\omega) t^b_I \quad , \\ H &= \frac{2}{\sqrt{q}}\left(\pi^{a(\eta)}_{IK} - e^{a(\eta)}_{IK}\right)\left(\pi^{b(\eta)}_{JL} - e^{b(\eta)}_{JL}\right)\eta^{KL} R_{ab}{}^{IJ}(\omega) - M^I D_a(\omega) t^a_I \quad . \end{aligned} \tag{7}$$

Since there are no velocities associated with the fields $N,\ N^a$ and ω_t^{IJ}, we have the constraints $H \approx 0,\ H_a \approx 0,\ G_{IJ} \approx 0$.

Next, we split the 18 spin-connection fields ω^{IJ}_a as:

$$A^i_a \;\equiv\; \omega_a^{(\eta)0i} = \omega^{0i}_a + \eta\tilde{\omega}^{0i}_a, \qquad K^i_a \;\equiv\; \omega^{0i}_a \quad . \tag{8}$$

The rationale behind such a choice is to make the SU(2) interpretation transparent, as can be understood by noting that A^i_a transforms as connection and K^i_a as adjoint representation under the SU(2) gauge transformations. Also, it is convenient (although not necessary) to work in the time gauge where the boost constraints are solved by the gauge choice $\chi_i = 0$. Thus, in this gauge, the symplectic form becomes:

$$\pi^a_{IJ}\partial_t\omega_a^{(\eta)IJ} \;+\; t^a_I\partial_t V^I_a \;=\; \hat{E}^a_i\partial_t A^i_a \;+\; \hat{F}^a_i\partial_t K^i_a \;+\; t^a_i\partial_t V^i_a \tag{9}$$

with

$$\hat{E}^i_a \;\equiv\; -\,\frac{2}{\eta}\,\tilde{\pi}^{a(\eta)}_{0i} \;\equiv\; -\,\frac{2}{\eta}\,(\tilde{\pi}^a_{0i} - \eta\pi^a_{0i}) \;=\; E^a_i - \frac{2}{\eta}\,\tilde{e}^{a(\eta)}_{0i}(A,K) \tag{10}$$

$$\hat{F}^a_i \;\equiv\; 2\left(\eta+\frac{1}{\eta}\right)\tilde{\pi}^a_{0i} \;=\; 2\left(\eta+\frac{1}{\eta}\right)\tilde{e}^a_{0i}(A,K) \tag{11}$$

Here, the fields V^i_a and its conjugate t^a_i are not independent; they obey the following second-class constraints:

$$V^i_a \;-\; \frac{1}{\sqrt{E}}\,E^i_a \;\equiv\; 0\;, \qquad t^a_i \;-\; \eta\epsilon^{abc}D_b(\omega)V^i_c \;=\; \epsilon^{abc}\left(\eta D_b(A)V^i_c - \epsilon^{ijk}K^j_b V^k_c\right) \tag{12}$$

Similarly, eq.(11) shows that the momenta $\hat{F}^a_i$ obey constraints of the form

$$\chi^a_i \;:=\; \hat{F}^a_i - f(A^j_b, K^k_c) \;\approx\; 0 \tag{13}$$

These imply secondary constraints:

$$[\chi^a_i(x), H(y)] \;\approx\; 0 \;=>\; t^a_i - \left(\frac{1+\eta^2}{\eta^2}\right)\left\{\eta\epsilon^{ijk}D_b(A)\left(\sqrt{E}E^a_jE^b_k\right) + \sqrt{E}E^{[a}_jE^{b]}_iK^j_b\right\} \;\approx\; 0 \tag{14}$$

The solution of (14) can be expressed in the form: $K^i_a - \kappa^i_a(A^j_b, E^c_k) \approx 0$. Since K^i_a and $\hat{F}^a_i$ are canonically conjugate, these constraints evidently form a second-class pair with (13). The constraints (12), (13) and (14), alongwith the constraints $G^{rot}_i \approx 0,\ H_a \approx 0\ ,\ H \approx 0$, completely characterise the canonical theory corresponding to the Lagrangian density (2).

Notice that for $\theta = 0$ and $\phi = 0$, the momenta $\hat{F}^a_i$ in (11) vanishes. This corresponds to the Barbero-Immirzi formulation. Thus, the effect of the addition of Pontryagin and Euler terms in the Lagrangian gets reflected through a richer symplectic structure characterised by a non-vanishing $\hat{F}^a_i$. Also, for non-vanishing θ and ϕ, the canonical conjugate of the connection A^i_a is $\hat{E}^a_i$, and not the densitized triad E^a_i as in the case for $\theta = 0,\ \phi = 0$.

3. SU(2) interpretation

The second-class constraints can all be implemented by using the corresponding Dirac brackets instead of the Poisson brackets. After imposing all the second-class pairs strongly, we are left with a set of seven first class constraints:

$$\begin{aligned} G_i^{rot} &\equiv \eta\, D_a(A)\hat{E}_i^a \;+\; \epsilon^{ijk}K_a^j\hat{F}_k^a \;\approx\; 0 \\ H_a &\equiv \hat{E}_i^b F_{ab}^i(A) \;+\; \hat{F}_i^b D_{[a}(A)K_{b]}^i \;-\; K_a^i D_b(A)\hat{F}_i^b - \eta^{-1}\, G_i^{rot}K_a^i \;\approx\; 0 \\ H &\equiv \frac{\sqrt{E}}{2\eta}\epsilon^{ijk}E_i^aE_j^bF_{ab}^k(A) - \left(\frac{1+\eta^2}{2\eta^2}\right)\sqrt{E}E_i^aE_j^bK_{[a}^iK_{b]}^j + \frac{1}{\eta}\,\partial_a\left(\sqrt{E}G_k^{rot}E_k^a\right) \approx\; 0 \end{aligned} \tag{15}$$

Evaluating the Dirac brackets of the rotation constraints G_i^{rot} with the basic fields, we find that they are the generators of the SU(2) gauge transformations:

$$\begin{aligned} \left[G_i^{rot}(x),\; \hat{E}_j^a(y)\right]_D &= \epsilon^{ijk}\hat{E}_k^a\;\delta^{(3)}(x,y)\;, \\ \left[G_i^{rot}(x),\; A_a^j(y)\right]_D &= -\eta\left(\delta^{ij}\partial_a \;+\eta^{-1}\;\epsilon^{ikj}A_a^k\right)\;\delta^{(3)}(x,y)\;\;. \end{aligned} \tag{16}$$

Thus, we have a SU(2) gauge theory of gravity with all three topological parameters. The Barbero-Immirzi parameter η^{-1} acts as the coupling constant of gauge field A_a^i whereas the other two parameters θ, ϕ enter in the definition of its conjugate $\hat{E}_i^a$. Since the topological densities are all functions of the geometric fields (i.e. tetrads and spin-connections), addition of matter coupling does not affect such a gauge theoretic interpretation of gravity.

4. Concluding remarks

With the Hamiltonian theory with all three topological densities in place, it is important to investigate what are the possible imports of these terms in non-perturbative quantum gravity, i.e., whether these terms imply non-trivial topological sectors and potential instanton effects in the quantum theory[7, 8]. Also, since the Barbero-Immirzi parameter is already known to appear in the area spectrum in Loop Quantum Gravity, there is no reason not to suspect a similar role of the other two parameters in the context of quantum geometry of spacetime.

We emphasize that in the real SU(2) formulation as presented here, the Dirac bracket between the phase space variables A_a^i and $\hat{E}_i^a$ is not a canonical one, unlike the Barbero-Immirzi formulation. Although this need not be an issue as far as the classical theory is concerned, quantization based on these canonical variables is not straightforward. However, as demonstrated in [2], it is possible to find a suitable canonical pair which leads to the standard bracket, thus providing a smooth passage towards the quantum theory.

Acknowledgments

It is a pleasure to thank Fernando Barbero and Madhavan Varadarajan for their critical remarks, Prof. Romesh Kaul for collaboration on this topic and the organisers of Loops-11 for the wonderful hospitality at CSIC, Madrid where this work was presented.

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