Pramāṇa, Vol. 7, No. 6, 1976, pp. 378-384. © Printed in India.

The effect of magnetic fields and boundary conditions on the shear flow of nematics

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MS received 16 March 1976; revised 30 July 1976

Abstract. The flow of a nematic liquid crystal between plane parallel plates, with one plate moving with uniform velocity relative to the other, is discussed. The apparent viscosity, orientation and velocity profiles are computed for *p*-azoxyanisole as functions of shear rate and magnetic field for symmetric and asymmetric molecular alignment at the plates. For symmetric homeotropic boundary condition, a magnetic field applied along the flow direction exhibits a threshold reminiscent of a Freedericksz transition in the hydrostatic case. In general the apparent viscosity for the asymmetric boundary condition is less than that for the symmetric case.

Keywords. Liquid crystal; *p*-azoxyanisole; shear flow.

1. Introduction

In previous papers (Kini and Ranganath 1975, Kini 1976) we discussed in terms of the Ericksen-Leslie theory (Ericksen 1962, Leslie 1968) the effect of magnetic fields and boundary conditions on Poiseuille flow and on Couette flow of nematic liquid crystals. In this paper we consider the problem of shear flow. Leslie (1971) has developed the general theory for the shear flow of nematics in the presence of a magnetic field and has predicted a change in the apparent viscosity when either the shear rate or the magnetic field is varied. Finlayson (1974) has investigated the effect of a magnetic field applied perpendicular to the flow with perpendicular molecular alignment at the walls. However, he has not considered the effect of a magnetic field applied along the flow direction. In the present communication we present detailed calculations for flow between parallel plates, with one plate moving at a constant velocity relative to the other. The apparent viscosity is studied as a function of the shear rate (or equivalently the relative velocity of the plates) as well as of magnetic fields applied along or perpendicular to the flow direction, for symmetric and asymmetric boundary conditions of molecular alignment at the plates. The Ericksen-Leslie equations governing the hydromechanics of liquid crystals can be summarized in the form

$$\rho \dot{v}_i = F_i + t_{ji,j}$$
$$\rho_1 \ddot{n}_i = G_i + g_i + \pi_{ji,j} \tag{1}$$

where v_i is the velocity, n_i the director, ρ the density, ρ_1 the moment of inertia of the director, F_i the external body force per unit volume, G_i the external director body force per unit volume, t_{ji} the stress tensor, π_{ji} the director surface stress and g_i the intrinsic director body force,

$$t_{ji} = -p\delta_{ij} - \frac{\partial W}{\partial n_{k,j}} n_{k,i} + \mu_1 d_{kp} n_k n_p n_i n_j + \mu_2 n_j N_i$$

$$+ \mu_3 n_i N_j + \mu_4 d_{ij} + \mu_5 n_j d_{ik} n_k + \mu_6 n_i d_{jk} n_k,$$

$$\pi_{ji} = \frac{\partial W}{\partial n_{i,j}},$$

$$g_i = -\frac{\partial W}{\partial n_i} + \gamma n_i + \lambda_1 N_i + \lambda_2 d_{ij} n_j$$

with

$$N_i = \dot{n}_i - w_{ik} n_k, \qquad w_{ik} = (v_{i,k} - v_{k,i})/2,$$

$$d_{ij} = (v_{i,j} + v_{j,i})/2, \qquad \lambda_1 = \mu_2 - \mu_3, \qquad \lambda_2 = \mu_5 - \mu_6$$

μ_1 to μ_6 are the viscosity coefficients introduced by Leslie (1968). W is the Frank elastic energy per unit volume

$$2W = K_{22} n_{i,j} n_{i,j} + (K_{11} - K_{22} - K_{24}) n_{i,i} n_{j,i}$$

$$+ (K_{33} - K_{22}) n_i n_j n_{k,i} n_{k,j} + K_{24} n_{i,j} n_{j,i}$$

with K_{11}, K_{22}, K_{33}, K_{24} as the elastic constants of a nematic liquid crystal. p and γ are arbitrary constants arising from constraints that the fluid is incompressible and that the director is of constant magnitude. The notation here differs from that used by Leslie (1971). In his paper Leslie (1971) assumes the nematic to be confined between two plane parallel plates which occupy the planes $y = 0$ and $y = 2h$, $2h$ being the sample thickness. The plate $y = 2h$ is assumed to move along the x direction with a uniform velocity V. A constant magnetic field H is applied in the plane of shear making an angle ψ with the x axis. The external body force acting on the fluid is ignored. Assuming a steady state solution

$$\vec{n} = [\cos\theta(y), \quad \sin\theta(y), \quad 0]$$

and

$$\vec{v} = [v(y), \qquad 0, \qquad 0], \tag{2}$$

Leslie obtains, by solving (1) for steady state, the following equations for the director orientation $\theta(y)$ and the velocity field $v(y)$

$$2f(\theta)\frac{d^2\theta}{dy^2} + \frac{df}{d\theta}\left(\frac{d\theta}{dy}\right)^2 + \frac{c(\lambda_1 + \lambda_2\cos 2\theta)}{g(\theta)}$$

$$+ (\triangle\chi) H^2 \sin(2\psi - 2\theta) = 0 \tag{3}$$

and

$$\frac{dv}{dy} = \frac{c}{g(\theta)} \tag{4}$$

where

$$f(\theta) = K_{11}\cos^2\theta + K_{33}\sin^2\theta$$

$$2g(\theta) = 2\mu_1 \sin^2\theta \cos^2\theta + (\mu_5 - \mu_2)\sin^2\theta + (\mu_6 + \mu_3)\cos^2\theta + \mu_4$$

and

$$t_{xy} = c = \text{a constant.}$$

We assume $\theta(0) = \theta_1$ and $\theta(2h) = \theta_2$ as boundary conditions for $\theta(y)$. For a given value of the shear rate c we integrate the eq. (3) to obtain θ as a function of y. We then integrate eq. (4) (*see* Leslie 1971) to obtain the velocity of the upper plate

$$V = c\int_0^{2h} \frac{dy}{g(\phi(y))}. \tag{5}$$

The apparent viscosity is calculated from the relation

$$\eta_{app} = \frac{2hc}{V}.$$

2. Results

Computations have been made for *p*-azoxyanisole (PAA). The numerical technique used has been seen to work for sample thicknesses up to 1000 μ but we present here only the results for a gap width of 50 μ. The elastic and viscosity coefficients have been assumed to be the same as those used by Tseng *et al* (1972; *see* table 1 of Kini and Ranganath 1975) which have been found to fit the experimental results of Fisher and Fredrickson (1969) very well, and the anisotropy of diamagnetic susceptibility $\triangle\chi$ has been taken to be $0{\cdot}136 \times 10^{-6}$ cgs (Gasparoux and Prost 1971). The equations have been solved by the orthogonal collocation method used by Finlayson (1972); Tseng *et al* (1972); *see* also Villadsen and Stewart (1967) which has an advantage over the finite difference method because it is faster and surer to yield meaningful results for values of $c > 1{\cdot}0$. We have chosen 16 collocation points corresponding to the zeroes of the sixteenth order Legendre polynomial P_{16}, with double precision arithmetic. Calculations have been repeated with 24 collocation points (being the zeroes of P_{24}) for cases involving large deformations to check the accuracy of calculations. η from the two calculations agrees to within 1%. Two types of boundary conditions for θ, which are easily realizable in practice, have been treated.

$\theta_1 = \pi/2 = \theta_2$: In this case the molecules are initially homeotropically aligned in the absence of a magnetic field. At low shear rates, in the absence of a magnetic field, θ remains almost constant at the Miesowicz value $\eta_A = (\mu_4 + \mu_5 - \mu_2)/2 = 0{\cdot}092$ poise for PAA (Miesowicz 1936, 1946). As the shear rate increases η decreases and approaches the Miesowicz value $\eta_{\parallel} = (\mu_3 + \mu_4 + \mu_6)/2 = 0{\cdot}024$ poise for PAA, corresponding to the molecules being oriented along the flow.

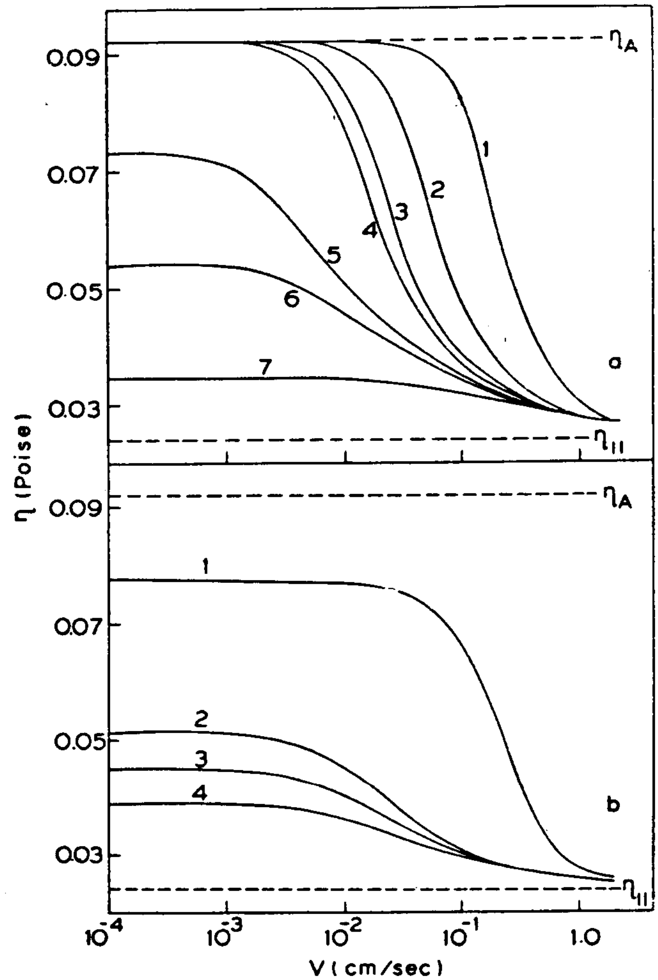

Figure 1. Variation of apparent viscosity with shear rate and magnetic field. (*a*) $\theta_1 = \pi/2 = \theta_2$; magnetic field normal to the plates H_y = (1) 5000 (2), 2000 (3) 0 gauss; magnetic field along the flow H_x = (4) 1000, (5) 1800 (6) 2000 (7) 3000; gauss;

(*b*) $\theta_1 = 0$; $\theta_2 = \pi/2$; H_y = (1) 5000 (2) 1000 (3) 0 gauss, (4) H_x=1000 gauss.

(The parameters of Tseng *et al* for PAA are such that $\cos\theta_0 = -\lambda_1/\lambda_2 = 1$, *i.e.*, the molecules are aligned along the flow direction at high shear rate. It must be admitted however that there is no conclusive experimental evidence that this is the case of PAA.) When a field H_y is applied normal to the plates there is a stabilizing effect and η decreases at first more slowly with the increase of shear rate and finally at large shear rates approaches $\eta_{\parallel}$.

In general η decreases in the presence of a field H_x applied along the flow direction (figure 1 *a*). At low shear rates η remains almost constant at η_A, until H_x attains a value $H_c \approx (\pi/2h)(K_{33}/\Delta\chi)^{\frac{1}{2}}$, which in the static case is the Freedericksz threshold and below which there is no deformation. Above H_c there is a fairly rapid change of η with field and for large H_x, η approaches $\eta_{\parallel}$ (figure 2 *a*). At low shear rates the value of η depends on whether H_x is greater or less than H_c but at high shear rates it approaches $\eta_{\parallel}$ regardless of the magnitude of H. At large shear rates η goes up from about $\eta_{\parallel}$ and attains η_A as H_y increases from a low to a high value (figure 2 *b*).

$\theta_1 = 0$; $\theta_2 = \pi/2$: The variation of η with shear rate and with magnetic fields in this configuration can also be explained in a similar manner. Two points may be emphasized: (*a*) the initial value of η at low shear rates and in the absence of a magnetic field is $0{\cdot}5\ \eta_A$, about half of what it is for $\theta_1 = \theta_2 = \pi/2$, because a deformation is present even in the absence of flow or field; as expected this initial value increases with increasing H_y attaining η_A at large values of H_y; (*b*) no threshold for H_x or H_y should be expected in this geometry.

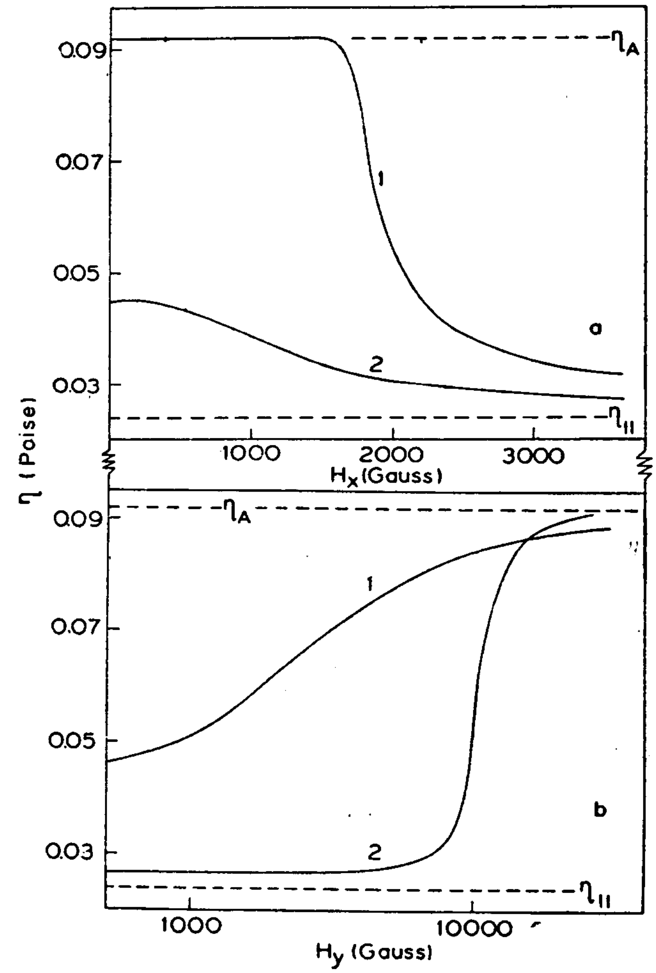

Figure 2. Variation of apparent viscosity with magnetic field.
(*a*) Field H_x along the flow; $c = 10^{-3}$ dynes cm^{-2} (1) $\theta_1 = \pi/2 = \theta_2$ (2) $\theta_1 = 0$. $\theta_2 = \pi/2$.
(*b*) Field H_y normal to the plates
(1) $\theta_1 = 0$; $\theta_2 = \pi/2$; $c = 10^{-3}$ dynes cm^{-2} (2) $\theta_1 = \pi/2 = \theta_2$; $c = 10{\cdot}0$ dynes cm^{-2}

It is quite evident from figures 3 and 4 that a magnetic field H_x applied along the direction of flow tends to decrease the apparent viscosity while a magnetic field H_y applied perpendicular to the flow tends to elevate it. At very high shear rates the θ and velocity profiles can show boundary layers. However in our present calculations we have not gone up to such high values of shear rate.

We have not treated the case in which $\theta_1 = \theta_2 = 0$ (whether with or without magnetic field) with the molecules aligned along the flow direction. The Leslie theory predicts that at high shear rates in the absence of magnetic fields all mole-

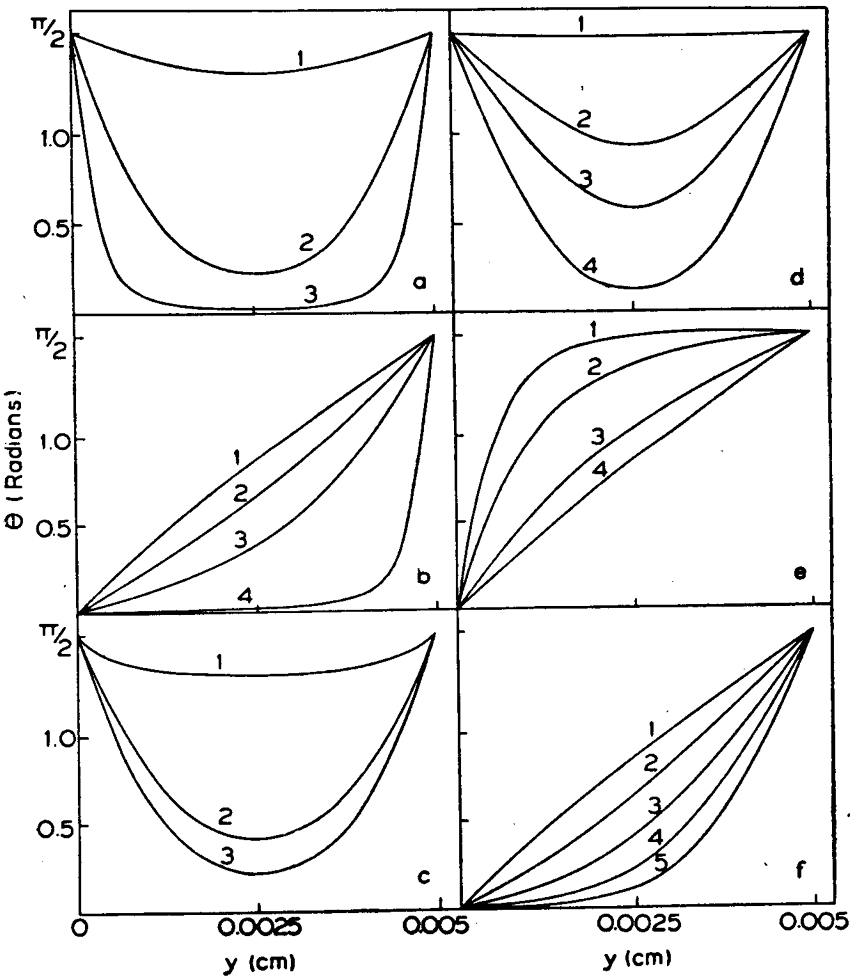

Figure 3. Orientation profiles for different shear rates and magnetic fields.
(a) $H = 0$; $\theta_1 = \pi/2 = \theta_2$; $c =$ (1) 0·1 (2) 1·0 (3) 10·0 dynes cm^{-2}.
(b) $H = 0$; $\theta_1 = 0$; $\theta_2 = \pi/2$; $c =$ (1) 10^{-3} (2) 0·1 (3) $10^{1/2}$ (4) 10·0 dynes cm^{-2},
(c) $c = 10{\cdot}0$ dynes cm^{-2}; $\theta_1 = \pi/2 = \theta_2$; $H_y =$ (1) 5000 (2) 2000 (3) 0 gauss.
(d) $c = 10^{-3}$ dynes cm^{-2}; $\theta_1 = \pi/2 = \theta_2$; $H_x =$ (1) 0 (2) 1800 (3) 2000 (4) 3000 gauss
(e) $c = 10^{-3}$ dynes cm^{-2}; $\theta_1 = 0$; $\theta_2 = \pi/2$; $H_y =$ (1) 5000 (2) 25000 (3) 1000 (4) 0 gauss.
(f) $c = 10^{-3}$ dynes cm^{-2}; $\theta_1 = 0$; $\theta_2 = \pi/2$; $H_x =$ (1) 0 (2) 1000 (3) 1500 (4) 2000 (5) 2500 gauss.

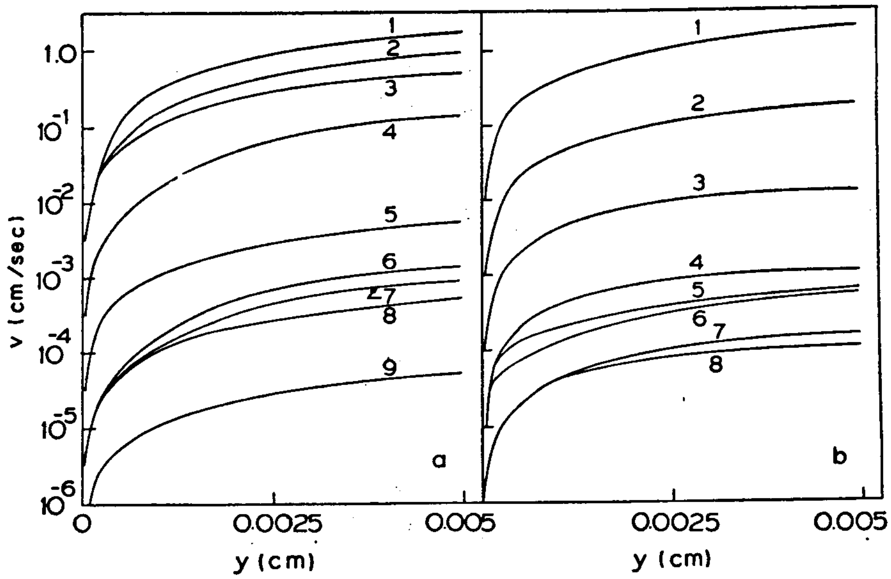

Figure 4. Velocity profiles for different shear rates and magnetic fields.
(*a*) $\theta_1 = \pi/2 = \theta_2$; $c = 10\cdot 0$ dynes cm^{-2}; $H_y =$ (1) 0 (2) 10000 (3) 25000 gauss $H = 0$; $c =$ (4) $1\cdot 0$ (5) $0\cdot 1$ dynes cm^{-2}; $c = 10^{-2}$ dynes cm^{-2}; $H_z =$ (6) 3000 (7) 2000 (8) 0 gauss (9) $c = 10^{-3}$ dynes cm^{-2}; $H = 0$.
(*b*) $\theta_1 = 0$; $\theta_2 = \pi/2$; $H = 0$; $c =$ (1) $10\cdot 0$ (2) $1\cdot 0$ (3) $0\cdot 1$ (4) 10^{-2} dynes cm^{-2}; $c = 10^{-2}$ dynes cm^{-2}; $H_y =$ (5) 5000 (6) 20000 gauss; $c = 10^{-3}$ dynes cm^{-2}; $H_z =$ (7) 2000 (8) 0 gauss.

cules will be aligned at an angle θ_0 given by $\cos 2\theta_0 = -\lambda_1/\lambda_2$. With our data for PAA, $\theta_0 = 0$ so that there will be no change of η with shear rate and the behaviour will be Newtonian.

Acknowledgements

The author thanks S Chandrasekhar and G S Ranganath for discussions. He also thanks a referee for helpful suggestions for the improvement of the manuscript.

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